A Some Historical Remarks

The development of mechanics over more than 2000 years was smoother, to some extent, than that of other fields of physics. At all times it received important impulses from mathematics, which was developing in parallel, and, vice versa, the analysis of mechanical phenomena fostered the development of mathematics. To the benefit of both, mathematics and physics, mechanics led to fundamental ideas and principles valid also for other fields of physics and mathematics and, in particular, helped shape our notions of spacetime and its properties.

The history of field theory, first classical and later quantum field theory, took place in a much shorter period of time, roughly two hundred years as compared to two thousand or more years of mechanics. In turn, it was even more dramatic, it seems to me, and followed a less straight path than that of mechanics. One of the most remarkable aspects of this history is that it started from a great number of disparate, seemingly unrelated phenomena and led towards more and more unified descriptions of fundamental interactions. In these historical remarks I would like to emphasize especially this aspect of *unification* that emerged step by step. Who would not dream of a unified theory that encompasses as many physical phenomena as possible? A unified description coined "world formula" in more popular writings, "ToE" (Theory of Everything) by the scholars.

Electricity and Magnetism and their Relation

Towards the end of the eighteenth century, the Age of Enlightenment, certain electric phenomena and a few facts about magnetism were known but the two types of phenomena seemed unrelated. Simple electrostatic experiments could be and were performed by laymen and the action of stationary currents was known (from Volta's batteries). These phenomena appeared as a whim of nature, curiosities which had no real relevance for daily life. Perhaps the only exception were the lightning rods invented by Benjamin Franklin (1706–1790) but even those were often rejected by ignorance or mistrust because people thought they would attract lightnings rather than to divert them.

F. Scheck, *Classical Field Theory*, Graduate Texts in Physics,
https://doi.org/10.1007/978-3-662-55579-8

Magnetism, in turn, was well-known in the form of natural magnetic material and was ascribed healing power in medicine. Franz Anton Mesmer (1734–1815), the famous doctor Mesmer, founded the lore of so-called animal magnetism, an early precursor of hypnotherapy. Note, however, that even in his time this treatment was not taken too seriously. No one less than Wolfgang Amadeus Mozart immortalized Mesmer and the "Mesmer stones" in his opera *Cosi fan tutte* (1789/1790) in which Despina, chamber maid of the ladies Fiordiligi and Dorabella, disguised as a medical doctor, tries to cure the enamoured gentlemen Guglielmo and Ferrando by magnetism, viz

> *Questo è quel pezzo di calamita, pietra Mesmerica,*
> *ch'ebbe l'origine n'ell Allemagna*
> *che poi si celebre in Francia fù.*

(W.A. Mozart, *Cosi fan tutte*, Act I, Scene 16.).[1]

In the preface to his book "The present state of music in France and Italy 1771" Charles Burney, the British musicologist (1726–1824) makes an interesting comparison between electricity and music,

> *Electricity is universally allowed to be a very entertaining and surprising phenomenon, but it has frequently been lamented that it has never yet, with much certainty, been applied to any very useful purpose.*
> *The same reflexion has often been made, no doubt, as to music. It is a charming resource, in an idle hour, to the rich and luxurious part of the world. But say the sour and the wordly, what is its use to the rest of mankind? …*
> *Music has indeed ever been the delight of accomplished princes, and the most elegant amusements of polite courts: but at present it is so combined with things sacred and important, as well as with our pleasures, that mankind seems wholly unable to subsist without it.*

And, indeed, in this diary of this and subsequent journeys he points to the great importance of music in 18th century life, from the noble people at European courts to the farmers on their fields.

Towards a Unified Theory: Maxwell's Equations

The fundamental law of the $1/r^2$–dependence of the force between two charges e_1 and e_2, and its sign, sign$(e_1 e_2)$, was discovered in 1785 by **Charles Augustin Coulomb, 1736–1806**. It took another 35 years before it became known in around 1820 that in reality electric and magnetic phenomena are closely related. The Danish physicist **Hans Christian Ørsted (1777–1851)** reported that electric currents circulating in conducting loops align magnetic needles in their neighbourhood. This first step towards unification of interactions attracted a great deal of attention and stimulated the subsequent quantitative investigations by Biot and Savart (**Jean-Baptiste**

[1] "This is that piece of magnet, the Mesmer stone, which had its origin in Germany, and then became so famous in France".

Biot, 1774–1862; Félix Savart, 1791–1841) culminating in the law that bears their names. These investigations were followed by the series of famous experiments of **André Marie Ampère (1775–1836)** who showed, among other effects, that small solenoids supporting electric current, behave like linear magnets in the magnetic field of the earth, and who first formulated the forces between current-carrying wires.

The great figures of the classical period of electrodynamics were **Michael Faraday (1791–1867)**) and **James Clerk Maxwell (1831–1879)**, the first of them primarily an eminent experimenter who discovered the key experiment of induction, the second of them the architect of the basic equations of electrodynamics in their universal local form. The induction law of 1831 established the first direct and explicit relationship between electric and magnetic fields but it needed Maxwell's concept of the displacement current (33 years later!), obtained from a nonstationary application of the Biot-Savart law, before he could formulate a closed and consistent theory of all electric and magnetic phenomena.[2]

Experimental Tests of the Equations

Maxwell's equations, formulated in 1864, found their most exciting and influential application in the experiments carried out in 1888 by **Heinrich Rudolph Hertz (1857–1894)** which proved the existence of electromagnetic waves. The tremendous development from then until the present time, from early wireless telegraphy to modern techniques of global positioning in ships, planes, and cars, and to modern telecommunications of all sorts, presumably is well known to the reader. So one might be tempted to repeat the last sentence of the quotation from Burney, but with modified subjects: *…at present it is so combined with things sacred and important, as well as with our pleasures, that mankind seems wholly unable to subsist without it.*

The unity of the spectra of all kinds of electromagnetic radiation started to emerge with the proof of the interference of X-rays by W. Friedrich and P. Knipping – following up an idea of Max von Laue. Nowadays we know that, though very different in appearance, X-rays, visible light, infrared radiation, all belong to the same spectrum of electromagnetic waves.

Gauge Invariance and the Unity of Space and Time

The notion of *vector potential* was essential for the discovery of *gauge invariance*, one of the fundamental principles of all gauge field theories. Its history is less straightforward and transparent. In their historical essay Jackson and Okun[3] showed that first hints are contained in the work of Franz E. Neumann and Wilhelm Weber around the middle of the 19th century but it was Gustav Kirchhoff (around 1857) and, about a decade or more later, Hermann von Helmholtz who formulated equations which relate scalar and vector potentials and, from a modern point of view, correspond to special choices of the gauge.

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[2] I recommend, in particular, the essay by Res Jost *Michael Faraday – 150 years after the discovery of electromagnetic induction,* in R. Jost, *Das Märchen vom elfenbeinernen Turm*, (Springer 1995).
[3] J.D. Jackson, L.B. Okun: *Historical roots of gauge invariance*, Rev. Mod. Phys. 73 (2001) 663.

The Danish physicist **Ludvig Valentin Lorenz** (1829–1891) was the first who wrote down retarded potentials of the kind of (4.30), i.e., in a notation in use today,

$$\Phi(t,\mathbf{x}) = \iiint \mathrm{d}^3x' \int \mathrm{d}t' \, \frac{\varrho(t',\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\delta(t'-t+\frac{1}{c}\left|\mathbf{x}-\mathbf{x}'\right|) \,, \tag{1a}$$

$$\mathbf{A}(t,\mathbf{x}) = \iiint \mathrm{d}^3x' \int \mathrm{d}t' \, \frac{\mathbf{j}(t',\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\delta(t'-t+\frac{1}{c}\left|\mathbf{x}-\mathbf{x}'\right|) \,, \tag{1b}$$

and who noticed that they fulfill the condition

$$\nabla \cdot \mathbf{A}(t,\mathbf{x}) + \frac{1}{c}\frac{\partial \Phi(t,\mathbf{x})}{\partial t} = 0 \,. \tag{2}$$

It seems as though the use of gauge transformations was familiar to him because he noted the equivalence of these potentials to those of the class $\nabla \cdot \mathbf{A} = 0$. It is a long-standing habit to ascribe the condition (2) to the Dutch physicist H. A. Lorentz (**Hendrik Antoon Lorentz, 1853–1928**). However, as L. V. Lorenz discovered and made use of it about a quarter of a century before H. A. Lorentz, it seems appropriate to correct this misassignment of many textbooks, without belittling the importance and the great achievements of the latter.[4]

Of utmost importance, from a modern point of view, is the covariance of Maxwell's equations under the Lorentz group that is based on the principle of the constancy of the speed of light in vacuum, and the discovery of special relativity by Albert Einstein. The qualified symmetry between three-dimensional space and time in special relativity, and the progress from Galilean space with its Newtonian absolute time to Minkowski space, brought about a very specific unification of space and time.

The notion of *gauge invariance* was coined in 1919 by Hermann Weyl (**Hermann Weyl, 1885–1955,** German mathematician and physicist). In his attempt to combine electrodynamics and gravitation he originally had in mind scale transformations of the metric

$$g_{\mu\nu} \longmapsto \mathrm{e}^{\lambda(x)} g_{\mu\nu}$$

with *real* functions $\lambda(x)$, i.e., a transformation by which coordinates were really "gauged" in the old sense of the word. Vladimir Fock[5] made an important discovery which often is not fully appreciated but which represents another important step of unification: The combination of U(1) transformations generated by real functions $\chi(t,\mathbf{x})$ on the one side, and the action of phases

$$\mathrm{e}^{\mathrm{i}\alpha(t,\mathbf{x})} \quad \text{with} \quad \alpha(t,\mathbf{x}) = \frac{e}{\hbar c}\chi(t,\mathbf{x})$$

[4]The names of the two of them are found in what is called the Lorenz-Lorentz effect in optics. This effect concerns the density dependence of the index of refraction. There is an analogue of this effect in the interaction of negative pions with nuclear matter, called the Ericson-Ericson effect, after Magda and Torleif Ericson.

[5]V. Fock, Z. Physik **38** (1926) 242 and **39** (1926) 226.

on wave functions of quantum theory, on the other, as worked out in Sect. 3.4.2, (3.38) and (3.39b), and in Sect. 5.2, (5.16). Here, the fundamental gauge principle of classical electrodynamics, and the characteristic freedom of phases of quantum theory, are combined to something new: A locally gauge invariant theory of radiation and matter in which the covariant derivative plays a special role in fixing the coupling between the two types of fields.

Unified Theory of the Fundamental Interactions

The next major step of unification is the combination of electrodynamics with the other fundamental interactions in the framework of the so-called minimal standard model of elementary particle physics. In Chap. 5 I collected the relevant steps of the development for the example of electrodynamics and the weak interactions on a *classical* basis. They lead to the widely ramified field of modern quantum field theory and to present-day research in elementary particle physics, for whose historical development I refer to the appendix of [QP].

General relativity which in a precise sense brings together the geometry of spacetime with the nature of gravitation, is a very special kind of unification of formerly unrelated notions. The other fundamental interactions are thought to be well described when formulated on an inert, given Euclidean spacetime. This, as we have seen, is not true for gravitation for which no consistent theory can be found on a pre-existing, flat spacetime. In its essential aspects, general relativity is the work of a single person, Albert Einstein. The genesis of this theory, the life of Albert Einstein, and much more, can be found, e. g., in the excellent biography by Abraham Pais [Pais 1982].

Exercises

B

Exercises: Chap. 1

1.1 If in $\mathbb{R}^3$ the cartesian basis $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ is replaced by a spherical basis

$$\hat{\mathbf{e}}_{\pm} := \mp\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_1 \pm \mathrm{i}\hat{\mathbf{e}}_2\right) , \; \hat{\mathbf{e}}_0 := \hat{\mathbf{e}}_3 , \tag{A.1}$$

the expansion of a vector reads $\mathbf{V} = \sum_{m=-1}^{+1} v^m \hat{\mathbf{e}}_m$. Write down the orthogonality relations for the base vectors $\hat{\mathbf{e}}_m$, the scalar product $\mathbf{V}\cdot\mathbf{W}$, and work out the difference between contravariant components v^m and the corresponding covariant components.

1.2 Show that the four-component current density (1.25) satisfies the continuity equation.

1.3 Estimate the mass of a Uranium nucleus in micrograms, knowing that it contains 92 protons and 143 neutrons.
Hint: $m_p c^2 \simeq m_n c^2 \simeq 939$ MeV.

1.4 Calculate the electric field in volt per meter that a muon feels in the $1s$-state of muonic lead.
Hints Bohr radius $a_B = \hbar c/(Z\alpha m_\mu c^2)$, $m_\mu c^2 = 105.6\,\text{MeV}$.

1.5 Prove the formula (1.48a), i. e.

$$\sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \; .$$

F. Scheck, *Classical Field Theory*, Graduate Texts in Physics,
https://doi.org/10.1007/978-3-662-55579-8

1.6 Prove the formula (1.52a) by means of the following construction: Consider two concentric spheres with radii R_{i} and R_{a}, respectively, and $R_{\mathrm{i}} < R_{\mathrm{a}}$ whose center is $\mathbf{x}$. Choose the reference point $\mathbf{x}'$ in the domain between the two spheres, and apply Green's second theorem to the volume enclosed by the spheres, for the functions ψ or ϕ equal to $1/r$ ($r = |\mathbf{x} - \mathbf{x}'|$). Let then R_{i} go to zero, R_{a} to infinity.

1.7 Determine the normalization factor N of the distribution

$$\varrho_{\text{Fermi}}(r) = \frac{N}{1 + \exp\{(r - c)/z\}} \tag{A.2}$$

such that ϱ integrated over the whole space gives 1.

1.8 Let η be the surface charge density on a given smooth surface F. Prove the relation (1.87a).
Hint: Choose a small "box" across the surface such that its base and its lid are parallel to the surface F and have the size $\mathrm{d}\sigma$ while its height is small of third order. Apply Gauss' theorem.

1.9 Prove: On a surface which carries the surface charge density η the tangential component of the electric field is continuous, (1.87b).
Hint: Choose a closed rectangular path which cuts the surface such that the edges perpendicular to the surface are much smaller than the edges parallel to the surface. Calculate the electromotive force along that path.

1.10 Prove the properties (1.97g) and (1.97h) using the explicit expressions (1.97a) for the spherical harmonics.

1.11 Derive the relation between the cartesian components Q^{ik} of the quadrupole, (1.111c), $i, k = 1, 2, 3$, and its spherical components $q_{2\mu}$.

1.12 Show that the space integral of the electric field strength of a dipole is proportional to the dipole moment,

$$\iiint_V \mathrm{d}^3x\, \mathbf{E}_{\text{Dipol}}(x) = -\frac{4\pi}{3}\mathbf{d}\,. \tag{A.3}$$

1.13 Given a capacitor consisting of two metallic plates and an electrically polarizable, insulating medium between them. Consider the process of discharge after short-circuiting the plates and calculate the displacement current in the medium.

1.14 Construct the additional term $F(\mathbf{x}, \mathbf{x}')$ in (1.90) which is needed for the Dirichlet Green function to vanish on the sphere.

1.15 A point-like electric dipole $\mathbf{d} = d\hat{\mathbf{e}}_3$ is placed in the center of a conducting sphere whose radius is R. Calculate the potential and the electric field inside the sphere. What is the field outside of the sphere? What is the charge density on (the surface of) the sphere?

1.16 A pointlike electric dipole is located in the point $\mathbf{x}^{(0)}$. Show: The potential that it creates as well as its energy in an external potential Φ_a can be described by the effective charge density

$$\varrho_{\text{eff}}(\mathbf{x}) = -\mathbf{d} \cdot \nabla\delta(\mathbf{x} - \mathbf{x}^{(0)}) \ .$$

1.17 A conducting sphere with total charge Q is placed in a homogeneous electric field $\mathbf{E}^{(0)} = E_0\hat{\mathbf{e}}_3$. How does the electric field change by the presence of the sphere? What is the distribution of the charge on (the surface of) the sphere?
Hint: Assume the potential to have the form

$$\Phi = f_0(r) + f_1(r)\cos\theta$$

and solve the Poisson equation. Can you give plausibility arguments for this ansatz?

1.18 Calculate the energy contained in the electric field of a spherically symmetric, homogeneous charge distribution (radius R, charge Q). Then calculate the self-energy

$$W = \frac{1}{2}\iiint \mathrm{d}^3x \ \varrho(\mathbf{x})\Phi(\mathbf{x})$$

of this charge distribution.

1.19 An electron located in the origin is assigned the charge distribution $\varrho = (-e)\delta(\mathbf{x})$. One considers a sphere of radius R, its center taken as the origin, and calculates the energy of the electric field outside the sphere. How must the radius be chosen if this energy is to be equal to the rest energy $m_e c^2$ of the electron? This radius is called the *classical electron radius.*
Answer: $R = e^2/(2m_e c^2) = \alpha\hbar c/(2m_e c^2)$.

1.20 A sphere with radius R is made of a homogeneous dielectric material with dielectric constant ε_1. The sphere is embedded in a medium which is homogeneous, too, and whose dielectric constant is ε_0. Furthermore, an external electric field $\mathbf{E} = E_0\hat{\mathbf{e}}_3$ is applied to it. Calculate the potential inside and outside the sphere. Sketch the equipotential surfaces for the special cases $(\varepsilon_0 \equiv \varepsilon, \varepsilon_1 = 1)$ and $(\varepsilon_0 = 1, \varepsilon_1 \equiv \varepsilon)$. In the second case let ε become very large and compare with the potential in Exercise 1.17.

1.21 Two positive charges $q = (e/2)$ and two negative charges $-q$ are placed in four points whose cartesian coordinates (x, y, z) are as given here

$$q_1 = \quad q : \ (a, 0, 0) \ , \qquad q_2 = q : \ (-a, 0, 0) \ ,$$
$$q_3 = -q : \ (0, b, 0) \ , \qquad q_4 = -q : \ (0, -b, 0) \ .$$

Write down the charge distribution by means of δ-distributions. What is the dipole moment of this distribution? Determine the quadrupole tensor $Q_{ij} = \iiint \mathrm{d}^3x \, [3x_i x_j - \mathbf{x}^2\delta_{ij}]\varrho(\mathbf{x})$ and the spectroscopic quadrupole moment

$$Q_0 := \sqrt{\frac{16\pi}{5}} \iiint \mathrm{d}^3x \ \varrho(\mathbf{x}) r^2 \ .$$

List also the moments $q_{\ell,m}$ for $\ell = 2$ (spherical basis).

1.22 A spherical shell with radius R which carries a constant surface charge density η, rotates with angular velocity ω about an axis through its center. What is the magnetic field it creates?
Hint: The surface current is given by the expression

$$\mathbf{K}(\theta) = \eta\,\boldsymbol{\omega} \times \mathbf{x} = \omega\eta R \sin\theta \hat{\mathbf{e}}_\phi \ .$$

1.23 A hollow ball with inner radius r and outer radius R, consists of a material with high magnetic permeability μ. This sphere is placed in an external induction field $\mathbf{B} = B_0\hat{\mathbf{e}}_3$. Calculate the field lines in the presence of this ball. In particular, study the special case $\mu \to \infty$.
Hint: As there is no current density one may derive the fields **H** and **B** from a magnetic potential Φ_{magn}. Use the multipole expansion for this potential.

Exercises: Chap. 2

2.1 By counting the base k-forms determine the dimension of the space $\Lambda^k(M)$ of k-forms over the manifold M.

2.2 Show: A symmetric tensor $S_{\mu\nu}$ of degree two contracted with another, antisymmetric tensor $A^{\mu\nu}$ of degree two, gives zero.

2.3 With $\varepsilon_{\alpha\beta\gamma\delta}$ the Levi-Civita symbol in dimension four, find summation formulae which correspond to the formulae (1.48a) and (1.48b).

2.4 Let $\mathbf{A}(t, \mathbf{x})$ be a given vector potential which is not subject to any special boundary condition. If one chooses the gauge function

$$\chi(t, \mathbf{x}) = \frac{1}{4\pi} \iiint \mathrm{d}^3 y \, \frac{1}{|\mathbf{x} - \mathbf{y}|} \boldsymbol{\nabla}_x \cdot \mathbf{A}(t, \mathbf{y})$$

in order to replace $\mathbf{A}$, what can be said about the divergence of the transformed vector potential $\mathbf{A}'$? In case there are no external sources what is the gauge function by means of which one obtains $A_0(t, \mathbf{x}) = 0$ without leaving the class of Coulomb gauges?

2.5 Since energy and momentum are conserved, a free electron cannot radiate a light quantum, $e \to e + \gamma$. Prove this by using relativistic kinematics.

Exercises: Chap. 3

3.1 Determine the physical dimensions of the quantities $u(t, \mathbf{x})$, (3.54a), $\mathbf{P}(t, \mathbf{x})$, (3.54b), $\mathbf{S}(t, \mathbf{x})$, (3.54c), and $T_j^{\,k}(t, \mathbf{x})$, (3.54d).

3.2 Show: In $\mathbb{R}^3$ both δ_{ij} and ε_{ijk} are tensors which are invariant under rotations $\mathbf{R} \in \mathrm{SO}(3)$. In Minkowski space what can you say about $\delta_{\mu\nu}$ and about $\varepsilon_{\mu\nu\sigma\tau}$ with regard to Lorentz transformations?

Exercises: Chap. 4

4.1 Which boundary conditions hold for electric fields and for induction fields at boundary surfaces? (See also Exercises 1.8 and 1.9).

4.2 A harmonically oscillating dipole souce is described by the current density

$$\mathbf{j}(t, \mathbf{x}) = -\mathrm{i}\omega \, \mathbf{d} \, \delta(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\omega t} \, .$$

Determine the corresponding charge density and the *physical* expressions for $\mathbf{j}$ and ϱ. Calculate the corresponding vector potential $\mathbf{A}_{\mathrm{E1}}$, including its harmonic time dependence. Calculate the electric field and the induction field.

4.3 Given two concentric rings made of a conducting material. The inner ring whose radius is a, carries the homogeneously distributed charge q, the outer ring with radius b carries the charge $-q$. Write down the charge density of this setup, expressed in cylinder coordinates where the z-axis goes through the center of the rings and is perpendicular to the plane spanned by them.

4.4 The setup of Exercise 4.3 is assumed to rotate about the z-axis with angular velocity ω. Derive the current density and calculate the magnetic dipole moment.

Exercises: Chap. 5

5.1 Solve the differential equation

$$\left(\Delta - \kappa^2\right)\phi(\mathbf{x}) = g\,\delta(\mathbf{x}) \tag{A.4}$$

in momentum space, i. e. by means of the ansatz

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \iiint \mathrm{d}^3k\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \widetilde{\phi}(\mathbf{k}) \; . \tag{A.5}$$

5.2 In a given representation the generators of a compact, simple Lie group have the property (omitting, for simplicity, the symbol $\mathbf{U}$ in $\mathbf{U}(\mathbf{T})$)

$$\mathrm{tr}\left\{\mathbf{T}_i, \mathbf{T}_j\right\} = \kappa\delta_{ij} \; . \tag{A.6}$$

Show that the constant κ, though depending on the representation, does not depend on i and j.

5.3 Construct a Lagrange density for the local gauge theory which is built on the structure group $G = \mathrm{SO}(3)$ and which contains a triplet of scalar fields.

5.4 A local gauge theory built on the structure group

$$G = \mathrm{SU}(p) \times \mathrm{SU}(q) \quad \text{with} \quad p, q > 1 \; ,$$

allows for two independent “charges” (coupling constants). Show this by constructing the gauge potential and the covariant derivative.

5.5 A major study project might be this: Study the group theoretical aspects of the publication on the self-gravitation of a rotating star quoted in Sect. 5.5. Investigate analytically the bifurcations reported in this work and illustrate by means of numerical examples.

5.6 Show that the matrices

$$M^{(k)}_{lm} = -\mathrm{i}C^{(m)}_{kl}$$

fulfill the Lie algebra (5.20).

Exercises: Chap. 6

6.1 The aim is to show that the $(n-1)$-sphere S_R^{n-1} which has radius R and is embedded in $\mathbb{R}^n$, is a smooth manifold.

Let $N = (0, \dots, 0, R)$ and $S = (0, \dots, 0, -R)$ be the north and south poles of the sphere, respectively. Let two charts be defined by the projection from N and from S, respectively, of the points $x \in S_R^{n-1}$ onto the equatorial hypersurface $x^n = 0$. The first chart applies to the subset $U_1 = S_R^{n-1} \backslash \{N\}$, the second applies to $U_2 = S_R^{n-1} \backslash \{S\}$. Specify the maps φ_i as well as their inverses for $i = 1, 2$. Derive the transition mapping from U_1 to U_2, show that this is a diffeomorphism on the intersection $U_1 \cap U_2$.

6.2 Gravitational redshift: Calculate the relative change of frequency $\Delta\omega/\omega$ of a photon which moves from the tip of a tower with height H to the ground. Compare the shift $\Delta\omega$ for the example $H = 22.5$ m with the natural line width Γ of a spectral line in ^{57}Fe for which $\omega/\Gamma = 3 \times 10^{12}$.

6.3 Let $X, Y, Z \in \mathfrak{X}(M)$ be smooth vector fields on the manifold M, $[X, Y]$ etc. their Lie brackets (commutators). Prove the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 .$$

Consider the example of $M = \mathbb{R}^2$ with $X = y\partial_x$ and $Y = x\partial_y$. What is their Lie bracket?

6.4 In general, tensor products do not commute. In order to illustrate this fact consider the examples $T^{(i)} \in \mathfrak{T}_2^0$, $i = 1, 2$, with

$$T^{(1)} = \mathrm{d}x^1 \otimes \mathrm{d}x^2 , \quad T^{(2)} = \mathrm{d}x^2 \otimes \mathrm{d}x^1 .$$

Calculate the functions $T^{(i)}(X, Y)$ for

$$X = a^1\partial_1 + a^2\partial_2 , \quad Y = b^1\partial_1 + b^2\partial_2$$

with constant coefficients $a^1, \dots, b^2$.

6.5 Another way to calculate the covariant derivative of tensor fields of type (0, 1) is the following. For the choice $V = \partial_\mu$ the covariant derivative D_V of a vector field X by V is known to be

$$X^\rho{}_{;\mu} = \partial_\mu X^\rho + \Gamma^\rho_{\mu\nu} X^\nu .$$

Use this formula to calculate $X_{\tau;\mu} = g_{\tau\rho} X^\rho{}_{;\mu}$ and make use of the coordinate expression of the Christoffel symbols.

6.6 The Christoffel symbols are not the components of a tensor field: The equations of motion of a free particle in Gaussian (or normal) coordinates at the point $x \in M$ read

$$\frac{\mathrm{d}^2 z^\mu}{\mathrm{d}\tau^2} = 0 \quad \text{with} \quad \mathrm{d}\tau^2 = \eta_{\mu\nu}\mathrm{d}z^\mu \mathrm{d}z^\nu \, ,$$

while in any other coordinates they are

$$\frac{\mathrm{d}^2 y^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}y^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}y^\sigma}{\mathrm{d}\tau} = 0 \, .$$

Prove the following formulae

$$g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial z^\alpha}{\partial y^\mu}\frac{\partial z^\beta}{\partial y^\nu} \, , \quad \Gamma^\mu_{\rho\sigma} = \frac{\partial y^\mu}{\partial z^\alpha}\frac{\partial^2 z^\alpha}{\partial y^\rho \partial y^\sigma} \, .$$

Derive the transformation formulae for Christoffel symbols under a diffeomorphism $\{y^\mu\} \mapsto \{y'^\mu\}$. The above conclusion follows from the result.

6.7 The semi-Riemannian manifold $(M, \mathbf{g})$ is assumed to have dimension n. Show that the contraction of the metric gives n and that for smooth functions $f \in \mathfrak{F}(M)$ the divergence of $f\mathbf{g}$ is equal to the exterior derivative of f, $\mathbf{div}(f\mathbf{g}) = \mathrm{d}f$.

6.8 A tensor field which is closely related to the Riemann tensor field $\mathbf{R}$ is the *Weyl tensor field.* It is a function of the Riemann tensor field $\mathbf{R}$, of the Ricci tensor field $\mathbf{R}^{(\text{Ricci})}$, and of the curvature scalar S. When written in components it is defined as follows

$$\begin{aligned} C_{\mu\nu\sigma\tau} :=\, & R_{\mu\nu\sigma\tau} + \frac{1}{6}S\left(g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma}\right) \\ & - \frac{1}{2}\left(g_{\mu\sigma}R^{(\text{Ricci})}_{\nu\tau} - g_{\mu\tau}R^{(\text{Ricci})}_{\nu\sigma} - g_{\nu\sigma}R^{(\text{Ricci})}_{\mu\tau} + g_{\nu\tau}R^{(\text{Ricci})}_{\mu\sigma}\right) \, . \end{aligned}$$

The tensor $C_{\mu\nu\sigma\tau}$ has the same symmetry properties as $R_{\mu\nu\sigma\tau}$. Show: All its contractions give zero. In dimension $n = 4$ it has ten independent components, in dimension $n = 3$ it is identically zero.

If M has dimension 4 and is endowed with a conformally flat metric, i.e. if $g_{\mu\nu} = \phi^2(x)\eta_{\mu\nu}$ holds with $\phi(x)$ a smooth function, the tensor field C is identically zero.

6.9 Prove the relation

$$L_X = \mathrm{d} \circ i_X + i_X \circ \mathrm{d}$$

between the Lie derivative L_X, the exterior derivative and the inner product.
Hint: It is sufficient to verify this relation for functions and for vector fields, or, alternatively, for functions and for one-forms.

C Selected Solutions of the Exercises

Chapter 1 (selected solutions)

1.1 The spherical base vectors $\hat{\mathbf{e}}_m$, $m = -1, 0, +1$, have the following properties (please verify!):

$$\hat{\mathbf{e}}_m^* = (-)^m \hat{\mathbf{e}}_{-m} \qquad \hat{\mathbf{e}}_m^* \cdot \hat{\mathbf{e}}_{m'} = \delta_{mm'} \ . \tag{1}$$

Expanding vectors in terms of these, $\mathbf{V} = \sum_{m=-1}^{+1} v^m \hat{\mathbf{e}}_m$, and recalling that $\mathbf{V}$ is real, there follows

$$\mathbf{V} = \sum v^{m\,*} \hat{\mathbf{e}}_m^* = \sum (-)^m v^{-m\,*} \hat{\mathbf{e}}_m \ . \tag{2}$$

Equations (1) and (2) show that the basis which is dual to the basis $\{\hat{\mathbf{e}}_m\}$ is given by $\vec{\hat{e}^m} = (-)^m \hat{\mathbf{e}}_{-m}$ and that one has $\mathbf{V} = \sum v_m \vec{\hat{e}^m}$ with $v_m = (-)^m v^{-m}$. The scalar product of two vectors reads

$$\mathbf{V} \cdot \mathbf{W} = \sum_{m=-1}^{+m} v^{m\,*} w^m = \sum_{m=-1}^{+1} v_m w^m \ . \tag{3}$$

Indeed, one verifies the scalar product by calculating

$$\begin{aligned} \sum v_m w^m &= \frac{1}{2}(v^1 - \mathrm{i}v^2)(w^1 + \mathrm{i}w^2) + v^3 w^3 + \frac{1}{2}(v^1 + \mathrm{i}v^2)(w^1 - \mathrm{i}w^2) \\ &= v^1 w^1 + v^2 w^2 + v^3 w^3 \ . \end{aligned}$$

This exercise shows that even in a Euclidean space one must distinguish covariant from contravariant transformation behaviour whenever one uses a complex basis instead of a real cartesian basis.

F. Scheck, *Classical Field Theory*, Graduate Texts in Physics,
https://doi.org/10.1007/978-3-662-55579-8

1.2 With a specific partition into space and time components one has

$$\begin{aligned}\partial_0 j^0 &= e\frac{\partial}{\partial t}\delta^{(3)}\big(\mathbf{y}-\mathbf{x}(t)\big) = e\dot{\mathbf{x}}\cdot\nabla_x\delta^{(3)}\big(\mathbf{y}-\mathbf{x}(t)\big)\\ &= -e\dot{\mathbf{x}}\cdot\nabla_y\delta^{(3)}\big(\mathbf{y}-\mathbf{x}(t)\big)\ ,\\ \partial_i j^i &= e\dot{\mathbf{x}}\cdot\nabla_y\delta^{(3)}\big(\mathbf{y}-\mathbf{x}(t)\big)\ .\end{aligned}$$

The sum of these expressions gives zero.

1.3 Except for binding effects one has

$$Mc^2 = 235\cdot 939\,\text{MeV} = 3.535\times 10^{-8}\,\text{J}\ .$$

Using the conversion formula (see, e. g., [QP] Appendix A.7) $1\,\text{eVc}^{-2} = 1.78266\times 10^{-27}\mu\,\text{g}$, one finds the approximate value $M = 3.9\cdot 10^{-16}\mu\,\text{g}$.

Remark 1 This value is still very small as compared to the Planck mass

$$M_{\text{Planck}} := \sqrt{\frac{\hbar c}{G_{\text{Newton}}}} = 22.2\mu\text{g}\ ,$$

which could be measured with a chemist's balance.

1.4 The muonic Bohr radius is smaller by the factor m_e/m_μ than the one of the electron. For the example of lead, i. e. with $Z = 82$, it is

$$a_{\text{B}}^{(\mu)}(Z=82) = \frac{\hbar c}{Z\alpha m_\mu c^2} = 3.12\times 10^{-15}\text{m}\ .$$

This value is smaller than the nuclear radius which is about 7×10^{-15}m. If instead the entire charge of the nucleus were localized in its center the magnitude of the electric field at the position of the muon would be

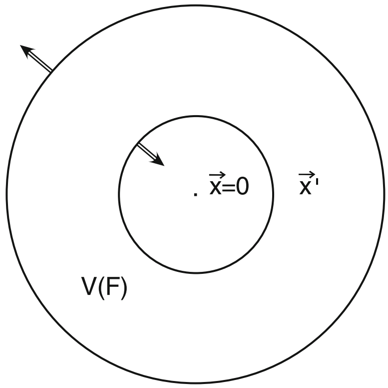
$$|\mathbf{E}| = \frac{Ze}{r^2} = 1.35\times 10^{12}\text{V m}^{-1}\ .$$

The realistic value which is smaller than this, can be estimated by a model of the lead nucleus in which the charge distribution is homogeneous and has the radius given above.

1.5 There are different ways to check the relation (1.48a).

(a) For fixed values of i and j also k is fixed. It cannot be equal to i or j. This holds also for l and m which cannot be equal to k. As they must be different from each other there remain the possibilities $(i = l, j = m)$ and $(i = m, j = l)$ only.

Fig. C.1



The first of these appears with the positive sign, the second appears with the negative sign.

(b) With $\{\hat{\mathbf{e}}_i\}$, $i = 1, 2, 3$ an orthonormal basis in $\mathbb{R}^3$, the first ε-symbol equals the scalar triple product $(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}_k$. Likewise the second ε-symbol equals $\hat{\mathbf{e}}_k \cdot (\hat{\mathbf{e}}_l \times \hat{\mathbf{e}}_m)$. As the sum $\sum_k |\hat{\mathbf{e}}_k\rangle\langle\hat{\mathbf{e}}_k|$ is equal to 1 the required expression is equal to

$$(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot (\hat{\mathbf{e}}_l \times \hat{\mathbf{e}}_m) \; .$$

This is the right-hand side of the equation that was to be proved.

1.6 The space between the two spheres of Fig. C.1 defines the volume $V(F)$. Its surface consists of the sphere with radius R_a whose normal points outwards, and the sphere with radius R_i whose normal points inwards. With $\Delta(1/r) = 0$ in the intermediate space and with $\Delta\Phi(\mathbf{x}) = -f(\mathbf{x})$ Green's second theorem yields

$$\iiint_{V(F)} \mathrm{d}^3x \; \frac{f(\mathbf{x})}{r} = \iint_F \mathrm{d}\sigma \left\{ -\Phi \frac{1}{r^2} - \frac{1}{r}\frac{\partial \Phi}{\partial r} \right\}$$

the center of the rings being the origin. Both spheres contribute to the right-hand side and we have $\mathrm{d}\sigma = r^2 \mathrm{d}\Omega$. The second term vanishes both in the limit $R_\mathrm{a} \to \infty$ and in the limit $R_\mathrm{i} \to 0$. While for $R_\mathrm{a} \to \infty$ the first term vanishes, too, for $R_\mathrm{i} \to 0$ it gives $4\pi\Phi(0)$. This is the answer that was to be proved.

1.7 In order to normalize the given distribution one must calculate the integral

$$I := 4\pi \int_0^\infty r^2 \mathrm{d}r \; \frac{1}{1 + \mathrm{e}^{(r-c)/z}} = 4\pi z^3 \int_0^\infty x^2 \mathrm{d}x \; \frac{1}{1 + \mathrm{e}^{(x-x_0)}} \qquad (4)$$

where $x = r/z$ and $x_0 = c/z$. The domain of integration is split into the intervals $[0, x_0)$ and $[x_0, \infty)$, such that in either case the integrand may be written as a geometric series, viz.

$$x < x_0 : \quad \frac{1}{1+\mathrm{e}^{(x-x_0)}} = 1 + \sum_{n=1}^{\infty} (-)^n \mathrm{e}^{-nx_0} \mathrm{e}^{nx} ,$$

$$x \geqslant x_0 : \quad \frac{1}{1+\mathrm{e}^{(x-x_0)}} = \sum_{n'=0}^{\infty} (-)^{n'} \mathrm{e}^{(n'+1)x_0} \mathrm{e}^{-(n'+1)x}$$

$$= -\sum_{n=1}^{\infty} (-)^n \mathrm{e}^{nx_0} \mathrm{e}^{-nx} ,$$

where in the last step $n = n' + 1$ was substituted.

The following two integration formulae are useful for the sequel and are easily derived:

$$I_< := \int_0^a \mathrm{d}x\; x^2 \mathrm{e}^x = \mathrm{e}^a \left(a^2 - 2a + 2\right) - 2 ,$$

$$I_> := \int_a^\infty \mathrm{d}x\; x^2 \mathrm{e}^{-x} = \mathrm{e}^{-a} \left(a^2 + 2a + 2\right) .$$

Using these formulae the required integral becomes

$$I = 4\pi z^3 \left\{ \frac{1}{3} x_0^3 + 4x_0 \sum_{n=1}^{\infty} (-)^{n+1} \frac{1}{n^2} - 2 \sum_{n=1}^{\infty} (-)^n \frac{1}{n^3} \mathrm{e}^{-nx_0} \right\} .$$

The infinite sum in the second term can be found, e.g., in [Abramovitz-Stegun; (23.2.19) and (23.2.24)]

$$\sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n^2} = -\frac{1}{2}\zeta(2) = \frac{\pi^2}{12} ,$$

where $\zeta(x)$ is Riemann's zeta function. As a result one obtains

$$I = \frac{4\pi c^3}{3} \left\{ 1 + \left(\frac{\pi z}{c}\right)^2 - 6\left(\frac{z}{c}\right)^3 \sum_{n=1}^{\infty} \frac{(-)^n}{n^3} \mathrm{e}^{-(nc)/z} \right\} \qquad (5)$$

and, from this, the formula given in the main text.

In charge distributions of atomic nuclei c, as a rule, is sensibly larger than z (typical values are $c = 6\,\text{fm}$, $z = 0.2\,\text{fm}$), i.e., $\exp\{-c/z\} \ll 1$. If one neglects the last term the distribution at $r = c$ takes about half the value it has at $r = 0$. The distance between the radii $r_{0.9}$ and $r_{0.1}$ where it is still 90 and 10% of its value at $r = 0$, respectively, is given by $t = 4\ln(3)z$. This parameter is customarily quoted as the *surface thickness* of the charge distribution.

Fig. C.2

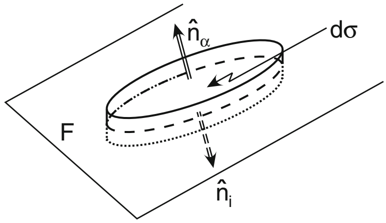
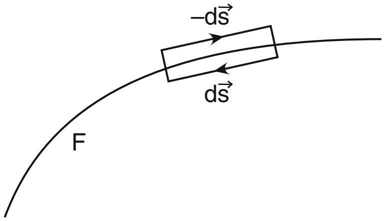


Fig. C.3



1.8 The integral over the volume of the "box" in Fig. C.2 gives 4π times the surface charge density. The hight being assumed to be small of third order, the surface integral receives contributions only from the two end faces of the box which differ by the direction of the normal. Thus, one obtains $(\mathbf{E}_a - \mathbf{E}_i) \cdot \hat{\mathbf{n}}$.

1.9 Choosing the closed path in Fig. C.3 such that it cuts through the surface with its short sides, one concludes

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{s} = \left(\mathbf{E}_a - \mathbf{E}_i\right) \cdot \hat{\mathbf{t}} = 0 \, .$$

This shows the continuity of the tangential component.

1.10 At first, the expression (1.97c) for the Legendre functions of the first kind holds only for $m \geqslant 0$. The following alternative representation[6]

$$P_\ell^m(z) = (-)^m \mathrm{e}^{-\mathrm{i}m\pi/2} \frac{(\ell+m)!}{2\pi \ell!} \cdot \int_{-\pi}^{+\pi} \mathrm{d}\psi \, \left(\cos\theta + \mathrm{i}\sin\theta\cos\psi\right)^\ell \cos(m\psi) \, , \qquad (z = \cos\theta)$$

holds for all values of m. From this one obtains the symmetry relation

$$P_\ell^{-m}(z) = (-)^m P_\ell^m(z) \frac{(\ell-m)!}{(\ell+m)!} \, . \tag{6}$$

[6]see e. g. N. Straumann, *Quantenmechanik,* Springer, Heidelberg 2002, (1.168). English Translation in preparation.

Inserting this into the formula (1.97a) one obtains the symmetry relation (1.97g).

For the derivation of the relation (1.97h) notice that with $z' = \cos(\pi - \theta) = -\cos\theta = -z$ one finds

$$e^{im(\phi+\pi)} = (-)^m e^{im\phi} ,$$
$$P_\ell^m(-z) = (-)^m (1-z^2)^{m/2} (-)^m \frac{d^m}{dz^m} P_\ell(-z) = (-)^{\ell-m} P_\ell^m(z) .$$

Here one has used that the Legendre polynomials produce a sign $(-)^\ell$ when $z \mapsto (-z)$. The composition of the two results yields the symmetry relation (1.97h).

1.11 The multipole moments are defined in (1.106d). Thus one has

$$\begin{aligned} q_{22} &= \iiint d^3x\, r^2 Y_{22}^*(\hat{\mathbf{x}})\varrho(\mathbf{x}) \\ &= \frac{\sqrt{15}}{4\sqrt{2\pi}} \iiint d^3x\, r^2 \sin^2\theta e^{-2i\phi} \varrho(\mathbf{x}) \\ &= \frac{\sqrt{15}}{4\sqrt{2\pi}} \iiint d^3x\, \left(x^1 - ix^2\right)^2 \varrho(\hat{\mathbf{x}}) \\ &= \frac{\sqrt{15}}{4\sqrt{2\pi}} \iiint d^3x\, \left\{x^1x^1 - 2ix^1x^2 - x^2x^2\right\} \varrho(\hat{\mathbf{x}}) \\ &= \frac{\sqrt{15}}{4\sqrt{2\pi}} \frac{1}{3} \left(Q^{11} - 2iQ^{12} - Q^{22}\right) . \end{aligned}$$

The two other components are calculated in the same manner

$$\begin{aligned} q_{21} &= -\frac{\sqrt{15}}{2\sqrt{2\pi}} \iiint d^3x\, x^3(x^1 - ix^2)\varrho(\mathbf{x}) = \frac{\sqrt{5}}{2\sqrt{6\pi}} \left(-Q^{13} + iQ^{23}\right) , \\ q_{20} &= \frac{\sqrt{5}}{4\sqrt{\pi}} \iiint d^3x\, \left(3x^3x^3 - r^2\right)\varrho(\hat{\mathbf{x}}) = \frac{\sqrt{5}}{4\sqrt{\pi}} Q^{33} . \end{aligned}$$

In this calculation the symmetry $Q^{ji} = Q^{ij}$ and the definition (1.111c) were utilized.

1.12 One calculates first the integral over the volume V of a ball with radius R and center at the position of the dipole. This integral is converted to a surface integral over the sphere by means of (1.6):

$$\begin{aligned} \iiint_V d^3x\, \mathbf{E}_D(\mathbf{x}) &= -\iiint_V d^3x\, \nabla\Phi_D(\mathbf{x}) = -R^2 \iint_{F(V)} d\sigma\, \Phi_D(\mathbf{x})\,\hat{\mathbf{n}} \\ &= -R^2 \frac{4\pi}{3} \iiint d^3x' \\ &\quad \cdot \iint_{F(V)} d\sigma\, \frac{r_<}{r_>^2} \sum_\mu Y_{1\mu}^*(\hat{x}) Y_{1\mu}(\hat{x}')\,\hat{\mathbf{n}} . \end{aligned}$$

Here the multipole expansion (1.105) was used of which, by the integration over the angular variables, only the term with $\ell = 1$ contributes. Consider the unit vector $\hat{\mathbf{n}}$, expanded in terms of spherical harmonics,

$$\hat{\mathbf{n}} = \sum_{m=-1}^{+1} a_m Y_{1m}(\hat{x}) \ .$$

The surface integral over the sphere then is found to be

$$\left(\iint_{F(V)} \mathrm{d}\sigma \ \sum_{\mu} Y^*_{1\mu}(\hat{x})\,\hat{\mathbf{n}} \right) Y_{1\mu}(\hat{x}') = \sum_m a_m Y_{1m}(\hat{x}') = \hat{\mathbf{n}}' \ .$$

As the dipole is localized, one has $r_< = r'$ and $r_> = R$ so that

$$\iiint_V \mathrm{d}^3x \ \mathbf{E}_{\mathrm{D}}(\mathbf{x}) = -R^2 \frac{4\pi}{3} \frac{1}{R^2} \iiint \mathrm{d}^3x' \ \varrho(\mathbf{x}') r' \hat{\mathbf{n}}' = -\frac{4\pi}{3} \mathbf{d} \ .$$

This derivation shows that the formula to be proved holds for every localized dipole distribution and, hence, for the pointlike dipole as a special case.

1.13 The capacitor consists of two identical plates in parallel orientation, their surface being F. In the initial state they carry the charges $+q$ and $-q$, respectively. The normal component of the displacement field $\mathbf{D} = \varepsilon \mathbf{E}$ fulfills the relation $\mathbf{D}_n = |\mathbf{D}| = 4\pi\eta$, where $\eta = q/F$ is the surface charge density (we are considering the example of the positively charged plate). Short-circuiting the plates causes a current

$$I = \frac{\mathrm{d}q}{\mathrm{d}t} = F \frac{\partial \eta}{\partial t} = \frac{F}{4\pi} \frac{\partial \mathbf{D}_n}{\partial t}$$

in the cable joining them. Thus, the current density of the displacement current is

$$\mathbf{j}_{\mathrm{V}} = \frac{1}{4\pi} \frac{\partial \mathbf{D}}{\partial t} \ .$$

These relations hold for the vector fields proper because they are perpendicular to the surface of the plates and, hence, are equal to their normal components.

1.14 This example illustrates the method of *image charges*. Choose two point charges $q^{(1)}$ and $q^{(2)}$ on a straight line through the origin and define this to be the 1-axis. The positions of the charges $\mathbf{x}^{(1)} = r^{(1)} \hat{\mathbf{e}}_1$ and $\mathbf{x}^{(2)} = r^{(2)} \hat{\mathbf{e}}_1$, respectively, are images of each other with respect to the sphere with radius R and center the origin which means that $r^{(1)}r^{(2)} = R^2$. Determine the charge $q^{(2)}$ such that the potential vanishes on the sphere. The potential at $|\mathbf{x}| = R$ is

$$\Phi(\mathbf{x})|_R = \left[\frac{q^{(1)}}{|\mathbf{x} - r^{(1)}\hat{\mathbf{e}}_1|} + \frac{q^{(2)}}{|\mathbf{x} - r^{(2)}\hat{\mathbf{e}}_1|}\right]_{|\mathbf{x}|=R}$$
$$= \frac{q^{(1)}}{R|\hat{\mathbf{x}} - (r^{(1)}/R)\hat{\mathbf{e}}_1|} + \frac{q^{(2)}}{r^{(2)}|(R/r^{(2)})\hat{\mathbf{x}} - \hat{\mathbf{e}}_1|} .$$

In this formula the two absolute values in the denominators are equal, viz.

$$\left|\hat{\mathbf{x}} - \left(\frac{r^{(1)}}{R}\right)\hat{\mathbf{e}}_1\right| = \left|\left(\frac{R}{r^{(2)}}\right)\hat{\mathbf{x}} - \hat{\mathbf{e}}_1\right| = 1 - 2\frac{r^{(1)}}{R}\hat{\mathbf{x}} \cdot \hat{\mathbf{e}}_1 + \left(\frac{r^{(1)}}{R}\right)^2 ,$$

Therefore the potential vanishes for $r = R$ if the second charge is chosen to be $q^{(2)} = -q^{(1)}(r^{(2)}/R)$. One can now replace the point charge in the interior of the sphere by the sphere with $\Phi|_R = 0$. This solves the problem.

1.15 Place the dipole in the origin. In the absence of the sphere the potential it creates would be $\Phi_{\mathrm{D}}(x) = d\ r \cos\theta/r^3$. The presence of the sphere causes an additive term which is such that the potential on the sphere is a constant. Without loss of generality one can choose this constant to be zero. Then the total potential in the interior of the sphere is

$$\Phi = d\cos\theta\frac{1}{r^2} + \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta)$$
$$= d\cos\theta\frac{1}{r^2} + a_1 r\cos\theta$$
$$= dr\cos\theta\left(\frac{1}{r^3} - \frac{1}{R^3}\right) .$$

Here use was made of the fact that only the term with $\ell = 1$ contributes, the boundary condition $\Phi(R) = 0$ fixes the coefficient a_1 to be $a_1 = -d/R^3$. The radial component and the θ-component of the electric field are, respectively,

$$E_r = -\frac{\partial\Phi}{\partial r} = d\cos\theta\left(\frac{2}{r^3} + \frac{1}{R^3}\right) ,$$
$$E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = d\sin\theta\left(\frac{1}{r^3} - \frac{1}{R^3}\right) .$$

At $r = R$ the θ-component vanishes. The discontinuity of the radial component follows from (1.87a),

$$(E_r)_{\mathrm{a}} - (E_r)_{\mathrm{i}} = (E_r)_{\mathrm{a}} - d\cos\theta\frac{3}{R^3} = 4\pi\eta .$$

By (1.92c) the induced surface charge density on the sphere is

$$\eta = \frac{1}{4\pi}\left.\frac{\partial \Phi}{\partial \hat{\mathbf{n}}}\right|_{r=R} = -\frac{d\cos\theta}{4\pi}\frac{3}{R^3}\;.$$

(Only the interior contributes to the derivative. The normal of the surface is the negative normal to the sphere, hence the minus sign.) One concludes $(E_r)_\mathrm{a} = 0$.

1.16 The potential created by the dipole is calculated as follows:

$$\begin{aligned}\Phi(\mathbf{x}) &= \iiint \mathrm{d}^3x\;\frac{\varrho_\mathrm{eff}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = -\mathbf{d}\cdot\iiint \mathrm{d}^3x\;\frac{\boldsymbol{\nabla}_{x'}\delta(\mathbf{x}-\mathbf{x}^{(0)})}{|\mathbf{x}-\mathbf{x}'|}\\ &= \mathbf{d}\cdot\iiint \mathrm{d}^3x\;\delta(\mathbf{x}-\mathbf{x}^{(0)})\boldsymbol{\nabla}_{x'}\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \mathbf{d}\cdot\frac{\mathbf{x}-\mathbf{x}^{(0)}}{|\mathbf{x}-\mathbf{x}^{(0)}|^3}\;.\end{aligned}$$

This expression coincides with (1.88c). The energy in the external potential is

$$\begin{aligned}W &= \iiint \mathrm{d}^3x\;\varrho_\mathrm{eff}(\mathbf{x})\Phi_\mathrm{a}(\mathbf{x})\\ &= -\mathbf{d}\cdot\iiint \mathrm{d}^3x\;\left(\boldsymbol{\nabla}_x\delta(\mathbf{x}-\mathbf{x}^{(0)})\right)\Phi_\mathrm{a}(\mathbf{x})\\ &= \mathbf{d}\cdot\iiint \mathrm{d}^3x\;\delta(\mathbf{x}-\mathbf{x}^{(0)})\boldsymbol{\nabla}_x\Phi_\mathrm{a}(\mathbf{x}) = -\mathbf{d}\cdot\mathbf{E}_\mathrm{a}(\mathbf{x}^{(0)})\;.\end{aligned}$$

This is the known expression for the energy of the electric dipole in an external electric field.

1.17 Without external field, $\mathbf{E}_0 = 0$, the potential would be the potential outside of a spherically symmetric charge distribution, $\Phi(r) = Q/r$; In the absence of the sphere it would be $\Phi(\mathbf{x}) = -E_0 z = -E_0 r\cos\theta$. As potentials are additive in the sources one is led to the suggested ansatz, i. e., the sum of a spherically symmetric term and a term with the characteristic $\cos\theta$-dependence of the homogeneous field, viz.

$$\Phi(\mathbf{x}) = f_0(r) + f_1(r)\cos\theta\;.$$

Inserting into the Laplace equation $\Delta\Phi(\mathbf{x}) = 0$ one finds

$$\begin{aligned}\Delta\Phi(\mathbf{x}) &= \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}f_0}{\mathrm{d}r}\right) + \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}f_1}{\mathrm{d}r}\right)\cos\theta\\ &\quad - \frac{f_1}{r^2\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin^2\theta\right)\\ &= \left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}f_0}{\mathrm{d}r}\right)\right] + \left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}f_1}{\mathrm{d}r}\right) - \frac{2f_1}{r^2}\right]\cos\theta = 0\;.\end{aligned}$$

This differential equation holds for all r and all θ. Therefore, the expressions in the big square brackets must both be zero individually:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}f_0}{\mathrm{d}r}\right)=0\,, \tag{7}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}f_1}{\mathrm{d}r}\right)-\frac{2f_1}{r^2}=0\,. \tag{8}$$

The first of these, (7), has the general solution $f_0(r) = A/r + B$. The second differential equation (8) reads $r^2 f_1'' + 2rf_1' - 2f_1 = 0$, its general solution is $f_1(r) = C/r^2 + Dr$. Thus the required solution has the form

$$\Phi(\mathbf{x}) = \frac{A}{r} + \left(\frac{C}{r^2} + Dr\right)\cos\theta + B\,. \tag{9}$$

The four constants are obtained from the boundary conditions:

(a) For $r \to \infty$ the term proportional to $r\cos\theta$ dominates. In this limit only the given external field is felt so that one must have $D = -E_0$.
(b) On the sphere the potential must be constant,

$$\Phi(\mathbf{x})|_{r=R} = \frac{A}{R} + \left(\frac{C}{R^2} - E_0 R\right)\cos\theta = \text{const.} \quad \forall \quad \theta\,;$$

from which follows $C = E_0 R^3$.
(c) Gauss' theorem yields the normalization condition

$$\iint_{r=R} \mathrm{d}\sigma\, \mathbf{E}\cdot\hat{\mathbf{n}} = -\iint_{r=R} \mathrm{d}\sigma\, \frac{\partial\Phi}{\partial r} = 4\pi Q\,.$$

By (9) the same integral is equal to $4\pi A$, giving $A = Q$.

Thus, the solution is

$$\Phi(\mathbf{x}) = \frac{Q}{r} + E_0\left(\frac{R^3}{r^2} - r\right)\cos\theta\,. \tag{10}$$

For the case $Q = 0$ the induced surface charge density is calculated to be

$$\eta(\theta) = -\frac{1}{4\pi}\frac{\partial\Phi}{\partial r} = \frac{3}{4\pi}E_0\cos\theta\,.$$

(Sketch a plane cutting the equipotential surfaces which contains the z-axis, as well as the electric field lines for the example $Q = 0$!)

Remark 2 The justification for our ansatz is an intuitive one. A more systematic approach makes use of the expansion of the potential in terms of spherical harmonics: By the axial symmetry of the problem the most general ansatz is

$$\Phi(\mathbf{x}) = \sum_{\ell=0}^{\infty} f_\ell(r) P_\ell(\cos\theta)$$

with $f_\ell = r^\ell$ or $f_\ell = r^{-\ell-1}$. The potential of the original homogeneous field is axially symmetric and is proportional to $P_1(\cos\theta)$. The added sphere does not modify the angular dependence and therefore causes an additive monopole term only. Thus, the solution must have the form

$$\Phi(\mathbf{x}) = \left(\frac{A}{r} + B\right) P_0(\cos\theta) + \left(\frac{C}{r^2} + Dr\right) P_1(\cos\theta)$$
$$\big(P_0(\cos\theta) = 1 \,, \quad P_1(\cos\theta) = \cos\theta\big) \,.$$

The constants are determined as before.

1.18 With $\varrho(r) = 3Q/(4\pi R^3)\Theta(R-r)$ the potential and the field strength inside and outside are, respectively,

$$r \leqslant R: \qquad \Phi_{\mathrm{i}}(r) = \frac{3Q}{2R^3}\left(R^2 - \frac{1}{3}r^2\right) , \qquad \mathbf{E}_{\mathrm{i}} = \frac{Q}{R^3} r\,\hat{\mathbf{e}}_r \,,$$
$$r > R: \qquad \Phi_{\mathrm{a}}(r) = \frac{Q}{r} , \qquad \mathbf{E}_{\mathrm{a}} = \frac{Q}{r^2}\hat{\mathbf{e}}_r \,.$$

Calculating the energy from the square of the electric field and integrating over the whole space,

$$W_{\mathrm{E}} = \frac{1}{8\pi}\iiint \mathrm{d}^3x\, \mathbf{E}^2 = \frac{1}{2}\int_0^\infty r^2 \mathrm{d}r\, \mathbf{E}^2$$
$$= \frac{1}{2}Q^2 \left\{ \frac{1}{R^6}\int_0^R \mathrm{d}r\, r^4 + \int_R^\infty \mathrm{d}r\, r^{-2} \right\} = \frac{3Q^2}{5R} ,$$

one obtains the same result as from the given formula, viz.

$$W = \frac{1}{2}\iiint \mathrm{d}^3x\, \varrho(r)\Phi(r) = \frac{9Q^2}{4R^6}\int_0^\infty r^2 \mathrm{d}r \left(R^2 - \frac{1}{3}r^2\right) = \frac{3Q^2}{5R} .$$

1.19 The energy contained in the field outside the electron is equal to

$$W_{\mathrm{a}} = \frac{1}{2}e^2 \int_R^\infty \mathrm{d}r\, \frac{1}{r^2} = \frac{e^2}{2R} .$$

Puting this equal to $m_e c^2$ one obtains the given value of R, i. e. the classical electron radius.

1.20 It follows from Maxwell's equations that on a surface carrying the charge density η the tangential component of the electric field is continuous while the normal component of the displacement field changes by the amount $4\pi\eta$. In the problem at stake one has $\eta = 0$, so that the normal component of $\mathbf{D}$ is continuous. Thus, the boundary conditions at $r = R$ are

$$\Phi_{\mathrm{i}} = \Phi_{\mathrm{a}} ,$$
$$\varepsilon_0 \frac{\partial \Phi_{\mathrm{a}}}{\partial r} = \varepsilon_1 \frac{\partial \Phi_{\mathrm{i}}}{\partial r} .$$

The spherical symmetry of the set-up is perturbed only by the external field which is axially symmetric and whose potential is $\Phi(r, \theta) = -E_0 P_1(\cos\theta)$. Thus, both in the inner and the outer regions the problem must have the general solution

$$\Phi_{\mathrm{a}} = \left(\frac{A}{r^2} + Br\right) P_1(\cos\theta) ,$$
$$\Phi_{\mathrm{i}} = \left(\frac{C}{r^2} + Dr\right) P_1(\cos\theta) .$$

Letting r go to infinity, $r \to \infty$, one sees that one must have $B = -E_0$; letting $r \to 0$ one concludes that $C = 0$.

The boundary conditions fix the remaining two constants as follows

$$A = \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1 + 2\varepsilon_0} E_0 R^3 ,$$
$$D = -\frac{3\varepsilon_0}{\varepsilon_1 + 2\varepsilon_0} E_0 .$$

The two special cases specified in the problem are as follows:

(a) $\varepsilon_0 \equiv \varepsilon$, $\varepsilon_1 = 1$: Here the potential is

$$\Phi_{\mathrm{a}} = \left[\frac{1-\varepsilon}{1+2\varepsilon}\frac{R^3}{r^3} - 1\right] r E_0 P_1(\cos\theta) ,$$
$$\Phi_{\mathrm{i}} = -\frac{3\varepsilon}{1+2\varepsilon} r E_0 P_1(\cos\theta) .$$

The field inside the sphere has the modulus

$$E_{\mathrm{i}} = \frac{3\varepsilon}{1+2\varepsilon} E_0 .$$

As $\varepsilon > 1$ it is *larger* than E_0.

(b) $\varepsilon_0 = 1$, $\varepsilon_1 \equiv \varepsilon$: The potential now reads

$$\Phi_a = \left[\frac{\varepsilon - 1}{\varepsilon + 2}\frac{R^3}{r^3} - 1\right] r E_0 P_1(\cos\theta) ,$$
$$\Phi_i = -\frac{3}{\varepsilon + 2} r E_0 P_1(\cos\theta) .$$

The modulus of the field inside is

$$E_i = \frac{3}{\varepsilon + 2} E_0$$

and hence is *smaller* than the external field.

Choosing in this case $\varepsilon \gg 1$, one has

$$\Phi_a \simeq \left[\frac{R^3}{r^3} - 1\right] r E_0 P_1(\cos\theta) , \quad \Phi_i \simeq 0 .$$

The field on the inside goes to zero, and one recovers the situation dealt with in Exercise 1.17 (with $Q = 0$).

1.21 The charge density created by the four point charges is

$$\varrho(x) = \frac{e}{2} \{[\delta(x-a) + \delta(x+a)]\,\delta(y)\delta(z)$$
$$- [\delta(y-b) + \delta(y+b)]\,\delta(x)\delta(z)\} .$$

One verifies at once that both the monopole moment

$$q_{00} = \frac{1}{\sqrt{4\pi}} \times \text{ total charge } = 0 ,$$

and all dipole moments

$$d_i = \iiint \mathrm{d}^3x\; x_i\,\varrho(\mathbf{x}) = 0$$

are equal to zero. The entries Q_{ij} of the quadrupole tensor are calculated to be

$$Q_{11} = \iiint \mathrm{d}^3x\; \left[2x^2 - y^2 - z^2\right] \varrho(\mathbf{x}) = e\left(2a^2 + b^2\right) ,$$
$$Q_{22} = \iiint \mathrm{d}^3x\; \left[2y^2 - z^2 - x^2\right] \varrho(\mathbf{x}) = -e\left(2b^2 + a^2\right) ,$$
$$Q_{33} = \iiint \mathrm{d}^3x\; \left[2z^2 - x^2 - y^2\right] \varrho(\mathbf{x}) = e\left(-a^2 + b^2\right) ,$$
$$Q_{12} = \iiint \mathrm{d}^3x\; 3xy\varrho(\mathbf{x}) = 0 ,$$

and, analogously, $Q_{13} = 0$, $Q_{23} = 0$.

Thus $\mathbf{Q} = e\,\mathrm{diag}(2a^2 + b^2, -2b^2 - a^2, -a^2 + b^2)$ and one confirms that $\mathbf{Q}$ has trace zero.

The spectroscopic quadrupole moment is

$$Q_0 = \iiint \mathrm{d}^3x\; r^2(3\cos^2\theta - 1)\varrho(\mathbf{x})$$
$$= \iiint \mathrm{d}^3x\; (2z^2 - x^2 - y^2) = Q_{33} = e(b^2 - a^2)\;.$$

In the spherical basis the moments are found to be

$$q_{22} = \frac{\sqrt{5}}{4\sqrt{6\pi}}(Q_{11} - 2\mathrm{i}Q_{12} - Q_{22}) = \frac{\sqrt{15}}{4\sqrt{2\pi}}e(a^2 + b^2)\;,$$
$$q_{21} = \frac{5}{2\sqrt{6\pi}}(-Q_{13} + \mathrm{i}Q_{23}) = 0\;,$$
$$q_{20} = \frac{\sqrt{5}}{4\sqrt{\pi}}Q_{33} = \frac{\sqrt{5}}{4\sqrt{\pi}}e(-a^2 + b^2)\;.$$

The moments $q_{2,-1}$ and $q_{2,-2}$ follow by the symmetry relations (1.107).

1.22 The current density is proportional to the surface charge density and to the tangential velocity at the point of reference,

$$\mathbf{j}(\mathbf{x}) = \eta\omega r\sin\theta\delta(r - R)\hat{\mathbf{e}}_\phi \equiv j_\phi\hat{\mathbf{e}}_\phi\;.$$

From this one calculates a vector potential by means of the formula (1.116). The unit vector $\hat{\mathbf{e}}_\phi$ is decomposed along the 1- and 2-directions, $\hat{\mathbf{e}}_\phi = -\sin\phi\hat{\mathbf{e}}_1 + \cos\phi\hat{\mathbf{e}}_2$. Writing the integral for $\mathbf{A}(r, \theta, \phi)$ in terms of spherical coordinates, one has

$$\mathbf{A}(r,\theta,\phi) = \eta\omega\frac{1}{c}\int_0^\infty r'^2\mathrm{d}r'\int \mathrm{d}\Omega'\; r'\delta(r' - R)\sin\theta'$$
$$\times\left(-\sin\phi'\hat{\mathbf{e}}_1 + \cos\phi'\hat{\mathbf{e}}_2\right)\sum_{\ell,m}\frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y^*_{\ell m}(\hat{\mathbf{x}}')Y_{\ell m}(\hat{\mathbf{x}})\;.$$

The calculation proceeds along the following lines: As the set-up is axially symmetric it is sufficient to calculate $\mathbf{A}$ for $\phi = 0$. On the other hand, the integral over ϕ' which is proportional to $\hat{\mathbf{e}}_1$, is equal to zero. This means that $\mathbf{A}(r, \theta, \phi = 0)$ is proportional to $\hat{\mathbf{e}}_2$ and, hence, is equal to the component A_ϕ. In the integrand make the replacement

$$\sin\theta'\cos\phi' = \sqrt{\frac{2\pi}{3}}\left(-Y_{11}(\hat{\mathbf{x}}') + Y_{1-1}(\hat{\mathbf{x}}')\right)$$

and calculate the angular integral. The induction field is obtained from the result $\mathbf{A} = A_\phi\hat{\mathbf{e}}_\phi$.

1.23 As there is neither a current density and nor a time dependent displacement current the field **H** is irrotational. Thus, it can be represented by a gradient field of a scalar magnetic potential Φ_M. In the inner space, inside the smaller of the two spheres, in the intermediate space, and in the outer space one writes down multipole expansions of Φ_M as follows,

$$\begin{aligned}\Phi_M^{(\text{inner})} &= \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta) \;, \\ \Phi_M^{(\text{inter})} &= \sum_{\ell=0}^{\infty} \left[c_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta) + d_\ell r^\ell P_\ell(\cos\theta) \right] \;, \\ \Phi_M^{(\text{outer})} &= \sum_{\ell=0}^{\infty} b_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta) + B_0 r P_1(\cos\theta) \;.\end{aligned} \tag{11}$$

In this ansatz we have made use of the fact that the potential must be regular in $r = 0$ and that it goes over into the potential of the homogeneous field as the argument goes to infinity, $r \to \infty$.

The boundary conditions are: The potential must be continuous at r and at R; at the two boundary surfaces the tangential component of **H** is continuous; furthermore, the normal component of **B** is continuous, i. e.

$$\Phi_M^{(1)} = \Phi_M^{(2)} \;, \tag{12a}$$

$$\frac{\partial \Phi_M^{(1)}}{\partial \theta} = \frac{\partial \Phi_M^{(2)}}{\partial \theta} \;, \tag{12b}$$

$$\mu_1 \frac{\partial \Phi_M^{(1)}}{\partial r} = \mu_2 \frac{\partial \Phi_M^{(2)}}{\partial r} \;, \tag{12c}$$

where the numbers 1 and 2 stand for any two neighbouring domains and where $\mu_{\text{inner}} = \mu_{\text{outer}} = 1$ in the inner and in the outer spaces, while in the intermediate space $\mu_{\text{inter}} = \mu$. One sees easily that the first two conditions (12a) and (12b) are equivalent. Therefore, it suffices to require continuity of the potentials only. Like in Exercise 1.17 one realizes that only the terms with $\ell = 1$ can contribute. Denoting by r the radius of the smaller, by R the radius of the bigger sphere one obtains the linear system of equations

$$\begin{aligned} a_1 r^3 &= c_1 + d_1 r^3 \;, \\ c_1 + d_1 R^3 &= b_1 - B_0 R^3 \;, \\ a_1 r^3 &= \mu \left[-2c_1 + d_1 r^3 \right] \;, \\ 2b_1 + B_0 R^3 &= \mu \left[2c_1 - d_1 R^3 \right] \;. \end{aligned}$$

The solution of this system of equations yields the coefficients in the ansatz (11), viz.

$$a_1 = \frac{9\mu R^3}{2(\mu-1)^2 r^3 - (\mu+2)(2\mu+1)R^3} B_0 \,, \tag{13a}$$

$$c_1 = \frac{3(\mu-1)r^3 R^3}{2(\mu-1)^2 r^3 - (\mu+2)(2\mu+1)R^3} B_0 \,, \tag{13b}$$

$$d_1 = \frac{3(2\mu+1)R^3}{2(\mu-1)^2 r^3 - (\mu+2)(2\mu+1)R^3} B_0 \,, \tag{13c}$$

$$b_1 = B_0 R^3 + 3R^3 \frac{(\mu-1)r^3 + (2\mu+1)R^3}{2(\mu-1)^2 r^3 - (\mu+2)(2\mu+1)R^3} B_0 \,. \tag{13d}$$

As a test of the result consider the case $\mu = 1$, for which the spherical shell is no longer seen. From (13a) to (13d) one obtains $a_1 = -B_0$, $c_1 = 0$, $d_1 = a_1$, $b_1 = 0$; any dependence on either r or R has disappeared.

The magnetic field is obtained by the generic formula

$$\Phi_{\mathrm{M}} = r^{\alpha} \cos\theta \,, \quad \mathbf{H} = -\nabla\Phi_{\mathrm{M}} = \alpha r^{\alpha-1} \cos\theta \hat{\mathbf{e}}_r + r^{\alpha-1}\hat{\mathbf{e}}_3 \,,$$

the magnetic induction is $\mathbf{B} = \mu\mathbf{H}$.

In the limit $\mu \to \infty$ the coefficients a_1, c_1, and d_1 go to zero, b_1 goes to $B_0 R^3$.

Chapter 2 (selected solutions)

2.1 The base-k-forms $\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}$ are totally antisymmetric, the indices i_1 to i_k can take all values from 1 up to the dimension n of the space. For fixed k there are

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

such base forms. This is shown as follows. In a first step one counts the number of possibilities to choose k different indices from the set $\{1, 2, \ldots, n\}$. The index i_1 can take any value from 1 to n, and thus there are n possibilities; for i_2 that has to be different from i_1, there are $(n-1)$ possible choices; for i_3 with $i_3 \neq i_1$ and $i_3 \neq i_2$, there remain $(n-2)$ possibilities; up to i_k which can take $(n-k+1)$ values. In total, the number of possible choices is

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!} \,.$$

As regards the ordering of k different indices there are $k!$ ways to do this, viz. as many as there are permutations of k elements. Only one of them fulfills the condition $i_1 < i_2 < \cdots < i_k$. Therefore, one must divide the number just obtained by $k!$, thus obtaining the dimension of the space $\Lambda^k(M)$.

2.2 It suffices to consider two fixed values μ and ν different from each other. Then one has

$$S_{\mu\nu}A^{\mu\nu} + S_{\nu\mu}A^{\nu\mu} = S_{\mu\nu}A^{\mu\nu} + S_{\mu\nu}\left(-A^{\mu\nu}\right) = 0 \,.$$

Equal values of μ and ν do not occur because $A^{\mu\mu} = 0$. The sum over all values of the two indices is the sum over all such pairs.

2.3 The indices α and β must be different from each other. Keeping α and β fixed, the other four indices have values which are not equal to one of these. This implies that one must have either $\sigma = \mu$ and $\tau = \nu$, or $\sigma = \nu$ and $\tau = \mu$. Taking the sum over α and β the term $\varepsilon^{\alpha\beta\sigma\tau}\varepsilon_{\alpha\beta\mu\nu}$ and the term $\varepsilon^{\beta\alpha\sigma\tau}\varepsilon_{\beta\alpha\mu\nu}$ give the same result. Hence the factor 2. With $\varepsilon_{0123} = 1$ one has $\varepsilon^{0123} = -1$, giving a minus sign in the following formula

$$\varepsilon^{\alpha\beta\sigma\tau}\varepsilon_{\alpha\beta\mu\nu} = -2\left(\delta^{\sigma}_{\mu}\delta^{\tau}_{\nu} - \delta^{\sigma}_{\nu}\delta^{\tau}_{\mu}\right) \,.$$

This is the analogue of (1.48a). Take now $\mu = \sigma$ and sum over this index to obtain

$$\varepsilon^{\alpha\beta\mu\tau}\varepsilon_{\alpha\beta\mu\nu} = -2(4-1)\delta^{\tau}_{\nu} = -6\delta^{\tau}_{\nu} \,.$$

This is the analogue of (1.48b).

2.4 One calculates the divergence of $\mathbf{A}' = \mathbf{A} + \nabla\chi$:

$$\nabla\cdot\mathbf{A}'(t,\mathbf{x}) = \nabla\cdot\mathbf{A}(t,\mathbf{x}) + \Delta_x\chi(t,\mathbf{x}) = \nabla\cdot\mathbf{A}(t,\mathbf{x}) - \nabla\cdot\mathbf{A}(t,\mathbf{x}) = 0 \,.$$

In the second step the equation $\Delta(1/\mathbf{x}-\mathbf{y}) = -4\pi\delta(\mathbf{x}-\mathbf{y})$ was used and the integral over $\mathbf{y}$ was done. Any further transformation with a gauge function ψ that satisfies the homogeneous differential equation $\Delta\psi = 0$, does not modify this result.

A gauge transformation $A''_{\mu} = A'_{\mu} - \partial_{\mu}\psi$ with

$$\psi(\mathbf{x}) = \int_0^{x^0} \mathrm{d}t' \; A^0(t',\mathbf{x})$$

leads to $A''_0 = 0$, as requested.

2.5 Perhaps the simplest argument is the following: The electron in the initial state has the four-momentum p_{i} which satisfies $p_{\mathrm{i}}^2 = m_e^2c^2$. In the final state it has the momentum p_{f}, the photon has the momentum k, with $p_{\mathrm{f}}^2 = m_e^2c^2$ and $k^2 = 0$. This is in contradiction to energy-momentum conservation which requires $p_{\mathrm{i}} = p_{\mathrm{f}} + k$: The condition $p_{\mathrm{f}}\cdot k = 0$ can only hold true if p_{f} is lightlike, i.e. if $p_{\mathrm{f}}^2 = 0$.

Chapter 3 (selected solutions)

3.1 The physical dimensions of the given quantities are

$$[\mathbf{S}] = \mathrm{MT}^3 = \frac{\text{energy}}{\text{surface} \times \text{time}} ,$$
$$[\mathbf{P}] = \mathrm{ML}^{-2}\mathrm{T}^{-1} = \frac{\text{momentum}}{\text{volume}} ,$$
$$[\mathbf{u}] = \frac{\text{energy}}{\text{volume}} .$$

3.2 Rotations are represented by orthogonal 3×3-matrices, i. e. one has $\mathbf{R}\mathbf{R}^{-1} = 1\!\!1_3$. Hence

$$\sum_{i,j} R_{mi} R_{nj} \delta_{ij} = \sum_i R_{mi} R^T_{in} = \delta_{mn} .$$

The transformation formula for the ε-tensor

$$\sum_{i,j,k} R_{mi} R_{nj} R_{pk} \varepsilon_{ijk} = \varepsilon'_{mnp}$$

is the determinant of $\mathbf{R}$ whenever (m, n, k) is an even permutation of $(1, 2, 3)$, and is equal to minus this determinant if (m, n, k) is an odd permutation. The determinant is invariant, its sign can be represented by ε_{mnp}. Hence, $\varepsilon'_{mnp} = \varepsilon_{mnp}$.

Chapter 4 (selected solutions)

4.1 From Maxwell's equations one derives the following boundary conditions: Given a surface separating two different media "1" and "2" which carries the surface charge density η, or in which flows the surface current density $\mathbf{j}$. Then the following relations hold for normal and tangential components of the fields, respectively,

$$\left(\mathbf{D}_2 - \mathbf{D}_1\right) \cdot \hat{\mathbf{n}} = 4\pi\eta , \tag{14a}$$
$$\left(\mathbf{B}_2 - \mathbf{B}_1\right) \cdot \hat{\mathbf{n}} = 0 , \tag{14b}$$
$$\left(\mathbf{E}_2 - \mathbf{E}_1\right) \times \hat{\mathbf{n}} = 0 , \tag{14c}$$
$$\left(\mathbf{H}_2 - \mathbf{H}_1\right) \times \hat{\mathbf{n}} = -\frac{4\pi}{c}\mathbf{j} . \tag{14d}$$

Here $\hat{\mathbf{n}}$ is the normal unit vector which is oriented such that it points from medium 1 to medium 2. Thus, when there are neither surface charge nor surface current densities the normal components of the fields $\mathbf{D}$ and $\mathbf{B}$ are continuous, the tangential components of $\mathbf{E}$ and the normal component of $\mathbf{H}$ are continuous.

4.2 This problem is closely related to Exercise 1.16. The charge density follows from the continuity equation. The vector potential follows from (4.30). The electric field and the magnetic induction field are obtained by means of the standard formulae.

4.3 and **4.4** The charge distribution is

$$\varrho(\mathbf{x}) = \frac{q}{2\pi}\Big[\frac{1}{a}\delta(r-a) - \frac{1}{b}\delta(r-b)\Big]\delta(z) \;,$$

where r denotes the radial coordinate in cylinder coordinates. With $\mathbf{v}(\mathbf{x}) = \omega|\mathbf{x}|\hat{\mathbf{e}}_\phi$ the current density reads

$$\mathbf{j}(\mathbf{x}) = \varrho(\mathbf{x})\mathbf{v}(\mathbf{x}) = \frac{q\omega}{2\pi}\big[\delta(r-a) - \delta(r-b)\big]\delta(z)\hat{\mathbf{e}}_\phi \;.$$

The magnetic dipole moment follows from formula (1.120b):

$$\begin{aligned}\boldsymbol{\mu} &= \frac{1}{2c}\iiint \mathrm{d}^3x\; \mathbf{x}\times\mathbf{j}(\mathbf{x}) \\ &= \frac{q\omega}{4\pi c}\int_0^\infty r\mathrm{d}r \int_{-\infty}^{+\infty}\mathrm{d}z\int_0^{2\pi}\mathrm{d}\phi\; \big(r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z\big)\times\hat{\mathbf{e}}_\phi\big[\delta(r-a)-\delta(r-b)\big]\delta(z) \\ &= \frac{q\omega}{2c}\big(a^2-b^2\big)\hat{\mathbf{e}}_r\times\hat{\mathbf{e}}_\phi \\ &= \frac{q\omega}{2c}\big(a^2-b^2\big)\hat{\mathbf{e}}_z \;.\end{aligned}$$

Chapter 5 (selected solutions)

5.1 Inserting $\phi(\mathbf{x})$ into the differential equation (A.4) converts it to an *algebraic* equation

$$\big(\mathbf{k}^2+\kappa^2\big)\widetilde{\phi}(\mathbf{k}) = -g\frac{1}{(2\pi)^{3/2}} \;,$$

which is easy to solve. The original function which is defined over position space, is obtained by the inverse transformation

$$\phi(\mathbf{x}) = -\frac{g}{(2\pi)^3}\iiint \mathrm{d}^3k\;\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2+\kappa^2} \;.$$

This integral can be calculated using spherical coordinates. The result is $-g\mathrm{e}^{-\kappa r}/(4\pi r)$.

5.2 We assume $\kappa \equiv \kappa_i$ to be dependent on the generator. By a suitable choice of the basis of the Lie algebra one can choose $\mathrm{tr}(T_i T_j)$ diagonal, i. e.

$$\mathrm{tr}(T_i T_j) = \kappa_i \delta_{ij} \ .$$

We define then the following totally antisymmetric quantity, with k fixed:

$$\mathcal{E}_{ijk} := \mathrm{tr}\Big(\big[T_i, T_j\big] T_k\Big) = \mathrm{tr}(T_i T_j T_k) - \mathrm{tr}(T_j T_i T_k) \ .$$

This quantity with fixed k can be computed by means of the commutators,

$$\mathcal{E}_{ijk} = \mathrm{i} \sum_n C_{ijn} \, \mathrm{tr}\big(T_n T_k\big) = \mathrm{i}\kappa_k C_{ijk} \ .$$

Exchange the indices j and k to obtain $\mathcal{E}_{ikj} = \mathrm{i}\kappa_j C_{ikj}$, again with fixed j. Both $\mathcal{E}_{ijk}$ and the structure constants C_{mnp} are antisymmetric. Comparison of the last two formulae shows that $\kappa_k = \kappa_j$ as long as the commutator $[T_j, T_k]$ is not equal to zero. Note that in a simple group any two generators are related by nonvanishing commutators. Therefore, all constants κ_i are equal and, hence, are independent of i.

5.3 The adjoint representation of SO(3) is three-dimensional. The gauge fields and the field strengths transform like vectors in $\mathbb{R}^3$. Therefore, the symbolic scalar product in (5.49) is the Euclidean scalar product. A triplet of scalar fields was treated in Example 5.21 so that it is straightforward to write down an SO(3)-gauge invariant Lagrange density (5.51) .

5.4 If the structure group is the direct product of two simple Lie groups, every generator of one group commutes with every generator of the other. Gauge potentials, field strengths, and covariant derivatives of the gauge groups SU(p) and SU(q) are independent of each other. Therefore they can be defined with independent coupling constants q_1 and q_2, respectively. Thus, according to (5.33b) one has for SU(p) and SU(q)

$$A = \mathrm{i}q_1 \sum_{k=1}^{N_p} \mathbf{T}_k \sum_{\mu=0}^{3} A_\mu^{(k)}(x)\mathrm{d}x^\mu \qquad (N_p = p^2 - 1) \ ,$$

$$B = \mathrm{i}q_2 \sum_{l=1}^{N_q} \mathbf{S}_l \sum_{\mu=0}^{3} A_\mu^{(l)}(x)\mathrm{d}x^\mu \qquad (N_q = q^2 - 1) \ .$$

In the Lagrange density (5.49) there are no interaction terms between the gauge bosons of one gauge group and those of the other, as all commutators $[\mathbf{T}_i, \mathbf{S}_k]$ vanish.

5.5 (see the reference quoted in Sect. 5.5 [Constantinescu, Michel, Radicati 1979].)

5.6 In the adjoint representation, using the summation convention for all paired indices, one has

$$\left[\mathbf{U}^{\mathrm{ad}}(\mathbf{T}_i), \mathbf{U}^{\mathrm{ad}}(\mathbf{T}_j)\right]_{ac} = +\mathrm{i}^2\left(C_{ia}^{b} C_{jb}^{c} - C_{ja}^{b} C_{ib}^{c}\right) .$$

Using the Jacobi relation (5.21) and the antisymmetry of the structure constants the expression in round brackets on the right-hand side can be rewritten as follows:

$$C_{ia}^{b} C_{jb}^{c} - C_{ja}^{b} C_{ib}^{c} = C_{ia}^{b} C_{jb}^{c} + C_{aj}^{b} C_{ib}^{c} = -C_{ji}^{k} C_{ka}^{c} = +C_{ij}^{k} C_{ka}^{c} .$$

On the other hand, writing the above commutator more explicitly, one obtains

$$\begin{aligned} &U_{ab}^{\mathrm{ad}}(\mathbf{T}_i) U_{bc}^{\mathrm{ad}}(\mathbf{T}_j) - U_{ab}^{\mathrm{ad}}(\mathbf{T}_j) U_{bc}^{\mathrm{ad}}(\mathbf{T}_i) \\ &= C_{ij}^{k} C_{ka}^{c} = i C_{ij}^{k}\left(-\mathrm{i} C_{ka}^{c}\right) = i C_{ij}^{k} U_{ac}^{\mathrm{ad}}(\mathbf{T}_k) . \end{aligned}$$

This yields what had to be shown.

Chapter 6 (selected solutions)

6.1 The construction of an atlas and the proof that the transition maps are diffeomorphisms are analogous to the case $S_R^2 \subset \mathbb{R}^3$. This example is worked out, e.g., in [ME], Sect. 5.2.3.

6.2 On the basis of the argument in the Example 6.3 in Sect. 6.1.3 one obtains $\Delta\omega/\omega \simeq Hg/c^2$. This yields

$$\frac{\Delta\omega}{\Gamma} = \frac{\Delta\omega}{\omega}\frac{\omega}{\Gamma} = \frac{22.5\,\mathrm{m} \cdot 10\,\mathrm{ms}^{-2}}{(3 \times 10^8\,\mathrm{ms}^{-1})^2} \simeq 0.7\% .$$

6.3 The left side of the supposed identity, when written out, reads

$$\begin{aligned} C := {} & XYZ - XZY + YZX - YXZ + ZXY - ZYX \\ & - YZX + ZYX - ZXY + XZY - XYZ + YXZ . \end{aligned}$$

The twelve terms cancel pairwise so that one finds $C = 0$, indeed.
For the given vector fields one has

$$\begin{aligned} XY &= y\partial_x(x\partial_y) = y\partial_y + yx\partial_x\partial_y , \\ YX &= x\partial_y(y\partial_x) = x\partial_x + xy\partial_y\partial_x , \\ XY - YX &= y\partial_y - x\partial_x . \end{aligned}$$

In this calculation one used the fact that the base fields commute.

6.4 Evaluating $T^{(1)}$ and $T^{(2)}$ on the two vector fields one finds

$$T^{(1)}\left(a^1\partial_1 + a^2\partial_2, b^1\partial_1 + b^2\partial_2\right) = a^1 b^2 \ ,$$
$$T^{(2)}\left(a^1\partial_1 + a^2\partial_2, b^1\partial_1 + b^2\partial_2\right) = a^2 b^1 \ .$$

In general, the answers are indeed different.

6.5 One has $X_{\sigma\,;\mu} = g_{\sigma\rho} X^{\rho}{}_{;\mu}$ where $X^{\rho}{}_{;\mu}$ is taken from the formula (6.57a). The obvious equation

$$\partial_\mu\left(g_{\sigma\rho} g^{\rho\tau}\right) = \partial_\mu\left(\delta^\tau_\sigma\right) = 0 = \partial_\mu\left(g_{\sigma\rho}\right) g^{\rho\tau} + g_{\sigma\rho}\partial_\mu\left(g^{\rho\tau}\right) \ , \tag{15}$$

and the coordinate formula (6.66) are utilized in the following calculation. One computes

$$g_{\sigma\rho} X^{\rho}{}_{;\mu} = g_{\sigma\rho}\left\{\partial_\mu\left(g^{\rho\tau} X_\tau\right) + \Gamma^\rho_{\mu\nu} g^{\nu\tau} X_\tau\right\} \ .$$

In differentiating the first term on the right-hand side by means of the product rule, one finds first the expected term $g_{\sigma\rho} g^{\rho\tau}\partial_\mu X_\tau = \partial_\mu X_\sigma$. The other term as well as the remaining terms must yield the Christoffel symbol $(-\Gamma^\tau_{\mu\sigma})$. This is verified as follows:

$$\begin{aligned}
&g_{\sigma\rho}\left(\partial_\mu g^{\rho\tau}\right) + g_{\sigma\rho}\Gamma^\rho_{\mu\nu} g^{\nu\tau} \\
&= g_{\sigma\rho}\left(\partial_\mu g^{\rho\tau}\right) + \frac{1}{2} g_{\sigma\rho} g^{\rho\alpha}\left[(\partial_\mu g_{\nu\alpha}) + (\partial_\nu g_{\mu\alpha}) - (\partial_\alpha g_{\mu\nu})\right] g^{\nu\tau} \\
&= g_{\sigma\rho}\left(\partial_\mu g^{\rho\tau}\right) - \frac{1}{2} g_{\sigma\rho}(\partial_\mu g^{\rho\alpha})\delta^\tau_\alpha - \frac{1}{2} g_{\sigma\rho}(\partial_\nu g^{\rho\alpha}) g_{\mu\alpha} g^{\nu\tau} - \frac{1}{2}\delta^\alpha_\sigma(\partial_\alpha g_{\mu\nu}) g^{\nu\tau} \ .
\end{aligned}$$

Up to here the relation (15) was used twice in order to shift the derivative to $g^{\rho\alpha}$. If one applies the same trick to the first three terms of the expression obtained in the last step, the first two can be combined so that one finds all in all

$$\begin{aligned}
&-\frac{1}{2}(\partial_\mu g_{\sigma\rho}) g^{\rho\tau} + \frac{1}{2}(\partial_\nu g_{\sigma\rho})\delta^\rho_\mu g^{\nu\tau} - \frac{1}{2}(\partial_\sigma g_{\mu\rho}) g^{\rho\tau} \\
&= -\frac{1}{2} g^{\rho\tau}\left[\partial_\mu g_{\sigma\rho} + \partial_\sigma g_{\mu\rho} - \partial_\rho g_{\sigma\mu}\right] = -\Gamma^\tau_{\mu\sigma} \ .
\end{aligned}$$

In the last but one line the summation index ν of the third term was replaced by ρ and, finally, the formula (6.66) was inserted. This proves the formula for the covariant derivative of a tensor of type (0, 1).

One realizes that the proof is sensibly more transparent if one applies the coordinate-free formula (6.40a) and introduces local coordinates there: Let $V = V^\mu \partial_\mu$ be a vector field, and $\omega = X_\lambda \mathrm{d}x^\lambda$ a one-form. According to (6.40a) one has

$$(D_V \omega)(W) = D_V(\omega(W)) - \omega(D_V(W)) \ .$$

Choosing now $V = \partial_\mu$ and $W = \partial_\sigma$ one obtains

$$\begin{aligned}\left(D_{\partial_\mu}(X_\lambda \mathrm{d}x^\lambda)\right)(\partial_\sigma) &= \partial_\mu X_\sigma - X_\lambda \mathrm{d}x^\lambda \left(\Gamma^\tau_{\mu\sigma}\partial_\tau\right) \\ &= \partial_\mu X_\sigma - \Gamma^\tau_{\mu\sigma} X_\tau \ .\end{aligned}$$

This is the same formula.

6.6 Consider two overlapping charts (U, φ) and (V, ψ) for the spacetime $(M, \mathbf{g})$ and let $x \in U \cap V$ be a point in their intersection. Using local coordinates the same point is represented by

$$\begin{aligned}\varphi(x) &= \left\{u^0(x), u^1(x), u^2(x), u^3(x)\right\} \ , \quad \text{and} \\ \psi(x) &= \left\{v^0(x), v^1(x), v^2(x), v^3(x)\right\}\end{aligned}$$

in two copies of $\mathbb{R}^4$. The transition maps $(\psi \circ \varphi^{-1})$ and $(\varphi \circ \psi^{-1})$ are diffeomorphisms so that $v^\mu(x) = (\psi \circ \varphi^{-1}(u))^\mu$. The local representations of the metric in these charts are related by

$$\mathbf{g} = g_{\mu\nu} \mathrm{d}u^\mu \otimes \mathrm{d}u^\nu = g'_{\alpha\beta} \mathrm{d}v^\alpha \otimes \mathrm{d}v^\beta \ .$$

The differentials fulfill the linear relation $\mathrm{d}v^\alpha = (\partial v^\alpha / \partial u^\mu)\mathrm{d}u^\mu$, so that

$$g_{\mu\nu} = \frac{\partial v^\alpha}{\partial u^\mu} \frac{\partial v^\beta}{\partial u^\nu} g'_{\alpha\beta} \ . \tag{16}$$

The comparison of arbitrary coordinates and of normal coordinates $v^\alpha \equiv z^\alpha$ yields the first of the formulae to be shown, viz.

$$g_{\mu\nu} = \frac{\partial z^\alpha}{\partial u^\mu} \frac{\partial z^\beta}{\partial u^\nu} \eta_{\alpha\beta} \ . \tag{17a}$$

For the inverse of the metric tensor this implies

$$g^{\mu\nu} = \frac{\partial u^\mu}{\partial z^\alpha} \frac{\partial u^\nu}{\partial z^\beta} \eta^{\alpha\beta} \ . \tag{17b}$$

Now one calculates the Christoffel symbols by means of the formula (6.66) and expresses them in terms of normal coordinates as follows:

$$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\tau} \left\{ \frac{\partial g_{\sigma\tau}}{\partial u^\rho} + \frac{\partial g_{\rho\tau}}{\partial u^\sigma} - \frac{\partial g_{\rho\sigma}}{\partial u^\tau} \right\}$$
$$= \frac{1}{2} \left(\frac{\partial u^\mu}{\partial z^{\bar\alpha}} \frac{\partial u^\tau}{\partial z^{\bar\beta}} \eta^{\bar\alpha\bar\beta} \right) \left\{ \frac{\partial}{\partial u^\rho} \left(\frac{\partial z^\alpha}{\partial u^\sigma} \frac{\partial z^\beta}{\partial u^\tau} \eta_{\alpha\beta} \right) \right.$$
$$\left. + \frac{\partial}{\partial u^\sigma} \left(\frac{\partial z^\alpha}{\partial u^\rho} \frac{\partial z^\beta}{\partial u^\tau} \eta_{\alpha\beta} \right) - \frac{\partial}{\partial u^\tau} \left(\frac{\partial z^\alpha}{\partial u^\rho} \frac{\partial z^\beta}{\partial u^\sigma} \eta_{\alpha\beta} \right) \right\} .$$

The tensor is constant and may be taken out of the derivatives. Recalling that all indices except μ, ρ, and σ are summation indices one realizes that the three terms in curly brackets can be combined to

$$\Gamma^{\mu}_{\rho\sigma} = \frac{\partial u^\mu}{\partial z^{\bar\alpha}} \left(\frac{\partial u^\tau}{\partial z^{\bar\beta}} \frac{\partial z^\beta}{\partial u^\tau} \right) \frac{\partial^2 z^\alpha}{\partial u^\rho \partial u^\sigma} \eta^{\bar\alpha\bar\beta} \eta_{\alpha\beta} = \frac{\partial u^\mu}{\partial z^\alpha} \frac{\partial^2 z^\alpha}{\partial u^\rho \partial u^\sigma} , \tag{18}$$

where the factor in parentheses is $\delta^{\beta}_{\bar\beta}$ so that only $\bar\alpha = \alpha$ contributes.

In (18) the z^α are normal coordinates, u^μ are arbitrary coordinates. This formula applies equally well to the transformation $u \mapsto z(u)$ as to the transformation $v \mapsto z(v)$. From this one derives the transformation formula for Christoffel symbols under the diffeomorphism $u \mapsto v$. One finds

$$\Gamma'^{\tau}_{\kappa\lambda} = \frac{\partial v^\tau}{\partial u^\mu} \frac{\partial u^\rho}{\partial v^\kappa} \frac{\partial u^\sigma}{\partial v^\lambda} \Gamma^{\mu}_{\rho\sigma} + \left(\frac{\partial v^\tau}{\partial u^\sigma} \frac{\partial^2 u^\sigma}{\partial v^\kappa \partial v^\lambda} \right) . \tag{19}$$

This affine transformation behaviour shows that the Christoffel symbols cannot be the components of a tensor field. Only antisymmetric combinations of them such as they appear in formula (6.80c) for the Ricci tensor or in formula (6.76) for the Riemann tensor, can be tensors. In these cases the inhomogeneous terms cancel.

6.8 In order to show that the Weyl tensor has the same symmetry properties as the Riemann tensor it is sufficient to verify this in the additional terms which contain the Ricci tensor or the curvature scalar. The property $C^{\nu}{}_{\nu\sigma\tau} = 0$ is easily confirmed by a little calculation. For other contractions use the symmetry properties.

Taking into account the symmetry properties of the Weyl tensor one sees that the property

$$C^{\nu}{}_{\nu\sigma\tau} = 0 \tag{20}$$

together with invariance under $\sigma \leftrightarrow \tau$, in dimension n, yields $n(n+1)/2$ constraints. The number of independent components of the Weyl tensor, using (6.78), is

$$N_{\rm C} = N_{\rm R} - \frac{1}{2}n(n+1) = \frac{1}{12}n^2(n^2-1) - \frac{1}{2}n(n+1)$$
$$= \frac{1}{12}n(n+1)(n+2)(n-3)\ . \tag{21}$$

In dimension 4 it has 10 independent components. In dimension 3 one has $N_{\rm C} = 0$, the Weyl tensor vanishes.

In the case of a conformally flat metric take $\phi = {\rm e}^f$ and calculate the Christoffel symbols by means of (6.66). One finds

$$\Gamma^\sigma_{\mu\nu} = (\partial_\mu f)\delta^\sigma_\nu + (\partial_\nu f)\delta^\sigma_\mu - (\partial_\rho f)\eta^{\sigma\rho}\eta_{\mu\nu}\ .$$

This is used to calculate $\mathbf{R}$, $\mathbf{R}^{(\rm Ricci)}$, and S, and, eventually, the Weyl tensor. One finds, indeed, that it vanishes.

6.9 We choose the second alternative of verifying the relation for functions and one-forms.

For a smooth function f one has

$$L_X f = X\,f, \quad i_X f = 0\,, \quad (i_X \circ {\rm d})\,f = {\rm d}f(X) = Xf\ . \tag{22}$$

With $\omega = \omega_\mu {\rm d}x^\mu$ a smooth one-form, one has (cf. (6.44c))

$$L_X\omega = \left\{X^\mu(\partial_\mu\omega_\sigma) + \omega_\mu(\partial_\sigma X^\mu)\right\}{\rm d}x^\sigma\,, \tag{23}$$
$${\rm d}\omega = (\partial_\tau\omega_\mu){\rm d}x^\tau \wedge {\rm d}x^\mu\,.$$

Using these formulae one calculates the combined action $(i_X \circ {\rm d})$ on the one-form ω:

$$(i_X \circ {\rm d})\,\omega = {\rm d}\omega\,(X,\cdot) = (\partial_\tau\omega_\mu)\Big\{X^\tau{\rm d}x^\mu - X^\mu{\rm d}x^\tau\Big\}\,, \tag{24a}$$

and, likewise, the action of $({\rm d} \circ i_X)$ on ω. With $i_X\omega = \omega(X) = \omega_\mu X^\mu$ one obtains

$${\rm d}\,(i_X\omega) = \Big\{(\partial_\tau\omega_\mu)X^\mu + \omega_\mu(\partial_\tau X^\mu)\Big\}{\rm d}x^\tau\,. \tag{24b}$$

Adding (24a) and (24b) the second term of the former cancels against the first term of the latter and one obtains upon comparison with (23)

$$\big({\rm d} \circ i_X + i_X \circ {\rm d}\big)\omega = L_X\omega\,.$$

Having proved the relation for functions and for one-forms, it follows for all smooth tensor fields.

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Subject Index

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https://doi.org/10.1007/978-3-662-55579-8

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F. Scheck, *Classical Field Theory*, Graduate Texts in Physics,
https://doi.org/10.1007/978-3-662-55579-8

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