Appendix A
Backgrounds on Related Topics

A.1 Symmetric Functions

This section presents minimum materials about symmetric functions used in this monograph. For a comprehensive discussion, see Macdonald's book [1] and Chap. 7 of Stanley's book [2].

Consider the ring $\mathbb{Z}[x_1, ..., x_k]$ of polynomials in independent variables $x_1, ..., x_k$ with rational integer coefficients. The symmetric group S_n acts on this ring by permutating the variables, and a polynomial is *symmetric* if it is invariant under this action. The symmetric polynomial form a subring

$$\Lambda_k = \mathbb{Z}[x_1, ..., x_k]^{S_k},$$

where Λ_k is a graded ring: we have $\Lambda_k = \oplus_{n \geq 0} \Lambda_k^n$, where Λ_k^n consists of the homogeneous symmetric polynomials of degree n, together with the zero polynomial. Let λ be a partition of length $l(\lambda) \leq k$. The polynomial

$$m_\lambda(x_1, ..., x_k) := \sum_\sigma \prod_{i=1}^k x_i^{\sigma_i}$$

summed over all distinct permutations σ of $\lambda = (\lambda_1, ..., \lambda_k)$ is called *monomial symmetric function*. The monomial symmetric functions such that $l(\lambda) \leq k$ and $|\lambda| = n$ form a basis of Λ_k^n. For example,

$$\begin{aligned} m_{(2,1)} &= x_1^2 x_2 + x_1^2 x_3 + x_1^2 x_4 + x_1 x_2^2 + x_2^2 x_3 + x_2^2 x_4 \\ &\quad + x_1 x_3^2 + x_2 x_3^2 + x_3^2 x_4 + x_1 x_4^2 + x_2 x_4^2 + x_3 x_4^2 \in \Lambda_4^3. \end{aligned}$$

For each $r \geq 1$ the r-th power sum is

S. Mano, *Partitions, Hypergeometric Systems, and Dirichlet Processes in Statistics*,
JSS Research Series in Statistics, https://doi.org/10.1007/978-4-431-55888-0

$$p_r := m_{(r)} = \sum_{i=1}^{k} x_i^r.$$

The *power sum symmetric function* is defined as

$$p_\lambda := p_{\lambda_1} \cdots p_{\lambda_{l(\lambda)}} \in \Lambda_Q := \mathbb{Q}[p_1, p_2, ...].$$

Section I.4 of [1] discusses orthogonality among symmetric functions. The Schur symmetric function is defined as

$$s_\lambda(x) := \frac{\det(x_i^{\lambda_j+k-j})_{1\le i,j\le k}}{\det(x_i^{k-j})_{1\le i,j\le k}}.$$

It is well known that the Schur symmetric functions satisfy Cauchy's identity:

$$\prod_{1\le i,j\le k} (1 - x_i y_j)^{-1} = \sum_{\{\lambda; l(\lambda)\le k\}} s_\lambda(x) s_\lambda(y). \tag{A.1}$$

In the theory of symmetric functions, the number of variables is usually irrelevant, provided that it is large enough, and it is often more convenient to work with symmetric functions in infinitely many variables. In the identity

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x) s_\lambda(y), \tag{A.2}$$

the sum is over all partitions. Let us introduce the orthonormality:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.$$

Here, $s_\lambda(x)$ such that $|\lambda| = n$ form an orthogonal basis of Λ^n, where Λ^n consists of homogeneous symmetric polynomials of degree n (refer to p. 18 of [1] for the definition). Using the power sum symmetric functions, the identity (A.2) is recast into

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} z_\lambda^{-1} p_\lambda(x) p_\lambda(y), \qquad z_\lambda := \prod_{i\ge 1} i^{c_i(\lambda)} c_i(\lambda)!,$$

and it follows that $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$, where p_λ form an orthogonal basis of Λ_Q.

The Jack symmetric function $P_\lambda^{(\alpha)}(x)$ is a generalization of the Schur symmetric function. Refer to Sect. VI.10 of [1] for the details. The Jack symmetric functions satisfy

$$\prod_{1\le i,j\le k} (1 - x_i y_j)^{-1/\alpha} = \sum_{\lambda} (z_\lambda \alpha^{l(\lambda)})^{-1} p_\lambda(x) p_\lambda(y).$$

Next, we introduce the following orthogonality relation:

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda,\mu} z_\lambda \alpha^{l(\lambda)}. \tag{A.3}$$

The partial order among partitions of the same weight is defined as

$$\lambda \geq \mu \qquad \Leftrightarrow \qquad \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i, \qquad \forall i \geq 1,$$

for partitions μ and λ. It can be shown that (p. 322 of [1]) for each partition λ, there is a unique symmetric function $P_\lambda^{(\alpha)}$ such that

$$P_\lambda^{(\alpha)} = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu}^{(\alpha)} m_\mu,$$

where

$$\langle P_\lambda^{(\alpha)}, P_\mu^{(\alpha)} \rangle_\alpha = 0, \qquad \lambda \neq \mu.$$

Here, the coefficient $u_{\lambda\mu}^{(1)}$ is called the Kostka number (see Sect. I.6 of [1]). The Jack symmetric functions $P_\lambda^{(\alpha)}$ such that $|\lambda| = n$ form an orthonormal basis of Λ^n. The inverse of the squared norm in the orthogonality relation (A.3) for each degree with normalization yields the Ewens sampling formula (2.12). In fact,

$$\sum_{\lambda \vdash n} \theta^{l(\lambda)} z_\lambda^{-1} = \sum_{\lambda \vdash n} \prod_{i=1}^{n} \left(\frac{\theta}{i}\right)^{c_i} \frac{1}{c_i!} = \frac{(\theta)_n}{n!}, \qquad \theta \equiv \frac{1}{\alpha},$$

and $n!\theta^{l(\lambda)}\{z_\lambda(\theta)_n\}^{-1}$ is the probability mass function of the Ewens sampling formula.

Remark A.1 The Jack symmetric function of $\alpha = 1$ is the Schur symmetric function, and that of $\alpha = 2$ with another normalization is known as the Zonal polynomial. The Zonal polynomial appears in integrations of the Haar measure of the orthogonal group, which appears in problems involving Wishart distributions [3]. Hashiguchi et al. discussed evaluation of the distribution function of the largest root of a Wishart matrix by using the holonomic gradient method discussed in Chap. 3 [4].

The Macdonald symmetric function is a further generalization of the Schur symmetric function. Chapter VI of [1] is devoted to this topic. The identity is

$$\prod_{i,j} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \sum_\lambda (z_\lambda(q,t))^{-1} p_\lambda(x) p_\lambda(y) \tag{A.4}$$

and the orthogonality relation is

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda(q,t), \tag{A.5}$$

where

$$z_\lambda(q,t) := z_\lambda \prod_{i\geq 1}\left(\frac{1-q^i}{1-t^i}\right)^{c_i}, \qquad (x;y)_n := \prod_{i=0}^{n-1}(1-xy^i).$$

When $q=t$, the Macdonald symmetric function reduces to the Schur symmetric function, and when $q=0$, it reduces to the Hall–Littlewood function. The Jack symmetric function appears in the limit $t=q^{1/\alpha}$, $q\to 1$.

The inverse of the squared norm in the orthogonality relation (A.5) for each degree with normalization yields a multiplicative measure induced by the exponential structure (2.16) with $w_i=(i-1)!(t^i-1)/(q^i-1)$. Setting $x_1=x$, $y_1=1$, and other variables to zero in the identity (A.4), we have

$$\frac{(tx;q)_\infty}{(x;q)_\infty} = \sum_{n=0}^{\infty}\sum_{\lambda\vdash n}(z_\lambda(q;t))^{-1}x^n.$$

From a q-analog of the negative binomial theorem (Theorem 12.2.5 in [5]):

$${}_1\phi_0(t;-;q,x) := \sum_{n=0}^{\infty}\frac{(t;q)_n}{(q;q)_n}x^n = \frac{(tx;q)_\infty}{(x;q)_\infty},$$

we have

$$\sum_{\lambda\vdash n}(z_\lambda(q;t))^{-1} = \sum_{\lambda\vdash n}\prod_{i=1}^{n}\left(\frac{t^i-1}{q^i-1}\frac{1}{i}\right)^{c_i}\frac{1}{c_i!} = \frac{(t;q)_n}{(q;q)_n}. \tag{A.6}$$

A.2 Processes on Partitions

Stochastic processes on partitions and measure-valued processes are closely related. Shimizu [6] discussed a measure-valued diffusion taking values in probability measures on Young tableaux. The Dirichlet process is the reversible measure of a measure-valued diffusion called the Fleming–Viot process [7], which appeared as a model of genetic diversity. It is one of the most studied measure-valued processes, whose theory was founded by Feller [8]. This section presents minimum materials about the Fleming–Viot process used in this monograph. Chapter 10 of [9] is a detailed introduction. Further developments can be found in [10, 11], and in [12] in Japanese. Related issues such as coagulation and fragmentation are discussed in [13, 14]. The roles in modeling of genetic diversity can be found in [15].

Consider a diffusion process with a generator

$$L = \sum_{i,j}\frac{1}{2}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j} + \sum_i b(x)\frac{\partial}{\partial x_i} \tag{A.7}$$

whose backward equation for the transition density ϕ has the form $\partial\phi/\partial t = L\phi$. The forward equation is $\partial\phi/\partial t = L^+\phi$, where L^+ is the adjoint operator of L:

$$L^+ \bullet \phi = \sum_{i,j} \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(x)\phi) - \sum_i \frac{\partial}{\partial x_i}(b(x)\phi).$$

For a test function f and a probability measure μ, let us introduce a notation $\langle f, \mu \rangle := \int f(x)d\mu(x)$. Let us assume existence of the unique stationary measure π for the diffusion. It should satisfy $L^+\pi = 0$, since

$$0 = \frac{d}{dt}\langle f, \pi \rangle = \langle Lf, \pi \rangle = \langle f, L^+\pi \rangle, \qquad \forall f. \tag{A.8}$$

Moreover, if π is *reversible*, π should satisfy

$$\langle Lf, g\pi \rangle = \langle Lg, f\pi \rangle, \qquad \forall f,\ g.$$

It can be observed that this condition is equivalent to $L_j^+\pi = 0$ for $\forall j$, where

$$L_j^+ \bullet \phi = \sum_i \frac{1}{2} \frac{\partial}{\partial x_j}(a_{ij}(x)\phi) - b_j(x)\phi, \qquad L^+ = \sum_j \frac{\partial}{\partial x_j} \bullet L_j^+.$$

Let us consider a diffusion process whose diffusion and drift coefficients are given by $a_{ij}(x) = x_i(\delta_{ij} - x_j)$ and $b_i(x) = \alpha(\sum_j x_j - mx_i)/2, \alpha > 0$, respectively. Consider a generator

$$L = \sum_{i=1}^m \sum_{j=1}^m \frac{x_i(\delta_{ij} - x_j)}{2} \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\alpha}{2} \sum_{i=1}^m \left(\sum_{j=1}^m x_j - mx_i \right) \frac{\partial}{\partial x_i} \tag{A.9}$$

in the state space $(x_1, ..., x_m) \in \Delta_{m-1}$. A diffusion process in the simplex with covariance diffusion coefficients is called *Wright–Fisher diffusion*. Since L^+ and L_j^+ annihilate the density (4.3) of the symmetric m-variate Dirichlet distribution of parameter α, the Dirichlet distribution is the reversible measure of the Wright–Fisher diffusion.

Remark A.2 The condition that L_j^+ annihilates the density is useful to obtain explicit expressions of the density of the reversible measure. A demonstration is given in [16]. Moreover, the condition is useful to construct a sampler from random partitions (see Sect. 5.1.4).

Remark A.3 For the Wright–Fisher diffusion (A.9), Griffiths [17] obtained an expansion of the transition density in terms of orthogonal polynomials of the form

$$f(x, y; t) = \pi_\alpha(y) \left\{ 1 + \sum_{i \geq 1} P_i(x)P_i(y) \exp\left(-\frac{i(i-1+m\alpha)}{2} t \right) \right\},$$

where π_α is the density of the symmetric m-variate Dirichlet distribution and $\{P_i(x), i \in \mathbb{N}\}$ are orthonormal Jacobi polynomials on the m-variate symmetric distribution scaled such that $\mathbb{E}_{\pi_\alpha}[P_i(X)P_j(X)] = \delta_{i,j}$, $i, j \in \mathbb{N}$. The symmetric kernel reflects the reversibility of the process.

Taking the monomial

$$q_n(x) = \frac{n!}{n_1! \cdots n_m!} x^n, \qquad x^n := \prod_{i=1}^{m} x_i^{n_i}, \tag{A.10}$$

as a test function, we obtain the Dirichlet-multinomial distribution (4.10):

$$p(n) := \langle q_n, \pi_\alpha \rangle = \binom{-m\alpha}{n}^{-1} \prod_{i=1}^{m} \binom{-\alpha}{n_i}.$$

The Dirichlet-multinomial distribution is an EPPF introduced in Sect. 4.3. The stationarity condition (A.8) yields the recurrence relation

$$\begin{aligned} p(n) &= \frac{n-1}{m\alpha + n - 1} \sum_{i=1}^{m} \frac{n_i - 1}{n-1} p(n - e_i) \\ &\quad + \frac{\alpha}{m\alpha + n - 1} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{n_j + 1 - \delta_{ij}}{n} p(n - e_i + e_j) \end{aligned} \tag{A.11}$$

with the boundary condition $p(e_i) = 1/m, i \in [m]$. Taking the limit $m \to \infty, \alpha \to 0$ with $\theta \equiv m\alpha$ in the Dirichlet-multinomial distribution gives the Ewens sampling formula (2.12) (Remark 4.6). Rewriting (A.11) in terms of size indices (see exponential structures in Sect. 2.1) and taking the limit, we have

$$\begin{aligned} \mu_n(c) &= \frac{n-1}{\theta + n - 1} \sum_{i=1}^{n-1} \frac{i(c_i + 1)}{n-1} \mu_{n-1}(c + e_i - e_{i+1}) \\ &\quad + \frac{\theta}{\theta + n - 1} \left\{ \frac{c_1}{n} \mu_n(c) + \sum_{i=2}^{n} \frac{i(c_i + 1)}{n} \mu_n(c - e_{i-1} + e_i - e_1) \right\} \end{aligned} \tag{A.12}$$

with the boundary condition $\mu_1(e_1) = 1$. The Ewens sampling formula satisfies this recurrence relation.

The above observation implies that the Dirichlet process is the reversible measure of an infinite-dimensional diffusion. Such a diffusion was formulated by Fleming and Viot [7], which is now called a Fleming–Viot diffusion. Let E be a compact metric space. Let $\mathscr{C}(E)$ be set of continuous real-valued functions on E and $\mathscr{P}(E)$ be the family of Borel probability measures on E. For $f \in \mathscr{B}(E)$ and $\mu \in \mathscr{P}(E)$, define

$$\phi(\mu) = F(\langle f_1, \mu \rangle, ..., \langle f_k, \mu \rangle) \in \mathscr{C}(\mathscr{P}(E))$$

for some $k \in \mathbb{N}$. A generator of the Fleming–Viot diffusion with a linear operator B on $\mathscr{C}(E)$ is defined as

$$G\phi(\mu) = \frac{1}{2}\sum_{i,j=1}^{k}(\langle f_i f_j, \mu\rangle - \langle f_i, \mu\rangle\langle f_j, \mu\rangle)F_{,ij}(\langle f_1, \mu\rangle, \cdots, \langle f_k, \mu\rangle)$$
$$+ \sum_{i=1}^{k}\langle Bf_i, \mu\rangle F_{,i}(\langle f_1, \mu\rangle, \cdots, \langle f_k, \mu\rangle), \tag{A.13}$$

where $F_{,i}(x_1, ..., x_k) = \partial F/\partial x_i$.

Example A.1 (*Dirichlet distribution*) Let $E = \{1/m, 2/m, ..., 1\}$ and define

$$Bf_i = \frac{\alpha}{2}\sum_{j=1}^{m}(f_j - f_i).$$

The solution of the martingale problem defined by the generator (A.13) is $\mu(t) = \sum_{i=1}^{m} x_i(t)\delta_{i/m}$, where $x(t)$ follows the Wright–Fisher diffusion governed by the generator (A.9). The transition density is given in Remark A.3 and the reversible measure is π_α.

Example A.2 (*Dirichlet process*) Let $E = [0, 1]$ and define

$$Bf(x) = \frac{\theta}{2}\int_0^1 \{f(y) - f(x)\}dy.$$

The reversible measure of the diffusion governed by the generator (A.13) has the form $\mu = \sum_{i\geq 1} x_i \delta_{V_i}$, $V_i \sim$ Unif.([0, 1]), where x follows the GEM distribution (Remark 4.3). Therefore, μ follows the Dirichlet process DP(θ; Unif.(E)), which appeared in Sect. 4.3.

The Ewens sampling formula is a random integer partition and a sample from the Dirichlet process. We have interest in stochastic processes on partitions which is related with the Fleming–Viot diffusion. Kingman discovered such a $\mathscr{P}_{\mathbb{N}}$-valued process, which is called *Kingman's coalescent*. It is a Markov chain on partitions with the following transition rule. Assume that the process is in the state $\{A_1, ..., A_l\}$. The only possible transitions are one of the $l(l-1)/2$ partitions obtained by merging parts A_i and A_j to form $A_i \cup A_j$ and leaving all other parts uncharged at rate one. The length of partition $(L_t; t \geq 0)$, $L_0 = n$, follows the death process whose transitions are $l \to l-1$ at rate $l(l-1)/2$. The process is eventually absorbed into the state of one. Let us consider this process as generating a tree upward from the leaves to the root. Time is vertical, and parts at a given time are located along a horizontal line. A merger is called *coalescence*. Let us introduce a Poisson process of marks, which is called *mutation*, along all branches of this tree at rate $\theta/2$ per unit length. Then, a

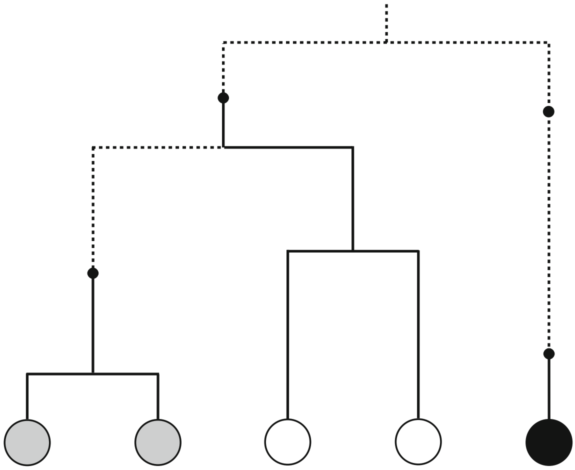

Fig. A.1 A realization of the coalescent tree of the partition (2, 2, 1)

random integer partition is generated by the equivalence relation $i \sim j$ if there is no mutation on the unique path in the tree that joins i to j, see Fig. A.1.

The relationship between the Fleming–Viot diffusion and the Kingman's coalescent can be understood by the notion of *duality* between Markov chains. The method of duality has been widely used in analyses of infinite particle systems. Many examples of the use can be found in [19]. If $(X_t; t \geq 0)$ with $X_0 = x$ and $(Y_t; t \geq 0)$ with $Y_0 = y$ are Markov processes in state spaces E_x and E_y, respectively, then the processes X_t and Y_t are said to be dual with respect to a kernel $k(x, y)$ if the identity

$$\mathbb{E}_x(k(X_t, y)) = \mathbb{E}_y(k(x, Y_t)), \qquad \forall x \in E_x, \ y \in E_y \tag{A.14}$$

holds. Consider generators of G_x for x_t and G_y for y_t. Then, the duality relationship (A.14) will be satisfied if the identity $G_x k(x, y) = G_y k(x, y)$ holds for all $x \in E_x$ and $y \in E_y$. Therefore, if we know G_x, we may identify the dual G_y. Let us consider the Wright–Fisher diffusion governed by the generator (A.9) and take the kernel $k(x, n) = x^n / \mathbb{E}_{\pi_\alpha}(x^n)$. We observe

$$Lk(x, n) = -\frac{n(n-1+m\alpha)}{2} k(x, n) + \frac{n(n-1+m\alpha)}{2} \sum_{i=1}^{m} \frac{n_i}{n} k(x, n - e_i) = G_n k(x, n),$$

and can lead the process $(N_t; t \geq 0)$, $N_0 = n$ from this expression. Taking the limit $m \to \infty$, $\alpha \to 0$ with $\theta \equiv m\alpha$, we can observe that N_t follows the infinite-dimensional death process whose transitions are $l \to l - e_i$ at late $l(l-1+\theta)/2 \times l_i/l$. This process is certainly generated by Kingman's coalescent; the rate $l(l-1)/2$ comes from the coalescence, while the rate $l\theta/2$ comes from the mutation.

Remark A.4 An extension of Kingman's coalescent, the Λ-coalescent, has been extensively studied. The fairly recent surveys are [20, 21]. The transition rule is as follows. Assume that the process is in the state $\{A_1, ..., A_b\}$. Then, k blocks merge with rates

$$\lambda_{b,k} = \int_0^1 x^{k-2}(1-x)^{b-k}\Lambda(dx), \qquad 2 \leq k \leq b,$$

where Λ is a nonnegative finite measure on [0, 1]. Pitman [22] showed that the array of rates $(\lambda_{b,k})$ is consistent if and only if $\lambda_{b,k} = \lambda_{b+1,k} + \lambda_{b+1,k+1}$. This condition originates from the de Finetti's representation theorem (Theorem 1.1), because the representation can be regarded as infinite exchangeable sequences of binomial random variables. Kingman's coalescent is the case of $\Lambda = \delta_0$. Another well-investigated example is the beta coalescent, whose Λ is the density of Beta$(2-\alpha, \alpha)$, $\alpha \in (0, 2)$. The diffusion and jump part of the generator of a Λ-Fleming–Viot process are

$$\begin{aligned} G\phi(\mu) =& \Lambda(\{0\})G_0\phi(\mu) \\ &+ \int_{(0,1]}\int_E \{\phi((1-x)\mu + x\delta_a) - \phi(\mu)\}\,\mu(da)\frac{\Lambda_0(dx)}{x^2}, \end{aligned}$$

where G_0 is the diffusion term in (A.13) and Λ_0 is the Λ on (0, 1].

References

1. Macdonald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. Oxford University Press, New York (1995)
2. Stanley, R.P.: Enumerative Combinatorics, vol. 2. Cambridge University Press, New York (1999)
3. Takemura, A.: Zonal Polynomials. Lecture Notes-Monograph Series, vol. 4. Michigan State University (1984)
4. Hashiguchi, H., Numata, Y., Takayama, N., Takemura, A.: The holonomic gradient method for the distribution function of the largest root of a Wishart matrix. J. Multivar. Anal. **117**, 296–312 (2013)
5. Ismail, M.E.H.: Classical and Quantum orthogonal polynomials in one variable. In: Encyclopedia of Mathematics and its Applications, vol. 98. Cambridge University Press, New York (2005)
6. Shimizu, A.: Stationary distribution of a diffusion process taking values in probability distributions on the partitions. In: Kimura, M., Kallianpur, G., Hida, T. (eds.) Stochastic Methods in Biology. Lecture Notes in Biomathematics, vol. 70, pp. 100–114. Springer, Berlin (1987)
7. Fleming, W.H., Viot, M.: Some measure valued Markov processes in population genetics theory. Indiana Univ. Math. J. **28**, 817–843 (1979)
8. Feller, W.: Diffusion processes in genetics. In: Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, pp. 227–246. University of Carfornia Press, Berkeley (1950)
9. Ethier, S.N., Kurtz, T.G.: Markov Processes: Characterization and Convergence. Wiley, New Jersey (1986)
10. Dawson, D.A.: Measure-valued Markov processes. In: Ecole d'Été de Probabilités de Saint Flour. Lecture Notes in Mathematics, vol. 1541. Springer, Berlin (1993)
11. Etheridge, A.M.: An Introduction to Superprocess. University Lecture Notes, vol. 20. American Mathematical Society, Providence (2000)

12. Handa, K.: Generalizations of Wright-Fisher diffusions and related topics. Proc. Inst. Stat. Math. **60**, 327–339 (2012)
13. Pitman, J.: Combinatorial Stochastic Processes. In: Ecole d'Été de Probabilités de Saint Flour, Lecture Notes in Mathematics, vol. 1875. Springer, Berlin (2006)
14. Bertoin, J.: Random Fragmentation and Coagulation Process. Cambridge University Press, Cambridge (2006)
15. Durrett, R.: Probability Models for DNA Sequence Evolution. Springer, New York (2008)
16. Handa, K.: Reversible distributions of multi-allelic Gillespie-Sato diffusion models. Ann. Inst. H. Poincaré Prob. Stat. **40**, 569–597 (2004)
17. Griffiths, R.C.: A transition density expansion for a multi-allele diffusion model. Adv. Appl. Probab. **11**, 310–325 (1979)
18. Kingman, J.F.C.: The coalescent. Stoch. Process Appl. **13**, 235–248 (1982)
19. Liggett, T.M.: Stochstic Interacting Systems: Contact, Voter and Exclusion Processes. Springer, Berlin (1999)
20. Berestycki, N.: Recent progress in coalescent theory. Ensaios Mathemáticos **16**, 1–193 (2009)
21. Birkner, M. Blath, J.: Measure-valued diffusions, general coalescents and population genetic inference. In: Blath, J., Mörters, P., Scheutzow, M. (eds.) Trends in Stochastic Analysis. London Mathematical Society Lecture Note Series, vol. 353, pp. 329–363. Cambridge University Press, Cambridge (2009)
22. Pitman, J.: Coalescent with multiple collisions. Ann. Probab. **27**, 1870–1902 (1999)

Index

S. Mano, *Partitions, Hypergeometric Systems, and Dirichlet Processes in Statistics*,
JSS Research Series in Statistics, https://doi.org/10.1007/978-4-431-55888-0