Part IV

Appendices

The results presented in the main text of the book are essentially based on notions and methods of the theory of C^*-algebras and K-theory of operator algebras. For the reader's convenience, some basic definitions and facts concerning these notions and methods are given below in Appendices A and B.

As a rule, the proofs of propositions and theorems are omitted. (They can be found in the relevant literature, including [6, 22, 35, 41, 62] etc.) The appendix does not claim to be a complete exposition of anything, and it should be treated not as an introduction to the topic but just as reference material intended possibly to decrease the reader's need to consult other sources when reading the book.

In Appendix C, we briefly describe the basic concepts of the theory of cyclic homology and cohomology. Although this material is practically not used in the book (except for the notions of enveloping differential algebra and closed cyclic traces), we still include it in the book, so as to give the reader the impetus to further profound study of the literature.

Appendix A

C^*-Algebras

A.1 Basic Notions

A.1.1 Definitions and Examples

Let H be a Hilbert space, and let $A = \mathcal{B}(H)$ be the set of bounded linear operators in H. This set has the following properties.

1. A is an algebra over the field $\mathbb{C}$ of complex numbers.
2. A is a Banach space (i.e., a complete normed vector space).

◀ The norm in $\mathcal{B}(H)$ is given by the formula

$$\|a\| = \sup_{\substack{u \in H \\ u \neq 0}} \frac{\|au\|}{\|u\|}, \qquad a \in \mathcal{B}(H).$$

It is called the *operator norm.* ▶

3. The multiplication in A is *norm continuous*, and moreover, the following inequality holds:

$$\|ab\| \leq \|a\| \, \|b\| \, , \qquad a, b \in A.$$

4. In the algebra A, an involution (i.e., an antilinear antihomomorphism) $a \mapsto a^*$ is defined, and moreover, $\|a\| = \|a^*\|$ for any $a \in A$.

◀ The involution in $\mathcal{B}(H)$ takes each operator to the adjoint operator. ▶

5. Each element $a \in A$ satisfies the relation

$$\|a\|^2 = \|a^* a\| \, .$$

◀ The norm satisfying such a relation is called a C^**-norm.* ▶

It is precisely properties 1–5 that characterize C^*-algebras. In addition, we give the following definition.

Definition A.1. An algebra A is called

1. A *Banach algebra* if properties 1–3 are satisfied.
2. An *involutive Banach algebra* if properties 1–4 are satisfied.
3. A C^**-algebra* if properties 1–5 are satisfied.

A subalgebra B of a C^*-algebra A is itself a C^*-algebra if it is closed and invariant under involution; in this case, it is called a C^**-subalgebra.* In what follows, unless otherwise specified, we only deal with C^*-algebras, and speaking of subalgebras, we always mean C^*-subalgebras.

The algebra $\mathcal{B}(H)$ and its subalgebras are examples of C^*-algebras. (They are called operator C^*-algebras.) It turns out that, in fact, there are no other examples.

Theorem A.2 (Gelfand–Naimark). *Any C^*-algebra is isomorphic to an operator C^*-algebra.*

Remark A.3. The Hilbert space H where the elements of a given C^*-algebra are realized as operators need not be separable.

Example A.4. We present several examples of C^*-algebras.

1. The algebra $\mathcal{K}(H)$ of all compact operators in a Hilbert space H. This is a subalgebra in $\mathcal{B}(H)$.

2. The algebra $M_n(\mathbb{C})$ of complex $n \times n$ matrices and its subalgebras. (Of course, these algebras are realized by operators in $\mathbb{C}^n$.)

3. More generally, the algebra $M_n(A)$ of $n \times n$ matrices whose entries are elements of some C^*-algebra A.

4. The algebra $C(X)$ of complex-valued bounded continuous functions on a locally compact space X and its subalgebra $C_0(X)$ consisting of functions tending to zero at infinity (i.e., of functions $f(x)$ such that for each $\varepsilon > 0$ the inequality $|f(x)| < \varepsilon$ holds outside some compact subset in X depending on ε). Under additional conditions, such an algebra can be realized by operators in the space $L^2(X, \mu)$ of functions on X square integrable with respect to a given measure μ on X.

A.1.2 Unital Algebras and Units

A C^*-algebra A is said to be *unital* if it contains a *unit* element, i.e., an (obviously, unique) element 1 such that

$$a1 = 1a \quad \text{for all } a \in A.$$

One can readily verify that, in addition, this element is also *self-adjoint*, i.e., satisfies the relation $1^* = 1$.

In the above examples, the algebras $M_n(\mathbb{C})$ and $C(X)$ are unital, and the algebras $\mathcal{K}(H)$ and $C_0(X)$ (in the case of an infinite-dimensional space H and noncompact space X) are not unital. The algebra $M_n(A)$ is unital if and only if the algebra A itself has this property.

Although a C^*-algebra may not contain a unit, it always contains so-called approximate units. An *approximate unit* is defined to be a net e_n of self-adjoint elements of a C^*-algebra A such that

$$\lim_{n\to\infty} e_n a = \lim_{n\to\infty} a e_n = a$$

for each element $a \in A$. Approximate units replace the "true" unit in the study of many problems.

If A is a possibly nonunital C^*-algebra, then it can always be transformed into a unital one by *adjoining* the unit to it, i.e., by considering the algebra A^+ whose elements are pairs (λ, a), where $\lambda \in \mathbb{C}$ and $a \in A$, and the algebraic operations are defined by the formulas

$$(\lambda, a) + (\lambda', a') = (\lambda + \lambda', a + a'), \qquad (\lambda, a)(\lambda', a') = (\lambda\lambda', \lambda' a + \lambda a' + a a'),$$
$$(\lambda, a)^* = (\overline{\lambda}, a^*).$$

The unit of the algebra A^+ is the element $(1, 0)$. The algebra A is naturally embedded in A^+ by the mapping $a \mapsto (0, a)$.

Proposition A.5. *On the algebra A^+, there exists a norm making it a C^*-algebra.*

If the algebra A itself is already unital, then A^+ is the direct sum of the algebras A and $\mathbb{C}$ in the sense of the following definition.

Definition A.6. The *direct sum* of C^*-algebras A and B is the algebra $A \oplus B$ whose elements are pairs (a, b), $a \in A$, $b \in B$, and the algebraic operations are defined componentwise.

Proposition A.7. *On the direct sum $A \oplus B$, there is a norm making it a C^*-algebra.*

If both algebras A and B are unital, then $A \oplus B$ is also unital; the unit in it is the element (e_A, e_B), which is the direct sum of the units in A and B.

A.1.3 Homomorphisms, Ideals, Quotient Algebras, and Extensions

A mapping $f\colon A \to B$ of a C^*-algebra A into a C^*-algebra B is called a *homomorphism* (of C^*-algebras) if it preserves the algebraic operations and the involution, i.e., if

$$f(a + \lambda b) = f(a) + \lambda f(b), \quad f(ab) = f(a) f(b), \quad f(a^*) = \bigl(f(a)\bigr)^*,$$
$$a, b \in A, \quad \lambda \in \mathbb{C}.$$

Note that we do not require the mapping f to be continuous, because, in fact, this property is satisfied automatically.

Proposition A.8. *An arbitrary homomorphism $f\colon A \to B$ of C^*-algebras does not increase the norm*:

$$\|f(a)\| \le \|a\|, \qquad a \in A.$$

Corollary A.9. *If A is a C^*-algebra, then the C^*-norm on A is uniquely determined.*

Let $f\colon A \to B$ be a homomorphism of C^*-algebras. The kernel

$$J = \{a \in A\colon f(a) = 0\}$$

of the homomorphism f is a closed linear subspace in A such that

$$aJ \subset J, \quad Ja \subset J \qquad \text{for any } a \in A,$$
$$J^* = J, \qquad \text{where } J^* = \{a^*\colon a \in J\}.$$

Any subspace $J \subset A$ satisfying the above conditions is called a *two-sided C^*-ideal in A*. (In what follows, for brevity, such subspaces will simply be called ideals.) Thus, the kernel of a C^*-algebra homomorphism is an ideal. Conversely, each ideal J in A is the kernel of a homomorphism into some C^*-algebra B. Namely, for B we can take the quotient algebra A/J determined as the set of cosets $a + J$ in the initial algebra A; this set is equipped with the algebraic operations

$$(a + J) + (a' + J) = (a + a') + J, \qquad (a + J)(a' + J) = aa' + J,$$
$$(a + J)^* = a^* + J.$$

Proposition A.10. *On the quotient algebra A/J, there exists a C^*-norm.*

Thus, A/J is naturally a C^*-algebra, and we obtain the *exact sequence*

$$0 \longrightarrow J \xrightarrow{\ f\ } A \xrightarrow{\ g\ } A/J \longrightarrow 0$$

of C^*-algebras, where f is the embedding of J in A and g is the natural projection onto the quotient algebra ($g(a) = a + J$). In general, if an exact sequence[1]

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0 \tag{A.1}$$

of C^*-algebras is given, then A is naturally identified with the image of itself, which is automatically an ideal in B, and C is identified with the quotient algebra of B by this ideal.

Definition A.11. The algebra B in the exact sequence (A.1) is called an *extension* of the algebra C by the algebra (ideal) A.

The sequence (A.1) is said to *split* if there exists a C^*-algebra homomorphism $h\colon C \to A$ such that $g \circ h = \mathrm{id}_C$ (the identity homomorphism of the algebra C). Obviously, in this case, $B = A \oplus C$, and the extension (A.1) is said to be *trivial.*

Extensions of C^*-algebras are considered in more detail in Sec. B.2.

[1] Recall that a sequence of homomorphisms is said to be *exact* if the kernel of each homomorphism in this sequence coincides with the image of the preceding homomorphism.

A.1.4 Commutative C^*-Algebras

Consider the structure of commutative C^*-algebras. Let A be a commutative C^*-algebra. An ideal $J \subset A$ is said to be *maximal* if it is *proper* (i.e., is nonzero and does not coincide with A) and is not contained in any other proper ideal of A. If $J \subset A$ is a maximal ideal, then the quotient algebra A/J is a *Banach field* (a commutative Banach algebra in which any nonzero element is invertible) and hence $A/J = \mathbb{C}$ by the Gelfand–Mazur theorem. Thus, for each maximal ideal $J \subset A$ there exists a naturally defined homomorphism

$$\varphi_J \colon A \longrightarrow \mathbb{C}.$$

(Note that the algebra $\mathbb{C}$ does not have nontrivial automorphisms.)

Let X be the set of all maximal ideals of the algebra A. Then to each element $a \in A$ one can assign a function $\widehat{a}$ on X by setting

$$\widehat{a}(J) = \varphi_J(A), \qquad J \in X.$$

The function $\widehat{a}$ is called the *Gelfand transform* of the element a.

Theorem A.12 (Gelfand–Naimark). *The set X can be equipped with the topology of a locally compact space such that the Gelfand transform is an isomorphism of the algebra A onto the algebra $C_0(X)$ of functions on X vanishing at infinity. The space X is compact if and only if the algebra A is unital*; *in this case, $C_0(X) = C(X)$, so that the Gelfand transform realizes an isomorphism of the algebra A onto the space of all continuous functions on X.*

A.1.5 Spectrum and Functional Calculus

Definition A.13. Let A be a unital C^*-algebra. The *spectrum* of an element $a \in A$ is defined to be the set of complex numbers λ for which the element $a - \lambda \equiv a - \lambda 1$ is not invertible:

$$\sigma(a) = \{\lambda \in \mathbb{C} \colon (a - \lambda)^{-1} \quad \text{does not exist}\}.$$

In what follows, we sometimes need an explicit indication of the algebra A with respect to which the spectrum of an element is considered. In this case, we write $\sigma_A(a)$ instead of $\sigma(a)$.

Definition A.14. Let A be a nonunital C^*-algebra. The *spectrum* of an element $a \in A$ is defined as the spectrum of a in the algebra A^+:

$$\sigma(a) = \sigma_{A^+}(a).$$

Note that if A is unital, then

$$\sigma_{A^+}(a) = \sigma_A(a) \cup \{0\}.$$

Theorem A.15 (Spectral invariance). *Let A be a unital C^*-algebra, and let $B \subset A$ be a unital subalgebra.*[2] *Then*

$$\sigma_A(a) = \sigma_B(a)$$

for each element $a \in B$. In particular, if an element $a \in B$ is invertible in A, then it is also invertible in B (i.e., $a^{-1} \in B$).

Let A be a unital C^*-algebra, let $a \in A$, and let $f(\lambda)$ be a function holomorphic in a neighborhood U of the spectrum of a on the complex plane. Further, let γ be a contour in the complex plane surrounding the spectrum $\sigma(a)$ in the positive sense and lying entirely in U. Then we can define the function $f(a)$ of the element a using the Cauchy integral by setting

$$f(a) = \frac{1}{2\pi i} \oint_\gamma f(\lambda)(\lambda - a)^{-1}\, d\lambda.$$

This integral gives a well-defined (i.e., independent of the choice of the contour) element $f(a)$ of the algebra A. This construction is referred to as *holomorphic functional calculus.*

Theorem A.16 (The spectral mapping theorem). *The following relation holds*:

$$\sigma(f(a)) = f(\sigma(a)).$$

For a given element $a \in A$, the mapping $f \mapsto f(a)$ is a homomorphism of the algebra of functions holomorphic in neighborhoods (*different for different functions*) *of the spectrum of a into the algebra A. Moreover,*

$$\bigl(f(a)\bigr)^* = \overline{f}(a^*),$$

where the holomorphic function $\overline{f}(\lambda)$ is defined by the formula

$$\overline{f}(\lambda) = \overline{f(\overline{\lambda})}.$$

If only self-adjoint elements $a^* = a$ of the algebra A are considered, then functional calculus can be defined for a much wider class of functions. In this case, the spectrum $\sigma(a)$ is a compact subset of the real axis, and the operator $f(a)$ can be defined for any continuous function f on the spectrum $\sigma(a)$. (To this end, one can use the Gelfand transform for the commutative C^*-subalgebra $B \subset A$ generated by the element a.)

Functional calculi and spectral mapping theorems can also be constructed for elements of nonunital algebras by passing from the algebra A to the algebra A^+. But in this case the condition $f(0) = 0$ ensuring that $f(a) \in A$ should be imposed on admissible functions f.

[2]In this case, we assumed that $1_B = 1_A$.

A.1.6 Local C^*-Algebras

In numerous problems of noncommutative geometry, it does not suffice to consider only C^*-algebras, since, for example, the (co)homology theory may well be trivial for them. Therefore, one should use a different, somewhat more complicated class of algebras, which inherit good properties of C^*-algebras but do not inherit their bad properties.

We mean the class of *local* C^*-algebras, for which we now give the corresponding definition. This definition has two versions, one of which can be used for unital algebras, while the other can be used for nonunital algebras.

Definition A.17. Let A be a C^*-algebra, and let $A_0 \subset A$ be a dense subalgebra invariant under the involution and equipped with the structure of a Fréchet algebra such that the embedding $A_0 \subset A$ is continuous. We say that A_0 is a *local* C^**-algebra* if A_0 is closed with respect to holomorphic functional calculus in A, i.e., if one of the following two conditions is satisfied:

1. A_0 is unital, and for any element $a \in A_0$ and function $f(z)$ holomorphic in a neighborhood of the spectrum $\sigma_A(a)$ of the element a in the algebra A, the element $f(a)$ lies in A_0.

2. A_0 is nonunital, and condition 1 holds under the additional assumption that $f(0) = 0$.

Remark A.18. In the unital case, condition 1 is equivalent to any of the following two conditions (e.g., see [67, 68]):

(i) If $a \in A_0$, then $e^a \in A_0$ as well.

(ii) If an element $a \in A_0$ is invertible in A, then $a^{-1} \in A_0$; i.e., it is also invertible in A_0.

Example A.19. By way of example, consider the C^*-algebra $C(X)$ of continuous functions on a smooth manifold X and the subalgebra $C^k(X)$ of k times continuously differentiable functions (or the subalgebra C^∞ of infinitely differentiable functions). This subalgebra is a local C^*-algebra in the sense of the above definition.

A.1.7 Positive Elements

An element a of a C^*-algebra A is said to be *positive* if it is self-adjoint (i.e., $a = a^*$) and $\sigma(a) \in [0, +\infty)$. We denote the set of positive elements of the algebra A by A_+. Each element of the form b^*b is positive, and each element $a \in A_+$ can be represented in the form $a = b^2$, where b is self-adjoint. Finally, each self-adjoint element $a \in A$ can be represented in the form $a = a_+ - a_-$, where $a_+, a_- \in A_+$ and $a_+a_- = a_-a_+ = 0$ (the Hahn decomposition).

A.1.8 Projections in C^*-Algebras

Let us present several technical results about projections in C^*-algebras. For details, see [22].

Definition A.20. An element $p \in A$ such that

$$p = p^* = p^2$$

is called a *projection* in the C^*-algebra A.

Obviously, any projection p lies in the set A_+, and $\sigma(p) \in \{0, 1\}$.

On the set of projections, there is a natural order. It is given by

$$p \leq q \iff q - p \in A_+.$$

Proposition A.21. *If p and q are projections in a C^*-algebra A, then the relation $p \leq q$ holds if and only if*

$$pq = qp = p.$$

In this case, $q - p$ is also a projection in A.

On the set of projections in a C^*-algebra A, the following three equivalence relations are introduced:

1. (Algebraic equivalence.) There exists an element $u \in A$ such that

   $$u^*u = p, \qquad uu^* = q.$$

   (Such an element u is called a *partial isometry*.)

2. (Similarity.) There exists a unitary element $u \in A$ such that

   $$u^*pu = q.$$

3. (Homotopy.) In the algebra, there exists a continuous family of projections $p(t)$, $t \in [0, 1]$, such that

   $$p(0) = p, \qquad p(1) = q.$$

Proposition A.22. *Each of the above equivalence relations is strictly stronger than the preceding one, namely*:

1. *If projections p and q are similar, then they are algebraically equivalent*; *the converse statement is generally not true.*

2. *If projections p and q are homotopic, then they are similar*; *the converse statement is generally not true.*

Moreover, if $\|p - q\| < 1$, then the projections p and q are homotopic.

Although these equivalence relations are not equivalent, it is possible to obtain converses of the above implications if we pass to the matrix algebra over A. Namely, the following statement holds.

Proposition A.23. *Let p and q be projections in a C^*-algebra A, and let P and Q be the projections determined by the formulas*

$$P = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$$

in the algebra $M_2(A)$.

1. *If p and q are algebraically equivalent, then P and Q are similar.*
2. *If p and q are similar, then P and Q are homotopic.*

Let us state two more results that can be useful when studying projections in A.

Let A be a unital C^*-algebra. By $GL_n(A)$ we denote the multiplicative subgroup of invertible elements in the algebra $M_n(A)$, and by $U_n(A)$ we denote the subgroup of unitary elements.

Proposition A.24. *Let $u, v \in GL_n(A)$. Then the elements*

$$uv \oplus 1, \qquad vu \oplus 1, \qquad u \oplus v, \qquad v \oplus u$$

are pairwise homotopic in $GL_{2n}(A)$.

Proposition A.25. *Let*

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\ \pi\ } A/J \longrightarrow 0$$

be an exact sequence of C^*-algebras. Then for an arbitrary element $u \in U_n(A/J)$ there exists an element $w \in U_{2n}(A)$ that is homotopic to the unit element and satisfies the condition $\pi(w) = u \oplus u^*$.

A.2 Representations of C^*-Algebras

A.2.1 Basic Definitions

Definition A.26. A homomorphism

$$\pi\colon A \longrightarrow \mathcal{B}(H)$$

of a C^*-algebra A into the algebra of bounded linear operators in a Hilbert space H is called a *representation* of the C^*-algebra A.

In what follows, we denote a representation by π or by (π, H) if we need to indicate the space where the representation acts. A representation is said to be *faithful* if $\pi(a) \neq 0$ for any nonzero element $a \in A$.

Let $K \subset H$ be a subspace in H invariant under a representation π. This means that

$$\pi(a)K \subseteq K \quad \text{for any } a \in A,$$

or, equivalently,

$$\pi(a)P_K = P_K\pi(a) \quad \text{for any } a \in A,$$

where P_K is the orthogonal projection in H onto the subspace K. Then the restriction $(\pi|_K, K)$ of the representation (π, H) to the subspace K is well defined; it is called the *subrepresentation* of the representation π, and the subspace K is also called a *reducing subspace* of the representation π. Obviously, if K is a reducing subspace, then its orthogonal complement $K^\perp$ in the space H is also a reducing subspace.

Let (π_j, H_j), $j \in \mathcal{J}$, be representations of the algebra A. The *direct sum* of these representations is defined to be the representation $(\bigoplus_j \pi_j, \bigoplus_j H_j)$ of the algebra A in the direct sum of the Hilbert spaces H_j, which is determined by the formula

$$\Big[\bigoplus_j \pi_j\Big](a)\Big(\sum_j \xi_j\Big) = \sum_j \pi_j(a)\xi_j, \qquad a \in A, \quad x_j \in H_j,\, j \in J.$$

Clearly, each of the representations-summands is a subrepresentation of the direct sum.

A representation π is said to be *nondegenerate* if for each vector $\xi \in H$ there exists an element $a \in A$ such that $\pi(a)\xi \neq 0$. Since A (and hence $\pi(A)$) is a C^*-algebra, this condition is equivalent to the fact that the closed linear span $\overline{\pi(A)H}$ of all vectors of the form $\pi(a)\xi$, where $\xi \in H$ and $a \in A$, coincides with H.

A representation π is said to be *cyclic* if $H = \overline{\pi(A)\xi}$ for some vector $\xi \in H$, which in this case is called a *cyclic vector* of the representation π. (Of course, the cyclic vector is not unique.) Clearly, any cyclic representation is nondegenerate. The converse statement is not true, but the following statement holds.

Proposition A.27. *Any nondegenerate representation is a direct sum of cyclic representations.*

If (π, H) is an arbitrary representation of the algebra A, then we have the orthogonal decomposition $H = H_0 \oplus H_1$ of the space H into the sum of invariant subspaces such that the restriction of the representation π to H_1 is the zero representation and its restriction to H_2 is a nondegenerate representation. Thus, an arbitrary representation of the C^*-algebra A is a direct sum of cyclic representations and possibly the zero representation.

Definition A.28. Two representations (π_1, H_1) and (π_2, H_2) of a C^*-algebra A are said to be *equivalent* (more precisely, *spatially* or *unitarily equivalent*) if there exists a unitary operator $U\colon H_1 \to H_2$ such that

$$\pi_1(a) = U^* \pi_2(A) U \quad \text{for any } a \in A.$$

The mapping $\xi \mapsto U\xi$ can be treated as a "change of coordinates" in the Hilbert space, and from this standpoint, equivalent representations are simply "forms of writing out a same representation in different coordinate systems," so that they need not be distinguished; accordingly, the main problem of representation theory of C^*-algebras is to study representations up to unitary equivalence.

The "simplest" representations are those that do not contain nontrivial subrepresentations (i.e., subrepresentations other than the representation itself and the zero representation in the zero-dimensional space), or, which is the same, nontrivial invariant subspaces.

Definition A.29. A representation (π, H) of a C^*-algebra A is said to be *irreducible* if the space H contains no subspaces invariant under $\pi(A)$ except for $\{0\}$ and H.

In the following statement, we describe several properties of a representation π which are equivalent to its irreducibility.

Proposition A.30. *Let (π, H) be a nonzero representation of a C^*-algebra A. It is irreducible if and only if any of the following conditions holds.*

1. *The commutant of the algebra $\pi(A)$ (i.e., the set of all operators in $\mathcal{B}(H)$ commuting with all operators in $\pi(A)$) consists of scalar operators.*
2. *The algebra $\pi(A)$ is strongly dense[3] in $\mathcal{B}(H)$.*
3. *Any nonzero vector in H is cyclic for the representation π.*
4. *If $\xi, \eta \in H$ and $\xi \neq 0$, then A contains an element a such that the operator $\pi(a)$ takes ξ to η; i.e., $\pi(a)\xi = \eta$.*

Suppose that (π_1, H_1) and (π_2, H_2) are two representations of a C^*-algebra A. A bounded linear operator $T\colon H_1 \longrightarrow H_2$ is called an *intertwining operator* for the representations π_1 and π_2 if

$$T\pi_1(a) = \pi_2(a)T \qquad \text{for every } a \in A.$$

Obviously, the intertwining operators for given representations π_1 and π_2 form a linear space, which we denote by $I(\pi_1, \pi_2)$.

Definition A.31. Representations (π_1, H_1) and (π_2, H_2) are said to be *disjoint* if $I(\pi_1, \pi_2) = \{0\}$ (consists only of the zero operator).

The following alternative holds for two arbitrary irreducible representations.

Proposition A.32. *Let (π_1, H_1) and (π_2, H_2) be two irreducible representations of a C^*-algebra A. Then they are either equivalent or disjoint.*

[3]Recall that a sequence of operators A_n converges to operator A *strongly* if $A_n x \to Ax$ as $n \to \infty$ for any $x \in H$.

A.2.2 Existence of Representations

We have not yet discussed the problem on the *existence* of representations (in particular, irreducible) of a C^*-algebra A. In this section, we present a method for constructing a sufficient supply of representations (and, in particular, for proving Theorem A.2). We consider only the case of a unital algebra A.

A linear functional φ on the algebra A (i.e., an element of the space A^* dual to the Banach space A) is said to be *positive* if

$$\varphi(a) \geq 0 \quad \text{for any } a \in A_+.$$

Definition A.33. A positive functional φ of unit norm is called a *state* on the C^*-algebra A. A state φ is said to be *pure* if any positive functional ψ on A such that $\varphi - \psi$ is also positive is proportional to φ, i.e., has the form

$$\psi = t\varphi \quad \text{for some } t \in [0, 1].$$

States can be used to construct representations of the C^*-algebra.

Theorem A.34 (The Gelfand–Naimark–Segal construction). *Let φ be a state on a C^*-algebra A. There exists a cyclic representation (π_φ, H_φ) of the algebra A with cyclic vector $\xi_\varphi \in H_\varphi$ such that*

$$(\pi_\varphi(a)\xi_\varphi, \xi_\varphi) = \varphi(a) \quad \textit{for any } a \in A.$$

If φ is a pure state, then the representation (π_φ, H_φ) is irreducible.

It turns out that there are "sufficiently many" pure states.

Proposition A.35. *If $a \in A$ is a positive element, then there exists a pure state φ such that $\varphi(a) = \|a\|$.*

Using this statement, the Hahn decomposition, and the Gelfand–Naimark–Segal construction, one can show that for each element $a \in A$ there exists an irreducible representation π_φ such that $\pi_\varphi(a) \neq 0$.

Corollary A.36. *An element a of a unital C^*-algebra A is invertible if and only if the operator $\pi(a)$ is invertible for any irreducible representation π of A.*

The direct sum

$$(\Pi, H) = \bigoplus_\varphi (\pi_\varphi, H_\varphi),$$

where φ runs over the set of all pure states on the algebra A, is called the *universal representation* of the C^*-algebra A. It follows from the preceding that the universal representation is faithful (and hence determines a realization of the C^*-algebra A as an operator algebra in a Hilbert space, which, however, can be nonseparable).

A.2.3 Representations of Ideals and Quotient Algebras

Let A be a C^*-algebra, and let J be an ideal in A, so that one has the exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0. \tag{A.2}$$

Now we shall see that studying representations (and especially irreducible representations) of the algebra A can be reduced to studying representations of the algebras J and A/J.

Proposition A.37. *Let (π, H) be a nondegenerate representation of the ideal J. Then there exists a unique representation $(\widetilde{\pi}, H)$ of A that continues the representation π, i.e., has the property*

$$\widetilde{\pi}(a) = \pi(a) \quad \textit{for all } a \in J.$$

If π is irreducible, then $\widetilde{\pi}$ is also irreducible.

Conversely, let a representation $\widetilde{\pi}$ of the algebra A be given. Obviously, its restriction to J determines a representation π of the ideal J.

Proposition A.38. *If the representation $\widetilde{\pi}$ is irreducible and its restriction π to the ideal J is nonzero, then the representation π of the ideal J is also irreducible.*

By $\widehat{A}$ we denote the set of equivalence classes of irreducible representations of a C^*-algebra A. (This set is called the *spectrum* of the algebra A.) It follows from the above that the following statement holds.

Proposition A.39. *One has*

$$\widehat{A} = \widehat{J} \cup \widehat{A/J}.$$

◀ Indeed, any irreducible representation of the quotient algebra A/J in composition with the natural projection onto the quotient algebra gives an irreducible representation of the algebra A. By Proposition A.37, the irreducible representations of J can be continued to irreducible representations of A. These representations are not equivalent to any representations lifted from the quotient algebra, because the latter are identically zero on J. Further, if π is an irreducible representation of the algebra A, then the following two cases are possible: either $\pi|_J \equiv 0$, and then π is the pullback of the representation $\widetilde{\pi}$ of A/J defined by the formula

$$\widetilde{\pi}(a + J) = \pi(a),$$

or $\pi|_J \neq 0$; in the latter case, by Proposition A.38, the representation $\pi|_J$ is an irreducible representation of the ideal J, and π is obtained from it by lifting according to Proposition A.37. ▶

An important special case is obtained if the ideal J is isomorphic to the algebra $\mathcal{K}(H)$ of compact operators in a Hilbert space H.

Proposition A.40. *Any irreducible representation of the algebra $\mathcal{K}(H)$ is equivalent to its identical representation.*

Corollary A.41. *Any irreducible operator C^*-algebra that contains at least one nonzero compact operator contains the entire algebra $\mathcal{K}(H)$.*

Corollary A.42. *All automorphisms of the algebra $\mathcal{B}(H)$ are given by conjugation by unitary operators in $\mathcal{B}(H)$, i.e., have the form*

$$a \longmapsto UaU^{-1}, \quad \textit{where } U \in \mathcal{B}(H) \textit{ is unitary.}$$

It follows from Proposition A.40 that if the ideal J in the exact sequence (A.2) is $\mathcal{K}(H)$, then

$$\widehat{A} = \widehat{A/\mathcal{K}(H)} \cup \{[\mathrm{id}_{\mathcal{K}(H)}]\},$$

where $[\mathrm{id}_{\mathcal{K}(H)}]$ is the equivalence class of the identity representation of the algebra of compact operators. In particular, if A is a subalgebra of the algebra $\mathcal{B}(H)$ and $J = \mathcal{K}(H)$, then the identity representation of the algebra A is irreducible and all "nontrivial" irreducible representations of the algebra A correspond to irreducible representations of the *Calkin algebra* $A/\mathcal{K}(H)$ of the algebra A.

A.2.4 Primitive Ideals

If A is a commutative C^*-algebra, then its spectrum $\widehat{A}$ coincides with the (locally) compact Hausdorff space X on which the Gelfand transforms of elements of A are defined, i.e., with the set of maximal ideals of the algebra A: the irreducible representation π_J corresponding to an element $J \in X$ is one-dimensional and is determined by the composition of the natural projection $A \to A/J$ and the isomorphism $A/J \simeq \mathbb{C}$. (In terms of the Gelfand transforms of elements of the algebra A, this representation is the evaluation mapping of functions at the point $J \in X$.) In this case, the ideal J is exactly the kernel of the representation π_J.

Maximal ideals do not play a similar role for irreducible representations of a *noncommutative* C^*-algebra A. Instead, primitive ideals should be considered. Namely, to each equivalence class $[\pi] \in \widehat{A}$ of irreducible representations of the C^*-algebra A one assigns the ideal $I = \ker \pi$ of the algebra A. (Note that the kernel of π is independent of the specific choice of π in the equivalence class.) Such ideals are said to be *primitive*.

Proposition A.43. *Any primitive ideal in a C^*-algebra A is simple.*

◀ Recall that a (two-sided) ideal I in the algebra A is said to be *simple* if the relation $xAy \subset I$ implies that either $x \in I$ or $y \in I$. The equivalent condition is as follows: if $I_1, I_2 \subset A$ are ideals such that $I_1 I_2 \subset I$, then either $I_1 \subset I$ or $I_2 \subset I$. ▶

If the algebra A is separable (i.e., if it contains a countable everywhere dense set), then the converse statement is also true.

Proposition A.44. *In a separable C^*-algebra A, any closed simple ideal is primitive.*

Thus, by definition, we have the mapping

$$\widehat{A} \longrightarrow \operatorname{Prim} A$$

of the set $\widehat{A}$ of equivalence classes of irreducible representations of the algebra A onto the set $\operatorname{Prim} A$ of primitive ideals. Generally speaking, this is not a one-to-one mapping (i.e., an irreducible representation is not determined by its kernel up to equivalence). But this is the case for a sufficiently wide class of algebras, which will be considered in the next section. Here we only note that in many cases it is more convenient to use the set $\operatorname{Prim} A$ rather than $\widehat{A}$, because there is a natural topology (the so-called *Jacobson topology*) on $\operatorname{Prim} A$. Closed sets in this topology have the form

$$\operatorname{hull}(I) = \{J \in \operatorname{Prim} A \colon J \supset I\},$$

where I runs over the set of C^*-ideals in A. Indeed, the mapping $I \mapsto \operatorname{hull}(I)$ is a bijection of the set of C^*-ideals in A onto the set of closed sets in the Jacobson topology on $\operatorname{Prim} A$. The inverse mapping is given by the formula

$$F \longmapsto \ker F = \bigcap_{J \in F} J, \qquad F = \overline{F} \subset \operatorname{Prim} A.$$

This is a consequence of the following assertion.

Proposition A.45. *Any ideal in a C^*-algebra A is the intersection of primitive ideals containing this ideal.*

We note that the Jacobson topology on $\operatorname{Prim} A$ is in general not Hausdorff (unless the algebra A is commutative). But in most cases of interest it is T_0. (Recall that in a T_0-space for each pair of distinct points there exists an open set that contains one of them and does not contain the other.)

A.2.5 Algebras of Type I

Definition A.46. A C^*-algebra A is called a CCR-*algebra* (CCR stands for completely continuous representations) if its image under any irreducible representation consists of compact operators.

If A is an arbitrary C^*-algebra, then by $\operatorname{CCR}(A)$ we denote the ideal in A consisting of elements whose images are compact for any irreducible representation of the algebra A. Obviously, this is the maximal CCR-ideal in A.

Definition A.47. A C^*-algebra A is said to be of *type* I if the ideal $\operatorname{CCR}(A/J)$ is nonzero for any proper ideal $J \subset A$.

Theorem A.48. *If A is a separable algebra of type* I, *then each equivalence class of irreducible representations of A is uniquely determined by the kernel of the representation.*

Thus, the mapping $\widehat{A} \to \operatorname{Prim} A$ is one-to-one for algebras of type I.

We present one more criterion for algebras to belong to type I.

Proposition A.49. *A C^*-algebra A is an algebra of type* I *if and only if its image under any nonzero irreducible representation contains at least one compact operator.*

A.3 Tensor Products and Nuclear Algebras

A.3.1 Minimal and Maximal Tensor Products

Let A and B be C^*-algebras. Consider the algebraic tensor product $A \otimes B$ of the linear spaces A and B. Recall that it is determined as the quotient space of the linear space W of finite formal linear combinations of elements $(a, b) \in A \times B$ by the subspace generated by elements of the form

$$(\lambda a, b) - \lambda(a, b), \qquad (a, \lambda b) - \lambda(a, b),$$
$$(a + a', b) - (a, b) - (a', b), \qquad (a, b + b') - (a, b) - (a, b'),$$

where $a, a' \in A$, $b, b' \in B$, and $\lambda \in \mathbb{C}$. In other words, the elements of the algebraic tensor product $A \otimes B$ are finite linear combinations of the form

$$\sum_{j=1}^{s} a_j \otimes b_j, \qquad a_j \in A, \quad b_j \in B,$$

and

$$(a + a') \otimes b = a \otimes b + a' \otimes b, \qquad a \otimes (b + b') = a \otimes b + a \otimes b',$$
$$(\lambda a) \otimes b = a \otimes (\lambda b) = \lambda(a \otimes b).$$

On $A \otimes B$, one can introduce the structure of an involutive algebra by setting

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb', \qquad (a \otimes b)^* = (a^* \otimes b^*)$$

on monomials and then by extending multiplication and involution to the entire tensor product by linearity. If the algebras A and B are unital, then $A \otimes B$ is also unital with unit $1_A \otimes 1_B$.

It turns out that the product $A \otimes B$ can be equipped with a C^*-norm, so that the completion of $A \otimes B$ with respect to this norm is a C^*-algebra. For example, this can be done as follows. Let π_A and π_B be faithful representations of the algebras A and B in Hilbert spaces H_A and H_B, respectively. The formula

$$\tau\biggl(\sum_{j=1}^{s} a_j \otimes b_j\biggr) = \sum_{j=1}^{s} \pi_A(a_j) \otimes \pi_B(b_j)$$

determines a representation τ of the algebra $A \otimes B$ in the Hilbert space $H_A \otimes H_B$, which is called the *tensor product* of the representations π_A and π_B. Now we can set

$$\|a\|_{\min} = \|\tau(a)\|, \qquad a \in A \otimes B. \tag{A.3}$$

This is a C^*-norm on $A \otimes B$ independent of the specific choice of the representations π_A and π_B, and the completion of the algebra $A \otimes B$ with respect to this norm is called the *minimal* (or *spatial*) tensor product of A and B and is denoted by $A \otimes_{\min} B$.

The C^*-norm on the product $A \otimes B$ is not unique in general. Any such norm satisfies the inequality

$$\|a\| \geq \|a\|_{\min}, \qquad a \in A \otimes B,$$

and is automatically a *cross-norm*, i.e., satisfies the relation

$$\|a \otimes b\| = \|a\| \, \|b\|, \qquad a \in A, \quad b \in B. \tag{A.4}$$

Therefore,

$$\left\| \sum_{j=1}^{s} a_j \otimes b_j \right\| \leq \sum_{j=1}^{s} \|a_j\| \, \|b_j\|,$$

which implies that the set of C^*-norms on $A \otimes B$ contains a maximal C^*-norm $\|\cdot\|_{\max}$ (which is equal to the supremum of all these norms). The completion of the algebra $A \otimes B$ with respect to this norm is called the *maximal* (or *projective*) tensor product of the algebras A and B and is denoted by $A \otimes_{\max} B$.

A.3.2 Nuclear Algebras

Definition A.50. A C^*-algebra A is said to be *nuclear* if there exists a unique C^*-norm on the product $A \otimes B$ (or, which is the same, $\|\cdot\|_{\min} = \|\cdot\|_{\max}$) for each C^*-algebra B. In this case, the completion of the product $A \otimes B$ with respect to this norm is simply called the *tensor product* of the algebras A and B and is denoted by $A \mathbin{\widehat{\otimes}} B$.

It turns out that the supply of nuclear C^*-algebras is rather wide.

Proposition A.51. *Any finite-dimensional C^*-algebra is nuclear.*

Proposition A.52. *If a C^*-algebra A has an inclusion-ordered family $\{A_j\}_{j \in \mathcal{J}}$ of nuclear subalgebras whose union is dense in A, then the algebra A itself is nuclear.*

Corollary A.53. *The algebra $\mathcal{K}(H)$ of compact operators in a Hilbert space H is nuclear.*

◀ Indeed, for the subalgebras A_j we can take the (finite-dimensional) algebras of operators in finite-dimensional subspaces of H. ▶

On the contrary, the algebra $\mathcal{B}(H)$ is not nuclear unless H is finite-dimensional.

Proposition A.54. *Any separable C^*-algebra of type I (in particular, any separable commutative C^*-algebra) is nuclear.*

A.3.3 Primitive Ideals in the Tensor Product

Let us find the structure of the primitive ideal space of the tensor product of C^*-algebras A and B. The tensor product of irreducible representations of A and B can be extended to an irreducible representation of the algebra $A \otimes_{\min} B$; therefore, there is a well-defined mapping

$$\iota\colon \operatorname{Prim} A \times \operatorname{Prim} B \longrightarrow \operatorname{Prim}(A \otimes_{\min} B), \tag{A.5}$$

where

$$\iota(\ker \pi_A, \ker \pi_B) = \ker(\pi_A \otimes_{\min} \pi_B). \tag{A.6}$$

This mapping turns out to be continuous (in the Jacobson topology), and its image is dense in $\operatorname{Prim}(A \otimes_{\min} B)$. If one of the algebras A and B is nuclear, then we have the following stronger statement.

Proposition A.55 (e.g., see [69]). *If A and B are C^*-algebras and at least one of them is nuclear, then the mapping* (A.5), (A.6) *is a homeomorphism of the direct product* $\operatorname{Prim} A \times \operatorname{Prim} B$ *onto* $\operatorname{Prim}(A \mathbin{\widehat{\otimes}} B)$.

Appendix B

K-Theory of Operator Algebras

B.1 Covariant K-Theory

Let X be a locally compact Hausdorff space. The correspondence $X \mapsto C_0(X)$ determines a contravariant functor from the category of locally compact Hausdorff spaces into the category of C^*-algebras, which is an isomorphism onto the subcategory of commutative C^*-algebras; the K-groups of a commutative C^*-algebra can be defined by the formula

$$K_*(A) = K^*(\widehat{A}), \qquad * = 0, 1, \tag{B.1}$$

where $\widehat{A}$ is the space of maximal ideals of the algebra A and K^* stands for the topological K-theory.

K-theory of C^*-algebras generalizes topological K-theory to the noncommutative case. In this theory, the K-groups of an arbitrary C^*-algebra A are defined in such a way that relation (B.1) is satisfied if the algebra A is commutative and the main results of topological K-theory (the six-term exact sequence and the Bott periodicity) also remain valid in the general case.

B.1.1 Topological K-Theory

We recall the construction of K-groups of topological spaces. First, let X be a compact Hausdorff space. The group $K^0(X)$ is defined as the Grothendieck group of the monoid whose elements are classes of isomorphic complex vector bundles over X with addition operation determined by the direct sum of vector bundles. By the Swan theorem, each vector bundle over X is a subbundle (and hence a direct summand) of a trivial bundle; therefore, the group $K^0(X)$ is generated by classes of stable equivalence of vector bundles over X. More precisely, two bundles $E_1, E_2 \in \operatorname{Vect} X$ define the same element in the group $K^0(X)$ if the bundles $E_1 \oplus \mathbb{C}^N$ and $E_2 \oplus \mathbb{C}^N$, where $\mathbb{C}^N$ is the trivial N-dimensional vector bundle over X, are isomorphic for some $N \geq 0$. If Y and X are compact Hausdorff spaces and $f\colon Y \to X$ is a continuous mapping, then there is a well-defined group homomorphism

$$f^*\colon K^0(X) \longrightarrow K^0(Y), \qquad [E] \longmapsto [f^*E], \quad E \in \operatorname{Vect} X. \tag{B.2}$$

Suppose that $f\colon Y \to X$ is an embedding. Then the *relative K^0-group* $K^0(X,Y)$ *of the pair* (X,Y) is defined as follows. Consider the Grothendieck group of the monoid whose elements are classes of isomorphic triples (E_1, E_2, φ), where $E_1, E_2 \in \operatorname{Vect} X$ and φ is an isomorphism of the restrictions of the bundles E_1 and E_2 to Y. Here an isomorphism of triples (E_1, E_2, φ) and (E_1', E_2', φ') is naturally determined as a pair of isomorphisms $\alpha_j\colon E_j \to E_j'$, $j = 1, 2$, such that the following diagram commutes over Y:

$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi} & E_2 \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ E_1' & \xrightarrow[\varphi']{} & E_2' \end{array}$$

A triple (E_1, E_2, φ) is trivial if φ can be extended to an isomorphism of E_1 and E_2. The relative K-group is the quotient of the Grothendieck group by the subgroup generated by trivial triples. In this case, we have the exact sequence of K-groups

$$\begin{array}{ccccc} K^0(X,Y) & \longrightarrow & K^0(X) & \xrightarrow{f^*} & K^0(Y), \\ [(E_1, E_2, \varphi)] & \longmapsto & [E_1] - [E_2]. & & \end{array} \tag{B.3}$$

The construction of the relative group is used to define the K-group of a locally compact Hausdorff space X: the one-point compactification $X^+ = X \cup \{pt\}$ of the space X is considered, and the group $K^0(X)$ is determined by the formula $K^0(X) = K(X^+, \{pt\})$; in this case, the corresponding sequence of the form (B.3) extends to give the exact sequence

$$0 \longrightarrow K^0(X, \{pt\}) \longrightarrow K^0(X^+) \xrightarrow{f^*} K^0(\{pt\}) \equiv \mathbb{Z} \longrightarrow 0,$$

so that $K^0(X)$ is simply the kernel of the homomorphism $K^0(X^+) \to K^0(\{pt\})$.

◀ This definition does not contradict the definition of the K-group in the case of a compact space X. Indeed, in this case, we have $X^+ = X \sqcup \{pt\}$ (the disjoint union); accordingly,

$$K^0(X^+) = K^0(X) \oplus K^0(\{pt\}),$$

the homomorphism f^* coincides with the projection onto the second summand, and its kernel is exactly $K^0(X)$. ▶

If X is a (locally) compact space, then the *suspension* over X is defined by the formula $SX = X \times \mathbb{R}$. The *odd K-group* of the space X is introduced by the formula

$$K^1(X) = K^0(SX).$$

The subsequent suspensions do not lead to new groups; namely, the following statement is true.

Theorem B.1 (Bott periodicity). *There is an isomorphism*

$$\beta\colon K^*(X) \longrightarrow K^*(S^2X), \qquad * = 0, 1.$$

◀ In terms of representatives, the isomorphism β takes the equivalence class $[E] \in K^0(X)$ of a bundle E on X to the equivalence class of the exterior tensor product $E \boxtimes \tau$, where τ is the (virtual) Bott bundle on $\mathbb{R}^2$, which can be described as follows. One has $K^0(\mathbb{R}^2) = K^0(\mathbb{D}^2, \mathbb{S}^1)$, where $\mathbb{D}^2$ is the unit disk in the plane $\mathbb{R}^2$ and $\mathbb{S}^1$ is its boundary, i.e., the unit circle. Then $\tau = (\mathbb{C}, \mathbb{C}, z)$, where $\mathbb{C}$ is the trivial one-dimensional bundle over $\mathbb{D}^2$ and z is the automorphism of restriction of this bundle to $\mathbb{S}^1$ determined by the multiplication by the function $z = x + iy$. (Here (x, y) are the standard coordinates in $\mathbb{R}^2$.) ▶

Theorem B.2 (Six-term exact sequence). *If X is a locally compact space and $Y \subset X$ is a closed subspace, then one has the natural exact sequence of K-groups*

$$\begin{array}{ccccc} K^0(X \setminus Y) & \longrightarrow & K^0(X) & \longrightarrow & K^0(Y) \\ \partial \big\uparrow & & & & \big\downarrow \partial \\ K^1(Y) & \longleftarrow & K^0(Y) & \longleftarrow & K^1(X \setminus Y). \end{array} \tag{B.4}$$

◀ Here we do not consider the definition of vertical arrows (the connecting homomorphisms ∂) and refer the reader to any standard text on topological K-theory. ▶

Bearing in mind the generalization of notions of K-theory to the case of C^*-algebras, which is described in the subsequent sections, it is useful to have an alternative description of K-groups, which appeals only to the algebra of continuous functions on the space X rather than directly to the space X. We give such a description for the case of a compact space X. Recall that, by the Swan theorem, each vector bundle E over X is a direct summand in some trivial bundle $\mathbb{C}^N$ over X and hence can be described as the image of some $N \times N$ matrix projection p whose entries are continuous functions on X. In other words, $p \in M_N(C(X))$. Thus, the elements of the group $K^0(X)$ can be treated as equivalence classes of projections in matrix algebras over $C(X)$. (Here we do not describe this equivalence relation, because this will be done later in general form.) To describe the elements of the group $K^1(X)$, we use the relation

$$K^1(X) = K^0(X \times \mathbb{R}) \simeq K^0\big(X \times [0,1], (X \times \{0\}) \cup (X \times \{1\})\big).$$

The elements of this group can be described as equivalence classes of quadruples $(E_1, E_2, \alpha, \beta)$, where $E_1, E_2 \in \mathrm{Vect}(X)$, α and β are isomorphisms of the bundles E_1 and E_2, and two quadruples are said to be equivalent if there exist families (with parameter $t \in [0,1]$) of isomorphisms of the corresponding bundles which conjugate the isomorphisms α for $t = 0$ and the isomorphisms β for $t = 1$. Obviously, in each equivalence class there is an element for which $\beta = \mathrm{id}$ and $E_1 = E_2 = \mathbb{C}^N$ for sufficiently large N. In this case, α turns out to be an invertible $N \times N$ matrix function on X, so that the elements of the group $K^1(X)$ can be treated as equivalence classes of invertible matrix functions on X, i.e., of invertible elements in matrix algebras over $C(X)$.

B.1.2 Group $K_0(A)$

Let A be a C^*-algebra. We define an embedding $j_{mn}\colon M_m(A) \to M_n(A)$ for $m, n \in \mathbb{Z}_+$, $m \le n$, by the formula

$$a \longmapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

(i.e., by adding $n - m$ zero columns and rows to the matrix a on the right and below).

The embeddings j_{mn} are isometric (which can readily be verified if we assume that A is an operator algebra on a Hilbert space) and satisfy the chain rule $j_{nl} \circ j_{mn} = j_{ml}$, $m \le n \le l$. Consider the inductive limit

$$M(A) = \liminf_{n\to\infty} M_n(A).$$

Its completion $\overline{M}(A)$ is a C^*-algebra (the inductive limit of the C^*-algebras $M_n(A)$). But we shall use only the algebra $M(A)$ itself, which can be described as the algebra of right and downward infinite matrices over A in which only finitely many entries are nonzero. For each n, the algebra $M_n(A)$ is naturally embedded into $M(A)$ as a subalgebra, and

$$M(A) = \bigcup_{n=0}^{\infty} M_n(A).$$

Now assume that A is unital. Let p, q be projections in the algebra $M(A)$. (In this case, one also says that p and q are projections *over* a.) Recall that this means that

$$p = p^* = p^2, \quad q = q^* = q^2.$$

The projections p and q are said to be *equivalent*, $p \sim q$, if for some n there exists a unitary element $u \in M_n(A)$ such that $u^{-1} p u = q$.

◀ Putting this differently, the projections p and q have finitely many nonzero entries, and hence $p \in M_m(A)$ and $q \in M_l(A)$ for some m and l. The projections p and q are equivalent if they can be bordered by zeros so that the resulting matrix projections have the same size $n \times n$ and are similar (unitary equivalent); i.e.,

$$u^{-1}\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}. \quad ▶$$

Remark B.3. Instead of similarity, we could use algebraic equivalence or homotopy, but the result would be the same because of Propositions A.22 and A.23.

Proposition B.4. *The equivalence relation introduced above is consistent with direct sums and homotopies of projections. Moreover,*

$$p \oplus q \sim q \oplus p$$

for any projections p and q over the algebra A.

Definition B.5. The *group* $K_0(A)$ of a unital C^*-algebra A is the Grothendieck group of the monoid of equivalence classes of projections over A with addition in the monoid being induced by the direct sum of projections.

If $f\colon A \to B$ is a homomorphism of unital C^*-algebras, then the induced homomorphism $M(A) \to M(B)$ obtained by applying the homomorphism f term by term takes direct sums to direct sums, projections to projections, and unitary elements in $M_n(A)$ to the unitary elements in the corresponding $M_n(B)$. Therefore, the induced homomorphism of K-groups is well defined:

$$f_*\colon K_0(A) \longrightarrow K_0(B).$$

Now we define the group $K_0(A)$ for a nonunital algebra A. We recall that in the commutative case a nonunital algebra corresponds to a noncompact locally compact space X and that the definition of the K-group of a noncompact space in topological K-theory is based on the use of the one-point compactification of the space X. An analog of the one-point compactification in C^*-algebras is given by unitization, which motivates the following construction. We add the unit to the algebra A and consider the exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow A & \longrightarrow & A^+ & \xrightarrow{\pi} & \mathbb{C} \longrightarrow 0, \\ a & \longmapsto & (0,a), & & \\ & & (\lambda, a) & \longmapsto & \lambda. \end{array}$$

Definition B.6. The kernel of the induced mapping

$$\pi_*\colon K_0(A_+) \longrightarrow K_0(\mathbb{C}) = \mathbb{Z}$$

is called the K_0-*group* of the algebra A and is denoted by $K_0(A)$.

One can readily see that with this definition we have the exact sequence

$$0 \longrightarrow K_0(A) \longrightarrow K_0(A^+) \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{B.5}$$

◀ Just as in the topological case, the two definitions do not contradict each other if A is actually unital. Indeed, in this case, $A^+ = A \oplus \mathbb{C}$, $K_0(A^+) = K_0(A) \oplus \mathbb{Z}$, and the homomorphism π_* is simply the projection onto the second factor, so that its kernel exactly coincides with $K_0(A)$. ▶

The general element of the group $K_0(A)$ can be represented as the difference $[p] - [I_n]$ of equivalence classes, where $p \in M(A^+)$ is a projection, $I_n \in M_n(A^+)$ is the unit $n \times n$ matrix, and $p - I_n \in M(A)$.

Proposition B.7. *If $A = C(X)$, where X is a locally compact Hausdorff space, then the following relation holds*:

$$K_0(A) = K^0(X).$$

◀ Indeed, in this case the above definition is an exact translation of the topological definition into the language of C^*-algebras. ▶

If

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

is a short exact sequence of C^*-algebras, then the corresponding three-term sequence

$$K_0(J) \longrightarrow K_0(A) \longrightarrow K_0(A/J) \tag{B.6}$$

is also exact. To continue it to either side as an exact sequence, one needs to introduce odd K-groups, just as in the topological case, and this will be done in the next section.

B.1.3 Group $K_1(A)$

In topology, the odd K-group is defined using the suspension, i.e., the passage from the space X to the product $X \times \mathbb{R}$. Let us translate this passage into the language of C^*-algebras. We have the relation

$$C_0(X \times \mathbb{R}) = C_0(X) \widehat{\otimes} C_0(\mathbb{R}) = C_0(\mathbb{R}, C_0(X)).$$

Here the tensor product of C^*-algebras $\widehat{\otimes}$ is unique, since the algebra $C_0(\mathbb{R})$ is nuclear. This relation motivates the following general definition.

Definition B.8. The *suspension* SA over a C^*-algebra A is defined to be the algebra

$$SA = C_0(\mathbb{R}) \widehat{\otimes} A = C_0(\mathbb{R}, A).$$

The K_1-*group* of the C^*-algebra A is determined by the formula

$$K_1(A) = K_0(SA).$$

Taking the above into account, we obtain the following statement.

Proposition B.9. *If $A = C(X)$, where X is a locally compact Hausdorff space, then*

$$K_1(A) = K^1(X).$$

Let us give an alternative description of elements of the group $K_1(A)$. For simplicity, let A be a unital algebra. By $U(A)$ we denote the direct limit of the groups $U_n(A)$ of unitary $n \times n$ matrices over A with respect to the embeddings $j'_{mn} \colon U_m(A) \to U_n(A)$,

$$j'_{mn}(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

The elements of the group $U(A)$ can be treated as right and downward infinite matrices of the form

$$\mathfrak{u} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \equiv u \oplus 1,$$

where $u \in U_n(A)$ for some n and 1 is the infinite identity matrix. The mapping $u \mapsto \mathfrak{u}$ gives a natural embedding $U_n(A) \to U(A)$. Two elements $\mathfrak{u}, \mathfrak{v} \in U(A)$ are said to be equivalent ($\mathfrak{u} \sim \mathfrak{v}$) if they are homotopic.

Theorem B.10. *The group $K_1(A)$ is isomorphic to the Grothendieck group of the monoid of equivalence classes of elements of the group $U(A)$ with respect to the equivalence relation described above. Here the operation of addition in the monoid is determined by the direct sum of elements: if $\mathfrak{u} = u \oplus 1$ and $\mathfrak{v} = v \oplus 1$ are two elements of group $U(A)$, then their direct sum is defined as $u \oplus v \oplus 1$.*

◀ Now let us explain the structure of this isomorphism. To this end, we need to assign some class in the group $K_0(SA)$ to each element $u \in U_n(A)$. This can be done as follows. According to Proposition A.25, in the group $U_{2n}(A)$ there exists a homotopy of the element $u \oplus u^*$ to the element 1. Let

$$w(t), \quad t \in [-\infty, \infty], \qquad w(-\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad w(\infty) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$$

be such a homotopy. Consider the projections p_1 and p_2 in the algebra $M_{2n}(SA^+)$ given by the formulas

$$p_1(t) = w(t)^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w(t), \qquad p_2(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(The second of the projections is a constant function of t.) Note that

$$p_1(-\infty) = p_1(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

so that

$$p_1 - p_2 \in M_{2n}(SA)$$

and the pair (p_1, p_2) gives a well-defined element $[p_1] - [p_2] \in K_0(SA)$. We can show that this element is independent of the choice of u in the equivalence class and of the ambiguity in the construction itself and that the mapping described above determines the desired isomorphism. ▶

Remark B.11. Suppose that elements $[u], [v] \in K_1(A)$ are represented by unitary matrices u, v of the same size. Then the class $[u] + [v]$ is equal to the class of composition uv (or vu). This follows from Proposition A.24.

B.1.4 Bott Periodicity

Just as in the topological case, further suspensions do not lead to new K-groups. Namely, the following statement holds.

Theorem B.12 (Bott periodicity in K-theory of operator algebras). *One has the isomorphism*

$$\beta\colon K_0(A) \longrightarrow K_0(S^2A). \tag{B.7}$$

◀ Let us describe the mapping

$$\beta\colon K_0(A) \longrightarrow K_0(S^2A).$$

It is convenient to do this by interpreting $K_0(S^2A)$ as $K_1(SA)$ and by using Theorem B.10. First, we assume that A is unital. Thus, on the level of representatives of equivalence classes, given a projection p over A, it is required to construct an invertible matrix function over SA^+, or, which is the same, an invertible matrix function $F_p(t)$ on $[-\infty, \infty]$ ranging in $U(A)$ and satisfying the condition

$$F_p(-\infty) = F_p(+\infty) = 1.$$

We define the mapping

$$z\colon [-\infty, \infty] \longrightarrow \mathbb{S}^1, \qquad t \longmapsto e^{i(\pi + 2\arctan t)}$$

and set

$$F_p(t) = z(t)p + \mathbf{1} - p.$$

(Here $\mathbf{1}$ is the identity element of the group $U(A)$.) One can readily verify that this is a unitary matrix function. Then the mapping β is given by the formula

$$\beta([p]) = [F_p].$$

For the case in which A is not unital, the desired function should range in $U(A^+)$, and the formula for the mapping β becomes

$$\beta([p] - [q]) = [F_p F_q^{-1}],$$

where $p, q \in M(A^+)$ are projections such that $p - q \in M(A)$. ▶

B.1.5 Long Exact Sequence in K-Theory

The six-term exact sequence (B.4) in topological K-theory can be generalized to K-theory of operator algebras, where the following result holds.

Theorem B.13 (six-term sequence in K-theory of algebras). *Let*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

be a short exact sequence of C^*-algebras. Then the three-term sequence (B.6) *is included in the exact sequence*

$$\begin{array}{ccccc} K_0(J) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/J) \\ \partial_1 \uparrow & & & & \downarrow \partial_2 \\ K_1(A/J) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J). \end{array} \tag{B.8}$$

As usual, the construction of *connecting homomorphisms* ∂_0 and ∂_1 in this sequence is nontrivial (and hence interesting). Let us describe these homomorphisms.

Homomorphism ∂_1

Let us describe the mapping

$$\partial_1 \colon K_1(A/J) \longrightarrow K_0(J).$$

Consider the element $[u] \in K_1(A/J)$ corresponding to some unitary element $u \in U_n(A/J)$. Using Proposition A.25, we lift the element $u \oplus u^* \in U_{2n}(A/J)$ to a unitary element $w \in U_{2n}(A)$ homotopic to the unit. Then the connecting homomorphism ∂_1 is given on the element $[u]$ by the formula

$$\partial_1[u] = \left[w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^* \right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0(J).$$

◀ To verify that this formula is well defined, note that the element w in the block 2×2 representation is diagonal modulo elements of $M_{2n}(J)$ and hence commutes modulo such elements with the projection $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. It follows that

$$w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^* \in M_{2n}(J^+)$$

and

$$w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^* - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_{2n}(J),$$

as desired. ▶

The mapping ∂_1 is called the *index mapping*, because it coincides with the usual (analytic) index of a Fredholm operator in the following situation. Let A be a subalgebra of $\mathcal{B}(H)$ containing the ideal $J = \mathcal{K}(H)$ of compact operators. Then $K_0(J) = \mathbb{Z}$, and the mapping

$$\partial_1 \colon K_1(A/J) \longrightarrow \mathbb{Z}$$

has the form

$$\partial_1([a]) = \operatorname{ind} \widehat{a},$$

where $a \in GL_n(A/J)$ is an invertible element and $\widehat{a}$ is an arbitrary lift of a to $M_n(A)$ (which is necessarily a Fredholm operator and whose index is independent of the choice of the lift).

Remark B.14. In this case, a can be treated as the "symbol" of the operator $\widehat{a}$.

Homomorphism ∂_0

The homomorphism

$$\partial_0 \colon K_0(A/J) \longrightarrow K_1(J)$$

is called the *exponential mapping*. Of course, one can describe it by using the already known description of the homomorphism ∂_1 and of the suspension homomorphism. Explicitly, the formula determining this homomorphism reads

$$\partial_0 \left([w] - \left[\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \right] \right) = [e^{2\pi i z}],$$

where w is a projection in $M((A/J)^+)$ such that

$$w - \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \in M(A/J)$$

and z is a lift of w to $M(A^+)$.

B.1.6 Stability of K-Groups

The following statement holds.

Theorem B.15. *For an arbitrary C^*-algebra A, the following isomorphisms hold*:

$$K_j(A) \simeq K_j(M_n(A)), \quad K_j(A) \simeq K_j(A \widehat{\otimes} \mathcal{K}(H)), \qquad j = 1, 2.$$

These isomorphisms are induced, respectively, by the natural embedding $A \to M_n(A)$ (as the left top corner) and by the embedding

$$A \longrightarrow A \widehat{\otimes} \mathcal{K}(H),$$
$$a \longmapsto a \otimes p,$$

where p is an arbitrary given projection of rank 1 in the Hilbert space H.

B.1.7 K-groups of Local C^*-Algebras

The theory described in the preceding sections can be transferred to local C^*-algebras without any essential changes. In this case, we have the following theorem.

Theorem B.16. *Let A be a C^*-algebra, and let $B \subset A$ be a dense local C^*-subalgebra in A. Then*

$$K_j(A) = K_j(B), \qquad j = 0, 1.$$

B.2 K-Homology

Just as in the case of covariant K-theory, the K-homology $K^*(A)$ of an arbitrary C^*-algebra A is a generalization of the K-homology of manifolds to the noncommutative case. Namely, it is defined in such a way that if the algebra A is commutative, then

$$K^*(A) = K_*(\widehat{A}), \qquad * = 0, 1,$$

where $\widehat{A}$ is the maximal ideal space of the algebra A. When passing to algebras, the variance of the functor changes to the opposite, so that, despite the name, the K-homology functor for algebras turns out to be *contravariant.* (Because of this, some authors think that this name is not too apt.) Thus, the reader should be careful here.

B.2.1 K-Homology of a Topological Space

First, let us briefly recall the definition of the K-homology of a compact Hausdorff space X. K-homology is dual to K-theory, and one can give a purely topological definition of the K-homology groups (for example, if X is a compact manifold, then we embed X in a sphere $\mathbb{S}^N$ of sufficiently large odd dimension N and set

$$K_*(X) = K^*(S^N \setminus U),$$

where U is a tubular neighborhood of the manifold X in $\mathbb{S}^N$.) But we are interested in definitions related to operators and operator algebras.

The K-homology classes of the space X can be defined in terms of *abstract elliptic operators* on X. The notion of abstract elliptic operator was first introduced by Atiyah [8]: an abstract elliptic operator is defined to be a triple (D, H_0, H_1), where H_0 and H_1 are Hilbert spaces equipped with actions π_0 and π_1 of the algebra $C(X)$ and $D\colon H_0 \to H_1$ is a Fredholm operator such that

$$D\pi_0(\varphi) - \pi_1(\varphi)D \in \mathcal{K}(H_0, H_1) \qquad \text{for any } \varphi \in C(X).$$

In other words, the operator D intertwines the actions of the algebra $C(X)$ on H_0 and H_1 modulo compact operators. The K-homology classes of the space X are equivalence classes of abstract elliptic operators with respect to a certain equivalence relation. Let us present Kasparov's formal definition.

Definition B.17. A triple (H, π, F), where H is a separable Hilbert space,

$$\pi\colon C(X) \to \mathcal{B}(H)$$

is a unital representation of the algebra $C(X)$ in the space H, and $F \in \mathcal{B}(H)$ is a self-adjoint linear operator such that

$$F^2 - 1 \in \mathcal{K}(H), \qquad [F, \pi(\varphi)] \in \mathcal{K}(H) \quad \text{for any } \varphi \in C(X), \tag{B.9}$$

is called a *Fredholm module* over the algebra $C(X)$. A Fredholm module such that the Hilbert space H is $\mathbb{Z}_2$-graded, the action of the algebra $C(X)$ preserves the grading, and the operator F is odd with respect to the grading, is called a *graded Fredholm module.*

A Fredholm module is said to be *degenerate* if

$$F^2 = 1, \qquad [F, \pi(\varphi)] = 0 \quad \text{for any } \varphi \in C(X).$$

Obviously, a direct sum of Fredholm modules (whose definition is obvious) is again a Fredholm module. Two Fredholm modules are said to be *equivalent* if, possibly after the addition of degenerate modules as direct summands and a unitary transformation, they are homotopic to each other. (A homotopy of Fredholm modules is understood as a norm-continuous homotopy of the operator F in the class of operators satisfying the condition (B.9).) For graded Fredholm modules, the definition of equivalence is similar but uses only graded degenerate modules, unitary transformations that preserve the grading, and homotopies in the class of graded modules.

By $K_1(X)$ we denote the set of equivalence classes of Fredholm modules over $C(X)$. By $K_0(X)$ we denote the set of equivalence classes of graded Fredholm modules over $C(X)$. One can readily show that these sets are Abelian groups with respect to the operation induced by the direct sum of modules, and the equivalence class of degenerate modules is the zero element.

Definition B.18. The groups $K_0(X)$ and $K_1(X)$ are called the *K-homology groups* of the space X.

Now let us explain how to assign elements of K-homology groups to abstract elliptic operators on X. If $D\colon H_0 \to H_1$ is an abstract elliptic operator, then one can define a graded Fredholm module by setting

$$H = H_0 \oplus H_1, \qquad F = \begin{pmatrix} 0 & D(P_{\ker D} + D^*D)^{-1/2} \\ (P_{\ker D} + D^*D)^{-1/2}D^* & 0 \end{pmatrix},$$

where $P_{\ker D}$ is the projection onto the kernel of the operator D. If $D\colon H \to H$ is a self-adjoint abstract elliptic operator, then one can define a Fredholm module by setting

$$F = 2P_+(D) - 1,$$

where $P_+(D)$ is the projection onto the positive spectral subspace of the operator D. Conversely, given a Fredholm module, one can construct an abstract elliptic operator on X. This operator is either F itself or (for graded Fredholm modules) the component of F acting from H_0 into H_1.

Thus, arbitrary abstract elliptic operators on X generate elements of the group $K_0(X)$, and self-adjoint abstract elliptic operators generate elements of the group $K_1(X)$.

Since a continuous mapping $f\colon X \to Y$ determines the algebra homomorphism

$$f^*\colon C(Y) \to C(X)$$

and thus turns any $C(X)$-module into a $C(Y)$-module, it is clear that the correspondence $X \mapsto K_*(X)$ is a covariant functor.

There are natural pairings between the K-groups and the respective K-homology groups of the space X. Let us describe these pairings.

Even Groups

Let $[E] \in K^0(X)$ be the equivalence class of some vector bundle E on the manifold X, and let $[D] \in K_0(X)$ be the equivalence class of an abstract elliptic operator $D\colon H_1 \to H_2$. The bundle E can be described as the image of some projection $P \in \mathrm{Mat}_N(C(X))$ in the trivial bundle with fiber $\mathbb{C}^N$ over X. The actions (which we denote by the same letters π_1 and π_2) of the algebra $\mathrm{Mat}_N(C(X))$ in the Hilbert spaces $H_1 \otimes \mathbb{C}^N$ and $H_2 \otimes \mathbb{C}^N$ are naturally defined, and we define the operator $D \otimes E$ ("the operator D with coefficients in the bundle E") by setting

$$D \otimes E = \pi_2(P)\bigl(D \otimes 1_{\mathbb{C}^N}\bigr)\colon \pi_1(P)(H_1 \otimes \mathbb{C}_N) \longrightarrow \pi_2(P)(H_2 \otimes \mathbb{C}_N).$$

Since the operator D "almost commutes" (i.e., commutes modulo compact operators) with the components of the projection P, it readily follows that $D \otimes E$ is a Fredholm operator. The pairing of the groups $K_0(X)$ and $K^0(X)$ is given by the formula

$$\begin{aligned} K_0(X) \times K^0(X) &\longrightarrow \mathbb{Z}, \\ ([D],[E]) &\longmapsto \operatorname{ind}(D \otimes E). \end{aligned} \tag{B.10}$$

Proposition B.19. *The pairing* (B.10) *is well defined and nondegenerate on the free parts of the groups $K_0(X)$ and $K^0(X)$.*

Odd Groups

Let $[f] \in K^1(X)$ be the equivalence class of some invertible $n \times n$ matrix function f on the space X, and let $[D] \in K_1(X)$ be the equivalence class of some self-adjoint abstract elliptic operator $D\colon H \to H$. Then the operator

$$P_+(D)f\colon P_+(D)H^n \longrightarrow P_+(D)H^n$$

(an *abstract Toeplitz operator* with symbol f) is a Fredholm operator (for an almost inverse operator one can take $P_+(D)f^{-1}$), and the pairing of the groups $K_1(X)$ and $K^1(X)$ is given by the formula

$$\begin{aligned} K_1(X) \times K^1(X) &\longrightarrow \quad \mathbb{Z}, \\ ([D],[f]) &\longmapsto \quad \operatorname{ind}(P_+(D)f). \end{aligned} \tag{B.11}$$

Proposition B.20. *The pairing* (B.11) *is well defined and nondegenerate on the free parts of the groups $K_1(X)$ and $K^1(X)$.*

B.2.2 K-Homology of Operator Algebras: Definitions

In this section, we present several different versions of the definition. From now on, we assume as a rule, without stipulating this explicitly, that the C^*-algebra A under study is separable and nuclear.

Fredholm Modules

Kasparov's definition, which is apparently the most technically convenient definition of K-homology in terms of Fredholm modules, provides a direct generalization of the corresponding definition given above for the algebra $C(X)$.

Definition B.21. Let A be a C^*-algebra. A triple (H, π, F), where H is a separable Hilbert space, $\pi\colon A \to \mathcal{B}(H)$ is a representation of A on H, and $F \in \mathcal{B}(H)$ is a linear self-adjoint operator such that

$$\begin{gathered} (F^2 - 1)\pi(a) \in \mathcal{K}(H), \quad [F, \pi(a)] \in \mathcal{K}(H) \\ \text{for any } a \in A, \end{gathered} \tag{B.12}$$

is called a *Fredholm module* over A. A Fredholm module such that the Hilbert space H is $\mathbb{Z}_2$-graded ($H = H_0 \oplus H_1$), the action of the algebra A preserves the grading ($AH_j \subset H_j$, $j = 0, 1$), and the operator F is odd with respect to the grading ($FH_j \subset H_{1-j}$, $j = 0, 1$), is called a *graded Fredholm module.*

◀ The definition is stated in this form so it can also be used in the case of a nonunital algebra A. Of course, the factor $\pi(a)$ in the condition $(F^2 - 1)\pi(a) \in \mathcal{K}(H)$ can be omitted if the algebra A is unital. ▶

Sometimes (if H and π are clear from the context or insignificant), we denote the Fredholm module simply by F.

Two Fredholm modules over A are said to be *homotopic* if the spaces H and representations π of these modules are the same and the operators F can be related by a norm-continuous homotopy such that all intermediate triples are Fredholm modules. For graded Fredholm modules, it is additionally required that all the intermediate triples be graded Fredholm modules.

Direct sums and *unitary equivalence* of Fredholm modules are naturally defined in both the graded and ungraded cases.

Definition B.22. We define *K-homology groups* $K^*(A)$, $* = 0, 1$, to be the Abelian group generated by the following generators and relations.

The generators of this group are the classes of unitary equivalence of graded Fredholm modules (for $* = 0$) or of Fredholm modules (for $* = 1$).

The relations have the form $[F_1] = [F_2]$ if F_1 and F_2 are homotopic, and

$$[F_1 \oplus F_2] = [F_1] + [F_2] \quad \text{for any } F_1 \text{ and } F_2.$$

A Fredholm module is said to be *degenerate* if

$$(F^2 - 1)\pi(a) = 0, \qquad [F, \pi(a)] = 0 \quad \text{for any } a \in A.$$

Proposition B.23. *The degenerate Fredholm modules determine the zero class in the K-homology of the algebra A.*

If $f\colon A \to B$ is a C^*-algebra homomorphism, then to each representation π of the algebra B we can assign the representation

$$f^*(\pi) = \pi \circ f$$

of the algebra A, and hence each Fredholm module over B becomes a Fredholm module over A. One can readily see that this correspondence gives rise to a group homomorphism

$$f^*\colon K^*(B) \longrightarrow K^*(A),$$

and the correspondence $A \mapsto K^*(A)$ turns out to be a contravariant functor from the category of C^*-algebras to the category of Abelian groups.

Dual Algebras

Another definition of K-homology can be given in terms of dual algebras. Let A be a C^*-algebra.

Definition B.24. A representation $\pi\colon A \to \mathcal{B}(H)$ of the algebra A in a Hilbert space H is said to be *ample* if

1. The representation π is nondegenerate.
2. The operator $\pi(a)$ is compact only if $a = 0$.

Suppose that an ample representation $\pi\colon A \to \mathcal{B}(H)$ of the algebra A is given.

Definition B.25. The subalgebra

$$\mathcal{D}(A) = \{T \in \mathcal{B}(H)\colon [T, \pi(a)] \in \mathcal{K}(H) \quad \text{for any } a \in A\}$$

is called the *dual algebra* of the algebra A.

Proposition B.26. *The algebra $\mathcal{D}(A)$ is well defined. Namely, Definition* B.25 *gives the same result (up to an isomorphism) for any ample representation of A.*

Now we can give the desired definition.

Definition B.27. The *K-homology groups* of a C^*-algebra A are determined by the formulas

$$K^j(A) = K_{1-j}(\mathcal{D}(A^+)), \qquad j = 0, 1. \tag{B.13}$$

Recall that here A^+ is the algebra obtained from A by adjoining a unit.

We present one more useful expression for the K-homology groups in terms of the dual algebra.

Definition B.28. Let J be an ideal in the C^*-algebra A. By $\mathcal{D}(A /\!\!/ J)$ we denote the ideal

$$\mathcal{D}(A /\!\!/ J) = \{T \in \mathcal{D}(A)\colon T\pi(a) \in \mathcal{K}(H) \quad \text{for any } a \in J\}$$

of the algebra $\mathcal{D}(A)$ consisting of the elements that become compact after multiplication by any element of J.

In particular, the ideal $\mathcal{D}(A /\!\!/ A)$ is called the ideal of *locally compact operators*.

◀ This terminology becomes clear if we consider the algebra $A = C_0(X)$ of functions vanishing at infinity on a locally compact Hausdorff space X. In this case, the elements of the ideal $\mathcal{D}(C_0(X) /\!\!/ C_0(X))$ are the operators that become compact after multiplication by any compactly supported function. ▶

Proposition B.29. *The following formula holds*:

$$K^j(A) = K_{1-j}\big(\mathcal{D}(A)/\mathcal{D}(A /\!\!/ A)\big), \qquad j = 0, 1. \tag{B.14}$$

This formula does not appeal to unitization (which is often important in practice if an ample representation precisely of the algebra A itself is originally given) and holds both for unital and nonunital algebras.

Remark B.30. Let M be a compact smooth manifold, and consider the natural representation $C(M) \to \mathcal{B}L^2(M)$ as operators of multiplication. Since $C(M)$ is unital, $\mathcal{D}(C(M) /\!\!/ C(M)) = \mathcal{K}$ and the K-homology group is by definition

$$K^0(C(M)) = K_1(\mathcal{D}(C(M))/\mathcal{K}).$$

Since K_1 is generated by homotopy classes of unitary elements, the latter formula recovers Atiyah's realization of K-homology classes by abstract elliptic operators.

Extensions of C^*-Algebras

The third (and the last) definition of K-homology, which we present here, is related to extensions of C^*-algebras. We consider only the definition of the group $K^1(A)$; the definition of the group $K^0(A)$ can be obtained by using the suspension (for details, see below).

Definition B.31. A short exact sequence of C^*-algebras of the form

$$0 \longrightarrow \mathcal{K}(H) \longrightarrow B \longrightarrow A \longrightarrow 0 \tag{B.15}$$

is called an *extension* of the C^*-algebra A (by the algebra of compact operators in a separable infinite-dimensional Hilbert space H). Two extensions of the algebra A are said to be *isomorphic* if there exists an isomorphism of the corresponding exact sequences identical in the term A. An extension is said to *split* if there exists a C^*-algebra homomorphism $A \to B$ splitting the sequence (B.15).

Definition B.32. A homomorphism of a C^*-algebra A into the Calkin algebra $\mathcal{Q}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ of a separable Hilbert space H is called a *quantization* of the algebra A. Two quantizations $\mu_j : A \to \mathcal{Q}(H_j)$, $j = 1, 2$, are said to be *unitarily equivalent* if there exists a unitary operator $U : H_1 \to H_2$ such that

$$\mu_2(a) = \operatorname{Ad}_U(\mu_1(a)), \qquad a \in A.$$

◀ The mapping

$$\operatorname{Ad}_U : \mathcal{B}(H_1) \to \mathcal{B}(H_2)$$

takes $\mathcal{K}(H_1)$ to $\mathcal{K}(H_2)$ and hence can be factorized to a mapping of Calkin algebras. ▶

Proposition B.33. *There exists a one-to-one correspondence between classes of isomorphic extensions and classes of unitarily equivalent quantizations of the algebra A. This correspondence can be described by the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}(H) & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \mu & & \\ 0 & \longrightarrow & \mathcal{K}(H) & \longrightarrow & \mathcal{B}(H) & \longrightarrow & \mathcal{Q}(H) & \longrightarrow & 0, \end{array}$$

where the upper row represents an arbitrary extension in a given equivalence class and the last vertical arrow μ represents some quantization (*depending on the choice of the extension*) *in the class of equivalent quantizations corresponding to it.*

Definition B.34. The equivalence classes of extensions of the algebra A form a semigroup with respect to the operation induced by the direct sum of representatives of these classes. This semigroup is denoted by $\operatorname{Ext}(A)$.

◀ Here we slightly digress from the traditional definition, where only injective extensions (extensions for which the corresponding quantization is injective) are considered. The point is that we deal only with separable algebras, for which the requirement to be injective is superfluous: any extension can be transformed into an injective extension by adding a direct summand that is a trivial injective extension, and by the following theorem, this addition does not change the equivalence class of the extension. ▶

Proposition B.35 (Voiculescu theorem). *If A is a separable C^*-algebra, then the equivalence class of any split extension is the zero element of the semigroup* $\operatorname{Ext}(A)$*. In particular, all split extensions are equivalent.*

Of all possible quantizations, the special class of so-called *Toeplitz quantizations* can be distinguished. Let $\pi\colon A \to \mathcal{B}(H)$ be a representation of a C^*-algebra A in a Hilbert space H, and let $P\colon H \to H$ be a projection such that

$$[P, \pi(a)] \in \mathcal{K}(H), \quad a \in A.$$

Then the mapping

$$a \longmapsto Pa\colon PH \longrightarrow PH$$

gives a well-defined homomorphism

$$\mu_P\colon A \longrightarrow \mathcal{Q}(PH),$$

which is called the *Toeplitz quantization corresponding to the projection P*.

Theorem B.36. *Let A be a separable nuclear C^*-algebra. Then the following statements hold*:

1. *The semigroup* $\operatorname{Ext}(A)$ *is a group, whose zero element is the equivalence class of split extensions.*
2. *In each class $\mu \in \operatorname{Ext}(\mathrm{A})$, there is a Toeplitz quantization.*
3. *The inverse element of the class $[\mu_P]$ of a Toeplitz quantization μ_P is the class $[\mu_{1-P}]$ of the Toeplitz quantization μ_{1-P} acting in the complementary subspace $(PH)^\perp = (1-P)H$.*

Now we can give the desired definition.

Definition B.37. The K^1-*homology group* of a separable nuclear C^*-algebra A is given by the formula

$$K^1(A) = \operatorname{Ext}(\mathrm{A})\,.$$

Equivalence of Different Definitions

The above three versions of the definition of K-homology of separable nuclear C^*-algebras are equivalent to each other. Let us show how the correspondence between the elements of the K-homology groups described in different definitions can be established.

Fredholm modules and duality. Let $[p] \in K_0(\mathcal{D}(A^+))$ be the equivalence class of some projection over $\mathcal{D}(A^+)$. Let us write out the Fredholm module determining the corresponding element in $K^1(A)$. Let $p \in M_n(\mathcal{D}(A^+))$. One can readily see that if the algebra $\mathcal{D}(A^+)$ is realized via an ample representation π of the algebra A^+ in the space H, then

$$M_n(\mathcal{D}(A^+)) = \{T \in \mathcal{B}(H \otimes \mathbb{C}^n)\colon [T, \pi(a) \otimes 1_n] \in \mathcal{K}(H \otimes \mathbb{C}^n), \quad a \in A^+\}.$$

But the representation $\pi(\,\cdot\,) \otimes 1_n$ of the algebra A^+ is also ample in $H \otimes \mathbb{C}^n$ (recall that A^+ is unital), and hence the algebra $M_n(\mathcal{D}(A^+))$ is isomorphic to the algebra $\mathcal{D}(A^+)$ by Proposition B.26. Thus, without loss of generality, we can assume that $p \in \mathcal{D}(A^+)$. Then the corresponding Fredholm module has the form $(H, \pi, 2p - 1)$. (For brevity, we simply write π, although, strictly speaking, we mean the restriction of the representation π to the subalgebra A.)

Now assume that $[u] \in K_1(\mathcal{D}(A^+))$ is the equivalence class of some unitary element over $\mathcal{D}(A^+)$. Let us write out the graded Fredholm module determining the corresponding element in $K^0(A)$. Just as above, without loss of generality, we can assume that $u \in \mathcal{D}(A^+)$. Then the corresponding Fredholm module has the form

$$\left(H, \pi, \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}\right).$$

Fredholm modules and the theory of extensions. Let us show how to assign an element of the group $K^1(A)$ described in terms of the Fredholm modules to an element of the group $\operatorname{Ext}(A)$. Here the construction is quite simple. By Theorem B.36, each element of the group $\operatorname{Ext}(A)$ is the equivalence class of some Toeplitz quantization μ_P, where P is a projection in the space H of the representation π of the algebra A and compactly commutes with the representation operators. This readily gives the Fredholm module $(H, \pi, 2P - 1)$.

B.2.3 Suspension and Bott Periodicity

Just as in the case of covariant K-theory, the following statements hold in K-theory of operator algebras.

Proposition B.38. *Let A be a separable nuclear C^*-algebra. Then*

1. (*Suspension.*)
$$K^0(SA) = K^1(A).$$

2. (*Bott periodicity.*)
$$K^*(S^2A) \simeq K(A), \quad * = 0, 1.$$

We do not explicitly write out the corresponding isomorphisms.

B.2.4 Long Exact Sequence in K-Homology

Theorem B.39 (Six-term sequence in K-homology of algebras). *Let*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

be a short exact sequence of C^*-algebras. Then there is a long exact sequence

$$\begin{array}{ccccc} K^0(J) & \longleftarrow & K^0(A) & \longleftarrow & K^0(A/J) \\ {\scriptstyle \delta_0}\downarrow & & & & \uparrow{\scriptstyle \delta_1} \\ K^1(A/J) & \longrightarrow & K^1(A) & \longrightarrow & K^1(J) \end{array} \tag{B.16}$$

of K-homology groups.

Remark B.40. Here we do not present any explicit formulas for the connecting homomorphisms δ_0 and δ_1. (For example, they can be found in [17] and [41].) We only note that for the special (commutative) case in which $A = C(X)$ is the algebra of continuous functions on a smooth compact manifold X with boundary ∂X, J is the ideal of functions vanishing on ∂X, and $A/J = C(\partial X)$ is the algebra of continuous functions on ∂X, the homomorphism δ_0 can be interpreted as follows. Let D be an elliptic first-order differential operator on X. Then a pseudodifferential operator T on X with symbol

$$\sigma(T) = \sigma(D)(1 + \sigma(D)^*\sigma(D))^{-1/2}$$

can be defined. This operator compactly commutes with the action of the algebra J and is elliptic in the interior of X and hence determines an element $[T]$ in $K^0(J)$. The element

$$\delta_0[T] \in K^1(A/J)$$

corresponds to the Calderón projection (e.g., see [42]) of the operator D, and its triviality is equivalent to the fact that the operator D admits (possibly, after stabilization) classical elliptic boundary value problems (Atiyah–Bott condition). More details about this can be found, e.g., in [16].

Remark B.41. Strictly speaking, in the exact sequence (B.16) the K-homology of the ideal J stands in place of the *relative K-homology*

$$K^j(A, A/J) = K_{1-j}\big(\mathcal{D}(A^+)/\mathcal{D}(A^+ /\!\!/ J)\big).$$

The two are equal by the following theorem.

Theorem B.42 (Excision in K-homology). *There is a natural isomorphism*

$$K^j(A, A/J) \simeq K^j(J).$$

B.2.5 Stability

Just as the K-groups, the K-homology of the C^*-algebra A does not change if the algebra A is replaced by $M_n(A)$ or $A \mathbin{\widehat{\otimes}} \mathcal{K}(H)$. Indeed, for example, in the second case the desired relation follows from the definition of K-homology with the use of duality theory and the following easy-to-verify relation:

$$\mathcal{D}(A \mathbin{\widehat{\otimes}} \mathcal{K}(H)) = \mathcal{D}(A) \mathbin{\widehat{\otimes}} \mathbb{C} \simeq \mathcal{D}(A).$$

B.2.6 Duality between K-Homology and K-Theory of Operator Algebras

There are natural pairings between the K-groups and the K-homology groups of the same parity of the algebra A. Let us describe these pairings.

Even Groups

Let $[p] \in K_0(A)$ be the equivalence class of a projection p in the algebra A, and let $[F] \in K^0(A)$ be the equivalence class of a graded Fredholm module (H, π, F). In the decomposition of the space H into graded components, the operator F has the form

$$F = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}.$$

The pairing of the groups $K_0(A)$ and $K^0(A)$ is given by the formula

$$\begin{aligned} K^0(A) \times K_0(A) &\longrightarrow \mathbb{Z}, \\ ([F], [p]) &\longmapsto \operatorname{ind}(pF\colon pH \longrightarrow pH). \end{aligned} \tag{B.17}$$

(The fact that the operator pF is Fredholm is obvious.)

The case in which p is the projection not in the algebra A itself but in some matrix algebra $M_n(A)$ can be considered in a similar way.

Proposition B.43. *The pairing* (B.17) *is well defined.*

Odd Groups

Let $[u] \in K_1(A)$ be the equivalence class of some unitary matrix u over the algebra A, and let $[F] \in K_1(A)$ be the equivalence class of some graded Fredholm module (H, π, F). For simplicity assume that u is a unitary element of the algebra A itself. Then the Toeplitz operator $P_+(F)u\colon P_+(F)H \to P_+(F)H$ is a Fredholm operator. The pairing of the groups $K_1(A)$ and $K^1(A)$ is given by the formula

$$\begin{aligned} K^1(A) \times K_1(A) &\longrightarrow \mathbb{Z}, \\ ([F], [u]) &\longmapsto \operatorname{ind}(P_+(F)u). \end{aligned} \tag{B.18}$$

Proposition B.44. *The pairing* (B.18) *is well defined.*

Appendix C

Cyclic Homology and Cohomology

C.1 Cyclic Cohomology

We start from the definition of cyclic cohomology of algebras, which is simpler than the definition of cyclic homology.

C.1.1 Enveloping Differential Algebra

Let A be a local C^*-algebra. For this algebra, we construct a *universal enveloping differential algebra* (Ω, d), i.e., a graded topological differential algebra (Ω, d),

$$\Omega = \bigoplus_{n=0}^{\infty} \Omega^n, \quad d(\Omega^n) \subset \Omega^{n+1},$$

along with a homomorphism

$$j\colon A \to \Omega^0$$

such that for each homomorphism

$$\rho\colon A \to {\Omega'}^0$$

into the degree zero term of a topological differential graded algebra (Ω', d') there exists a unique homomorphism

$$\widetilde{\rho}\colon (\Omega, d) \to (\Omega', d')$$

of topological differential graded algebras for which the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ j\ } & \Omega^0 \\ \Big\| & & \Big\downarrow{\scriptstyle\widetilde{\rho}} \\ A & \xrightarrow[\ \rho\ ]{} & \Omega'^0 \end{array}$$

commutes.

It turns out that the universal enveloping differential algebra exists, is unique up to an isomorphism, and can be described by the following explicit construction. We set

$$\Omega^n \equiv \Omega^n(A) = A^+ \otimes \underbrace{A \otimes \cdots \otimes A}_{n \text{ factors}}.$$

(This is the projective tensor product of Fréchet spaces. Recall that A^+ is obtained from A by adding a unit even if A already has one.) Thus, the decomposable elements in $\Omega^n(A)$ become

$$\widetilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_n \equiv (a_0 + \lambda_0 \mathbf{1}) \otimes a_1 \otimes \cdots \otimes a_n,$$

where $a_j \in A$, $j = 0, 1 \ldots, n$, $\lambda \in \mathbb{C}$, and $\mathbf{1}$ is the adjoined unity and, from now on, to preserve the notation, we write $\widetilde{a} = a + \lambda \mathbf{1}$.

The space Ω^n, whose elements will be called *noncommutative n-forms*, is a right A-module with respect to the action defined on the decomposable elements by the formula

$$\big(\widetilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_n\big)a = \sum_{j=0}^{n-1} (-1)^{n-j} \widetilde{a}_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a + \widetilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_n a. \quad \text{(C.1)}$$

We equip the direct sum

$$\Omega = \bigoplus_{n=0}^{\infty} \Omega^n \quad \text{(C.2)}$$

with the multiplication (inductively) described on the decomposable elements by the formula

$$\omega \cdot \big(\widetilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_n\big) \stackrel{\text{def}}{=} \omega a_0 \otimes a_1 \otimes \cdots \otimes a_n. \quad \text{(C.3)}$$

This multiplication is associative and makes Ω a graded topological algebra. We define the differential (of degree +1) on Ω by the formula

$$d\big(\widetilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_n\big) = \mathbf{1} \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n. \quad \text{(C.4)}$$

In particular, $d(\mathbf{1} \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) = 0$.

The above description of the universal enveloping differential algebra looks rather cumbersome, and the formulas defining the product seem to be artificial. Therefore, we give an alternative description of this algebra. To this end, to each element $a \in A$ we assign the element

$$da = \mathbf{1} \otimes a \in \Omega^1.$$

It turns out that each element $\omega \in \Omega^n$ can be expressed as a linear combination of elements of the form

$$\widetilde{a}_0 \, da_1 \, da_2 \cdots da_n;$$

for such elements, the formal rules of multiplication and differentiation are the same as for the usual de Rham differential forms (with the noncommutativity of the algebra taken into account). In particular,

$$d\bigl(\widetilde{a}_0 \, da_1 \cdots da_n\bigr) = da_0 \, da_1 \cdots da_n,$$
$$da \cdot b = d(ab) - a \, db,$$

etc.

Remark C.1. A similar universal algebra can be constructed in the category of unital algebras and unital maps. In this case, one defines the space of n-forms as

$$\Omega^n = A \otimes (A/\mathbb{C}1)^n,$$

where $1 \in A$ is the unit. In this case, we have $d1 = 0$ (unlike the previous construction for general algebras, where $d1 \neq 0$.)

C.1.2 Graded Traces and Cyclic Functionals

Let

$$\widehat{\tau} \colon \Omega^n \longrightarrow \mathbb{C}$$

be a continuous linear functional possessing the cyclic invariance property

$$\widehat{\tau}(\omega_1\omega_2) = (-1)^{kl}\widehat{\tau}(\omega_2\omega_1), \qquad \omega_1 \in \Omega^k, \quad \omega_2 \in \Omega^l, \quad k + l = n. \tag{C.5}$$

Each functional of this form will be called a *graded trace of degree* n on Ω. To the trace τ we can assign a continuous $(n+1)$-linear functional on the algebra A by the formula

$$\tau(a_0, a_1, \ldots, a_n) = \widehat{\tau}(a_0 \, da_1 \cdots da_n). \tag{C.6}$$

We also assume that the trace $\widehat{\tau}$ is *closed*, i.e., satisfies

$$\widehat{\tau}(d\omega) = 0, \qquad \omega \in \Omega^{n-1}. \tag{C.7}$$

If this is the case, then the functional τ is *cyclic*, i.e., satisfies

$$\tau(a_n, a_0, a_1, \ldots, a_{n-1}) = (-1)^n \tau(a_0, a_1, \ldots, a_n). \tag{C.8}$$

Indeed,

$$
\begin{aligned}
\widehat{\tau}(a_n\,da_0\cdots da_{n-1}) &= \widehat{\tau}(d(a_na_0)\,da_1\cdots da_{n-1}) - \widehat{\tau}(da_n\cdot a_0\,da_1\cdots da_{n-1})\\
&= -\widehat{\tau}(da_n\cdot a_0\,da_1\cdots da_{n-1}) \quad \text{(because of the closedness)}\\
&= (-1)^n\widehat{\tau}(a_0\,da_1\cdots da_n) \quad \text{(because of the cyclic invariance).}
\end{aligned}
$$

C.1.3 Cochains, Cyclic Cochains, and the Hochschild Differential

The set of all cyclic continuous $(n+1)$-linear functionals (cyclic n-cochains) on A is a linear space, which we denote by $C^n_\lambda = C^n_\lambda(A)$. (The subscript λ stands for graded invariance with respect to the cyclic permutation operator, which we denote by the same letter.) There is a theorem stating that all such functionals can be obtained from closed graded traces. We define the *Hochschild differential*

$$b\colon C^n_\lambda \longrightarrow C^{n+1}_\lambda$$

by the formula

$$
\begin{aligned}
(b\tau)(a_0,a_1,\dots,a_{n+1}) = \sum_{j=0}^{n} \tau(a_0,a_1,\dots,a_ja_{j+1},\dots,a_{n+1})\\
+ (-1)^{n+1}\tau(a_{n+1}a_0,a_1,\dots,a_n).
\end{aligned} \tag{C.9}
$$

The Hochschild differential on the space $C^n = C^n(A)$ of *all* (not only cyclic) cochains can be defined by the same formula. Generally speaking, the relation

$$b\lambda + \lambda b = 0$$

does not hold on $C^n(A)$, but it already holds on the subspace $C^n_\lambda(A) \subset C^n(A)$ of cyclic cochains, so that $b(C^n_\lambda) \subset C^{n+1}_\lambda$ and Definition (C.9) makes sense.

C.1.4 Cyclic Cohomology and Hochschild Cohomology

A straightforward verification shows that $b^2 = 0$. Thus, the Hochschild differential defines the two complexes

$$0 \longrightarrow C^0_\lambda(A) \xrightarrow{\;b\;} C^1_\lambda(A) \xrightarrow{\;b\;} C^1_\lambda(A) \xrightarrow{\;b\;} \cdots, \tag{C.10}$$

$$0 \longrightarrow C^0(A) \xrightarrow{\;b\;} C^1(A) \xrightarrow{\;b\;} C^1(A) \xrightarrow{\;b\;} \cdots. \tag{C.11}$$

The cohomology groups of the complexes (C.10) and (C.11) are called the *cyclic cohomology* and the *Hochschild cohomology*, respectively, of the algebra A and are denoted by

$$HC^n(A) = Z^n_\lambda(A)/B^n_\lambda(A), \qquad HH^n(A) = Z^n(A)/B^n(A),$$

with the usual notation for the corresponding spaces of cocycles and coboundaries.

C.1.5 Example

Now let us consider a simple example. Let M be a smooth oriented n-dimensional manifold, and consider the algebra

$$A = \mathrm{Mat}_N(C^\infty(M))$$

of *matrix* C^∞-functions. What multilinear functionals on A can be indicated? Suppose that we wish to describe a k-cochain, i.e., a $(k+1)$-linear functional. The following simple method suggests itself: to a differential form ω of degree $n-k$ on M, we assign the functional

$$\tau_\omega(a_0, a_1, \ldots, a_k) = \int_M \mathrm{tr}\big(a_0\, da_1 \wedge \cdots \wedge da_k\big) \wedge \omega, \tag{C.12}$$

where d is the usual de Rham differential and tr is the matrix trace. For an arbitrary form ω, the functional (C.12) need not be cyclic, but it is necessarily Hochschild closed: $b\tau_\omega = 0$. But if $d\omega = 0$ (the form ω is closed), then the functional τ_ω is already cyclic, which can be shown by the following straightforward computations:

$$\begin{aligned}\tau_\omega(a_0, a_1, \ldots, a_k) &= (-1)^{k-1} \int_M \mathrm{tr}\big(da_k \wedge a_0\, da_1 \wedge \cdots \wedge da_{k-1}\big) \wedge \omega \\ &= (-1)^{k-1} \int_M \mathrm{tr}\big(d(a_k a_0) \wedge da_1 \wedge \cdots \wedge da_{k-1}\big) \wedge \omega \\ &\quad + (-1)^{k} \int_M \mathrm{tr}\big(a_k\, da_0 \wedge da_1 \wedge \cdots \wedge da_{k-1}\big) \wedge \omega \\ &= (-1)^k \tau_\omega(a_k, a_0, \ldots, a_{k-1}),\end{aligned}$$

since the first term in the sum is an integral of an exact form and hence is zero.

Thus, the functional τ_ω assigns a number to each differential form $a_0\, da_1 \wedge \ldots da_k$, and hence the above argument suggests that

$$HC^k(C^\infty(M)) \sim H_k(M).$$

(And, in a similar way,

$$HH^k(C^\infty(M)) \sim \mathrm{Currents}_k(M),$$

where $\mathrm{Currents}_k(M)$ is the space of de Rham *currents* of degree k on M.) But this is not true. The point is that there also exist other cyclic multilinear functionals noncohomological to the above-described functionals.

Indeed, take a form ω of degree $n-m$, where $m = k - s < k$, and define a k-linear functional on A by the formula

$$\tau_\omega(a_0, \ldots, a_k) = \sum_{\substack{1 \le j_1 < j_2 \\ < \cdots < j_s \le k}} (\pm) \int_M \mathrm{tr}\big(a_0\, da_1 \wedge \cdots \wedge \widehat{da_{j_1}} \wedge \cdots \wedge \widehat{da_{j_s}} \wedge \cdots \wedge da_k\big) \wedge \omega, \tag{C.13}$$

where the "hat" over a factor da_j means that this factor is replaced by a_j and the signs $\pm$ are chosen in a specific way. (Here we do not describe the choice of the signs.) *It turns out that if $s = k - m$ is even and ω is closed, then, under an appropriate choice of the signs, we obtain a cyclic cocycle.* The right answer has the form [26]

$$HC^k(C^\infty(M)) \simeq \ker b \oplus H_{k-2}(M,\mathbb{C}) \oplus H_{k-4}(M,\mathbb{C}) \oplus \cdots , \tag{C.14}$$

where b is the de Rham differential on currents of degree k and $H_l(M,\mathbb{C})$ is the usual homology of manifold M.

C.1.6 Cup Product in Cyclic Cohomology

Let A and B be two algebras, and let $\widehat{\tau}'$ and $\widehat{\tau}''$ be closed graded traces of degrees n' and n'' on $\Omega^{n'}(A)$ and $\Omega^{n''}(B)$, respectively. We wish to define the cup product $\widehat{\tau}'\#\widehat{\tau}''$, which should be a closed graded trace on $\Omega(A\otimes B)$ (more precisely, on $\Omega^{n'+n''}(A\otimes B)$, since on the other components this is 0). Although $\Omega(A\otimes B) \neq \Omega(A)\otimes\Omega(B)$ in the general case, this does not prevent us from presenting a well-posed definition. Namely, we set

$$\widehat{\tau}(\omega'\otimes\omega'') = (-1)^{n'n''}\widehat{\tau}'(\omega')\widehat{\tau}''(\omega''); \tag{C.15}$$

this determines the trace $\widehat{\tau} \stackrel{\text{def}}{=} \widehat{\tau}'\#\widehat{\tau}''$ on the graded tensor product $\Omega(A)\otimes\Omega(B)$. Further, we use the universal property of the differential enveloping algebra, which implies that there is a unique homomorphism π of differential algebras making the diagram below commute:

$$\begin{array}{ccc} A\otimes B & = & A\otimes B \\ {\scriptstyle j_{A\otimes B}}\big\downarrow & & \big\downarrow{\scriptstyle j_A\otimes j_B} \\ \Omega(A\otimes B) & \xrightarrow[\pi]{} & \Omega(A)\otimes\Omega(B). \end{array}$$

Using this homomorphism, we lift the trace τ to $\Omega(A\otimes B)$.

The same can be stated in the language of cyclic functionals: if

$$\varphi \in Z^m_\lambda(A), \quad \psi \in Z^n_\lambda(B),$$

then we determine $\varphi\#\psi \in Z^{m+n}_\lambda(A\otimes B)$ by the formula

$$\big(\varphi\#\psi\big)^\wedge = \pi^*\big(\widehat{\varphi}\otimes\widehat{\psi}\big) \equiv \big(\widehat{\varphi}\otimes\widehat{\psi}\big)\circ\pi.$$

One can show that this product gives a well-posed definition of *cup product*

$$\#\colon HC^*(A)\times HC^*(B) \to HC^*(A\otimes B)$$

in cyclic cohomology.

C.1.7 Periodicity in Cyclic Cohomology

Periodicity is a key operation connecting the cyclic cohomology groups of different dimensions, which serves as an analog of the Bott isomorphism in topological K-theory.

This operation is related to the tensor multiplication of an algebra by $\mathbb{C}$; therefore, we first compute the cyclic cohomology of the algebra $\mathbb{C}$ of complex numbers. We denote the unit in $\mathbb{C}$ by $e \in \mathbb{C}$, and by $\mathbf{1}$, as usual, we denote the adjoined unit in the algebra $\mathbb{C}^+$ obtained from $\mathbb{C}$ by unitization. For each n, the space $\Omega^n(\mathbb{C})$ is the two-dimensional complex space $\mathbb{C}^2$ with the basis

$$\underbrace{de\,de\cdots de}_{n\text{ factors}},\quad e\,\underbrace{de\,de\cdots de}_{n\text{ factors}}. \tag{C.16}$$

Let us find out what closed graded traces of degree n exist on $\Omega(\mathbb{C})$. If $\widehat{\tau}$ is such a trace, then

$$\widehat{\tau}(de\,de\cdots de) = 0$$

owing to closedness, and hence the trace is completely determined by its value on the second basis element in (C.16). One can readily verify the following identities:

$$e^2 = e \qquad \text{(the former unit is an idempotent)}, \tag{C.17}$$

$$e\,de + de\,e = de \qquad \text{(differentiate (C.17))}, \tag{C.18}$$

$$e\,de\,e = 0 \qquad \text{(multiply (C.18) by } e \text{ on the right and on the left)}, \tag{C.19}$$

$$e\,de\,de = de\,de\,e \qquad \text{(differentiate (C.19))}. \tag{C.20}$$

Using these formulas, we readily compute

$$\begin{aligned}
\widehat{\tau}(e\,de\,de\cdots de) &= \widehat{\tau}(e^2\,de\,de\cdots de) && \text{by (C.17)}\\
&= (-1)^n\widehat{\tau}(e\,de\,de\cdots de\,e) && \text{(by cyclicity)}\\
&= \begin{cases} \widehat{\tau}(e\,de\,e\,de\cdots de) = 0 & \text{for } n = 2k+1,\\ \widehat{\tau}(e\,de\,de\cdots de) & \text{for } n = 2k \end{cases} && \text{by (C.20) and (C.19).}
\end{aligned}$$

One can show that for even n, the value attained by τ can indeed be arbitrary, so that the space of closed graded traces of degree n is $\mathbb{C}$ for even n and $\{0\}$ for odd n. Then the same statement also holds for the spaces of cyclic cochains. This implies that the Hochschild differential is zero in this case, and finally,

$$HC^n(\mathbb{C}) = \begin{cases} \mathbb{C}, & n = 2k,\\ \{0\}, & n = 2k+1. \end{cases} \tag{C.21}$$

Since $\mathbb{C}\otimes\mathbb{C} = \mathbb{C}$, we see that $HC^*(\mathbb{C})$ is equipped with a multiplication making the cyclic cohomology space of the algebra of complex numbers a ring with a single multiplicative generator $\sigma \in HC^2(\mathbb{C})$. We eliminate the ambiguity in the choice of this generator by using the normalization condition

$$\sigma(e\, de\, de) = 1.$$

Theorem C.2. *For any local C^*-algebra A, the space $HC^*(A)$ is an $HC^*(\mathbb{C})$-module.* (*The module structure is induced by the tensor multiplication $A \otimes \mathbb{C} = A$, as was explained above.*)

We now define the *periodicity operator*

$$S\colon HC^n(A) \longrightarrow HC^{n+2}(A),$$
$$\tau \longmapsto \tau \# \sigma,$$

where $\sigma \in HC^2(\mathbb{C})$ is the element constructed earlier. This operator enables us to define the *periodic cyclic cohomology groups*

$$HP^*(A) = \lim_{n\to\infty} HC^{n+*}(A), \qquad \text{where } * = 0, 1,$$

as the direct limit with respect to the mappings given by operator S.

Example C.3. Cyclic cohomology of the algebra of C^∞ functions on a smooth compact manifold is (see Eq. (C.14))

$$HC^k(C^\infty(M)) \simeq \ker b \oplus H_{k-2}(M, \mathbb{C}) \oplus H_{k-4}(M, \mathbb{C}) \oplus \cdots .$$

It can be shown that S preserves this decomposition and, moreover, periodic cyclic cohomology in this case is

$$HP^0(C^\infty(M)) \simeq \bigoplus_{k\ge 0} H_{2k}(M, \mathbb{C}),$$

$$HP^1(C^\infty(M)) \simeq \bigoplus_{k\ge 0} H_{2k+1}(M, \mathbb{C}).$$

Let us write out a formula for the periodicity operator at the level of representatives. Let τ be a cyclic n-cocycle on A corresponding to a closed graded trace $\widehat{\tau}$ of degree n. We act on the class $[\tau] \in HC^n(A)$ by the operator S. The operator S is given by the tensor multiplication by σ; therefore, to write out a representative $S\tau$ of the class $S[\tau]$, we should compute the corresponding tensor product. We have

$$\widehat{\tau \otimes \sigma} = \widehat{\tau} \otimes \widehat{\sigma}.$$

(In this case, the preimage under π^* is not required, since the isomorphism $\Omega(A \otimes \mathbb{C}) \equiv \Omega(A) \sim \Omega(A) \otimes \Omega(\mathbb{C})$ occurs under the multiplication by $\mathbb{C}$.) Next,

$$S\tau(a_0, a_1, \dots, a_{n+2}) = \big(\widehat{\tau} \otimes \widehat{\sigma}\big)\big((a_0 \otimes e)\, d(a_1 \otimes e) \wedge \cdots \wedge d(a_{n+2} \otimes e)\big). \quad \text{(C.22)}$$

Using the identity

$$d(a \otimes e) = da \otimes e + a \otimes de$$

and the linearity of the graded trace $\widehat{\tau} \otimes \widehat{\sigma}$, we multiply out on the right-hand side in (C.22). Then the sum contains only terms with precisely two occurrences of de. Moreover, it follows from identities (C.17) and (C.18) that only the terms containing these two occurrences of de in neighboring positions can "survive." In these positions, the corresponding a_j are not differentiated, and, as a result, for $S\tau$ we obtain the formula

$$\begin{aligned}(S\tau)(a_0, a_1, \dots, a_n, a_{n+1}, a_{n+2}) &= \tau(a_0 a_1 a_2, a_3, \dots, a_{n+2}) \\ &- \tau(a_0, a_1 a_2 a_3, a_4, \dots, a_{n+2}) + \tau(a_0, a_1, a_2 a_3 a_4, \dots, a_{n+2}) \\ &- \cdots + (-1)^n \tau(a_0, a_1, \dots, a_n a_{n+1} a_{n+2}) \\ &+ (-1)^{n+1} \tau(a_{n+1} a_{n+2} a_0, a_1, \dots, a_n). \quad \text{(C.23)}\end{aligned}$$

C.1.8 Example

Now let us describe how the operator S acts in the case of algebra of C^∞ functions in terms of differential forms. Thus, let M be an n-dimensional manifold, and let $A = \mathrm{Mat}_L(C^\infty(M))$. Let ω be a closed differential $(n-k)$-form on M determining the functional $\tau = \tau_\omega$ by formula (C.12):

$$\tau(a_0, a_1, \dots, a_k) = \int_M \mathrm{tr}\big(a_0\, da_1 \wedge \cdots \wedge da_k\big) \wedge \omega,$$

so that $[\tau] \in HC^k(C^\infty(M))$. We apply the operator S to the class $[\tau]$. The representative $S\tau$ of the class $S[\tau]$ is given using (C.23) by the formula

$$\begin{aligned}&\big(S\tau\big)(a_0, a_1, \dots, a_{k+2}) \\ &\quad = \sum_{j=1}^{n+1} \int_M \mathrm{tr}\big(a_0\, da_1 \wedge \cdots \wedge da_{j-1}\, a_j a_{j+1}\, da_{j+2} \wedge \cdots da_{k+2}\big) \wedge \omega. \quad \text{(C.24)}\end{aligned}$$

C.2 Cyclic Homology

C.2.1 Definition of Cyclic Homology

Again let A be a local C^*-algebra. (We assume that it is unital, i.e., adjoin the unit if necessary.) Set

$$\Omega_n(A) = A^{\otimes(n+1)} \equiv \underbrace{A \otimes \cdots \otimes A}_{n+1 \text{ factors}}.$$

The elements of this space are called *Hochschild chains*. A chain is said to be *cyclic* if it changes or does not change the sign (according to whether the number n is even) under a cyclic permutation of the factors:

$$\omega = \sum_j a_0^j \otimes a_1^j \otimes \cdots \otimes a_n^j = (-1)^n \sum_j a_n^j \otimes a_0^j \otimes \cdots \otimes a_{n-1}^j.$$

We denote the set of cyclic chains by $C_n(A) \subset \Omega_n(A)$ and introduce the boundary operator

$$b\colon \Omega_n(A) \longrightarrow \Omega_{n-1}(A)$$

by the formula

$$b(a_0 \otimes \cdots \otimes a_n) = a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n - a_0 \otimes a_1 a_2 \otimes a_3 \otimes \cdots \otimes a_n + \cdots + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

Under this definition, we have $b(C_n(A)) \subset C_{n-1}(A)$, and the cyclic homology of the algebra A is determined as usual as the homology of the complex

$$\cdots \xrightarrow{\ b\ } C_n(A) \cdots \xrightarrow{\ b\ } C_2(A) \xrightarrow{\ b\ } C_1(A) \xrightarrow{\ b\ } C_0(A) \longrightarrow 0. \tag{C.25}$$

Namely,

$$HC_n(A) = Z_n(A)/B_n(A), \qquad n = 0, 1, 2, \dots . \tag{C.26}$$

In particular,

$$b(a_0 \otimes a_1) = a_0 a_1 - a_1 a_0 \equiv [a_0, a_1]$$

is the commutator of the elements a_0 and a_1, so that the homology at the first step measures the noncommutativity of the algebra A. Namely,

$$HC_0(A) = A/[A, A].$$

C.2.2 Periodicity

Just as in the case of cyclic cohomology, there is a (dual) periodicity operator

$$S\colon HC_n(A) \longrightarrow HC_{n-2}(A), \tag{C.27}$$

which is an *epimorphism.* The projective limits

$$\operatorname{limproj}_S HC_n(A) = \begin{cases} HP_0(A) & \text{for even } n, \\ HP_1(A) & \text{for odd } n \end{cases} \tag{C.28}$$

are called the *periodic cyclic homology* of the algebra A. In the case of the algebra $A = C^\infty(M)$, there are natural isomorphisms

$$HP_0(A) = \oplus H^{2k}(M, \mathbb{C}), \qquad HP_1(A) = \oplus H^{2k+1}(M, \mathbb{C}).$$

Here the formulas for the periodicity operator S are more complicated than those in the case of the cohomology, and we do not write them out because of their cumbersomeness.

C.2.3 Pairing between Cyclic Homology and Cohomology

There is a natural pairing between cyclic homology and cohomology,

$$HC_n(A) \times HC^n(A) \longrightarrow \mathbb{C}, \tag{C.29}$$

generated on the level of representatives by the mapping

$$(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \times \tau(\cdot, \ldots, \cdot) \longmapsto \tau(a_0, a_1, \ldots, a_n).$$

(We consider the case of a unital algebra A.)

C.2.4 Morita Invariance

There exist natural isomorphisms

$$HC^n(A) = HC^n(M_N(A)), \tag{C.30}$$
$$HC_n(A) = HC_n(M_N(A)), \tag{C.31}$$

$N = 1, 2, \ldots$, where $M_N(A)$ is the algebra of $N \times N$ matrices over A. For example, the isomorphism (C.30) can be given by the formula

$$\tau(a_0 \otimes m_0, a_1 \otimes m_1, \ldots, a_n \otimes m_n) \overset{\text{def}}{=} \tau(a_0, a_1, \ldots, a_n)\,\mathrm{tr}(m_0 m_1 \cdots m_n),$$

where $m_j \in M_N(\mathbb{C})$ and $\mathrm{tr}(\cdots)$ is the usual matrix trace.

C.2.5 Chern Characters in Homology

Let us present mappings of K-theory into cyclic homology. These mappings are called *Chern characters.*

First, we describe the Chern character on the odd K-group

$$\mathrm{ch}_1^n\colon K_1(A) \longrightarrow HC_{2n-1}(A).$$

The descriptions look somewhat different in the case of unital algebras and in the case of algebras without unity.

1. Let A be unital. Then for elements $a \in GL(A)$ we define the Chern character by the formula

$$K_1(A) \ni [a] \longmapsto [c_n \underbrace{a^{-1} \otimes a \otimes \cdots \otimes a^{-1} \otimes a}_{2n \text{ factors}}] \in HC_{2n-1}(A), \tag{C.32}$$

where $c_n = 2^{-2n+1}\Gamma(n + 1/2)$ is the normalization constant. In this formula we also used stability of cyclic homology $HC_{2n-1}(M(A)) \simeq HC_{2n-1}(A)$.

2. Let A be nonunital. Then, by definition, $K_1(A)$ consists of the equivalence classes of invertible matrices a with elements in A^+ such that $a - \mathbf{1} \in M(A)$. We set

$$\mathrm{ch}_1^n[a] = c_n\big[(a^{-1} - \mathbf{1}) \otimes (a - \mathbf{1}) \otimes \cdots \otimes (a - \mathbf{1})\big]. \tag{C.33}$$

Now we describe the Chern character on the even K-group,

$$\mathrm{ch}_0^n\colon K_0(A) \longrightarrow HC_{2n}(A).$$

For simplicity, we consider only the case of unital algebras. The elements of the group $K_0(A)$ are the equivalence classes $[P]$ of matrix projections P with entries in A. We specify the mapping

$$\mathrm{ch}_0^n\colon K_0(A) \longrightarrow HC_{2n}(A) \tag{C.34}$$

by the formula

$$[P] \longmapsto c'_n \underbrace{P \otimes \cdots \otimes P}_{2n+1 \text{ factors}}, \tag{C.35}$$

where $c'_n = (-1)^n (2n)!/(n!)$ is the normalization constant.

Theorem C.4. *The mapping* (C.34), (C.35) *is well defined and is a group homomorphism.*

Remark C.5. The normalization constants are chosen in such a way that the Chern characters defined for different values of n are compatible with the periodicity operator S in cyclic homology:

$$\mathrm{ch}_*^{n-1} = S \circ \mathrm{ch}_*^n, \quad * = 0, 1.$$

Thus, these Chern characters induce Chern character with values in periodic cyclic homology.

Concise Bibliographical Remarks

Chapter 1

A detailed presentation of the theory of crossed products, as well as further references, can be found, for example, in the monographs [62, 77]. An important role in the theory of crossed products is played by the isomorphism theorem (see [1, 3]). Smooth subalgebras in group algebras of polynomial growth and in the corresponding crossed products were studied in [67, 68]. We note that there are also known local subalgebras for other classes of groups, for example, for hyperbolic groups (e.g., see [43, 45] and the recent survey in [44]).

Chapter 2

The notion of symbol for pseudodifferential operators with shifts was introduced in [3]. The Fredholm property of elliptic operators generated by an amenable group of shifts was also proved there. The use of the isomorphism theorem is a key point in this proof. Note that in our book we do not consider problems related to studying the ellipticity of the symbol (i.e., its invertibility). This is a very interesting and meaningful field of studies closely related to the theory of dynamical systems. For example, in the case of an action of the group $\mathbb{Z}$ (i.e., for nonlocal operators generated by iterations of a given diffeomorphism of a manifold), the symbol invertibility condition is equivalent to the hyperbolicity condition for a linear extension of the diffeomorphism. This class of problems was studied in [1], where the reader can find further references.

Chapter 3

The theory of (local) elliptic operators over C^*-algebras was constructed in [57]. Further development of this theory can be found in [48] and in the monographs [53, 72]. As far as we know, nonlocal operators over C^*-algebras have not been studied earlier.

Chapter 4

Atiyah and Singer [10] showed that the group of stable homotopy classes of elliptic operators on a smooth manifold is isomorphic to the K-group of the cotangent bundle of the manifold. In fact, the isomorphism is induced by the symbol mapping. In this chapter, this result is generalized to the case of nonlocal operators.

Chapter 5

In the case of a nonlocal operator corresponding to a finite group Γ of shifts, Antonevich [5] proved that such an operator determines a new Γ-invariant operator. Passing to the Γ-index of the latter operator, we see that nonlocal operators have an invariant ranging in the virtual representation ring of the group Γ. In this chapter, we define the corresponding invariant in the case of infinite groups. In this case, this invariant ranges in the K-group of the C^*-algebra of the group Γ. In fact, the Fredholm index is contained in this invariant.

Chapter 6

In this chapter, we establish the Bott periodicity theorem for isometric actions of polynomial growth groups. This theorem is new. By the way, we note that there is a close generalization of the periodicity theorem presented in [40]. These authors state the periodicity theorem in terms of noncommutative algebras including functions of anticommuting variables. The statement and the proof of the periodicity theorem presented in our book are closer to the classical version. In particular, we construct the mapping providing the inverse of the Bott homomorphism in terms of indices of some operators.

Chapter 7

Here we consider the direct image and the index formula in K-theory. We express the index of an elliptic operator as the direct image (under the projection of the manifold into a point) of an element in K-theory corresponding to the symbol of the operator. But, in contrast to the classical proof of the index theorem due to Atiyah and Singer [10], we do not use the axiomatic approach but successively transform the elliptic operator and simultaneously prove that its analytical index is preserved under these transformations. Note that a proof, close in spirit, of an index theorem was given in [54].

Let us also mention another approach to index formulas based on K-theory [48]. In this approach, an arbitrary elliptic operator is first reduced to the Dolbeault operator on the cotangent bundle of the manifold twisted by the bundle determined by the symbol of the original operator. In our situation, there would be no obstacle to such a realization and establishing the corresponding reduction to the Dolbeault operator, but then, to obtain a cohomology formula for the index, we would have to calculate the equivariant Chern–Connes character of the Dolbeault operator. Here the calculations by means of the local Connes–Moscovici index theorem [31] encounter some technical difficulties related to small denominators, and one has to impose unnecessary additional conditions; see [59]. In the approach based on the K-theory index formula in terms of the direct image, these additional conditions do not arise.

Chapter 8

The definition of the Chern character in noncommutative geometry goes back to Connes and Karoubi (see the monographs [27,46] and the references therein). Here the Chern character ranges in the cyclic homology group or in the noncommutative de Rham cohomology, respectively. In the case of actions of discrete groups, the Chern character was also constructed in [15]. We use Karoubi's approach and construct a Chern character ranging in the cohomology of fixed point sets of the action. (To define such a Chern character, we have to choose some closed graded trace over a smooth group algebra.)

Chapter 9

The Todd class contained in our index formula arises already in the classical Lefschetz formula for the index of equivariant operators [9]; see also [15]. The index formulas on noncommutative toric manifolds [28, 29], in particular, on the noncommutative sphere [50] are specific cases of the index formula established in this chapter.

Chapter 10

To obtain numerical invariants from the index of an operator over a C^*-algebra, it was proposed in [30] to couple the index with closed graded traces over a local subalgebra. Formulas expressing such a coupling in topological terms are called "higher index formulas."

Chapter 11

The index of (local) elliptic operators over C^*-algebras was calculated in [57] modulo finite-order elements. In [73], the so-called "exact" index formula was obtained in terms of direct image in K-theory. Other approaches to this theory, in particular, to the cohomology formulas, were given in [18, 19, 65, 74, 76]. As far as we know, the index of nonlocal operators over C^*-algebras is considered in this book for the first time.

Chapter 12

Elliptic theory on the noncommutative torus constructed in [24, 25, 27] has served as the starting point for our studies of nonlocal equations. These papers present ellipticity conditions for the corresponding nonlocal operators, and an index formula was obtained there. A beautiful example is given by a first-order operator defined in terms of the Rieffel projection [64].

Chapter 13

In the situation of local elliptic operators, the theory of the index of almost projection operators (or self-adjoint operators), in which the index is an element of an odd K-group, was constructed in [12]. In fact, this theory can be reduced to the usual elliptic theory by using the suspension operation.

Chapter 14

As was already noted, the index of nonlocal operators with a finite group of shifts was calculated in [5]. For Dirac operators, the calculation of the index in this and similar situations was also performed in [61], [14], and [63].

Appendices

Appendix A. There are numerous good textbooks concerned with the theory of C^*-algebras. Details of the proofs and further information can be found, for example, in the monographs [6, 35, 37, 58]

Appendix B. The reader can find a more detailed presentation of K-theory of C^*-algebras in the monographs [22, 33, 41].

Appendix C. The reader can find a more detailed description of the cyclic (co)homology theory in the monographs [27, 46, 51] and the survey papers [20, 32, 75].

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