Appendix A

Fourier Series

Fourier coefficients – Expansion in Fourier series

A.1 Fourier coefficients

Let u be a $2T$-periodic function in $\mathbb{R}$ and assume that u can be expanded in a trigonometric series as follows:

$$u(x) = U + \sum_{k=1}^{\infty} \{a_k \cos k\omega x + b_k \sin k\omega x\} \tag{A.1}$$

where $\omega = \pi/T$.

First question: how u and the coefficients U, a_k and b_k are related to each other? To answer, we use the following so called *orthogonality relations*, whose proof is elementary:

$$\int_{-T}^{T} \cos k\omega x \ \cos m\omega x \ dx = \int_{-T}^{T} \sin k\omega x \ \sin m\omega x \ dx = 0 \qquad \text{if } k \neq m$$

$$\int_{-T}^{T} \cos k\omega x \ \sin m\omega x \ dx = 0 \quad \text{for all } k, m \geq 0.$$

Moreover

$$\int_{-T}^{T} \cos^2 k\omega x \ dx = \int_{-T}^{T} \sin^2 k\omega x \ dx = T. \tag{A.2}$$

Now, suppose that the series (A.1) converges *uniformly* in $\mathbb{R}$. Multiplying (A.1) by $\cos n\omega x$ and integrating term by term over $(-T, T)$, the orhogonality relations and (A.2) yield, for $n \geq 1$,

$$\int_{-T}^{T} u(x) \cos n\omega x \ dx = Ta_n$$

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or

$$a_n = \frac{1}{T}\int_{-T}^{T} u(x)\cos n\omega x \; dx. \tag{A.3}$$

For $n = 0$ we get

$$\int_{-T}^{T} u(x) \; dx = 2UT$$

or, setting $U = a_0/2$,

$$a_0 = \frac{1}{T}\int_{-T}^{T} u(x) \; dx \tag{A.4}$$

which is coherent with (A.3) as $n = 0$.

Similarly, we find

$$b_n = \frac{1}{T}\int_{-T}^{T} u(x)\sin n\omega x \; dx. \tag{A.5}$$

Thus, if u has the uniformly convergent expansion (A.1), the coefficients a_n, b_n (with $a_0 = 2U$) must be given by the formulas (A.3) and (A.5). In this case we say that the trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \{a_k \cos k\omega x + b_k \sin k\omega x\} \tag{A.6}$$

is the *Fourier series of* u and the coefficients (A.3), (A.4) and (A.5) are called the *Fourier coefficients* of u.

- *Odd and even functions.* If u is an *odd* function, i.e. $u(-x) = -u(x)$, we have $a_k = 0$ for every $k \geq 0$, while

$$b_k = \frac{2}{T}\int_{0}^{T} u(x)\sin k\omega x \; dx.$$

Thus, if u is odd, its Fourier series is a *sine* Fouries series:

$$u(x) = \sum_{k=1}^{\infty} b_k \sin k\omega x.$$

Similarly, if u is *even*, i.e. $u(-x) = u(x)$, we have $b_k = 0$ for every $k \geq 1$, while

$$a_k = \frac{2}{T}\int_{0}^{T} u(x)\cos k\omega x \; dx.$$

Thus, if u is even, its Fourier series is a *cosine* Fouries series:

$$u(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega x.$$

• *Fourier coefficients of a derivative.* Let $u \in C^1(\mathbb{R})$ be $2T-$periodic. Then we may compute the Fourier coefficients a_k' and b_k' of u'. We have, integrating by parts, for $k \geq 1$:

$$\begin{aligned} a_k' &= \frac{1}{T}\int_{-T}^{T} u'(x)\cos k\omega x \, dx \\ &= \frac{1}{T}\left[u(x)\cos k\omega x\right]_{-T}^{T} + \frac{k\omega}{T}\int_{-T}^{T} u(x)\sin k\omega x \, dx \\ &= \frac{k\omega}{T}\int_{-T}^{T} u(x)\sin k\omega x \, dx \\ &= k\omega b_k \end{aligned}$$

and

$$\begin{aligned} b_k' &= \frac{1}{T}\int_{-T}^{T} u'(x)\sin k\omega x \, dx \\ &= \frac{1}{T}\left[u(x)\sin k\omega x\right]_{-T}^{T} - \frac{k\omega}{T}\int_{-T}^{T} u(x)\cos k\omega x \, dx \\ &= -\frac{k\omega}{T}\int_{-T}^{T} u(x)\cos k\omega x \, dx \\ &= -k\omega a_k. \end{aligned}$$

Thus, the Fourier coefficients a_k' and b_k' are related to a_k and b_k by the following formulas:

$$a_k' = k\omega b_k, \quad b_k' = -k\omega a_k. \tag{A.7}$$

• *Complex form of a Fourier series.* Using the Euler identities

$$e^{\pm ik\omega x} = \cos k\omega x \pm i \sin k\omega x$$

the Fourier series (A.6) can be expressed in the complex form

$$\sum_{k=-\infty}^{\infty} c_k e^{ik\omega x},$$

where the complex Fourier coefficients c_k are given by

$$c_k = \frac{1}{2T}\int_{-T}^{T} u(z)\, e^{-ik\omega z} dz.$$

The relations among the real and the complex Fourier coefficients are:

$$c_0 = \frac{1}{2}a_0$$

and

$$c_k = \frac{1}{2}(a_k - b_k), \quad c_{-k} = \bar{c}_k \qquad \text{for } k > 0.$$

A.2 Expansion in Fourier series

In the above computations we started from a function u admitting a uniform convergent expansion in Fourier series. Adopting a different point of view, let u be a $2T-$periodic function and assume we can compute its Fourier coefficients, given by formulas (A.3) and (A.5). Thus, we can *associate* with u its Fourier series and write

$$u \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \{a_k \cos k\omega x + b_k \sin k\omega x\}.$$

The main questions are now the following:

1. Which conditions on u do assure "the convergence" of its Fourier series? Of course there are several notions of convergence (e.g pointwise, uniform, least squares).

2. If the Fourier series is convergent in some sense, does it always have sum u?

A complete answer to the above questions is not elementary. The convergence of a Fourier series is a rather delicate matter. We indicate some basic results (for the proofs, see e.g. *Rudin,* 1964 and 1974, *Royden,* 1988, or *Zygmund and Wheeden,* 1977.

• *Least squares or L^2 convergence.* This is perhaps the most natural type of convergence for Fourier series (see subsection 6.4.2). Let

$$S_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} \{a_k \cos k\omega x + b_k \sin k\omega x\}$$

be the $N-$partial sum of the Fourier series of u. We have

Theorem A.1 *Let u be a square integrable function*[1] *on $(-T,T)$. Then*

$$\lim_{N\to+\infty} \int_{-T}^{T} [S_N(x) - u(x)]^2 \, dx = 0.$$

Moreover, the following Parseval relation holds:

$$\frac{1}{T}\int_{-T}^{T} u^2 = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2\right). \tag{A.8}$$

Since the numerical series in the right hand side of (A.8) is convergent, we deduce the following important consequence:

Corollary A.1 (Riemann-Lebesgue).

$$\lim_{k\to+\infty} a_k = \lim_{k\to+\infty} b_k = 0$$

• *Pointwise convergence.* We say that u satisfies the *Dirichlet conditions* in $[-T,T]$ if it is continuous in $[-T,T]$ except possibly at a finite number of points

[1] That is $\int_{-T}^{T} u^2 < \infty$.

of jump discontinuity and moreover if the interval $[-T, T]$ can be partitioned in a finite numbers of subintervals such that u is monotone in each one of them.

The following theorem holds.

Theorem A.2. *If u satisfies the Dirichlet conditions in $[-T, T]$ then the Fourier series of u converges at each point of $[-T, T]$. Moreover*[2]:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \{a_k \cos k\omega x + b_k \sin k\omega x\} = \begin{cases} \dfrac{u(x+) + u(x-)}{2} & x \in (-T, T) \\ \dfrac{u(T-) + u(-T+)}{2} & x = \pm T \end{cases}$$

In particular, under the hypotheses of Theorem A.2, at every point x of continuity of u the Fourier series converges to $u(x)$.

• *Uniform convergence.* A simple criterion of uniform convergence is provided by the Weierstrass test (see Section 1.4). Since

$$|a_k \cos k\omega x + b_k \sin k\omega x| \leq |a_k| + |b_k|$$

we deduce: *If the numerical series*

$$\sum_{k=1}^{\infty} |a_k| \quad \text{and} \quad \sum_{k=1}^{\infty} |b_k|$$

are convergent, then the Fourier series of u is uniformly convergent in $\mathbb{R}$, with sum u.

This is the case, for instance, if $u \in C^1(\mathbb{R})$ and is $2T$ periodic. In fact, from (A.7) we have for every $k \geq 1$,

$$a_k = -\frac{1}{\omega k} b'_k \quad \text{and} \quad b_k = \frac{1}{\omega k} a'_k.$$

Therefore

$$|a_k| \leq \frac{1}{\omega k^2} + (b'_k)^2$$

and

$$|b_k| \leq \frac{1}{\omega k^2} + (a'_k)^2.$$

Now, the series $\sum \frac{1}{k^2}$ is convergent. On the other hand, also the series

$$\sum_{k=1}^{\infty} (a'_k)^2 \quad \text{and} \quad \sum_{k=1}^{\infty} (b'_k)^2$$

are convergent, by Parseval's relation (A.8) applied to u' in place of u. The conclusion is that *if $u \in C^1(\mathbb{R})$ and $2T$ periodic, its Fourier series is uniformly convergent in $\mathbb{R}$ with sum u.*

[2] We set $f(x\pm) = \lim_{y \to \pm x} f(y)$.

Another useful result is a refinement of Theorem A.2.

Theorem A.3 *Assume u satisfies the Dirichlet conditions in $[-T, T]$. Then:*

a) If u is continuous in $[a, b] \subset (-T, T)$, then its Fourier series converges uniformly in $[a, b]$.

b) If u is continuous in $[-T, T]$ and $u(-T) = u(T)$, then its Fourier series converges uniformly in $[-T, T]$ (and therefore in $\mathbb{R}$).

Appendix B

Measures and Integrals

Lebesgue Measure and Integral

B.1 Lebesgue Measure and Integral

B.1.1 A counting problem

Two persons, that we denote by $\mathcal{R}$ and $\mathcal{L}$, must compute the total value of M coins, ranging from 1 to 50 cents. $\mathcal{R}$ decides to group the coins arbitrarily in piles of, say, 10 coins each, then to compute the value of each pile and finally to sum all these values. $\mathcal{L}$, instead, decides to partition the coins according to their value, forming piles of 1-cent coins, of 5-cents coins and so on. Then he computes the value of each pile and finally sums all their values.

In more analytical terms, let

$$V : M \to \mathbb{N}$$

a *value function* that associates to each element of M (i.e. each coin) its value. $\mathcal{R}$ partitions the **domain** of V in disjoint subsets, sums the values of V in such subsets and then sums everything. $\mathcal{L}$ considers each point p in the **image** of V (the value of a single coin), considers the inverse image $V^{-1}(p)$ (the pile of coins with the same value p), computes the corresponding value and finally sums over every p.

These two ways of counting correspond to the strategy behind the definitions of the integrals of Riemann and Lebesgue, respectively. Since V is defined on a discrete set and is integer valued, in both cases there is no problem in summing its values and the choice is determined by an efficiency criterion. Usually, the method of $\mathcal{L}$ is more efficient.

In the case of a real (or complex) function f, the "sums of its values" corresponds to an integration of f. While the construction of $\mathcal{R}$ remains rather elementary, the one of $\mathcal{L}$ requires new tools.

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Let us examine the particular case of a *bounded* and *positive* function, defined on an interval $[a, b] \subset \mathbb{R}$. Thus, let

$$f : [a, b] \to [\inf f, \sup f] .$$

To construct the Riemann integral, we partition $[a, b]$ in subintervals $I_1, ..., I_N$ (the piles of $\mathcal{R}$), then we choose in each interval I_k a point ξ_k and we compute $f(\xi_k) \, l(I_k)$, where $l(I_k)$ is the length of I_k, (i.e. the value of the $k-th$ pile). Now we sum the values $f(\xi_k) \, l(I_k)$ and set

$$(\mathcal{R}) \int_a^b f = \lim_{\delta \to 0} \sum_{k=1}^{N} f(\xi_k) \, l(I_k),$$

where $\delta = \max \{l(I_1), ..., l(I_N)\}$. If the limit is finite and moreover is independent of the choice of the points ξ_k, then this limit defines the Riemann integral of f in $[a, b]$.

Now, let us examine the Lebesgue strategy. This time we partition the interval $[\inf f, \sup f]$ in subintervals $[y_{k-1}, y_k]$ (the values of each coin for $\mathcal{L}$) with

$$\inf f = y_0 < y_1 < ... < y_{N-1} < y_N = \sup f.$$

Then we consider the inverse images $E_k = f^{-1}([y_{k-1}, y_k])$ (the piles of homogeneous coins) and we would like to compute their .... *length*. However, in general E_k is *not* an interval or a union of intervals and, in principle, it could be a very irregular set so that it is not clear what is the "length" of E_k.

Thus, the need arises to associate with every E_k a *measure,* which replaces the length when E_k is an irregular set. This leads to the introduction of the *Lebesgue measure* of (practically every) set $E \subseteq \mathbb{R}$, denoted by $|E|$.

Once we know how to measure E_k (the number of coins in the $k-th$ pile), we choose an arbitrary point $\overline{\alpha}_k \in [y_{k-1}, y_k]$ and we compute $\overline{\alpha}_k |E_k|$ (the value of the $k-th$ pile). Then, we sum all the values $\overline{\alpha}_k |E_k|$ and set

$$(L) \int_a^b f = \lim_{\rho \to 0} \sum_{k=1}^{N} \overline{\alpha}_k |E_k| .$$

where ρ is the maximum among the lengths of the intervals $[y_{k-1}, y_k]$. It can be seen that under our hypotheses, the limit exists, is finite and is independent of the choice of $\overline{\alpha}_k$. Thus, we may always choose $\overline{\alpha}_k = y_{k-1}$. This remark leads to the definition of the Lebesgue integral in subsection B.3: the number $\sum_{k=1}^{N} y_{k-1} |E_k|$ is nothing else that the integral of a *simple function*, which approximates f from below and whose range is the finite set $y_0 < ... < y_{N-1}$. The integral of f is the supremum of these numbers.

The resulting theory has several advantages with respect to that of Riemann. For instance, the class of integrable functions is much wider and there is no need to distinguish among bounded or unbounded functions or integration domains.

Especially important are the convergence theorems presented in subsection B.1.4, which allow the possibility of interchanging the operation of limit and integration, under rather mild conditions.

Finally, the construction of the Lebesgue measure and integral can be greatly generalized as we will mention in subsection B.1.5.

For the proofs of the theorems stated in this Appendix, the interested reader can consult *Rudin,* 1964 and 1974, *Royden,* 1988, or *Zygmund and Wheeden*, 1977.

B.1.2 Measures and measurable functions

A measure in a set Ω is a *set function*, defined on a particular class of subsets of Ω called *measurable set* which "behaves well" with respect to union, intersection and complementation. Precisely:

Definition B.1 *A collection $\mathcal{F}$ of subsets of Ω is called $\sigma-$algebra if:*

(i) $\varnothing, \Omega \in \mathcal{F}$;

(ii) $A \in \mathcal{F}$ *implies* $\Omega \backslash A \in \mathcal{F}$;

(iii) if $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ *then also* $\cup A_k$ *and* $\cap A_k$ *belong to* $\mathcal{F}$.

Example B.1. If $\Omega = \mathbb{R}^n$, we can define the smallest $\sigma-$algebra containing all the open subsets of $\mathbb{R}^n$, called the *Borel $\sigma-$algebra*. Its elements are called *Borel sets*, typically obtained by countable unions and/or intersections of open sets.

Definition B.2 *Given a $\sigma-$algebra $\mathcal{F}$ in a set Ω, a measure on $\mathcal{F}$ is a function*

$$\mu : \mathcal{F} \to \mathbb{R}$$

such that:

(i) $\mu(A) \geq 0$ *for every* $A \in \mathcal{F}$;

(ii) if $A_1, A_2, ...$ *are pairwise disjoint sets in* $\mathcal{F}$, *then*

$$\mu\left(\cup_{k\geq 1} A_k\right) = \sum_{k\geq 1} \mu(A_k) \qquad (\sigma - \textit{additivity}).$$

The elements of $\mathcal{F}$ are called measurable sets.

The Lebesgue measure in $\mathbb{R}^n$ is defined on a $\sigma-$*algebra* $\mathcal{M}$ containing the Borel $\sigma-$algebra, through the following theorem.

Theorem B.1 *There exists in $\mathbb{R}^n$ a $\sigma-$algebra $\mathcal{M}$ and a measure*

$$|\cdot|_n : \mathcal{M} \to [0, +\infty]$$

with the following properties:

1. *Each open and closed set belongs to $\mathcal{M}$.*
2. *If $A \in \mathcal{M}$ and A has measure zero, every subset of A belongs to $\mathcal{M}$ and has measure zero.*

3. *If*
$$A = \{\mathbf{x} \in \mathbb{R}^n : a_j < x_j < b_j;\ j = 1, ..., n\}$$
then $|A| = \prod_{j=1}^{n} (b_j - a_j)$.

The elements of $\mathcal{M}$ are called *Lebesgue measurable sets* and $|\cdot|_n$ (or simply $|\cdot|$ if no confusion arises) is called the *n−dimensional Lebesgue measure.* Unless explicitly said, from now on, *measurable* means *Lebesgue measurable* and the measure is the Lebesgue measure.

Not every subset of $\mathbb{R}^n$ is measurable. However, the nonmeasurable ones are quite ... pathological[1] !

The sets of measure zero are quite important. Here are some examples: all countable sets, e.g. the set $\mathbb{Q}$ of rational numbers; straight lines or smooth curves in $\mathbb{R}^2$; straight lines, hyperplanes, smooth curves and surfaces in $\mathbb{R}^3$.

Notice that a straight line segment has measure zero in $\mathbb{R}^2$ but, of course not in $\mathbb{R}$.

We say that a *property holds almost everywhere in* $A \in \mathcal{M}$ (in short, a.e. in A) *if it holds at every point of* A *except that in a subset of measure zero.*

For instance, the sequence $f_k(x) = \exp\left(-n \sin^2 x\right)$ converges to zero a.e. in $\mathbb{R}$, a Lipschitz function is differentiable a.e. in its domain (Rademacher's Theorem 1.1).

The Lebesgue integral is defined for *measurable* functions, characterized by the fact that the inverse image of every closed set is measurable.

Definition B.3 *Let* $A \subseteq \mathbb{R}^n$ *be measurable, and* $f : A \to \mathbb{R}$. *We say that* f *is measurable if*
$$f^{-1}(C) \in \mathcal{F}$$
for any closed set $C \subseteq \mathbb{R}$.

If f is continuous, is measurable. The sum and the product of a finite number of measurable functions is measurable. The pointwise limit of a sequence of measurable functions is measurable.

If $f : A \to \mathbb{R}$, is measurable, we define its *essential supremum* or *least upper bound* by the formula:
$$\operatorname{ess\,sup} f = \inf \{K : f \leq K \quad \text{a.e. in } A\}.$$
Note that, if $f = \chi_{\mathbb{Q}}$, the characteristic functions of the rational numbers, we have $\sup f = 1$, but $\operatorname{ess\,sup} f = 0$, since $|\mathbb{Q}| = 0$.

Every measurable function may be approximated by **simple functions**.A function $s : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be **simple** if its range is constituted by a *finite number* of values $s_1, ..., s_N$, attained respectively on measurable sets $A_1, ..., A_N$, contained in A. Introducing the characteristic functions χ_{A_j}, we may write
$$s = \sum_{j=1}^{N} s_j \chi_{A_j}.$$

[1] See e.g. *Rudin,* 1974.

We have:

Theorem B.2. *Let $f : A \to \mathbb{R}$, be measurable. There exists a sequence $\{s_k\}$ of simple functions converging pointwise to f in A. Moreover, if $f \geq 0$, we may choose $\{s_k\}$ increasing.*

B.1.3 The Lebesgue integral

We define the Lebesgue integral of a measurable function on a measurable set A. For a simple function $s = \sum_{j=1}^{N} s_j \chi_{A_j}$ we set:

$$\int_A s = \sum_{j=1}^{N} s_j \, |A_j|$$

with the convenction that, if $s_j = 0$ and $|A_j| = +\infty$, then $s_j \, |A_j| = 0$.

If $f \geq 0$ is measurable, we define

$$\int_A f = \sup \int_A s$$

where the supremum is computed over the set of all simple functions s such that $s \leq f$ in A.

In general, if f is measurable, we write $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ are the positive and negative parts of f, respectively. Then we set:

$$\int_A f = \int_A f^+ - \int_A f^-$$

under the condition that at least one of the two integrals in the right hand side is finite.

If both these integrals are finite, the function f is said to be **integrable** or **summable** in A. From the definition, it follows immediately that a measurable functions f is *integrable if and only if* $|f|$ *is integrable.*

All the functions Riemann integrable in a set A are Lebesgue integrable as well. An interesting example of non integrable function in $(0, +\infty)$ is given by $h(x) = \sin x / x$. In fact[2]

$$\int_0^{+\infty} \frac{|\sin x|}{x} dx = +\infty.$$

On the contrary, it may be proved that

$$\lim_{N \to +\infty} \int_0^N \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

and therefore the improper Riemann integral of h is finite.

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[2] We may write

$$\int_0^{+\infty} \frac{|\sin x|}{x} dx = \sum_{k=1}^{\infty} \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x} dx \geq \sum_{k=1}^{\infty} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x| \, dx = \sum_{k=1}^{\infty} \frac{2}{k\pi} = +\infty.$$

The set of the integrable functions in A is denoted by $L^1(A)$. If we identify two functions when they agree a.e. in A, $L^1(A)$ becomes a Banach space with the norm[3]

$$\|f\|_{L^1(A)} = \int_A |f|.$$

We denote by $L^1_{loc}(A)$ the set of *locally summable functions*, i.e. of the functions which are summable in every compact subset of A.

B.1.4 Some fundamental theorems

The following theorems are among the most important and useful in the theory of integration.

Theorem B.3 (Dominated Convergence Theorem). *Let $\{f_k\}$ be a sequence of summable functions in A such that $f_k \to f$ a.e.in A. If there exists $g \geq 0$, summable in A and such that $|f_k| \leq g$ a.e. in A, then f is summable and*

$$\|f_k - f\|_{L^1(A)} \to 0 \quad \text{as } k \to +\infty.$$

In particular

$$\lim_{k\to\infty} \int_A f_k = \int_A f.$$

Theorem B.4 *Let $\{f_k\}$ be a sequence of summable functions in A such that $\|f_k - f\|_{L^1(A)} \to 0$ as $k \to +\infty$. Then there exists a subsequence $\{f_{k_j}\}$ such that $f_{k_j} \to f$ a.e. as $j \to +\infty$.*

Theorem B.5 (Monotone Convergence Theorem). *Let $\{f_k\}$ be a sequence of nonnegative, measurable functions in A such that*

$$f_1 \leq f_2 \leq ... \leq f_k \leq f_{k+1} \leq \cdots.$$

Then

$$\lim_{k\to\infty} \int_A f_k = \int_A \lim_{k\to\infty} f_k.$$

Example B.2. A typical situation we often encounter in this book is the following. Let $f \in L^1(A)$ and, for $\varepsilon > 0$, set $A_\varepsilon = \{|f| > \varepsilon\}$. Then, we have

$$\int_{A_\varepsilon} f \to \int_A f \qquad \text{as } \varepsilon \to 0.$$

This follows from Theorem B.4 since, for every sequence $\varepsilon_j \to 0$, we have $|f| \chi_{A_{\varepsilon_j}} \leq |f|$ and therefore

$$\int_{A_{\varepsilon_j}} f = \int_A f\chi_{A_{\varepsilon_j}} \to \int_A f \qquad \text{as } \varepsilon \to 0.$$

[3] See Chapter 6.

Let $C_0(A)$ be the set of continuous functions in A, compactly supported in A. An important fact is that any summable function may be approximated by a function in $C_0(A)$.

Theorem B.6. *Let $f \in L^1(A)$. Then, for every $\delta > 0$, there exists a continuous function $g \in C_0(A)$ such that*

$$\|f - g\|_{L^1(A)} < \delta.$$

The fundamental theorem of calculus extends to the Lebesgue integral in the following form:

Theorem B.7. (Differentiation). *Let $f \in L^1_{loc}(\mathbb{R})$. Then*

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad \text{a.e. } x \in \mathbb{R}.$$

Finally, the integral of a summable function can be computed via iterated integrals in any order. Precisely, let

$$I_1 = \{\mathbf{x} \in \mathbb{R}^n : -\infty \leq a_i < x_i < b_i \leq \infty;\ i = 1, ..., n\}$$

and

$$I_2 = \{\mathbf{y} \in \mathbb{R}^m : -\infty \leq a_j < y_j < b_j \leq \infty;\ j = 1, ..., m\}.$$

Theorem B.8 (Fubini). *Let f be summable in $I = I_1 \times I_2 \subset \mathbb{R}^n \times \mathbb{R}^m$. Then*

1. *$f(\mathbf{x}, \cdot) \in L^1(I_2)$ for a.e. $\mathbf{x} \in I_1$, and $f(\cdot, \mathbf{y}) \in L^1(I_1)$ for a.e. $\mathbf{y} \in I_2$,*
2. *$\int_{I_2} f(\cdot, \mathbf{y})\,d\mathbf{y} \in L^1(I_1)$ and $\int_{I_1} f(\mathbf{x}, \cdot)\,d\mathbf{x} \in L^1(I_2)$,*
3. *the following formulas hold:*

$$\int_I f(\mathbf{x}, \mathbf{y})\,d\mathbf{x}d\mathbf{y} = \int_{I_1} d\mathbf{x} \int_{I_2} f(\mathbf{x}, \mathbf{y})\,d\mathbf{y} = \int_{I_2} d\mathbf{x} \int_{I_1} f(\mathbf{x}, \mathbf{y})d\mathbf{y}.$$

B.1.5 Probability spaces, random variables and their integrals

Let $\mathcal{F}$ be a $\sigma-$*algebra* in a set Ω. A *probability measure P on $\mathcal{F}$* is a measure in the sense of definition B.2, such that $P(\Omega) = 1$ and

$$P : \mathcal{F} \to [0, 1].$$

The triplet $(\Omega, \mathcal{F}, P)$ is called a *probability space*. In this setting, the elements ω of Ω are *sample points*, while a set $A \in \mathcal{F}$ has to be interpreted as an *event*. $P(A)$ is the probability of (occurrence of) A.

A typical example is given by the triplet

$$\Omega = [0, 1],\ \mathcal{F} = \mathcal{M} \cap [0, 1],\ P(A) = |A|$$

which models a *uniform random choice* of a point in $[0, 1]$.

A 1−*dimensional random variable in* $(\Omega, \mathcal{F}, P)$ is a function

$$X : \Omega \to \mathbb{R}$$

such that X is $\mathcal{F}$−*measurable,* that is

$$X^{-1}(C) \in \mathcal{F}$$

for each closed set $C \subseteq \mathbb{R}$.

Example B.3. The *number k of steps to the right after N steps* in the random walk of Section 2.4 is a random variable. Here Ω is the set of walks of N steps.

By the same procedure used to define the Lebesgue integral we can define the integral of a random variable with respect to a probability measure. We sketch the main steps.

If X is *simple*, i.e. $X = \sum_{j=1}^{N} s_j \chi_{A_j}$, we define

$$\int_\Omega X \, dP = \sum_{j=1}^{N} s_j P(A_j).$$

If $X \geq 0$ we set

$$\int_\Omega X \, dP = \sup \left\{ \int_\Omega Y \, dP : Y \leq X, \, Y \text{ simple} \right\}.$$

Finally, if $X = X^+ - X^-$ we define

$$\int_\Omega X \, dP = \int_\Omega X^+ \, dP - \int_\Omega X^- \, dP$$

provided at least one of the integral on the right hand side is finite.

In particular, if

$$\int_\Omega |X| \; dP < \infty,$$

then

$$E(X) = \langle X \rangle = \int_\Omega X \, dP$$

is called *the expected value* (*or mean value* or *expectation*) of X, while

$$\mathrm{Var}(X) = \int_\Omega (X - E(X))^2 \; dP$$

is called the *variance of* X.

Analogous definitions can be given componentwise for n−*dimensional random variables*

$$\mathbf{X} : \Omega \to \mathbb{R}^n.$$

Appendix C

Identities and Formulas

Gradient, Divergence, Curl, Laplacian – Formulas

C.1 Gradient, Divergence, Curl, Laplacian

Let $\mathbf{F}$ be a smooth vector field and f a smooth real function, in $\mathbb{R}^3$.

Orthogonal cartesian coordinates

1. *gradient*:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

2. *divergence* ($\mathbf{F} = F_1\mathbf{i} + F_1\mathbf{j} + F_3\mathbf{k}$):

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3$$

3. *laplacian*:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

4. *curl*:

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Cylindrical coordinates

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z \qquad (r > 0,\ 0 \le \theta \le 2\pi)$$

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \quad \mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}, \quad \mathbf{e}_z = \mathbf{k}.$$

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1. *gradient*:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z$$

2. *divergence* ($\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{k}$):

$$\text{div }\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial}{\partial \theta}F_\theta + \frac{\partial}{\partial z}F_z$$

3. *laplacian*:

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

4. *curl*:

$$\text{curl }\mathbf{F} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ F_r & rF_\theta & F_z \end{vmatrix}$$

Spherical coordinates

$$x = r\cos\theta\sin\psi,\ y = r\sin\theta\sin\psi,\ z = r\cos\psi \qquad (r > 0,\ 0 \le \theta \le 2\pi,\ 0 \le \psi \le \pi)$$

$$\begin{aligned} \mathbf{e}_r &= \cos\theta\sin\psi\mathbf{i} + \sin\theta\sin\psi\mathbf{j} + \cos\psi\mathbf{k} \\ \mathbf{e}_\theta &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \\ \mathbf{e}_\psi &= \cos\theta\cos\psi\mathbf{i} + \sin\theta\cos\psi\mathbf{j} - \sin\psi\mathbf{k}. \end{aligned}$$

1. *gradient*:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r\sin\psi}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r}\frac{\partial f}{\partial \psi}\mathbf{e}_\psi$$

2. *divergence* ($\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\psi\mathbf{e}_\psi$):

$$\text{div }\mathbf{F} = \underbrace{\frac{\partial}{\partial r}F_r + \frac{2}{r}F_r}_{\text{radial part}} + \underbrace{\frac{1}{r}\left[\frac{1}{\sin\psi}\frac{\partial}{\partial \theta}F_\theta + \frac{\partial}{\partial \psi}F_\psi + \cot\psi F_\psi\right]}_{\text{spherical part}}$$

3. *laplacian*:

$$\Delta f = \underbrace{\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}}_{\text{radial part}} + \frac{1}{r^2}\underbrace{\left\{\frac{1}{(\sin\psi)^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \psi^2} + \cot\psi\frac{\partial f}{\partial \psi}\right\}}_{\text{spherical part (Laplace-Beltrami operator)}}$$

4. *curl*:

$$\text{rot }\mathbf{F} = \frac{1}{r^2\sin\psi}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\psi & r\sin\psi\mathbf{e}_\theta \\ \partial_r & \partial_\psi & \partial_\theta \\ F_r & rF_\psi & r\sin\psi F_z \end{vmatrix}.$$

C.2 Formulas

Gauss' formulas

In $\mathbb{R}^n$, $n \geq 2$, let:

- Ω be a bounded smooth domain and and $\boldsymbol{\nu}$ the outward unit normal on $\partial\Omega$;
- $\mathbf{u}, \mathbf{v}$ be vector fields of class $C^1\left(\overline{\Omega}\right)$;
- φ, ψ be real functions of class $C^1\left(\overline{\Omega}\right)$;
- $d\sigma$ be the area element on $\partial\Omega$.

1. $\int_\Omega \text{div}\, \mathbf{u}\; d\mathbf{x} = \int_{\partial\Omega} \mathbf{u} \cdot \boldsymbol{\nu}\; d\sigma$ (Divergence Theorem)
2. $\int_\Omega \nabla\varphi\; d\mathbf{x} = \int_{\partial\Omega} \varphi\boldsymbol{\nu}\; d\sigma$
3. $\int_\Omega \Delta\varphi\; d\mathbf{x} = \int_{\partial\Omega} \nabla\varphi \cdot \boldsymbol{\nu}\; d\sigma = \int_{\partial\Omega} \partial_\nu\varphi\; d\sigma$
4. $\int_\Omega \psi\; \text{div}\mathbf{F}\; d\mathbf{x} = \int_{\partial\Omega} \psi\mathbf{F} \cdot \boldsymbol{\nu}\; d\sigma - \int_\Omega \nabla\psi \cdot \mathbf{F}\; d\mathbf{x}$ (Integration by parts)
5. $\int_\Omega \psi\Delta\varphi\; d\mathbf{x} = \int_{\partial\Omega} \psi\partial_\nu\varphi\; d\sigma - \int_\Omega \nabla\varphi \cdot \nabla\psi\; d\mathbf{x}$ (Green's identity I)
6. $\int_\Omega (\psi\Delta\varphi - \varphi\Delta\psi)\; d\mathbf{x} = \int_{\partial\Omega} (\psi\partial_\nu\varphi - \varphi\partial_\nu\psi)\; d\sigma$ (Green's identity II)
7. $\int_\Omega \text{curl}\, \mathbf{u}\; d\mathbf{x} = -\int_{\partial\Omega} \mathbf{u} \times \boldsymbol{\nu}\; d\sigma$
8. $\int_\Omega \mathbf{u} \cdot \text{curl}\, \mathbf{v}\; d\mathbf{x} = \int_{\boldsymbol{\Omega}} \mathbf{v} \cdot \text{curl}\, \mathbf{u}\; \mathbf{dx} - \int_{\partial\Omega} (\mathbf{u} \times \mathbf{v}) \cdot \boldsymbol{\nu}\; d\sigma.$

Identities

1. $\text{div curl}\, \mathbf{u} = 0$
2. $\text{curl}\, \nabla\varphi = \mathbf{0}$
3. $\text{div}\, (\varphi\mathbf{u}) = \varphi\, \text{div}\, \mathbf{u} + \nabla\varphi \cdot \mathbf{u}$
4. $\text{curl}\, (\varphi\mathbf{u}) = \varphi\, \text{curl}\, \mathbf{u} + \nabla\varphi \times \mathbf{u}$
5. $\text{curl}\, (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\, \mathbf{u} - (\mathbf{u} \cdot \nabla)\, \mathbf{v} + (\text{div}\, \mathbf{v})\, \mathbf{u} - (\text{div}\, \mathbf{u})\, \mathbf{v}$
6. $\text{div}\, (\mathbf{u} \times \mathbf{v}) = \text{curl}\mathbf{u} \cdot \mathbf{v} - \text{curl}\mathbf{v} \cdot \mathbf{u}$
7. $\nabla\, (\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times \text{curl}\, \mathbf{v} + \mathbf{v} \times \text{curl}\, \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{u}$
8. $(\mathbf{u} \cdot \nabla)\, \mathbf{u} = \text{curl}\mathbf{u} \times \mathbf{u} + \frac{1}{2}\nabla\, |\mathbf{u}|^2$
9. $\text{curl curl}\, \mathbf{u} = \nabla(\text{div}\, \mathbf{u}) - \Delta\mathbf{u}.$

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