Appendix A
Solutions to the Exercises

For many exercises extensive calculations are necessary that are difficult to perform by pencil and paper. It is recommended to apply the userfunctions provided on the website www.alltypes.de in those cases; a short description is given in Appendix E.

Chapter 1

1.1. For $y'' + a_1 y' + a_2 y = 0$ the answer is

$$a_1 = -\frac{1}{W^{(2)}}\begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}, \quad a_2 = -\frac{1}{W^{(2)}}\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} \quad \text{where} \quad W^{(2)} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

For $y''' + a_1 y'' + a_2 y' + a_1 y = 0$ the answer is

$$a_1 = -\frac{1}{W^{(3)}}\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}, \quad a_2 = \frac{1}{W^{(3)}}\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1''' & y_2''' & y_3''' \end{vmatrix},$$

$$a_3 = -\frac{1}{W^{(3)}}\begin{vmatrix} y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} \quad \text{where} \quad W^{(3)} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

1.2. Go to the ALLTYPES user interface and define

```
e1:=Df(z_1,x)-z_2;
e2:=Df(z_2,x)-z_3+a_1*z_2+a_2*z_1;
e3:=Df(z_3,x)-a_3*z_1+a_1*z_3;
T==|LDFMOD(DFRATF(Q,{a_1,a_2,a_3},{x},GRLEX),
                        {z_3,z_2,z_1},{x},LEX)|;
JanetBasis({e1,e2,e3}|T|);
```

The result of Example 1.1 is returned. The term orders

```
T==|LDFMOD(DFRATF(Q,{a_1,a_2,a_3},{x},GRLEX),
                        {z_1,z_2,z_3},{x},LEX)|;
JanetBasis({e1,e2,e3}|T|);
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F. Schwarz, *Loewy Decomposition of Linear Differential Equations*, Texts & Monographs in Symbolic Computation, DOI 10.1007/978-3-7091-1286-1,

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T==|LDFMOD(DFRATF(Q,{a_1,a_2,a_3},{x},GRLEX),
                       {z_3,z_1,z_2},{x},LEX)|;
JanetBasis({e1,e2,e3}|T|);
```

yield the associated equations for z_3 and z_2 respectively.

1.3. Let the given fourth-order equation be

$$y'''' + a_1 y''' + a_2 y'' + a_3 y' + a_4 y = 0. \tag{A.1}$$

In addition to the functions z_1, z_2, and z_3 defined in Example 1.1 the functions

$$z_4 = \begin{vmatrix} y_1 & y_2 \\ y_1''' & y_2''' \end{vmatrix}, \quad z_5 = \begin{vmatrix} y_1' & y_2' \\ y_1''' & y_2''' \end{vmatrix}, \quad z_6 = \begin{vmatrix} y_1'' & y_2'' \\ y_1''' & y_2''' \end{vmatrix}.$$

are required. They obey the system

$$z_1' = z_2,\quad z_2' = z_3 + z_4,\quad z_3' = z_5,\quad z_4' = z_5 - a_1 z_4 - a_2 z_2 + a_4 z_1,$$
$$z_5' = z_6 - a_1 z_5 - a_2 z_3 + a_4 z_1,\quad z_6' = -a_1 z_6 + a_3 a_3 + a_4 z_2.$$

A Janet basis in lex term order with $z_1 \succ \ldots \succ z_6$ has the form

$$z_1^{(VI)} + \sum_{k=0}^{5} r_k(a_1, \ldots, a_4) z_1^{(k)},\quad z_2 = z_1',\quad \text{and}\quad z_i = \sum_{k=0}^{5} f_{i,k}(a_1, \ldots, a_4) z_1^{(k)}$$

for $i = 3, \ldots, 6$. The r_k and $f_{i,k}$ are differential functions of the coefficients $a_1, \ldots, a_4$. In order to determine the coefficients of a second-order factor (compare the discussion in Example 1.3) it suffices to find a solution with rational logarithmic derivative of the first equation for z_1.

1.4. Substituting $y = \phi z$ into the given equation yields

$$z'' + \Big(2\frac{\phi'}{\phi} + p\Big)z' + \Big(\frac{\phi''}{\phi} + p\frac{\phi'}{\phi} + q\Big)z = 0.$$

The coefficient of z' vanishes if $\frac{\phi'}{\phi} = -\frac{1}{2}p$; hence $\phi = C \exp\big(-\frac{1}{2}\int p dx\big)$, C a constant, is the most general transformation with this property. Substitution into the coefficient of z leads to $r = -\frac{1}{2} - \frac{1}{4}p^2 + q$.

1.5. The first-order right factors of a $\mathscr{L}_3^2$ type decomposition have the form $l^{(1)}(C) = D - p - \frac{r'}{r + C}$, where p and r originate from the solution of the Riccati equation (notation as in (B.1)), and C is a constant. The $Lclm$ for two operators of this form is

$$L \equiv Lclm\big(l^{(1)}(C_1), l^{(1)}(C_2)\big) = D^2 - \Big(\frac{r''}{r'} + 2p\Big)D + \frac{r''}{r'}p - p' + p^2.$$

By division it is shown that any operator $l^{(1)}(C)$ is a divisor of L, i.e. it is contained in the left intersection ideal generated by it.

1.6. For $\mathscr{L}_1^2$, the first solution y_1 is obtained from $(D + a_1)y = 0$, the second from $(D + a_1)y = \bar{y}_2$ with $\bar{y}_2$ a solution of $(D + a_2)y = 0$.

For $\mathscr{L}_2^2$ the two solutions are obtained from $(D + a_i)y = 0$. Linear dependence over the base field would imply a relation $q_1 y_1 + q_2 y_2 = 0$ with q_1, q_2 from the base field. Substituting the solutions this would entail $\frac{q_1}{q_2} = -\exp \int (a_2 - a_1)dx$. Due to the non-equivalence of a_1 and a_2, its difference is not a logarithmic derivative; thus the right hand side cannot be rational.

For $\mathscr{L}_3^2$ the equation $(D + a(C))y = 0$ has to be solved, then C is specialized to $\bar{C}$ and $\bar{\bar{C}}$. Substituting $a_1 = \dfrac{r'}{r + \bar{C}} + p$ and $a_2 = \dfrac{r'}{r + \bar{\bar{C}}} + p$ in the above quotient, the integration may be performed with the result $\dfrac{q_1}{q_2} = -\dfrac{r + \bar{\bar{C}}}{r + \bar{C}}$ which is rational.

1.7. Define $\Delta \equiv \sqrt{A^2 - 4B}$. Two cases are distinguished. If $\Delta \neq 0$, two first-order right factors are $l_{1,2} = D + \frac{1}{2}A \pm \frac{1}{2}\Delta$; L has the decomposition $L = Lclm(l_1, l_2)$ of type $\mathscr{L}_2^2$; a fundamental system is $y_{1,2} = \exp\big((-\frac{1}{2}A \pm \frac{1}{2}\Delta)x\big)$. If $\Delta = 0$, the type $\mathscr{L}_3^2$ decomposition is $L = Lclm\big(D + \frac{1}{2}A - \dfrac{1}{x + C}\big)$, C a constant; it yields the fundamental system $y_1 = \exp\big(\frac{1}{2}A\big)$, $y_2 = x\exp\big(\frac{1}{2}A\big)$. This result shows: A second-order lode with constant coefficients is always completely reducible.

1.8. According to Lemma 1.1, case (i), the coefficient a of a first-order factor $D + a$ has to satisfy

$$a'' - 3a'1 + \big(1 - \frac{1}{x}\big)a' + a^3 - \big(1 - \frac{1}{x}\big)a^2 + \big(x - \frac{2}{x}\big)a - \frac{2}{x^2} = 0.$$

This second-order Riccati equation does not have a rational solution; thus a first-order right factor does not exist. According to case (ii) of the same lemma, the coefficient b of a second-order factor $D^2 + bD + c$ follows from

$$b'' - 3b'b + \big(2 - \frac{2}{x}\big)b' + b^3 - \big(2 - \frac{2}{x}\big)b^2 + \big(x + 1 - \frac{4}{x} + \frac{2}{x^2}\big)b - x = \frac{2}{x} - \frac{2}{x^2} = 0.$$

Its single rational solution $b = 1$ leads to $c = x - \frac{1}{x}$ and yields the second-order factor given in Example 1.3.

1.9. Let y_1, y_2, and y_3 be a fundamental system for the homogeneous equation and W its Wronskian. The general solution may be written as

$$y = C_1 y_1 + C_2 y_2 + C_3 y_3 + y_1 \int \frac{r}{W}(y_2 y_3' - y_2' y_3)dx$$
$$-y_2 \int \frac{r}{W}(y_1 y_3' - y_1' y_3)dx + y_3 \int \frac{r}{W}(y_1 y_2' - y_1' y_2)dx$$

where C_1, C_2 and C_3 are constants. For $y''' = r$ with the fundamental system $y_1 = 1$, $y_2 = x$ and $y_3 = x^2$ there follows

$$y = C_1 + C_2 x + C_3 x^2 + \tfrac{1}{2}\int x^2 r dx - x\int x r dx + \frac{1}{2}x^2\int r dx.$$

1.10. A simple calculation shows that the commutator between $l_1 = D + a_1$ and $l_2 = D + a_2$ vanishes if $a_1' - a_2' = 0$, i.e. if a_1 and a_2 differ by a constant. If this is true, the representation (1.17) simplifies to $L = D^2 + (a_1 + a_2)D + a_1 a_2 + a_1'$; furthermore $L = l_1 l_2 = l_2 l_1$. An example of this case is

$$D^2 - 4xD + 4x^2 - 3 = Lclm(D - 2x + 1, D - 2x - 1).$$

An example of non-commutative first-order factors is

$$D^2 - \big(1 + \frac{1}{x}\big)D + \frac{1}{x} = Lclm\big(D - 1, D - 1 + \frac{1}{x+1}\big).$$

Chapter 2

2.1. The product of $l_1 \equiv \partial_x + a_1\partial_y + b_1$ and $l_2 \equiv \partial_x + a_1\partial_y + b_2$ is

$$l_1 l_2 = \partial_{xx} + (a_1 + a_2)\partial_{xy} + a_1 a_2 \partial_{yy} + (b_1 + b_2)\partial_x$$
$$+(a_{2,x} + a_1 a_{2,y} + a_1 b_2 + a_2 b_1)\partial_y + b_{2,x} + a_1 b_{2,y} + b_1 b_2.$$

The product $l_2 l_1$ is obtained from this expression by interchange of all indices 1 and 2. Comparing those coefficients that are not symmetrical under this permutation leads to

$$a_{2,x} + a_1 a_{2,y} = a_{1,x} + a_2 a_{1,y}, \quad b_{2,x} + a_1 b_{2,y} = b_{1,x} + a_2 b_{1,y}.$$

Upon rearrangement, the conditions for case (i) given in Lemma 2.1 are obtained. The calculation for case (ii) is similar.

2.2. By definition of the Hilbert-Kolchin polynomial, the lc of an ideal with differential dimension $(0, k)$ is k, the dimension of the solution space of the corresponding system of pde's. Hence (2.8) for this special case reduces to the well known relation

$$dim\ V_{I+J} + dim\ V_{I\cap J} = dim\ V_I + dim\ V_J.$$

2.3. The third-order terms of the intersection ideal are

$$\partial_{xxx} - (a_1^2 + a_1 a_2 + a_2^2)\partial_{xyy} - (a_1 + a_2)a_1 a_2 \partial_{yyy}, \quad \partial_{xxy} + (a_1 + a_2)\partial_{xxy} + a_1 a_2 \partial_{yyy}.$$

2.4. The generator of the intersection ideal is

$$\partial_{xy} + \partial_{yy} + b_2\partial_x + (a_1 b_2 + b_1)\partial_y + b_{2,x} + (a_1 b_2 + a_1)_y + b_1 b_2.$$

2.5. The solutions of the three equations $l_i z_i = 0$ are $z_1 = f(y)\exp(-2x)$, $z_2 = g(y - x)\exp(-x)$ and $z_3 = h(y - 2x)$; f, g and h are undetermined functions of the respective argument. The system corresponding to the $Gcrd$ is $z_x + 2z = 0$, $z_y - z = 0$ with the solution $z = C\exp(y - 2x)$, C a constant. The undetermined functions have to be specialized as follows in order to obtain this latter solution:

$$f(y) = C\exp(y),\quad g(y-x) = C\exp(y-x),\quad h(y-2x) = C\exp(y-2x).$$

2.6. For constant coefficients the constraints (2.41), (2.46), and (2.47) are always satisfied. Thus, for subcase (b) the generator for the principal intersection ideal is

$$\partial_{xxx} + \sum a_i\partial_{xxy} + \sum_{i<j} a_i a_j \partial_{xyy} + \prod a_i \partial_{yyy} + \sum b_i \partial_{xx} + \sum_{i\neq j} a_i b_j \partial_{xy}$$
$$+ \sum_{i\neq j} a_i a_j b_k \partial_{yy} + \sum_{i<j} b_i b_j \partial_x + \sum_{i\neq j<k} a_i b_j b_k \partial_y + \prod b_i.$$

All indices run from 1 to 3. In order to satisfy the condition for subcase (a), $b_2 = (a_2 - a_3)b_1 + \frac{a_1 - a_2}{a_1 - a_3} b_3$ may be chosen. The third-order terms are not changed. The remaining expression is

$$\Big(b_1 + b_3 + \frac{a_2 - a_3}{a_1 - a_3} b_1 + \frac{a_1 - a_2}{a_1 - a_3} \Big) \partial_{xx}$$
$$+ \frac{2}{a_1 - a_3}\Big[(a_1a_2 - a_3^2)b_1 + (a_1^2 - a_2a_3)b_3 \Big] \partial_{xy}$$
$$+ \Big[a_2(a_1b_3 + a_3b_1) + \frac{a_1a_3}{a_1 - a_3}[(a_2 - a_3)b_1 + (a_1 - a_2)b_3 \Big] \partial_{yy}$$
$$+ \Big[2b_1b_3 + \frac{a_2 - a_3}{a_1 - a_3} b_1^2 + \frac{a_1 - a_2}{a_1 - a_3} b_3^2 \Big] \partial_x$$
$$+ \Big[2a_2b_1b_3 + \frac{a_2 - a_3}{a_1 - a_3} a_3 b_1^2 + \frac{a_1 - a_2}{a_1 - a_3} a_1 b_3^2 \Big] \partial_y$$
$$+ \frac{b_1b_3}{a_1 - a_3}\Big[(a_2 - a_3)b_1 + (a_1 - a_2)b_3 \Big].$$

2.7. The number of derivatives in the ideal generated by the leading derivatives of I is $\frac{1}{2}(n-4)(n-5) + 2n - 7 = \frac{1}{2}n^2 - \frac{5}{2}n + 3$. Thus $H_I = 6n - 9$ and $d_I = (1, 6)$.

2.8. There is a single coherence condition

$$a_{1,y} - a_{2,y}c - a_{3,y}d - (b_{1,x} - b_{2,x}c - b_{3,x}d) + c_x b_2 - c_y a_2 + d_x b_3 - d_y a_3 = 0.$$

2.9. In order to determine the coherence conditions for the ideal $\mathbb{J}_{xxx}$ open the interactive user interface on the ALLTYPES website and define

```
L1:=Df(z,x,3)+A_1*Df(z,x,y,2)+A_2*Df(z,y,3)
          +A_3*Df(z,x,2)+A_4*Df(z,x,y)+A_5*Df(z,y,2)
          +A_6*Df(z,x)+A_7*Df(z,y)+A_8*z;
L2:=Df(z,x,2,y)+B_1*Df(z,x,y,2)+B_2*Df(z,y,3)
          +B_3*Df(z,x,2)+B_4*Df(z,x,y)+B_5*Df(z,y,2)
          +B_6*Df(z,x)+B_7*Df(z,y)+B_8*z;
T==|LDFMOD(DFRATF(Q,A_1,A_2,A_3,A_4,A_5,A_6,A_7,A_8,
           B_1,B_2,B_3,B_4,B_5,B_6,B_7,B_8, {x,y},GRLEX),
          {z},{x,y},GRLEX)|;
```

Then the coherence conditions given in Lemma 2.2 are displayed by submitting

```
cs:=IntegrabilityConditions({L1,L2}|T|);
```

A Janet basis representation of this system of conditions in various term orderings may be obtained by defining for example

```
Tp==|DFPOLID(Q,{B_8,B_7,B_6,B_5,B_4,B_3,B_2,B_1,
          A_8,A_7,A_6,A_5,A_4,A_3,A_2,A_1},{x,y},LEX)|;
jb:=JanetBasis(cs|Tp|);
```

The result comprises two alternatives; in either case there are two constraints for the $A's$, and a system from which the $B's$ may be determined. You may experiment with varying term orders and try to obtain a system of constraints as simple as possible. The calculations for the ideal $\mathbb{J}_{xxy}$ are similar.

2.10. The two third-order operators must be divisible by $\partial_x + a_i\partial_y + b_i$, $i = 1, 2$. The conditions that this division be exact yields two linear algebraic systems for the p_i and the q_i. The result for two highest coefficients is

$$p_1 = -(a_1^2 + a_1a_2 + a_2^2),\ \ p_2 = -a_1a_2(a_1 + a_2),\ \ q_1 = a_1 + a_2,\ \ q_2 = a_1a_2. \quad \text{(A.2)}$$

The expressions for the remaining coefficients are too large to be given here.

2.11. Continuing the calculation given in the proof of Theorem 2.3 the following generator is obtained.

$$\partial_{xy} + a_1\partial_{yy} + b_2\partial_x + (a_{1,y} + b_1 + a_1b_2)\partial_y + b_{2,x} + (a_1b_2)_y + b_1b_2.$$

2.12. The calculation is similar as for Exercise 2.10; the result is $p_1 = -a_1$, $p_2 = b_2$, $q_1 = a_1$ and $q_2 = 0$.

Chapter 3

3.1. The coefficients a_1, a_2 and a_3 have to satisfy

$$\begin{aligned} a_1z_{1,x} + a_2z_{1,y} + a_3z_1 &= -z_{1,xx},\\ a_1z_{2,x} + a_2z_{2,y} + a_3z_2 &= -z_{2,xx},\\ a_1z_{3,x} + a_2z_{3,y} + a_3z_3 &= -z_{3,xx}. \end{aligned} \quad \text{(A.3)}$$

The systems for the coefficients b_i and c_i are obtained by replacing the second-order derivatives $z_{i,xx}$ by $z_{i,xy}$ or $z_{i,yy}$ respectively. Solving system (A.3) yields for the a_i

$$a_1 = -\frac{1}{w}\begin{vmatrix} z_{1,xx} & z_{1,y} & z_1\\ z_{2,xx} & z_{2,y} & z_2\\ z_{3,xx} & z_{3,y} & z_3 \end{vmatrix},\quad a_2 = \frac{1}{w}\begin{vmatrix} z_{1,x} & z_{1,xx} & z_1\\ z_{2,x} & z_{2,xx} & z_2\\ z_{3,x} & z_{3,xx} & z_3 \end{vmatrix},$$

$$a_3 = -\frac{1}{w}\begin{vmatrix} z_{1,x} & z_{1,y} & z_{1,xx}\\ z_{2,x} & z_{2,y} & z_{2,xx}\\ z_{3,x} & z_{3,y} & z_{3,xx} \end{vmatrix} \quad \text{where} \quad w = \begin{vmatrix} z_{1,x} & z_{1,y} & z_1\\ z_{2,x} & z_{2,y} & z_2\\ z_{3,x} & z_{3,y} & z_3 \end{vmatrix}.$$

For the b_i and c_i there follows

$$b_1 = -\frac{1}{w}\begin{vmatrix} z_{1,xy} & z_{1,y} & z_1 \\ z_{2,xy} & z_{2,y} & z_2 \\ z_{3,xy} & z_{3,y} & z_3 \end{vmatrix}, \quad b_2 = \frac{1}{w}\begin{vmatrix} z_{1,x} & z_{1,xy} & z_1 \\ z_{2,x} & z_{2,xy} & z_2 \\ z_{3,x} & z_{3,xy} & z_3 \end{vmatrix}, \quad b_3 = -\frac{1}{w}\begin{vmatrix} z_{1,x} & z_{1,y} & z_{1,xy} \\ z_{2,x} & z_{2,y} & z_{2,xy} \\ z_{3,x} & z_{3,y} & z_{3,xy} \end{vmatrix},$$

$$c_1 = -\frac{1}{w}\begin{vmatrix} z_{1,yy} & z_{1,y} & z_1 \\ z_{2,yy} & z_{2,y} & z_2 \\ z_{3,yy} & z_{3,y} & z_3 \end{vmatrix}, \quad c_2 = \frac{1}{w}\begin{vmatrix} z_{1,x} & z_{1,yy} & z_1 \\ z_{2,x} & z_{2,yy} & z_2 \\ z_{3,x} & z_{3,yy} & z_3 \end{vmatrix}, \quad c_3 = -\frac{1}{w}\begin{vmatrix} z_{1,x} & z_{1,y} & z_{1,yy} \\ z_{2,x} & z_{2,y} & z_{2,yy} \\ z_{3,x} & z_{3,y} & z_{3,yy} \end{vmatrix}.$$

3.2. System (3.4) has to satisfy the coherence conditions (B.11) given in Appendix B. Proper substitution of the variables $A_1, \ldots, B_3$ by the coefficients of (3.4) yields the following conditions.

$$B_{1,yy} - 2B_{2,y} - A_{1,x} - ((A_1B_1)_y = 0,$$
$$B_{2,yy} + (B_{2,y} - A_2B_1)A_1 - A_{2,x} - (B_{1,y} - A_1B_1)A_2 = 0.$$

Upon reduction w.r.t. the integrability conditions for the ideal of type $\mathbb{J}_1^{(0,3)}$ in Proposition 2.1 they vanish. The proof for system (3.5) is similar. This example shows the importance of knowing the coherence conditions for any system of pde's; without knowing them such a system is meaningless.

3.3. Depending on the divisor I_1 the left factor M may be of type $\mathbb{M}_1^{(0,1)}$ or $\mathbb{M}_2^{(0,1)}$ respectively.

3.4. Define $z_1 \equiv z_x + az$ and $z_2 \equiv z_y + bz$. For the type $\mathscr{L}^1_{yy,x}$ decomposition the syzygy and the quotient yield the system

$$z_{1,y} - z_{2,x} = 0, \;\; z_{2,y} + (A_1 - b)z_2 = 0, \;\; z_1 + B_1z_2 = 0.$$

In $grlex$, $z_1 \succ z_2$, $x \succ y$ term order the Janet basis for the exact quotient module is obtained in the form

$$z_{2,x} + (B_{1,y} - A_1B_1 + B_1b)z_2 = 0, \;\; z_{2,y} + (A_1 - b)z_2 = 0, \;\; z_1 + B_1z_2 = 0.$$

With the notation of Proposition 3.2 the decomposition may be written as

$$I = \left\langle \begin{pmatrix} 0 & \partial_x + B_{2,y} + A_1B_2 - B_2 \\ 0 & \partial_y + A_1 - r \\ 1 & B_1 \end{pmatrix} \begin{pmatrix} \partial_x + B_2 - B_1r \\ \partial_y + r \end{pmatrix} \right\rangle.$$

For the type $\mathscr{L}^1_{xx,y}$ decomposition a similar calculation yields

$$z_{1,x} + (A_1 - a)z_1 = 0, \;\; z_{1,y} = 0, \;\; z_2 = 0$$

and the decomposition

$$I = \left\langle \begin{pmatrix} \partial_x + A_1 - r & 0 \\ \partial_y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_x + r \\ \partial_y + B \end{pmatrix} \right\rangle.$$

3.5. The system (3.6) on page 70 for this ideal is

$$a_x - a^2 + \tfrac{4}{x}a - \tfrac{2}{x^2} = 0, \;\; a_y - ab + \tfrac{1}{x}b = 0,$$
$$b_x - ab + \tfrac{1}{x}b = 0, \;\; b_y - b^2 - \tfrac{x}{y^2}a + \tfrac{1}{y}b + \tfrac{2}{y^2} = 0.$$

The first equation is a Riccati ode for the x-dependence of a with the general solution $a = \frac{1}{x}\frac{Cx-2}{Cx-1}$. Two special solutions for $C \to \infty$ and $C = 0$ are $a = \frac{1}{x}$ and $a = \frac{2}{x}$ respectively. In the latter case the only choice for b is $b = 0$. In the former the equations $b_x = 0$ and $b_y - b^2 + \frac{1}{y}b + \frac{1}{y} = 0$ for b are obtained. Two special solutions are $b = \pm\frac{1}{y}$. Thus there are three divisors

$$l_1 = \left\langle\!\left\langle \partial_x + \frac{1}{x}, \partial_y + \frac{1}{y} \right\rangle\!\right\rangle, \;\; l_2 = \left\langle\!\left\langle \partial_x + \frac{1}{x}, \partial_y - \frac{1}{y} \right\rangle\!\right\rangle, \;\; l_3 = \left\langle\!\left\langle \partial_x + \frac{2}{x}, \partial_y \right\rangle\!\right\rangle;$$

the given ideal may be represented as $Lclm(l_1, l_2, l_3)$.

3.6. The system to be solved is $z_{yyy} + \frac{3}{x}z_{yy} = 0$, $z_x + \frac{y}{x}z_y = 0$. Two first-order right divisors determined in Example 3.9 are $\langle\!\langle \partial_x, \partial_y \rangle\!\rangle$ and $\langle\!\langle \partial_x + \frac{1}{x}, \partial_y - \frac{1}{y} \rangle\!\rangle$; they yield the basis elements 1 and $\frac{y}{x}$. The exact quotient module generates the system

$$z_{1,x} + \left(\frac{2}{x} - \frac{3y}{x^2}\right)z_1 = 0, \quad z_{1,y} + \frac{3}{x}z_1 = 0, \;\; z_2 = 0.$$

The special solution $z_1 = \frac{1}{x^2}\exp\left(-\frac{3y}{x}\right)$, $z_2 = 0$ leads to the inhomogeneous system $z_{yy} = \frac{1}{x^2}\exp\left(-\frac{3y}{x}\right)$, $z_x + \frac{y}{x}z_y = 0$; its special solution $\exp\left(-\frac{3y}{x}\right)$ yields the third basis element, i.e. a fundamental system is $\left\{1, \frac{y}{x}, \exp\left(-\frac{3y}{x}\right)\right\}$.

Chapter 4

4.1. From (4.4) the two equations $a_i^2 - A_1a_i + A_2 = 0$ for $i = 1, 2$ are obtained; they yield $A_1 = a_1 + a_2$ and $A_2 = a_1a_2$. Equation 4.5 leads to

$$a_{i,x} + (A_1 - a_i)a_{i,y} + A_3a_i + (A_1 - 2a_i)b_i = A_4 \;\text{ for }\; i = 1, 2.$$

These equations may be solved for A_3 and A_4 with the result

$$A_3 = b_1 + b_2 - \frac{(a_1 - a_2)_x}{a_1 - a_2} - \frac{a_{1,y}a_2 - a_{2,y}a_1}{a_1 - a_2},$$

$$A_4 = a_1 b_2 - a_2 b_1 + a_1 a_2 \frac{b_1 - b_2}{a_1 - a_2} - \frac{a_{1,x}a_2 - a_{2,x}a_1}{a_1 - a_2} - \frac{a_{1,y}a_1^2 - a_{2,y}a_1^2}{a_1 - a_2}.$$

Substituting these values into (4.6), e.g. for $i = 1$, there follows

$$A_5 = b_1 b_2 - b_2 \frac{(a_1 - a_2)_x}{a_1 - a_2} - b_2 \frac{a_{1,y}a_2 - a_{2,y}a_1}{a_1 - a_2} + b_{2,x} + a_1 b_{2,y}.$$

In addition there is a constraint for solvability of the linear system for $A_1, \ldots, A_5$. It is most easily obtained by taking the difference of the two equations obtained from (4.6) for $i = 1, 2$ and substituting $A_1, \ldots, A_4$ obtained above. It leads to the constraint

$$\frac{(b_1 - b_2)_x}{a_1 - a_2} - \frac{b_1 - b_1}{a_1 - a_2}\frac{(a_1 - a_2)_x}{a_1 - a_2} = \frac{(a_1 b_2 - a_2 b_1)_y}{a_1 - a_2} - \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}\frac{(a_1 - a_2)_y}{a_1 - a_2}.$$

By means of a few simple manipulations it may be shown that it is identical to the condition of case (i) in Theorem 2.2, i.e. it proves Lemma 2.6 for this special case.

4.2. Substituting $z = f_0 F(y) + f_1 F'(y) + g_0 G(x) + g_1 G'(x)$ into the equation $z_{xy} + A_1 z_x + A_2 z_y + A_3 z = 0$ yields an expression that is linear and homogeneous in F, G and its derivatives. It vanishes if all coefficients vanish. This leads to the following system of equations involving f_0 and f_1.

$$\begin{aligned} f_{1,x} + A_2 f_1 &= 0, \\ f_{0,x} + f_{1,xy} + A_1 f_{1,x} + A_2(f_0 + f_{1,y}) + A_3 f_1 &= 0, \\ f_{0,xy} + A_1 f_{0,x} + A_2 f_{0,y} + A_3 f_0 &= 0. \end{aligned} \tag{A.4}$$

There is a similar system involving g_0 and g_1.

$$\begin{aligned} g_{1,y} + A_1 g_1 &= 0, \\ g_{0,y} + g_{1,xy} + A_1(g_0 + g_{1,x}) + A_2 g_{1,y} + A_3 g_1 &= 0, \\ g_{0,xy} + A_1 g_{0,x} + A_2 g_{0,y} + A_3 g_0 &= 0. \end{aligned} \tag{A.5}$$

System (A.4) has the solution $A_2 = -\dfrac{f_{1,x}}{f_1}$,

$$A_1 = \frac{f_0}{f_1} + \frac{1}{\begin{vmatrix} f_0 & f_1 \\ f_{0,x} & f_{1,x} \end{vmatrix}} \left(\frac{f_{1,x}}{f_1} \begin{vmatrix} f_0 & f_1 \\ f_{0,y} & f_{1,y} \end{vmatrix} - \begin{vmatrix} f_0 & f_1 \\ f_{0,xy} & f_{1,xy} \end{vmatrix} \right),$$

$$A_3 = -\frac{f_{0,x}}{f_1} + \frac{1}{\begin{vmatrix} f_0 & f_1 \\ f_{0,x} & f_{1,x} \end{vmatrix}} \left(\frac{f_{1,x}}{f_1} \begin{vmatrix} f_{0,x} & f_{1,x} \\ f_{0,y} & f_{1,y} \end{vmatrix} - \begin{vmatrix} f_{0,x} & f_{1,x} \\ f_{0,xy} & f_{1,xy} \end{vmatrix} \right).$$

If these values for A_1, A_2 and A_3 are substituted into (A.5), constraints between the f_0, f_1, g_0 and g_1 are obtained.

4.3. If the single integrability condition $\partial_x L = \partial_y \mathfrak{l}_{2,xx}$ is reduced w.r.t. L and $\mathfrak{l}_{2,xx}$, the following system of equations is obtained.

$$\begin{gathered} a_1 A_1 A_2 + a_1 A_{2,y} - a_1 A_3 + A_{1,x} A_2 + A_1 A_{2,x} - 2A_1 A_2^2 \\ + A_{2,xy} - 2A_{2,y} A_2 + 2A_2 A_3 - A_{3,x} = 0, \\ a_{1,y} - A_{1,x} + A_1 A_2 - A_3 = 0, \quad a_2 - a_1 A_2 - A_{2,x} + A_2^2 = 0. \end{gathered}$$

A few elimination steps yield

$$\begin{aligned} a_1 &= 2A_2 - \log\left(A_{2,y} + A_1 A_2 - A_3\right)_x, \\ a_2 &= A_{2,x} + A_2^2 - A_2 \log\left(A_{2,y} + A_1 A_2 - A_3\right)_x \end{aligned}$$

and the condition

$$2A_{2,y} - \log\left(A_{2,y} + A_1 A_2 - A_3\right)_{xy} - A_{1,x} + A_1 A_2 - A_3 = 0.$$

Performing the same reduction steps to $L - z_1$ and $\mathfrak{l}_{2,xx} - z_2$, the relation

$$z_{1,x} + (a_1 - A_2) z_1 - z_{2,y} - A_1 z_2 = 0$$

follows, i.e. there is a single syzygy $(\partial_x + a_1 - A_2, -\partial_y - A_1)$. The exact quotient module is generated by $((1,0), (\partial_x + a_1 - A_2, -\partial_y - A_1))$; the corresponding Janet basis is $((1,0), (0, \partial_y + A_1))$.

4.4. Define $\Delta \equiv \sqrt{A_1^2 - 4A_2}$. The two roots of (4.4) are $a_{1,2} = \frac{1}{2}A_1 \pm \frac{1}{2}\Delta$. If $\Delta \neq 0$ there follows $b_{1,2} = \frac{1}{2}A_3 \pm \frac{1}{2\Delta}(A_1 A_3 - 2A_4)$. The constraint (4.6) is $A_5(A_1^2 - 4A_2) + (A_2 A_3 - A_1 A_4) A_3 + A_4^2 = 0$. If it is satisfied there are two factors

$$l_{1,2} = \partial_x + \tfrac{1}{2}(A_1 \pm \Delta)\partial_y + \tfrac{1}{2}A_3 \pm \frac{1}{2\Delta}(A_1 A_3 - 2A_4).$$

If $\Delta = 0$ there follows $a_1 = a_2 = \frac{1}{2}A_1$; condition (4.2) is $A_4 - \frac{1}{2}A_1 A_3 = 0$. If it is satisfied, b has to be determined from $b_x + \frac{1}{2}A_1 b_y - b^2 + A_3 b - A_5 = 0$. Applying the results of Exercise B.2, the following factorizations are obtained. If $A_3^2 - 4A_5 \neq 0$ there are the two factors

$$l_{1,2} = \partial_x + \tfrac{1}{2}A_1 \partial_y + \tfrac{1}{2}A_3 \pm \tfrac{1}{2}(4A_5 - A_3^2).$$

Finally, if $A_3^2 - 4A_5 = 0$ there is the right factor

$$l = \partial_x + \tfrac{1}{2}A_1\partial_y + \tfrac{1}{2}A_3 - \frac{1}{F\big(y - \frac{1}{2}xA_1\big) + x}.$$

4.5. By Proposition 4.2 and Corollary 4.2 the type $\mathscr{L}_{xy}^3$ decomposition $L = Lclm\big(\partial_x + y, \partial_y + x + \frac{1}{y}\big)$ is obtained.

Chapter 5

5.1. The given equation factorizes according to

$$(\partial_x - x\partial_y - 1)(\partial_x + (y^2 - x)\partial_y + 1)z = 0,$$

i.e. the order of the factors of the equation in Example 5.2 is reversed. Because the right factor equation now does not allow a Liouvillian solution, the full second order equation does not have one either. The left-factor equation $w_x + xw_y - w = 0$ has the solution $w = e^x F(y + \frac{1}{2}x^2)$. Therefore a second element of a fundamental system is determined by

$$z_x + (y^2 - x)z_y + z = e^x F\big(y + \tfrac{1}{2}x^2\big).$$

5.2. The same steps as in Example 5.18 are performed. The inhomogeneity has to satisfy $r_y + \dfrac{2}{x-y}r = y$; its general solution is

$$r = C(x - y)^2 + (x - y)^2 \log(x - y) + (x - y)y.$$

Choosing $C = 0$ the equation

$$z_{xx} - \frac{2}{x-y}z_x + \frac{2}{(x-y)^2}z = (x - y)^2 \log(x - y) + (x - y)y$$

has to be solved. Substituting its general solution

$$z = C_1(x-y) + C_2x(x-y) + \tfrac{1}{6}(x-y)^2 \log(x - y) - \tfrac{1}{36}x(x-y)(5x^2 - 33xy + 6y^2)$$

into the given equation leads to the constraint $C_{1,y} + yC_{2,y} - C_2 + \frac{1}{3}y^2 = 0$. A special solution is $C_1 = 0$, $C_2 = -\frac{1}{3}y^2$, it yields

$$z_0(x, y) = \tfrac{1}{6}(x - y)^4 \log(x - y) - \tfrac{1}{36}x(x - y)(x - 6y)(5x - 3y).$$

5.3. The difference of any two special solutions is a solution of the corresponding homogeneous equation. For the three alternatives in Example 5.19 these differences are

$$x^2, \quad \frac{x^2y^4 + 2xy^2 - 1}{y^3}, \quad \frac{x^2y^3(y-1) + 2xy^2 - 1}{y^3}.$$

They satisfy the homogeneous equation $z_{xy} + xyz_x - 2yz = 0$. The general solution of this equation corresponding to the Laplace divisor has been determined in Example 5.10, Eq. (5.3). It must be possible to obtain the above differences by suitable choices of the undetermined function $F(y)$. To this end, they may be considered as inhomogeneities of a linear ordinary differential equation for F. For the first alternative this leads to

$$F'' + \frac{2xy^2 - 1}{y}F' + x^2y^2F = x^2y^2;$$

its special solution $F_0 = 1$ is the desired result. For the remaining two cases the result is $F_0 = y$ and $F_0 = y - 1$ respectively.

5.4. The equation $z_{xy} + xyz_x - 2yz = \frac{x^2+1}{y}$ has the special solution

$$z_0(x,y) = (y^4 - 4xy^2 + 8)\frac{1}{y^6}\log x - \frac{1}{y^6}\left(\tfrac{3}{2}x^2y^4 - 12\right)$$
$$-\frac{16}{y^6}\exp\left(-\tfrac{1}{2}xy^2\right)\int \exp\left(\tfrac{1}{2}xy^2\right)\frac{dy}{y}.$$

The equation $z_{xy} + xyz_x - 2yz = \frac{1}{xy}$ has the special solution

$$z_0(x,y) = \left(\tfrac{1}{8}x^2y^4 + xy^2 + 1\right)\exp\left(\tfrac{1}{2}xy^2\right)\int \exp\left(-\tfrac{1}{2}xy^2\right)\int \exp\left(\tfrac{1}{2}xy^2\right)\frac{dy}{y}\frac{dx}{x}$$
$$+\left(\tfrac{1}{4}xy^2 + \tfrac{3}{2}\right)\int \exp\left(\tfrac{1}{2}xy^2\right)\frac{dy}{y} + \tfrac{1}{8}\left(xy^2 + 7\right)\exp\left(\tfrac{1}{2}xy^2\right).$$

5.5. In general, the coefficients $A_1, \ldots, B_5$ of

$$I \equiv \langle \partial_{xx} + A_1\partial_{yy} + A_2\partial_x + A_3\partial_y + A_4, \partial_{xy} + B_1\partial_{yy} + B_2\partial_x + B_3\partial_y + B_4\rangle$$

generated by a Janet basis in $grlex$, $x \succ y$ term order are related by $A_1 + B_1^2 = 0$ and

$$A_{4,y} - B_{4,x} + B_{4,y}B_1 - A_2B_4 + A_4B_2 - B_1B_2B_4 + B_3B_4 = 0,$$
$$A_{3,y} - B_{3,x} + B_{3,y}B_1 - A_2B_3 + A_3B_2 + A_4 - B_1B_2B_3 + B_1B_4 + B_3^2 = 0,$$
$$A_{2,y} - B_{2,x} + B_{2,y}B_1 - B_1B_2^2 + B_2B_3 - B_4 = 0,$$
$$A_{1,y} - B_{1,x} + B_{1,y}B_1 + A_1B_2 - A_2B_1 + A_3 - B_1^2B_2 + 2B_1B_3 = 0.$$

If $A_1, \ldots, A_4$ are given, this system determines $B_1, \ldots, B_4$ such that the integrability conditions are satisfied. For the special values $A_1 = 0$, $A_2 = -1$, $A_3 = -\frac{2}{x}$ and $A_4 = A_5 = 0$ the solutions are $B_1 \pm 1$, $B_2 = \pm\frac{1}{x}$ and $B_3 = B_4 = 0$. They yield the two ideals

$$I_1 \equiv \left\langle\!\left\langle \partial_{xx} - \partial_{xy} - \tfrac{2}{x}\partial_x, \partial_{xy} + \partial_{yy} + \tfrac{1}{x}\partial_x \right\rangle\!\right\rangle \quad \text{and}$$
$$I_2 \equiv \left\langle\!\left\langle \partial_{xx} - \partial_{xy} - \tfrac{2}{x}\partial_x, \partial_{xy} - \partial_{yy} - \tfrac{1}{x}\partial_x \right\rangle\!\right\rangle$$

such that $I = I_1 \cap I_2$; I_1 and I_2 correspond to the components involving F and G respectively in the solution $Lz = 0$ (compare Example 5.23).

5.6. The equation may be represented as $Lclm\big(\partial_x - \frac{1}{x}\partial_y, \partial_x - y\partial_y\big)z = 0$. This yields $z_1(x,y) = F\big(xe^y\big)$ and $z_2(x,y) = G\big(ye^x\big)$; F and G are undetermined functions.

5.7. The homogeneous part of the equation allows a type $\mathscr{L}_{xx}^1$ factorization, it yields the representation

$$\Big(\partial_x - \frac{y}{x}\partial_y - \frac{1}{x}\Big)\big(\partial_x + \partial_y\big)z = -xy.$$

Applying Proposition 5.1, case (i), the differential fundamental system

$$z_1(x,y) = F(y-x), \quad z_2(x,y) = \int xG(xy)\big|_{y=\bar{y}+x}dx\big|_{\bar{y}=y-x}$$

is obtained; F and G are undetermined functions. Proposition 5.4, case (i), yields the special solution

$$z_0(x,y) = \tfrac{1}{12}x^3\big(x\log x - 4y\log x - \tfrac{7}{12}x + \tfrac{4}{3}y\big).$$

5.8. The equation does not have any first-order factor. Introducing new variables by $u = x + iy$ and $v = x - iy$ the equation $w_{uv} + \dfrac{2w}{(1+uv)^2} = 0$ for $w(u,v) \equiv f(x,y)$ is obtained. The corresponding operator has the two Laplace divisors

$$\left\langle\!\!\left\langle \partial_{uv} + \frac{2}{(1+uv)^2}, \partial_{uu} + \frac{2v}{1+uv}\partial_u \right\rangle\!\!\right\rangle \text{ and } \left\langle\!\!\left\langle \partial_{uv} + \frac{2}{(1+uv)^2}, \partial_{vv} + \frac{2u}{1+uv}\partial_v \right\rangle\!\!\right\rangle.$$

This type $\mathscr{L}_{uv}^4$ decomposition yields the solutions

$$w_1(u,v) = \frac{2v}{1+uv}f(u) - f'(u) \text{ and } w_2(u,v) = \frac{2u}{1+uv}g(v) - g'(v);$$

f and g are undetermined functions. Upon substitution of the variables u and v the corresponding expressions for $z_1(x,y)$ and $z_2(x,y)$ are obtained.

5.9. Dividing out the factor ∂_y leads to the representation $L = \big(\partial_x - \frac{1}{x+y}\big)\partial_y$. The left-factor equation $w_x - \dfrac{1}{x+y}w = 0$ has the solution $w(x,y) = (x+y)G(y)$ where $G(y)$ is an undetermined function of y. The inhomogeneous right-factor equation $z_y = (x+y)G(y)$ yields the second element of a fundamental system $z_2(x,y) = \int yG(y)dy + x\int G(y)dy$. The identifications $C_1(y) = \int yG(y)dy$ and $C_2(y) = \int G(y)dy$ show that it is identical to $z_2(x,y)$ obtained in Example 5.11 as it was to be expected. However, the representation of $z_2(x,y)$ obtained from the single factor ∂_y is unnecessarily complicated due to the occurrence of integrals. In general, it is not obvious how to simplify a solution obtained from an incomplete factorization.

5.10. The Loewy decomposition is

$$Lz = \begin{pmatrix} 1 & 0 \\ 0 & \partial_y - \frac{2}{y} \end{pmatrix} \begin{pmatrix} w_1 \equiv z_{xy} - \frac{2}{y} z_x - y z_y \\ w_2 \equiv z_{xx} - 2y z_x + y^2 z \end{pmatrix}.$$

The equation $z_{xx} - 2yz_x + y^2 z = 0$ has the solution $z = C_1 \exp(xy) + C_2 x \exp(xy)$; C_1 and C_2 are undetermined functions of y. Substitution into $Lz = 0$ leads to $yC_2' - yC_1 - 2C_2 = 0$, i.e. $C_1 = C_2' - \frac{2}{y}C_2$. This yields

$$z_1(x,y) = \exp(xy)\big(F'(y) + (x - \frac{2}{y})F(y)\big)$$

where $F \equiv C_2$. The equation $w_{2,y} - \frac{2}{y}w_2 = 0$ has the solution $w_2 = G(x)y^2$; G is an undetermined function of x. A special solution of the inhomogeneous equation $z_{xx} - 2yz_x + y^2 z = Gy^2$ yields the second member of a fundamental system

$$z_2(x,y) = xy^2 \exp(xy) \int \exp(-xy)G(x)dx - y^2 \exp(xy) \int x \exp(-xy)G(x)dx.$$

5.11. The Loewy decomposition is

$$Lz = \begin{pmatrix} 1 & 0 \\ 0 & \partial_y \end{pmatrix} \begin{pmatrix} w_1 \equiv z_{xy} + (xy-1)z_y + 2xz \\ w_2 \equiv z_{xx} + \big(2xy - 2 - \frac{1}{x}\big)z_x + \big(x^2y^2 - 2xy + 1 + \frac{1}{x}\big)z \end{pmatrix}.$$

Proceeding similar as in the preceding problem the fundamental system

$$z_1(x,y) = \exp\big(x - \tfrac{1}{2}x^2 y\big)\big(2F'(y) - x^2 F(y)\big),$$

$$z_2(x,y) = \exp(x - \tfrac{1}{2}x^2y)\Big(x^2 \int \exp\big(\tfrac{1}{2}x^2 y - x\big)G(x)\frac{dx}{x} - \int \exp\big(\tfrac{1}{2}x^2y - x\big)G(x)x\,dx\Big)$$

is obtained. F and G are undetermined functions.

5.12. The operator L is completely reducible, its Loewy decomposition of type $\mathscr{L}^4_{xy}$ is

$$L = Lclm\big(\langle\!\langle L, \partial_{xx}\rangle\!\rangle, \langle\!\langle L, \partial_{yy} + \frac{2}{y}\partial_y\rangle\!\rangle\big).$$

It yields the fundamental system

$$z_1(x,y) = \frac{xy+1}{y}F'(y) - \frac{2x}{y}F(y), \quad z_2(x,y) = \frac{xy+1}{xy}G'(x) - \frac{3xy+1}{x^2y}G(x);$$

F and G are undetermined functions.

5.13. Substituting $z = \varphi(x,y)w$ into the equation $z_{xx} + Az_x + Bz_y + Cz = 0$ yields

$$w_{xx} + \Big(2\frac{\varphi_x}{\varphi} + A\Big)w_x + Bw_y + \Big(\frac{\varphi_{xx}}{\varphi} + A\frac{\varphi_x}{\varphi}\Big) + B\frac{\varphi_y}{\varphi} + Cw = 0.$$

The coefficient of w_x vanishes if $\frac{\varphi_x}{\varphi} = \frac{1}{2}A$, i.e. if $\varphi = \exp\big(-\frac{1}{2}\int A dx\big)$. Substituting it into the coefficient of w yields the result of case (i).

In case (ii), the same substitution into the equation $z_{xy} + Az_x + Bz_y + Cz = 0$ yields

$$w_{xy} + \big(A + \frac{\varphi_y}{\varphi}\big)w_x + \big(B + \frac{\varphi_x}{\varphi}\big)w_y + \big(C + B\frac{\varphi_y}{\varphi} + A\frac{\varphi_x}{\varphi} + \frac{\varphi_{xy}}{\varphi}\big)w = 0.$$

The derivative w_x does not occur in this equation if φ is a solution of the equation $\varphi_y + A\varphi = 0$; a special solution is $\varphi = \exp\big(-\int A dy\big)$. Substituting this expression into the above equation for w yields the second case of Corollary 5.4.

In order that the derivative w_y vanishes as well, φ has to obey the system $\varphi_x + B\varphi = 0$, $\varphi_y + A\varphi = 0$. In order that a nontrivial solution exists, the condition $A_x = B_y$ must be valid. Then $w_{xy} + (C - B_y - AB)w = 0$.

5.14. Substituting $z = \varphi w$ into the given equation yields

$$\begin{aligned} &w_{xx} + A_1 w_{xy} + A_2 w_{yy} \\ &\quad + \Big(2\frac{\varphi_x}{\varphi} + A_1\frac{\varphi_y}{\varphi} + A_3\Big)w_x + \Big(A_1\frac{\varphi_x}{\varphi} + 2A_2\frac{\varphi_y}{\varphi} + A_4\Big)w_y \\ &\quad + \Big(\frac{\varphi_{xx}}{\varphi} + A_1\frac{\varphi_{xy}}{\varphi} + A_2\frac{\varphi_{yy}}{\varphi} + A_3\frac{\varphi_x}{\varphi} + A_4\frac{\varphi_y}{\varphi} + A_5\Big)w = 0. \end{aligned}$$

The two first-order derivatives disappear if

$$2\frac{\varphi_x}{\varphi} + A_1\frac{\varphi_y}{\varphi} + A_3 = 0, \quad A_1\frac{\varphi_x}{\varphi} + 2A_2\frac{\varphi_y}{\varphi} + A_4 = 0;$$

if $A_1^2 - 4A_2 \neq 0$ this algebraic system has the solution

$$\frac{\varphi_x}{\varphi} = \frac{A_1A_4 - 2A_2A_3}{A_1^2 - 2A_2} \equiv p \quad \text{and} \quad \frac{\varphi_y}{\varphi} = \frac{A_1A_3 - 4A_2}{A_1^2 - 4A_2} \equiv q;$$

its integrability condition is $p_y = q_x$. If it is satisfied, integration yields the solution given in case (i).

If $A_1^2 - 4A_2 = 0$, the above algebraic system is consistent if $2A_4 - A_1A_3 = 0$ and the equation

$$\varphi_x + \tfrac{1}{2}A_2\varphi_y + \tfrac{1}{2}A_3\varphi = 0$$

for φ follows. According to Corollary B.1 its solution is

$$\varphi(x, y) = \Phi(\bar{y}) \exp\Big(-\tfrac{1}{2}\int A_3\big(x, \bar{\psi}(x, \bar{y})\big)dx\Big|_{\bar{y}=\psi(x,y)}\Big);$$

$\psi(x, y)$ is a first integral of $\dfrac{dy}{dx} = \tfrac{1}{2}A_1$; $\bar{y} \equiv \psi(x, y)$ and $y = \bar{\psi}(x, \bar{y})$. This is case (ii).

5.15. Substitution of $z = \varphi(x, y)w$ into the given equation yields

$$w_{xy} + \big(A_1\frac{\varphi_y}{\varphi}\big)w_x + \big(A_2\frac{\varphi_y}{\varphi}\big)w_y + \big(A_3 + A_2\frac{\varphi_y}{\varphi} + A_1\frac{\varphi_x}{\varphi} + \frac{\varphi_{xy}}{\varphi}\big)w = 0.$$

Solvability of the conditions $\frac{\varphi_x}{\varphi} + A_2 = 0$ and $\frac{\varphi_y}{\varphi} + A_1 = 0$ requires $A_{1,x} = A_{2,y}$. If it is satisfied, the transformed equation is

$$w_{xy} + (A_3 - A_1A_2 - A_{2,y})w = 0.$$

Obviously its coefficients are in the base field of the originally given equation.

Chapter 6

6.1. The derivatives up to order three are

$$z = F + Ge^{-x}, \quad z_x = -xF' - Ge^{-x}, \quad z_y = F' + G'e^{-x},$$
$$z_{xx} = -F' + x^2F'' + Ge^{-x}, \quad z_{xy} = -xF'' - G'e^{-x}, \quad z_{yy} = F'' + G''e^{-x},$$
$$z_{xxx} = 3xF'' - x^3F''' - Ge^{-x}, \quad z_{xxy} = -F'' + x^2F''' + G'e^{-x},$$
$$z_{xyy} = -xF''' - G''e^{-x}, \quad z_{yyy} = F''' + G'''e^{-x}.$$

The $'$ means the derivative w.r.t. the argument of F or G. These ten equations may be considered as an inhomogeneous linear system for the eight indeterminates $F, G, F', \ldots G'''$; it may be solved by Gauss elimination. The two consistency conditions yield the desired generators of the $Lclm$.

6.2. The answer is obvious from the representations

$$L_1 = \big(\partial_{xx} - x^2\partial_{yy} + 2\partial_x + (2x+3)\partial_y\big)(\partial_x + 1)$$
$$= \big(\partial_{xx} - x\partial_{xy} + 3\partial_x - (x-1)\partial_y + 2\big)(\partial_x + x\partial_y)$$

and

$$L_2 = \Big(\partial_{xy} + x\partial_{yy} - \tfrac{1}{x}\partial_x + \big(1 + \tfrac{1}{x}\big)\partial_y\Big)(\partial_x + 1)$$
$$= \Big(\partial_{xy} - \tfrac{1}{x}\partial_x + \big(1 - \tfrac{1}{x}\big)\partial_y - \tfrac{1}{x}\Big)(\partial_x + x\partial_y)$$

6.3. The Loewy decomposition is

$$L = \begin{pmatrix} 1 & 1 \\ 0 & \partial_x - (x-1)\partial_y + 2 \end{pmatrix}\begin{pmatrix} L_1 \\ L_2 \end{pmatrix};$$

L_1 and L_2 are defined in (6.20), i.e. the operator L is in the same ideal as Blumberg's operator (6.17). L may be represented as $L = L_1 + L_2$.

6.4. By Proposition 6.3 there is no first-order right factor. Up to order $m = 10$ a Laplace divisor $\mathbb{L}_{y^n}(L)$ does not exist. However, there is a Laplace divisor $\mathbb{L}_{x^2}(L)$ corresponding to $\mathfrak{l}_2 \equiv \partial_{xx} + \frac{2}{x}\partial_x$; it yields the type $\mathscr{L}^{14}_{xyy}$ Loewy decomposition

$$L = \begin{pmatrix} 1 & 1 \\ 0 & \partial_{xy} - (xy - y)\partial_y - \frac{2}{y} \end{pmatrix}$$
$$\begin{pmatrix} \partial_{xyy} + (xy - y)\partial_{xy} + \frac{1}{x}\partial_{yy} - \frac{2}{y}\partial_x - \frac{y}{x}\partial_y - \frac{2}{xy} \\ \partial_{xx} + \frac{2}{x}\partial_x \end{pmatrix}.$$

Chapter 7

7.1. The solutions $z_1(x,y)$ and $z_2(x,y)$ are easily shown to annihilate the left-hand side of (7.2) by substitution. Let $\varphi = \log(y+1) - x$ and $\psi = \exp(\bar{y} + x) - 1$. Define

$$\bar{H}(x,\bar{y}) \equiv H(y)\big|_{y=\psi(x,\bar{y})} \quad \text{and} \quad z \equiv \int (x-1)\bar{H}(x,\bar{y})dx\big|_{\bar{y}=\varphi(x,y)}.$$

Applying the chain rule, the following expressions for the derivatives of z are obtained.

$$z_x = (x-1)H(y) + \varphi_x \int (x-1)\bar{H}_{\bar{y}}(x,\bar{y})dx\big|_{\bar{y}=\varphi(x,y)},$$

$$z_y = \varphi_y \int (x-1)\bar{H}_{\bar{y}}(x,\bar{y})dx\big|_{\bar{y}=\varphi(x,y)},$$

$$z_{xx} = H(y) + \varphi_{xx} \int (x-1)\bar{H}_{\bar{y}}(x,\bar{y})dx\big|_{\bar{y}=\varphi(x,y)}$$
$$+(x-1)\varphi_x \bar{H}_{\bar{y}}(x,\bar{y})\big|_{\bar{y}=\varphi(x,y)} + \varphi_x^2 \int (x-1)\bar{H}_{\bar{y}\bar{y}}(x,\bar{y})dx\big|_{\bar{y}=\varphi(x,y)},$$

$$z_{xy} = (x-1)H_y(y) + \varphi_{xy} \int (x-1)\bar{H}_{\bar{y}}(x,\bar{y})dx\big|_{\bar{y}=\varphi(x,y)}$$
$$+\varphi_x\varphi_y \int (x-1)\bar{H}_{\bar{y}\bar{y}}(x,\bar{y})dx\big|_{\bar{y}=\varphi(x,y)}.$$

Substitution into (7.2) yields

$$z = xH(y) + (x-1)\big(\varphi_x\psi_{\bar{y}}|_{\bar{y}=\varphi(x,y)} + y + 1\big)H_y(y)$$
$$+\big(\varphi_x^2 + (y+1)\varphi_x\varphi_y\big) \int (x-1)\bar{H}_{\bar{y}\bar{y}}(x,\bar{y})dx\big|_{\bar{y}=\varphi(x,y)}$$
$$+\big(\varphi_{xx} + (y+1)\varphi_{xy} + \varphi_x + (y+1)\varphi_y\big) \int (x-1)\bar{H}_{\bar{y}}(x,\bar{y})dx\big|_{\bar{y}=\varphi(x,y)}.$$

If the above expressions for φ and ψ are substituted, the coefficients of H_y and the integrals vanish, i.e. only the first term $xH(y)$ remains; this is the right hand side of (7.2).

7.2. The first-order right factor in the second line of (6.19) yields $z_1 = F\big(y - \frac{1}{2}x^2\big)$. The two first-order factors in the argument of the $Lclm$ yield

$$w_1 = G(y)e^{-x} \quad \text{and} \quad w_2 = H(y)xe^{-x}.$$

They lead to the inhomogeneous equations $z_x + xz_y = w_i$ with special solutions

$$z_2(x,y) = \int G\big(\bar{y} + \tfrac{1}{2}x^2\big)e^{-x}dx\Big|_{\bar{y}=y-\frac{1}{2}x^2} \quad \text{and}$$
$$z_3(x,y) = \int G\big(\bar{y} + \tfrac{1}{2}x^2\big)xe^{-x}dx\Big|_{\bar{y}=y-\frac{1}{2}x^2}$$

respectively. The differential fundamental system obtained in this way is more complicated than the one obtained in Example 7.8.

7.3. Introducing new variables $u \equiv \varphi(x,y) = y$ and $v \equiv \psi(x,y) = y - \frac{1}{2}x^2$ into the left factor $\partial_{xx} + x\partial_{xy} + \partial_x + (x+2)\partial_y$ yields, according to case (i) of Lemma 5.4,

$$D_{uv} \equiv \partial_{uv} - \frac{\sqrt{2(u-v)}+2}{2(u-v)}\partial_u - \frac{1}{2(u-v)}\partial_v.$$

According to Proposition 4.2 a first-order right factor does not exist. However, there is the operator $D_{uu} \equiv \partial_{uu} + \frac{1}{2(u-v)}\partial_u$ such that $\langle D_{uv}, D_{uu}\rangle$ is a Laplace divisor generated by a Janet basis; it yields the solution $w_1 = F(v) - \sqrt{2(u-v)}F'(v)$.

Up to order 5 there is no Laplace divisor involving the variable v. Therefore the exact quotient module

$$\left\langle\!\left\langle (1,0), \left(0, \partial_v - \frac{\sqrt{2(u-v)}+2}{2(u-v)}\right)\right\rangle\!\right\rangle$$

is set up. The equation corresponding to the latter operator has the solution $\frac{H(u)}{u-v}\exp\left(-\sqrt{2(u-v)}\right)$. It leads to the inhomogeneous equation

$$w_{2,uu} + \frac{1}{2(u-v)}w_{2,u} = \frac{H(u)}{u-v}\exp\left(-\sqrt{2(u-v)}\right)$$

with the solution

$$w_2 = \sqrt{u-v}\int \frac{H(u)}{\sqrt{u-v}}\exp\left(-\sqrt{2(u-v)}\right)du - \int H(u)\exp\left(-\sqrt{2(u-v)}\right)du.$$

In the original variables the solutions are

$$z_1 = w_1\big|_{u=y,v=y-\frac{1}{2}x^2} = F(y-\tfrac{1}{2}x^2) - xF'(y-\tfrac{1}{2}x^2) \text{ and } z_2 = w_2\big|_{u=y,v=y-\frac{1}{2}x^2}.$$

Again, this result is more complicated than that given in Example 7.8.

7.4. The operator

$$L \equiv \partial_{xxx} - x^2\partial_{xyy} + 3\partial_{xx} + (2x+3)\partial_{xy} - x^2\partial_{yy} + 2\partial_x + (2x+3)\partial_y$$

defined by the given equation has the right factors $\partial_x + 1$ and $\partial_x + x\partial_y$, i.e. it is contained in the ideal (6.20) generated by L_1 and L_2. With the notation of Example 6.9 there follows $L = L_1$, i.e. it is contained in the same ideal as Blumberg's example but is a different combination of the two generators L_1 and L_2. Its Loewy decomposition is

$$L = \begin{pmatrix} 1 & 0 \\ 0 & \partial_x - x\partial_y + 2 + \frac{1}{x} \end{pmatrix}\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}.$$

Equations 7.5 yield $w_1 = 0$ and $w_2 = \frac{1}{x}\exp(-2x)F\left(y + \frac{1}{2}x^2\right)$; F is an undetermined function. The system $L_1 z = w_1$ and $L_2 z = w_2$ is solved by Theorem 5.1. The constraint (5.24) is satisfied. The system (5.26) is

$$r_{xy} - \tfrac{1}{x}r_x + \big(1 + \tfrac{2}{x}\big)r_y - \tfrac{1}{x}r = \tfrac{1}{x}\exp(-2x)F\big(y + \tfrac{1}{2}x^2\big),$$
$$r_{yy} - \tfrac{2}{x}r_y + \frac{1}{x^2}r = \frac{1}{x^2}\exp(-2x)F\big(y + \tfrac{1}{2}x^2\big)$$

with the special solution

$$r_0(x,y) = \frac{1}{x^2}\exp\big(\tfrac{y}{x} - 2x\big)\Big(y\int \exp\Big(-\frac{y}{x}\Big)F\big(y + \tfrac{1}{2}x^2\big)dy$$
$$- \int y\exp\Big(-\frac{y}{x}\Big)F\big(y + \tfrac{1}{2}x^2\big)dy\Big).$$

Substitution into (5.25) finally yields

$$z_3(x,y) = \exp(x)\int r_0(x,y)\exp(-x)dx - \int r_0(x,y)dx.$$

7.5. In general, a differential type 0 solution may be obtained from a system $z_x + az = 0$, $z_y + bz = 0$, $a_y = b_x$, that reduces the given third order equation to zero. For the problem at hand, this leads to the following constraint for the coefficients a and b.

$$b_{xx} + xb_{xy} - \frac{1}{y+1}a_x + b_x + b_y - \frac{1}{y+1}(a+b) - a_xb - 2b_xa - 2xb_xb$$
$$+\frac{1}{y+1}a^2 - ab - b^2 - xab_y + xab^2 + a^2b = 0.$$

A solution scheme for this nonlinear second-order equation does not seem to exist. A special solution may be obtained from the requirement that a solution with no dependence on x is searched. It yields the result $z = (y+1)^2$ that cannot be obtained by specialization of F in the solution given in Example 7.19. The corresponding values of a and b are $a = 0$ and $b = -\dfrac{2}{y+1}$; they obey the above equation.

7.6. The three alternatives may be described as follows.

$$L_1 \equiv (\partial_x + y)(\partial_y - x)(\partial_y - y)$$
$$= \partial_{xyy} - (x+y)\partial_{xy} + y\partial_{yy} + (xy-1)\partial_x + (xy + y^2 + 1)\partial_y + xy^2;$$

$L_1z = 0$ has fundamental system

$$z_1(x,y) = F(x)\exp\big(\tfrac{1}{2}y^2\big), \quad z_2(x,y) = G(x)\exp\big(\tfrac{1}{2}y^2\big)\int \exp\big(xy - \tfrac{1}{2}y^2\big)dy,$$

$$z_3(x,y) = \exp\big(\tfrac{1}{2}y^2\big)\Big(\int \exp\big(xy - \tfrac{1}{2}y^2\big)dy\int \exp(-2xy)H(y)dy$$
$$- \int \exp(-2xy)H(y)\int \exp\big(xy - \tfrac{1}{2}y^2\big)dydy\Big).$$

$$L_2 \equiv (\partial_y - x)(\partial_y - y)(\partial_x + y)$$
$$= \partial_{xyy} - (x+y)\partial_{xy} + y\partial_{yy} + (xy-1)\partial_x + (xy+y^2-2)\partial_y + xy^2 - x - 2y;$$

$L_2 z = 0$ has fundamental system

$$z_1(x,y) = F(y)\exp(-xy), \quad z_2(x,y) = \exp\left(\tfrac{1}{2}y^2 - xy\right)\int \exp(xy)G(x)dx,$$

$$z_3(x,y) = \exp(-xy)\int \exp\left(xy + \tfrac{1}{2}y^2\right)H(x)\int \exp\left(xy - \tfrac{1}{2}y^2\right)dydx.$$

$$L_3 \equiv (\partial_y - x)(\partial_x + y)(\partial_y - y)$$
$$= \partial_{xyy} - (x+y)\partial_{xy} + y\partial_{yy} + (xy-1)\partial_x + (xy+y^2-1)\partial_y + xy^2 - 2y;$$

$L_3 z = 0$ has fundamental system

$$z_1(x,y) = F(x)\exp\left(\tfrac{1}{2}y^2\right), \quad z_2(x,y) = \exp\left(\tfrac{1}{2}y^2\right)\int \exp\left(-xy - \tfrac{1}{2}y^2\right)G(y)dy,$$

$$z_3(x,y) = \exp\left(\tfrac{1}{2}y^2\right)\int \exp\left(-xy - \tfrac{1}{2}y^2\right)\int \exp(2xy)H(x)dxdy.$$

Despite the fact that all six alternatives considered in Examples 7.21–7.23 have identical symbols, the structure of the solutions differs significantly. There are three different elements $z_1(x,y)$ corresponding to the three possible rightmost factors; hence, there are three pairs of permutations with identical elements $z_1(x,y)$. There is not a single pair with identical elements $z_2(x,y)$; the most significant distinguishing feature is the occurrence of the undetermined functions under the integral sign or not. In the element $z_3(x,y)$ there occurs always a repeated integral over x and y or a twofold integral over y. The undetermined function occurs always under the integral sign, either under the innermost integration or the outermost one.

7.7. The operator $L \equiv \partial_{xyy} + (x+1)\partial_{xy} - \partial_y$ allows the factor ∂_y and in addition the Laplace divisor $\langle\!\langle \partial_{xyy} + (x+1)\partial_{xy} - \partial_y, \partial_{xx} \rangle\!\rangle$; its type $\mathscr{L}^{15}_{xyy}$ Loewy decomposition is

$$L = \begin{pmatrix} 0, & 1 \\ \partial_y + x + 1, & 0 \end{pmatrix} \begin{pmatrix} K_1 \equiv \partial_{xxy} \\ K_2 \equiv \partial_{xyy} + (x+1)\partial_{xy} - \partial_y \end{pmatrix}. \qquad \square$$

It leads to the fundamental system is $z_1(x,y) = F(x)$, $z_2(x,y) = (x+1)G(y) + G'(y)$, $z_3(x,y) = \int\int H(x)\exp\big(-(x+1)y\big)dxdx$; F, G and H are undetermined functions.

Appendix B

B.1. If $\Delta \equiv \sqrt{a^2-4b} \neq 0$ there are two special solutions $z_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\Delta$. They yield the two representations of the general solution

$$z(x) = \frac{\pm\Delta \exp\big(\mp \Delta x\big)}{C - \exp\big(\mp \Delta x\big)} - \tfrac{1}{2}a \pm \tfrac{1}{2}\Delta;$$

C is a constant. For $C = 0$ and $C \to \infty$ the special solutions z_1 and z_2 respectively are obtained; they are the only rational solutions in this case.

If $\Delta = 0$ the single solution $z_1 = z_2 = -\frac{1}{2}a$ follows. It yields the general rational solution $z(x) = -\frac{1}{2}a + \dfrac{1}{C+x}$.

B.2. Define $\Delta \equiv \sqrt{c^2-4bd}$ now. If $\Delta \neq 0$ there are two special rational solutions $z_{1,2} = -\dfrac{c}{2b} \pm \dfrac{1}{2b}\Delta$. The general solution is

$$z(x,y) = \frac{1}{b}\frac{\pm\Delta \exp\big(\mp \Delta x\big)}{F(y-ax) - \exp\big(\mp \Delta x\big)} - \frac{c}{2b} \pm \frac{1}{2b}\Delta.$$

F is an undetermined function. If $\Delta = 0$ the general solution is

$$z(x,y) = -\frac{c}{2b} + \frac{1}{b}\frac{1}{F(y-ax)+x}.$$

Appendix C

C.1. Applying $z = \lambda(x,y)w$ to the expression E defined at the beginning of Appendix C yields an equation of the form $w_{xy} + \bar{a}w_x + \bar{b}w_y + \bar{c}w = 0$ where

$$\bar{a} = a + \frac{\partial \log\lambda}{\partial y}, \quad \bar{b} = b + \frac{\partial \log\lambda}{\partial x}, \quad \bar{c} = c + a\frac{\partial \log\lambda}{\partial x} b\frac{\partial \log\lambda}{\partial y} + \frac{1}{\lambda}\frac{\partial^2\lambda}{\partial x\partial y}.$$

Substituting these expressions into $\bar{a}_x + \bar{a}\bar{b} - \bar{c}$ and $\bar{b}_y + \bar{a}\bar{b} - \bar{c}$ makes the invariance explicit.

C.2. Forsyth's example. The coefficients are $a = \dfrac{2}{x-y}$, $b = -\dfrac{2}{x-y}$, and $c = -\dfrac{4}{(x-y)^2}$. The invariants $h = k = -\dfrac{2}{(x-y)^2}$ and $h_1 = k_{-1} = 0$ explain the solution given in Exercise 4.9.

Imschenetzky's example $z_{xy} + xyz_x - 2yz = 0$. Here $a = xy$, $b = 0$ and $c = -2y$. The transformed equations are

$$z_{1,xy} + \big(xy - \frac{1}{y}\big)z_{1,x} - 3yz_1 = 0, \quad z_{2,xy} + \big(xy - \frac{2}{y}\big)z_{2,x} - 4yz_2 = 0$$

and in general

$$z_{n,xy} + \big(xy - \frac{n}{y}\big)z_{n,x} - (n+2)yz_n = 0$$

for any positive integer n. For negative indices the corresponding equations are

$$z_{-1,xy}+xyz_{-1,x}-yz_{-1}=0,\ z_{-2,xy}+xyz_{-2,x}=0,\ z_{-3,xy}+xyz_{-3,x}+yz_{-3}=0,$$

$$z_{-n,xy}+xyz_{-n,x}-(n+2)yz_{-n}=0 \ \text{ for } \ n\geq 4.$$

The invariants $k_0=2y$, $k_{-1}=y$, and $k_{-2}=0$ explain the existence of solution z_1 in Example 5.10. The two lowest h−invariants are $h_0=3y$ and $h_1=4y$. Due to the fact that they depend only on y the recurrence (C.4) simplifies to $h_{i+1}=2h_i-h_{i-1}$. The solution satisfying the above initial conditions for $i=0$ and $i=1$ is $h_i=(i+1)y$; it does not vanish for $i\geq 2$. Hence, for this particular case it is proved that a Laplace divisor $\langle\langle L,\mathfrak{k}_n\rangle\rangle$ does not exist for any n.

Appendix D

D.1. By Proposition 4.2, the generic equation $z_{xy}+A(x,y)z_y+B(x,y)z=0$ has a right factor with leading derivative ∂_x if $B(x,y)=A_y(x,y)$; then $\partial_y(z_x+A(x,y)z)=0$, i.e. the left hand side is a total derivative. There is a right factor with leading derivative ∂_y if $B(x,y)=0$, i.e. $(\partial_x+A(x,y))z_y=0$. There are both right factors if $B(x,y)=0$ and $A_y(x,y)=0$, i.e. if A does not depend on y. In this case $Lclm(\partial_x+A(x),\partial_y)z=0$, i.e. the equation is completely reducible.

In special cases one may get even further. In cases (i) and (iii) the above condition for a right factor requires $A=y+C$, C a constant; it yields the factorization $\partial_{xy}+(y+C)\partial_y+1=\partial_y(\partial_x+y+C)$. If $B=A_y$ does not hold, there may be a Laplace divisor, e.g. $\langle\langle\partial_{xy}-(y+C)\partial_y+1,\partial_{yy}\rangle\rangle$; it may be found with the help of the function `LaplaceDivisor` provided by the ALLTYPES system.

In case (iv), for $A=B$ the factorization $\partial_y\big(\partial_x+\frac{A}{x-y}\big)$ is obtained. A nontrivial Loewy-decomposition allows finding the general solution of the corresponding differential equation involving two undetermined functions. A symmetry yields only so-called similarity solutions of less generality.

Appendix B
Solving Riccati Equations

Abstract Historically Riccati equations were the first non-linear ordinary differential equations that have been systematically studied. A good account of these efforts may be found in the book by Ince [29]. Originally they were of first order, linear in the first derivative, and quadratic in the dependent variable. Its importance arises from the fact that they occur as subproblems in many more advanced applications. Later on, ordinary Riccati equations of higher order have been considered. *Partial Riccati equations* are introduced as a straightforward generalization of the ordinary ones. All derivatives are of first order and occur only linearly, whereas the dependent variables may occur quadratically.

In the first section a few results for ordinary Riccati equations of first and second order are given. Thereafter various generalizations to partial Riccati-equations or Riccati-like systems of equations are considered.

B.1 Ordinary Riccati Equations

In the subsequent lemma the following terminology is applied. Two rational functions $p, q \in \mathbb{Q}(x)$ are called equivalent if there exists another function $r \in \mathbb{Q}(x)$ such that $p - q = \frac{r'}{r}$ holds, i.e. if p and q differ only by a logarithmic derivative of another rational function. This defines an equivalence relation on $\mathbb{Q}(x)$. A *special rational solution* does not contain a constant.

Lemma B.1. *If a first order Riccati equation $z' + z^2 + az + b = 0$ with $a, b \in \mathbb{Q}(x)$ has rational solutions, one of the following cases applies.*

(i) *The general solution is rational and has the form*

$$z = \frac{r'}{r + C} + p \tag{B.1}$$

where $p, r \in \bar{\mathbb{Q}}(x)$; $\bar{\mathbb{Q}}$ is a suitable algebraic extension of $\mathbb{Q}$, and C is a constant.

(ii) *There is only one, or there are two inequivalent special rational solutions.*

F. Schwarz, *Loewy Decomposition of Linear Differential Equations*, Texts & Monographs in Symbolic Computation, DOI 10.1007/978-3-7091-1286-1,

Analogous results for Riccati equations of second order are given next.

Lemma B.2. *If a second order Riccati equation*

$$z'' + 3zz' + z^3 + a(z' + z^2) + bz + c = 0$$

with $a, b, c \in \mathbb{Q}(x)$ has rational solutions, one of the following cases applies.

(i) *The general solution is rational and has the form*

$$z = \frac{C_2 u' + v'}{C_1 + C_2 u + v} + p \tag{B.2}$$

where $p, u, v \in \bar{\mathbb{Q}}(x)$; $\bar{\mathbb{Q}}$ is a suitable algebraic extension of $\mathbb{Q}$, C_1 and C_2 are constants.

(ii) *There is a single rational solution containing a constant, it has the form shown in equation (B.1).*

(iii) *There is a rational solution containing a single constant as in the preceding case, and in addition a single special rational solution.*

(iv) *There is only a single one, or there are two or three special rational solutions that are pairwise inequivalent.*

The proofs of Lemma B.1 and B.2 may be found in Chap. 2 of the book by Schwarz [61].

B.2 Partial Riccati Equations

At first general first-order linear pde's in x and y are considered; they may be obtained as specializations of a Riccati pde if the quadratic term in the unknown function is missing.

Lemma B.3. *Let the first-order linear pde $z_x + az_y + bz = c$ for $z(x, y)$ be given where $a, b, c \in \mathbb{Q}(x, y)$. Define $\varphi(x, y) = const$ to be a rational first integral of $\frac{dy}{dx} = a(x, y)$; assign $\bar{y} = \varphi(x, y)$ and the inverse $y = \psi(x, \bar{y})$ which is assumed to exist. Define*

$$\mathscr{E}(x, y) \equiv \exp\Big(- \int b(x, y)|_{y=\psi(x,\bar{y})} dx \Big)\Big|_{\bar{y}=\varphi(x,y)}. \tag{B.3}$$

The general solution $z = z_1 + z_0$ of the given first-order pde is

$$z_1(x, y) = \mathscr{E}(x, y)\Phi(\varphi), \quad z_0 = \mathscr{E}(x, y) \int \frac{c(x, y)}{\mathscr{E}(x, y)}\Big|_{y=\psi(x,\bar{y})} dx \Big|_{\bar{y}=\varphi(x,y)}; \tag{B.4}$$

Φ is an undetermined function.

Proof. Introducing a new variable $\bar{y} = \varphi(x, y)$ as defined above leads to the first-order ode $\bar{z}_x + \bar{b}(x, \bar{y})\bar{z} = \bar{c}(x, \bar{y})$. Upon substitution of $\bar{y}$ into its general solution, the solution (B.4) in the original variables is obtained. □

For $b = 0$ or $c = 0$ the expressions (B.4) simplify considerably as shown next.

Corollary B.1. *With the same notations as in the preceding lemma, the homogeneous equation $z_x + az_y + bz = 0$ has the solution*

$$z_1(x, y) = \mathscr{E}(x, y)\Phi(\varphi). \tag{B.5}$$

The equation $z_x + az_y = c$ has the solution $z = z_1 + z_0$ where

$$z_1(x, y) = \Phi(\varphi), \quad z_0(x, y) = \int c(x, y)|_{y=\psi(x,\bar{y})} dx \Big|_{\bar{y}=\varphi(x,y)} \tag{B.6}$$

with $\bar{y}$ and Φ as defined in Lemma B.3.

It should be noticed that Lemma B.3 in general does not allow solving a linear pde algorithmically. To this end, a rational first integral of $\frac{dy}{dx} = a(x, y)$ is required. This problem is discussed below. The subject of the next lemma is the general first-order Riccati pde in x and y.

Lemma B.4. *If the partial Riccati equation*

$$z_x + az_y + bz^2 + cz + d = 0 \tag{B.7}$$

where $a, b, c, d \in \mathbb{Q}(x, y)$ has rational solutions, two cases may occur.

(i) *The general solution is rational and has the form*

$$z = \frac{1}{a}\left(\frac{r_x(x, \bar{y})}{r(x, \bar{y}) + \Phi(\bar{y})} + p(x, \bar{y})\right)\Big|_{\bar{y}=\varphi(x,y)} \tag{B.8}$$

where $\varphi(x, y)$ is a rational first integral of $\frac{dy}{dx} = a(x, y)$, r and p are rational functions of its arguments and Φ is an undetermined function.

(ii) *There is a single rational solution, or there are two inequivalent rational solutions which do not contain undetermined elements.*

Proof. Introducing the new dependent variable w by $z = \frac{w}{b}$, (B.7) is transformed into

$$w_x + aw_y + w^2 + \left(c - \frac{b_x}{b} - a\frac{b_y}{b}\right)w + bd = 0. \tag{B.9}$$

Assume that the first integral $\varphi(x, y) \equiv \bar{y}$ of $\frac{dy}{dx} = a(x, y)$ is rational and the inverse $y = \bar{\varphi}(x, \bar{y})$ exists. Replacing y by $\bar{y}$ leads to

$$\bar{w}_x + \bar{w}^2 + \left(\bar{c} - \frac{\bar{b}_x}{\bar{b}} - \bar{a}\frac{\bar{b}_y}{\bar{b}}\right)\bar{w} + \bar{b}\bar{d} = 0$$

where $\bar{w}(x,\bar{y}) \equiv w(x,y)|_{y=\bar{y}}$, $\bar{a}(x,\bar{y}) \equiv a(x,y)|_{y=\bar{y}}$ and similar for the other coefficients. This is an ordinary Riccati equation for $\bar{w}$ in x with parameter $\bar{y}$. If its general solution is rational, it has the form $\frac{r_x}{r+\Phi(\bar{y})}+p$ where r and p are rational functions of x and $\bar{y}$. Back substitution of the original variables yields (B.8). If the general solution is not rational, one or two special rational solutions may exist leading to case (ii). □

B.3 Partial Riccati-Like Systems

In several problems discussed in the main part of this monograph there occur special systems of first-order pde's in two independent, and one or two dependent variables, involving quadratic terms of the latter. They have been baptized *partial Riccati-like systems* by Li et al. [43]; see also Chap. 2 of the book by Schwarz [61]. At first a system for a single function is considered.

Theorem B.1. *The first order Riccati-like system of pde's*

$$e_1 \equiv z_x + A_1 z^2 + A_2 z + A_3 = 0, \quad e_2 \equiv z_y + B_1 z^2 + B_2 z + B_3 = 0 \tag{B.10}$$

is coherent and its general solution depends on a single constant if its coefficients satisfy the constraints

$$\begin{aligned} A_{1,y} - B_{1,x} - A_1 B_2 + A_2 B_1 &= 0,\\ A_{2,y} - B_{2,x} - 2A_1 B_3 + 2B_1 A_3 &= 0,\\ A_{3,y} - B_{3,x} - A_2 B_3 + A_3 B_2 &= 0. \end{aligned} \tag{B.11}$$

Let A_k, $B_k \in \mathbb{Q}(x,y)$, $A_1 \neq 0$, $B_1 \neq 0$ and system (B.11) be satisfied. If (B.10) has a rational solution, one of the following alternatives applies.

(i) *The general solution is rational and has the form*

$$z = \frac{1}{A_1}\frac{r_x}{r+C} + p = \frac{1}{B_1}\frac{r_y}{r+C} + p \tag{B.12}$$

where p, $r \in \mathbb{Q}(x,y)$ and C is the integration constant.

(ii) *There is a single one, or there are two inequivalent special rational solutions.*

The proof may be found in the above quoted literature. A system with two dependent variables is

$$\begin{aligned} e_1 &\equiv z_{1,x} + z_1^2 + A_1 z_2 + A_2 z_1 + A_3 = 0,\\ e_2 &\equiv z_{1,y} + z_1 z_2 + B_1 z_2 + B_2 z_1 + B_3 = 0,\\ e_3 &\equiv z_{2,x} + z_1 z_2 + B_1 z_2 + B_2 z_1 + B_3 = 0,\\ e_4 &\equiv z_{2,y} + z_2^2 + D_1 z_2 + D_2 z_1 + D_3 = 0. \end{aligned} \tag{B.13}$$

It is coherent if the coefficients $A_1, \ldots, D_3$ satisfy

$$\begin{gathered} A_{1,y} - B_{1,x} - A_2 B_1 + A_1 B_2 - A_1 D_1 + A_3 + B_1^2 = 0, \\ A_{2,y} - B_{2,x} - A_1 D_2 + B_1 B_2 - B_3 = 0, \\ A_{3,y} - B_{3,x} - A_2 B_3 - A_1 D_3 + A_3 B_2 + B_1 B_3 = 0 \\ B_{1,y} - D_{1,x} + A_1 D_2 - B_1 B_2 + B_3 = 0, \\ B_{2,y} - D_{2,x} + A_2 D_2 - B_2^2 + B_2 D_1 - B_1 D_2 - D_3 = 0, \\ B_{3,y} - D_{3,x} + A_3 D_2 - B_2 B_3 - B_1 D_3 + B_3 D_1 = 0. \end{gathered} \tag{B.14}$$

The rational solutions of the system (B.13) are described next.

Theorem B.2. *If the coherent system (B.13) has a rational solution, one of the following alternatives applies with* $r, s, p, q \in \mathbb{Q}(x, y)$ *and* $p_y = q_x$.

(i) *The general solution is rational and contains two constants. It may be written in the form*

$$z_1 = \frac{C_2 r_x + s_x}{C_1 + C_2 r + s} + p, \quad z_2 = \frac{C_2 r_y + s_y}{C_1 + C_2 r + s} + q.$$

(ii) *There is a rational solution containing a single constant, it may be written in the form*

$$z_1 = \frac{r_x}{C + r} + p, \quad z_2 = \frac{r_y}{C + r} + q.$$

(iii) *There is a solution as described in the preceding case, and in addition there is a special rational solution not equivalent to it.*

(iv) *There is a single one, or there are two or three special rational solutions which are pairwise inequivalent.*

The proof may be found in the above references [43] and [61].

B.4 First Integrals of Differential Equations

In several places above solving first-order ode's occurred as a subproblem. Usually however the solution is not required explicitly but in terms of a first integral of the form $\varphi(x, y) = C$ where x and y are the independent and dependent variable, and C is a constant. The relevant features of this problem are discussed in the remaining part of this appendix. It is based on articles by Prelle and Singer [56] and Man and MacCallum [73, 74].

Let a first-order differential equation

$$\frac{dy}{dx} = \frac{Q}{P} \quad \text{or equivalently } Q dx - P dy = 0 \tag{B.15}$$

be given with polynomials $P, Q \in \mathbb{Q}[x, y]$. A first integral of (B.15) is a function $F(x, y)$ that is constant on its solutions, i.e. for which

$$dF = \partial_x F + y' \partial_y F = (P\partial_x + Q\partial_y)F = 0.$$

Prelle and Singer [56] prove that if a differential equation (B.15) has an elementary first integral then it must have a very special form.

Theorem B.3. *(Prelle and Singer [56]) If a differential equation (B.15) has an elementary first integral, it has the form*

$$F(x, y) = w_0(x, y) + \sum c_i \log w_i(x, y) \tag{B.16}$$

where the c_i are constants and the w_i are algebraic functions of x and y.

This result provides strong restrictions on the form of an elementary first integral, if there is any. There remains the question how to determine a first integral explicitly for any given equation (B.15). The following procedure has been given by Man [73], based on the results of Prelle and Singer. In addition to the operator $D \equiv P\partial_x + Q\partial_y$ an upper bound for the degree d of the polynomials occurring in the numerators and denominators of the $w_i(x, y)$ in (B.16) must be provided as input. It is a semi-decision procedure for the existence of an elementary first integral; up to degree d the correct answer is guaranteed if a first integral is known to exist; beyond it there is no conclusion. To make it into an algorithm a bound for the polynomials involved is required. Despite considerable efforts [10, 16], an answer seems not to be known at present (Singer, 2006, private communication).

Algorithm PrelleSinger(D,d) Given an operator $D = P\partial_x + Q\partial_y$ where $P, Q \in \mathbb{Q}[x, y]$, and $d \in \mathbb{N}$, $d \geq 1$, an elementary first integral of order not higher than d is returned if it exists, or "$failed$" otherwise. Assign $N = 0$.

$S1$: Set $N = N + 1$. If $N > d$ return "failed";

$S2$: Find all monic irreducible polynomials $f_i \in \mathbb{Q}[x, y]$ such that $deg\ f_i \leq N$ and $f_i | Df_i$; define $g_i = \dfrac{Df_i}{f_i}$.

$S3$: Decide if there are constants n_i, not all zero, such that $\sum n_i g_i = 0$. If such n_i exist return $\prod f_i^{n_i}$.

$S4$: Decide if there are constants n_i, not all zero, such that $\sum n_i g_i = -(P_x + Q_y)$. If such n_i exist, set $R = \prod f_i^{n_i}$ and return $\oint RP dx + RQ dy$, otherwise goto S1.

The polynomials f_i determined in step $S2$ with the property that f_i divides Df_i are called *Darboux polynomials*.

If this procedure is left in step $S3$ there are no logarithmic terms, i.e. a rational first integral has been found. This case is most interesting for the applications in this monograph; two examples are given next.

Example B.1. Consider the equation $(x^2 + x + 2y)y' + 3x^2 + 2xy + y = 0$. It defines the operator $D = (x^2 + x + 2y)\partial_x - (3x^2 + 2xy + y)\partial_y$. For $n = 2$ the following Darboux polynomials are obtained.

Table B.1 The polynomials f and g for the differential equation in Example B.3 for degree of f not higher than 5

f_i	g_i	Range
x^k	k	$1 \le k \le 5$
y^k	$-k(3x+y)y$	$1 \le k \le 5$
$x^k y$	$-(3x+y)y+k$	$1 \le k \le 4$
xy^k	$ky(3x+y)+1$	$2 \le k \le 4$
x^2y^2	$-2y(3x+y)+k$	$2 \le k \le 3$
x^2y^2	$-3y(3x+y)+1$	

$$f_1 = (x+y)^2, \quad g_1 = -4x+2, \quad f_2 = x^2+y,$$
$$g_2 = 2x-1, \quad f_3 = x+y, \quad g_3 = -2x+1.$$

The constraint $n_1g_1 + n_2g_2 + n_3g_3 = 0$ yields $n_2 = 2n_1 + n_3$; it leads to the first integral $F(x,y) = (x+y)^{2n_1}(x^2+y)^{n_2}(x+y)^{n_3} = (x+y)(x^2+y)$ where in the last step the substitution $n_2 = 1$ has been made. □

Example B.2. Equation $x^2y' + x^2y^2 + 4xy + 2 = 0$, no. 1.140 of Chap. C.1 in the collection by Kamke [32], defines the operator $D = x^2\partial_x - (x^2y^2 + 4xy + 2)\partial_y$. For $n = 2$ the Darboux polynomials

$$f_1 = xy+2, \; g_1 = -x^2y-x, \quad f_2 = xy+1, \; g_2 = -x^2y-2x, \quad f_3 = x, \; g_3 = x$$

are obtained. The constraint $n_1g_1 + n_2g_2 + n_3g_3 = 0$ yields $n_2 = n_3 = -n_1$; it leads to the first integral

$$F(x,y) = (xy+2)^{n_1}(xy+1)^{n_2}x^{n_3} = \frac{xy+2}{x(xy+1)};$$

in the last step the substitution $n_2 = n_3 = -1$ has been made. □

The answers in the preceding two examples are easily obtained because there does exist a rational first integral of low order. This is different if a rational first integral does not seem to exist as shown in the next example.

Example B.3. The Abel equation $xy' + y^3 + 3xy^2 = 0$, no. 1.111 of Chap. C.1 in the collection by Kamke [32], defines the operator $D \equiv x\partial_x - (3xy^2 + y^3)\partial_y$. Up to order $n \le 5$, the Darboux polynomials listed in Table B.1 are obtained. The f's cover all monomials of total degree up to order 5. Any system that may be constructed from these polynomials $\sum n_i g_i = 0$ has only the trivial solution $n_i = 0$ for all i; consequently, a nontrivial rational first integral up to order 5 does not exist. In order to prove non-existence for any order, it has to be shown that the structure of these system remains the same as shown in Table B.1. □

In the preceding example it may be possible to show that a rational first integral does not exist for any given order n. However, for a generic first order equation there is only a partial answer. If it is known that a rational first integral exists, the above procedure of Prelle and Singer returns the result; otherwise it does not provide a conclusion.

The situation is similar as for solving diophantine equations. These problems are semi-decidable, i.e if an equation is known to have any integer solution it can always be found, otherwise in general there is no answer. Even more: It has been shown by Matyasevich [48] that in general diophantine problems are undecidable. It remains an open problem to obtain a conclusive answer for the existence of a rational first integral of a generic first order differential equation, either by designing an algorithm for its solution or proving that it is undecidable.

B.5 Exercises

Exercise B.1. Discuss the solutions of the first-order ordinary Riccati equation $z' + z^2 + az + b = 0$ if a and b are constant.

Exercise B.2. The same problem for the first-order partial Riccati equation $z_x + az_y + bz^2 + cz + d = 0$ if a, b, c and d are constant.

Appendix C
The Method of Laplace

This appendix follows closely Chap. 2 of the book by Darboux [14] and Chap. 5 of the book by Goursat [18]; both are based on the original work of Laplace [40]. The method known under his name deals with equations of the form

$$E \equiv z_{xy} + az_x + bz_y + cz = 0;$$

a, b and c are functions of x and y without further specification. This equation may be written as

$$(\partial_x + b)(\partial_y + a)z = 0 \ \text{ if } \ a_x + ab = c.$$

In this case the general solution is

$$z = \exp\Big(-\int a dy\Big)\ \Big(\ F(x) + \int G(y)\exp\Big(\int a dy - \int b dx\Big) dy\ \Big)$$

where F and G are undetermined functions of its argument. Similarly, if

$$(\partial_y + a)(\partial_x + b)z = 0 \ \text{ if } \ b_y + ab = c$$

the general solution is

$$z = \exp\Big(-\int b dx\Big)\ \Big[\ G(y) + \int F(x)\exp\Big(\int b dx - \int a dy\Big) dx\ \Big].$$

If both conditions $a_x + ab = c$ and $b_y + ab = c$ hold, the general solution is

$$z = F(x)\exp\Big(-\int a dy\Big) + G(y)\exp\Big(-\int b dx\Big).$$

For the further discussion it is advantageous to define the two quantities

$$h_0 \equiv h \equiv a_x + ab - c \ \text{ and } \ k_0 \equiv k \equiv b_y + ab - c. \tag{C.1}$$

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In Exercise C.1 it will be shown that h and k are invariants w.r.t. to transformations $z = \lambda(x, y)w$ of the Laplace equation E.

If both $h \neq 0$ and $k \neq 0$, the above factorizations do not apply. According to Laplace one may proceed as follows. A new dependent variable z_1 is introduced by $z_1 \equiv z_y + az$. Using it, the original equation may be written as $z_{1,x} + bz_1 = hz$. Eliminating z from the latter and substituting it into the definition of z_1, the equation

$$E_1 \equiv z_{1,xy} + a_1 z_{1,x} + b_1 z_{1,y} + c_1 z_1 = 0$$

of the same type as equation E is obtained where

$$a_1 = a - \frac{h_y}{h}, \quad b_1 = b, \quad c_1 = c - a_x + b_y - b\frac{h_y}{h}. \tag{C.2}$$

The corresponding invariants are

$$h_1 \equiv a_{1,x} + a_1 b_1 - c_1 = 2h - k - (\log h)_{xy}, \quad k_1 = h.$$

On the other hand, a new dependent variable z_{-1} may be introduced by $z_{-1} \equiv z_x + bz$. Using it, the original equation may be written as $z_{-1,y} + az_{-1} = kz$. Eliminating z from the latter and substituting it into the definition of z_{-1}, the equation

$$E_{-1} \equiv z_{-1,xy} + a_{-1} z_{-1,x} + b_{-1} z_{-1,y} + c_{-1} z_{-1} = 0$$

follows where

$$a_{-1} = a, \quad b_{-1} = b - \frac{k_x}{k}, \quad c_{-1} = c - b_y + a_x - a\frac{k_x}{k}. \tag{C.3}$$

Its invariants are

$$h_{-1} \equiv a_{1,x} + a_1 b_1 - c_1 = k, \quad k_{-1} = 2k - h - (\log k)_{xy}.$$

This proceeding may be repeated, generating a sequence of equations

$$\ldots \; E_{-2}, E_{-1}, E_0 \equiv E, E_1, E_2 \; \ldots$$

as long as the corresponding invariants are different from zero. They are related by

$$h_{i+1} = 2h_i - h_{i-1} - (\log h_i)_{xy}, \quad k_{i+1} = h_i. \tag{C.4}$$

Solving for h_i and k_i yields

$$h_i = k_{i+1}, \quad k_i = 2k_{i+1} - h_{i+1} - (\log k_{i+1})_{xy}.$$

This sequence of equations terminates for a positive integer i if $h_i = 0$; if this is true the corresponding equation E_i may be solved as described above. By back substitution a special solution of the original equation E may be obtained. It has the form

$$z = F(x)r_0(x, y) + F'(x)r_1(x, y) + \ldots + F^{(i)}(x)r_i(x, y);$$

F is an undetermined function of x, the $r_k(x, y)$ are functions of x and y which are determined by the problem.

Similarly, if the sequence of equations terminates for a negative value j because $k_j = 0$, a special solution of the form

$$z = G(y)s_0(x, y) + G'(y)s_1(x, y) + \ldots + G^{(j)}(y)s_j(x, y)$$

is obtained. Now G is an undetermined function of y; the $s_k(x, y)$ are functions of x and y which are determined by the originally given equation.

Obviously the success of the method hinges on the question whether any invariant vanishes for sufficiently large value of its index i or j. Right now, an upper bound for these values does not seem to exist. Consequently, the existence of a Laplace divisor is semi-decidable; if for a particular equation it is known to exist it may always be found, otherwise in general it remains an open question and may even turn out to be undecidable.

Laplace' method has been generalized to certain equations with leading derivative $\frac{\partial^{n+1}}{\partial x \partial y^n}$ for $n \geq 2$ by [53]. Another generalization has been given by [70].

C.1 Exercises

Exercise C.1. Show that the expressions h and k defined by (C.1) are invariants w.r.t. the transformations $z = \lambda(x, y)w$.

Exercise C.2. Determine the transformed equations $E_{\pm i}$ and the invariants h_i and k_i for the equations considered in Examples 5.9 and 5.10. To this end, apply the user-functions `LaplaceTransformation` and `LaplaceInvariant` provided on the website www.alltypes.de. Use the result to explain the solution behavior.

Appendix D
Equations with Lie Symmetries

Symmetries are special transformations of a differential equation leaving its form invariant; the totality of symmetries forms a group, the *symmetry group* of the respective equation. Good introductions into the subject may be found in the books by Olver [51] or Bluman and Kumei [4]; for symmetries of ode's see also the book by Schwarz [61].

Sophus Lie [44] discussed his symmetry methods for second-order linear pde's in two independent variables in full detail. These results are given in this appendix without proofs. He considers the general homogeneous equation for a function $z(x, y)$ in the form

$$R(x, y)z_{xx} + S(x, y)z_{xy} + T(x, y)z_{yy} + P(x, y)z_x + Q(x, y)z_y + Z(x, y)z = 0.$$

The coefficients $R, S, \ldots, Z$ are functions of the independent variables x and y without specifying its function field. According to Lemma 5.4 two cases with different leading derivatives are distinguished. If $R(x, y) = T(x, y) = 0$ there is only the mixed leading derivative z_{xy}; this case is considered first.

Theorem D.1. *Any equation $z_{xy} + A(x, y)z_y + B(x, y)z = 0$ has symmetry generators $U_1 = z\partial_z$ and $U_\infty = \varphi(x, y)\partial_z$ where φ is a solution of the given differential equation; its commutator is $[U_1, U_\infty] = -U_\infty$. There are four special cases with larger symmetry groups and corresponding canonical forms.*

(i) *An equation with canonical form $z_{xy} + A(y)z_y + z = 0$ has symmetry generators $U_1 = z\partial_z$, $U_2 = \partial_x$, and U_∞ with non-vanishing commutator $[U_2, U_\infty] = (\log\varphi)_x U_\infty$.*

(ii) *An equation with canonical form $z_{xy} + P(x - y)z_y + Q(x - y)z = 0$, P and Q undetermined functions of its single argument $x - y$, has symmetry generators $U_1 = z\partial_z$, $U_2 = \partial_x + \partial_y$, and U_∞ with non-vanishing commutator $[U_2, U_\infty] = \big((log\varphi)_x + (\log\varphi)_y\big)U_\infty$.*

(iii) *An equation with canonical form $z_{xy} + Cyz_y + z = 0$, C a constant, has symmetry generators $U_1 = z\partial_z$, $U_2 = \partial_x$, $U_3 = x\partial_x - y\partial_y$, $U_4 = \partial_y - Cxz\partial_z$,*

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and U_∞ *with non-vanishing commutators*

$$[U_2, U_3] = U_2, \quad [U_2, U_4] = -CU_1, \quad [U_3, U_4] = -U_4.$$

(iv) *An equation with canonical form* $z_{xy} + \frac{A}{x-y} z_y + \frac{B}{(x-y)^2} z = 0$*,* A *and* B *constants, has symmetry generators*

$$U_1 = z\partial_z, \; U_2 = \partial_x + \partial_y, \; U_3 = x\partial_x + y\partial_y, \; U_4 = x^2\partial_x + y^2\partial_y - Axz\partial_z$$

and U_∞ *with non-vanishing commutators*

$$[U_2, U_3] = U_2, \quad [U_2, U_4] = 2U_3 - AU_1, \quad [U_3, U_4] = U_4.$$

If $S(x, y) = T(x, y) = 0$ the only second-order derivative is z_{xx}; the corresponding equations are considered next.

Theorem D.2. *Any equation* $z_{xx} + A(x, y)z_y + B(x, y)z = 0$ *has symmetry generators* $U_1 = z\partial_z$ *and* $U_\infty = \varphi(x, y)\partial_z$ *where* φ *is a solution of the differential equation; its commutator is* $[U_1, U_\infty] = -U_\infty$*. There are three special cases with larger symmetry groups and corresponding canonical forms.*

(i) *An equation with canonical form* $z_{xx} + z_y + B(x)z = 0$ *has symmetry generators* $U_1 = z\partial_z$*,* $U_2 = \partial_y$*, and* U_∞*; there are no non-vanishing commutators.*

(ii) *An equation with canonical form* $z_{xx} + z_y = 0$ *has symmetry generators*

$$U_1 = z\partial_z, \; U_2 = \partial_x, \; U_3 = \partial_y, \; U_4 = x\partial_x + 2y\partial_y,$$
$$U_5 = 2y\partial_x + xz\partial_z, \; U_6 = xy\partial_x + y^2\partial_y + (\tfrac{1}{4}x^2 - \tfrac{1}{2}y)z\partial_z, \quad \text{and} \quad U_\infty;$$

its non-vanishing commutators are

$$[U_2, U_4] = U_2, \quad [U_2, U_5] = U_1, \quad [U_2, U_6] = \tfrac{1}{2}U_5, \quad [U_3, U_4] = 2U_3,$$
$$[U_3, U_5] = 2U_2, \quad [U_3, U_6] = U_4 - \tfrac{1}{2}U_1, \quad [U_4, U_5] = U_5, \quad [U_4, U_6] = 2U_6.$$

(iii) *An equation with canonical form* $z_{xx} + z_y + \frac{C}{x^2} z = 0$*,* C *a constant, has symmetry generators*

$$U_1 = z\partial_z, \quad U_2 = \partial_y, \quad U_3 = x\partial_x + 2y\partial_y,$$
$$U_4 = xy\partial_x + y^2\partial_y + (\tfrac{1}{4}x^2 - \tfrac{1}{2}y)z\partial_z, \quad \text{and} \quad U_\infty;$$

its non-vanishing commutators are

$$[U_2, U_3] = 2U_2, \quad [U_2, U_4] = U_3 - \tfrac{1}{2}U_1, \quad [U_3, U_4] = 2U_4.$$

There arises the question as to what the relation between the symmetries of an equation and its factorization properties are, if any. For equations with leading derivative z_{xx} considered in the preceding theorem the answer is easy; from Proposition 4.1 it follows immediately that there are no factorizations related to the enlarged symmetry groups of cases (i), (ii) or (iii). This is different for equations with leading derivative z_{xy}; a few examples are discussed in the subsequent exercise.

D.1 Exercises

Exercise D.1. Investigate the factorization properties of the equations considered in Theorem D.1 and discuss the results.

Appendix E
ALLTYPES in the Web

Many calculations described in this monograph cannot be performed by pencil and paper because they are too voluminous. The website www.alltypes.de provides a collection of user functions for this purpose. In addition to an interactive environment for applying them, a short documentation for each function is given, including some examples. This appendix contains a list of those functions that are particularly relevant for the subject of this book; details may be found on the above website.

`Adjoint(u)`. The single arguments `u` is a linear ode; returns adjoint equation.

`Commutator(u,v)`. The two arguments `u` and `v` are ordinary or partial differential operators; returns commutator.

`ExactQuotient(u,v)`. The two arguments `u` and `v` are ordinary or partial differential operators; returns exact quotient or error message if it does not exist.

`FirstIntegrals(u,n)`. The first argument `u` represents a system of first-order ode's; returns polynomial or quasipolynomial first integrals up to order `n`.

`FirstOrderRightFactors(u)`. The single argument `u` represents an ordinary or partial differential operator in the plane; returns all first-order right factors with rational function coefficients.

`Gcrd(u,v)`. The two arguments `u` and `v` may be individual ordinary or partial differential operators in the plane, or generators of an ideal; returns generators of greatest common right divisor or sum ideal.

`IntegrabilityConditions(u)`. The single argument is a set of generators for a left ideal in the ring of differential operators or module over such a ring; returns integrability conditions for the coefficients.

`JanetBasis(u)`. The single argument `u` represents a list of generators of an ideal or a module of differential operators. Returns Janet basis generators in the same term ordering as applied for input.

`LaplaceDivisor(u,k)`. The first argument `u` is a partial differential operator in the plane, the second argument `k` a natural number; returns Laplace divisor of order not higher than `k` if it exists.

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`LaplaceInvariants(u,k)`. The first argument `u` is a partial differential operator in the plane, the second argument `k` a natural number. Returns Laplace invariant of order not higher than `k`.

`Lclm(u,v)`. The two arguments `u` and `v` are ordinary or partial differential operators in the plane; returns generators of least common left multiple or left intersection ideal.

`LoewyDecomposition(u)`. The single argument `u` is an ordinary or partial differential operator in the plane; returns Loewy decomposition of `u`.

`LoewyDivisor(u)`. The single argument `u` is an ordinary or partial differential operator in the plane; returns largest completely reducible right component,

`RationalSolutions(u)`. The single argument `u` may be a linear ode or a linear pde in the plane.

`Solve(u)`. The single argument `u` may be a linear or nonlinear ode, a first- or second-order linear pde in the plane, or a system of linear pde's of differential type zero.

List of Notation

Symbol	Meaning	Page of definition
$\prec, \preceq, \succ, \succeq$	Order relation between terms	23
$Gcrd$	Greatest common right divisor	26
$Lclm$	Least common left multiple	26
$o(z_{x^m y^n})$	Subsumes all terms not higher than argument	29
$\langle l_1, l_2, \dots \rangle$	Ideal generated by $l_1, l_2, \dots$	22
$\langle\!\langle l_1, l_2, \dots \rangle\!\rangle$	Ideal generated by Janet basis $l_1, l_2, \dots$	25
$\langle \partial_{xx}, \partial_{xy} \rangle_{LT}$	Ideal with leading derivatives ∂_{xx} and ∂_{xy}	30
$\mathscr{D}$	Ring of differential operators	22
H_I	Hilbert-Kolchin polynomial of ideal I	28
(k, j)	Differential dimension of ideal or module	29
$\mathbb{J}^{(0,k)}$	Type of ideals of differential dimension $(0, k)$	30
$\mathbb{M}^{(0,k)}$	Type of modules of differential dimension $(0, k)$	32
$\mathbb{L}_{x^m}(L)$	Laplace divisor for L of order m in x	34
$\mathfrak{l}_m$	Generator of Laplace divisor $\mathbb{L}_{x^m}(L)$	34
$\mathbb{L}_{y^n}(L)$	Laplace divisor for L of order n in y	34
$\mathfrak{k}_n$	Generator of Laplace divisor $\mathbb{L}_{y^n}(L)$	34
$\mathbb{J}_{xxx}, \mathbb{J}_{xxy}$	Generic intersection ideals for first-order operators	45
$\mathscr{E}_i(x, y)$	Shifted exponential integral for $\partial_x + a_i \partial_y + b_i$	92
$\varepsilon_i(x, y)$	Exponential integral for $\partial_x + b_i$	95
$\mathscr{L}_k^2$	Type of Loewy decomposition of second-order ode	7
$\mathscr{L}_k^3$	Type of Loewy decomposition of third-order ode	8

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Symbol	Meaning	Page of definition
$\mathscr{L}_{xx}^k$	k-th decomposition type for operator with leading derivative ∂_{xx}	83
$\mathscr{L}_{xy}^k$	k-th decomposition type for operator with leading derivative ∂_{xy}	86
$\mathscr{L}_{xxx}^k$	k-th decomposition type for operator with leading derivative ∂_{xxx}	123
$\mathscr{L}_{xxy}^k$	k-th decomposition type for operator with leading derivative ∂_{xxy}	132
$\mathscr{L}_{xyy}^k$	k-th decomposition type for operator with leading derivative ∂_{xyy}	140

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$$\sum_{i=0}^{n=1} A_{1i}(x,y)\frac{\partial^{i+1}z}{\partial x\partial y^i} + \sum_{i=0}^{n} A_{0i}(x,y)\frac{\partial^i z}{\partial y^i} = 0.$$

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