Appendix
Stresses and Strains in Non-uniformly Heated Rods, Plates, and Trusses

The Appendix gives examples of analytical solutions of some thermal strength problems for basic elements of mechanical structures (rods of a different kind and plates and trusses), which can be useful for demonstrating some sides of non-uniformly heated structure behavior and testing of numerical programs. There is a very vast literature on analytical solutions of thermal strength problems (see references to Monograph Chapters). Only the most necessary additional information is stated below.

A.1 Non-uniformly Heated Rods

A.1.1 Prismatic Rods

A.1.1.1 Tension and Bending

A prismatic rod of constant cross-section A is stretched along the axis z by longitudinal force F_z and bent in two planes XZ and YZ by moments M_ξ and M_η (Fig. A.1a). The principal axes of rod cross sections have identical directions. A temperature field $T(\xi, \eta)$, arbitrary over the section and constant along length of the rod, is set. Fields of the elasticity modulus $E(\xi, \eta)$ and temperature expansion $\varepsilon_T(\xi, \eta)$ corresponding to the temperature field are also known.

According to the flat cross-section hypothesis, a total longitudinal strain of a rod along its axis z is presented by the equation

$$\varepsilon(\xi, \eta) = \varepsilon_0 + \chi_\xi \eta - \chi_\eta \xi, \tag{A.1}$$

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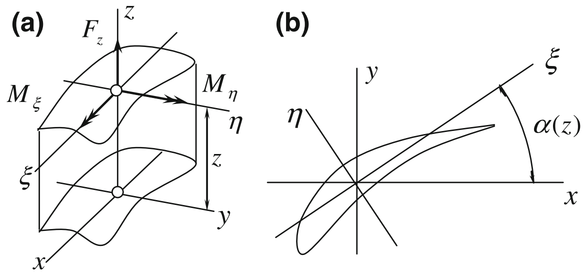

Fig. A.1 Tension and bending of a rod; **a** loads, **b** section of a pre-twisted rod

where ε_0 is a strain at the coordinates ξ, η origin, and χ_ξ and χ_η are components of the axis elastic curvature in the planes ηz and ξz.

Equation (A.1), as well as other kinematic relations

$$\varepsilon_0 = u_z', \chi_\xi = -u_\eta'', \chi_\eta = u_\xi'', \tag{A.2}$$

do not depend on material properties.

Here, u_ξ, u_η and u_z are displacements of the point of coordinates ξ, η origin for the current section along the corresponding axes, and $f' = df/dz$ is a derivative of any function f in coordinate z.

Total strain is equal to the sum of the elastic, temperature, and additional (inelastic) parts. In case of uniaxial stress state it is

$$\varepsilon(\xi, \eta) = \frac{\sigma}{E} + \varepsilon_T + \varepsilon^\circ. \tag{A.3}$$

Having input stress from Eq. (A.3) into three equations of the rod equilibrium

$$\int_A \sigma dA = F_z, \quad \int_A \sigma \xi dA = -M_\eta, \quad \int_A \sigma \eta dA = M_\xi,$$

and having defined the coordinates origin and direction of axes ξ, η from conditions

$$\int_A E\eta dA = 0, \int_A E\xi dA = 0, \int_A E\xi\eta dA = 0,$$

we obtain an expression for longitudinal normal stress at any point of a non-uniformly heated cross-section under tension and bending

$$\sigma(\xi, \eta) = \bar{E}\left(\frac{F + \tilde{F}}{A} + \frac{M_\xi + \tilde{M}_\xi}{J_\xi^*}\eta - \frac{M_\eta + \tilde{M}_\eta}{J_\eta^*}\xi - E_m\tilde{\varepsilon}\right), \tag{A.4a}$$

where elastic-geometrical characteristics are defined by integration of the function $E(\xi, \eta)$ over the entire area of the cross section:

$$E_m = \frac{1}{A}\int_A E dA, \quad J_\xi^* = \int_A \bar{E}\eta^2 dA, \quad J_\eta^* = \int_A \bar{E}\xi^2 dA, \quad \bar{E} = E/E_m,$$
$$\tilde{F} = \int_A E\tilde{\varepsilon} dA, \quad \tilde{M}_\xi = \int_A E\tilde{\varepsilon}\eta dA, \quad \tilde{M}_\eta = -\int_A E\tilde{\varepsilon}\xi dA. \tag{A.4b}$$

For thermal expansion it is accepted that $\tilde{\varepsilon} = \varepsilon_T$, and for additional deformations of any origin is $\tilde{\varepsilon} = \varepsilon^\circ$.

At constant temperature there is $\bar{E} = 1$, the origin of the coordinates ξ, η coincides with the center of cross-section mass, and parameters J_ξ^* и J_η^*—with the principal moments of inertia.

For statically determinable mechanical systems the force F_z and the bending moments M_ξ and M_η induced by external loads can be defined for any cross section due to equilibrium conditions

$$F_z' + q_z = 0, \quad M_\xi'' + q_\eta = 0, \quad M_\eta'' - q_\xi = 0\ , \tag{A.5}$$

where q_ξ, q_η, q_z are outer loads distributed on a rod length in corresponding directions.

Generally, Eqs. (A.1)–(A.3) yield

$$u_z' = \frac{F_z + \tilde{F}_z}{E_m A}, \quad u_\xi'' = \frac{M_\eta + \tilde{M}_\eta}{E_m J_\xi^*}, \quad u_\eta'' = -\frac{M_\xi + \tilde{M}_\xi}{E_m J_\eta^*}. \tag{A.6}$$

Statically, indeterminable problems can be solved separately for each of the directions by integrating Eqs. (A.5) and (A.6) under the set boundary conditions.

A.1.1.2 Torsion

The relative cross-section rotation angle θ of a longitudinally non-uniformly heated rod induced by a torque M_z is defined as

$$\theta = \frac{d\alpha}{dz} = \frac{M_z + \tilde{M}_z}{G_m J_z^*}, \tag{A.7a}$$

where J_z^* is a reduced rigidity in torsion, and $\tilde{M}_z$ is a torque moment from additional deformations.

For a circular cylinder with an axisymmetric temperature field there is

$$J_z^* = \int_A \bar{G}(r)r^2 dA, \tilde{M}_z = \int_A \bar{G}(r)\tilde{\theta}(r)r^2 dA, \bar{G} = G/G_m \approx \bar{E}. \quad \text{(A.7b)}$$

In this case thermal expansion does not cause shear strains and rotation angles of the cross sections ($\tilde{\theta} = \theta_T = 0$). But non-uniform heating of a rod of noncircular cross sections leads, under torsion, to their deplanation (warping) with point displacement along the axis z, which can lead to torsion deformation.

A.1.2 Pre-twisted Rods

For gas-dynamic reasons, gas and steam turbines blades are produced with an installation angle $\alpha(z)$ varying along the length of the blades profile (see above Fig. A.1b), i.e., are pre-twisted. Practical interest is represented by the rods of elongated profile, in which the "length" b of a cross section along one of the axis ξ essentially exceeds "thickness" δ along the other axis η (Fig. A.2a).

Let us designate initially an angle of rotation of the cross-section principal axes related to unit of the rod length (relative pre-twist) through $\theta_{in} = d\alpha(z)/dz$ and enter a parameter $\gamma = \theta_{in}b^2/\delta$ that can largely vary. At distance $R = (\xi^2 + \eta^2)^{0.5}$ from the rod axis, an inclination of the longitudinal "fiber" on a small angle $\beta_{in}(R) \approx \theta_{in}R \ll 1$ corresponds to the pre-twist θ_{in} (Fig. A.2b). If the twist increases in size θ due to rod deformation, the inclination increases in $\beta(R) \approx \theta R$ which causes the fiber lengthening in $\varepsilon \approx \beta_{in}\beta \approx \theta_{in}\theta R^2$ (as shown by bold site Δ) and appearence in Eq. (A.1) of a corresponding additional member.

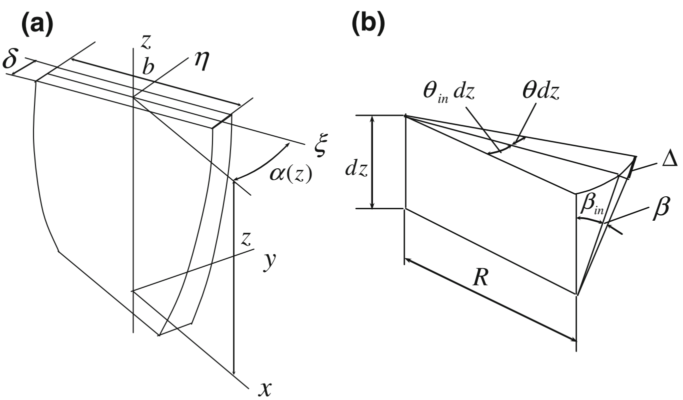

Fig. A.2 Scheme of a pre-twisted rod—(**a**), and elongation of an inclined fiber—(**b**)

For the elongated sections there are $R \approx \xi$, $R_{\max} \approx 0.5b$, and $\gamma = 2(\beta_{in})_{\max}(b/\delta)$, where the angle$(\beta_{in})_{\max} = 0.5\theta_{in}b$ refers to the inclined fiber, most removed from the rod axis. Analysis shows that the influence of pre-twist on the stress-strain state and frequency characteristics of rods of the elongated profile can be essential, despite angles β_{in} and β being small.

Equation (A.1), obtained by kinematic relations, is applicable at any temperature fields, additional deformations, and material properties. From Eq. (A.3) with an additional member $E(\xi,\eta)\theta\theta_{in}R_*^2$ it follows

$$\sigma(\xi,\eta) = E(\xi,\eta)\left[\varepsilon_0 + \chi_\xi\eta - \chi_\eta\xi + \theta\theta_{in}R^2 - \tilde{\varepsilon}_T(\xi,\eta)\right], \tag{A.7c}$$

and an additional member $E(\xi,\eta)\,\theta\theta_{in}R_*^2$ occurs in Eq. (A.4a, b) where

$$R_*^2 = R^2 - \frac{j_p^*}{A} - \frac{J_{p\xi}^*}{J_\xi^*}\eta - \frac{j_{p\eta}^*}{J_\eta^*}\xi, \tag{A.8}$$

$J_p^* = \int_A \bar{E}R^2 dA$, $J_{p\xi}^* = \int\limits_A \bar{E}R^2\eta\, dA$, $J_{p\eta}^* = \int\limits_A \bar{E}R^2\xi\, dA$.

The generalized coordinate R_*^2 can accept both positive and negative values. In the pre-twisted rod, a torque moment M_z is counterbalanced by the sum of the moment of tangential stresses M_τ, and the moment M_σ of projections of normal stresses in inclined fibers onto a cross-section plane

$$M_z = M_\tau + M_\sigma = G_m J_t^*\theta + \theta_{in}\int\limits_A \sigma R^2 dA, \tag{A.9}$$

where the torsion rigidity for a rod of elongated cross-section is

$$J_t^* \approx \frac{1}{3}\int\limits_b \bar{G}(\xi)\delta^3(\xi)d\xi, \quad \bar{G} \approx \bar{E}.$$

Having substituted the expression for normal stress σ from Eq. (A.7c) in Eq. (A.9), we find a relative angle of elastic twist

$$\theta = \frac{1}{G_m J_t^{**}}\left[M_z - \theta_{in}\left(\frac{F+\tilde{F}}{A}J_p^* + \frac{M_\xi + \tilde{M}_\xi}{J_\xi^*}J_{p\xi}^* - \frac{M_\eta + \tilde{M}_\eta}{J_\eta^*}J_{p\eta}^* - \tilde{B}^*\right)\right], \tag{A.10}$$

where $J_t^{**} = J_t^* + \theta_{in}^2\frac{E_m}{G_m}J_r^*$, at $J_r^* = J_{r0}^* - \frac{(J_p^*)^2}{A} - \frac{(J_{p\xi}^*)^2}{J_\xi^*} - \frac{(J_{p\eta}^*)^2}{J_\eta^*}$, $J_{r0}^* = \int\limits_A \bar{E}R^4 dA$,
$\tilde{B}^* = \int\limits_A E\tilde{\varepsilon}R^2 dA$.

Tangential stresses of torsion at a profile contour are defined approximately as $\tau_{\xi z} = G(\xi)\delta(\xi)\theta$.

The basic features of the pre-twisted rods are the coupling of longitudinal, torsion and bending deformations, increase of torsion, and reduction of longitudinal rigidities with pre-twist increasing, and nonlinear distribution of normal stresses over the cross section.

Temperature expansion of a material causes additional effects.

At uniform heating of a rod, when displacements are restricted only in longitudinal direction, normal compressing stress at the rod axis will stay the same, as in prismatic one, while the stresses at the endpoints of the cross sections can become tensile.

Additional temperature loads, defined by Eqs. (A.4a, b), (A.9), and (A.10), arise at uneven heating.

A.2 Non-uniformly Heated Circular Plates

A.2.1 Axisymmetric Tension/Compression of Plates

A.2.1.1 Thin Plates

A thin plate is usually considered as being in generalized flat axis-symmetric stress state with lateral stress $\sigma_z \approx 0$. Then physical relations presented in polar coordinates r—in radial and φ—in circumference directions look like

$$\varepsilon_r = \frac{\sigma_r - \mu\sigma_\varphi}{E} + \tilde{\varepsilon}_r, \varepsilon_\varphi = \frac{\sigma_\varphi - \mu\sigma_r}{E} + \tilde{\varepsilon}_\varphi, \varepsilon_z = -\mu\frac{\sigma_r + \sigma_\varphi}{E} + \tilde{\varepsilon}_z, \quad \text{(A.11a)}$$

where σ_r, ε_r are radial, and σ_φ, ε_φ—circumferential stresses and strains.

Additional strains $\tilde{\varepsilon}_r, \tilde{\varepsilon}_\varphi, \tilde{\varepsilon}_z$ can include temperature expansion ε_T and inelastic strains.

Invert correlations are

$$\sigma_r = E\frac{\varepsilon_r + \mu\varepsilon_\varphi - (\tilde{\varepsilon}_r + \mu\tilde{\varepsilon}_\varphi)}{1-\mu^2}; \sigma_\varphi = E\frac{\varepsilon_\varphi + \mu\varepsilon_r - (\tilde{\varepsilon}_\varphi + \mu\tilde{\varepsilon}_r)}{1-\mu^2}. \quad \text{(A.11b)}$$

As well as the above, for temperature expansion we accept $\tilde{\varepsilon}_r = \tilde{\varepsilon}_\varphi = \varepsilon_T$, for inelastic strains—$\tilde{\varepsilon}_r = \varepsilon_r^\circ$, $\tilde{\varepsilon}_\varphi = \varepsilon_\varphi^\circ$. Generally, $E = E(r)$ at $\mu \approx \text{const}$.

Total strains are connected with radial displacement $u(r)$ by geometrical relations

$$\varepsilon_r = \frac{du}{dr}, \ \varepsilon_\varphi = \frac{u}{r}. \quad \text{(A.12)}$$

The equilibrium equation in a radial direction looks like

$$\frac{d(\sigma_r r\delta)}{dr} - \sigma_\varphi \delta + q_r r = 0, \tag{A.13}$$

where q_r is a distributed radial load referred to a plate thickness δ.

For an elastic plate of constant thickness with a constant elasticity modulus, solutions can be obtained in the closed kind. In this case, it follows from Eqs. (A.11a, b)–(A.13) a differential equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = \frac{d}{dr}\left[\frac{1}{r}\frac{d(ru)}{dr}\right] = -\frac{1-\mu^2}{E\delta}q_r + (1+\mu)\frac{d\varepsilon_T}{dr}, \tag{A.14a}$$

with general solution at $q_r = 0$

$$u(r) = \frac{1+\mu}{r}\int_{r_0}^{r} r\varepsilon_T(r)dr + Ar + \frac{B}{r}. \tag{A.14b}$$

The constants A and B must be defined by the set boundary conditions at internal and external plate radii $\sigma_r(r_0) = \sigma_{r0}$ and $\sigma_r(R) = \sigma_R$.

A.2.1.2 Plate with Central Hole

For a plate with a central hole, temperature stresses $\sigma_r(r)$ and $\sigma_\varphi(r)$ are defined by equations

$$\begin{aligned} \sigma_r(r) &= \frac{E}{r^2}\left[\frac{r^2 - r_0^2}{R^2 - r_0^2}\int_{r_0}^{R} r\varepsilon_T(r)dr - \int_{r_0}^{r} r\varepsilon_T(r)dr\right], \\ \sigma_\varphi(r) &= \frac{E}{r^2}\left[\frac{r^2 + r_0^2}{R^2 - r_0^2}\int_{r_0}^{R} r\varepsilon_T(r)dr + \int_{r_0}^{r} r\varepsilon_T(r)dr - \varepsilon_T r^2\right] \end{aligned} \tag{A.15a}$$

and temperature displacements are

$$u(r) = \frac{1}{r}\left[(1+\mu)\int_{r_0}^{r} r\varepsilon_T(r)dr + \frac{(1-\mu)r^2 + (1+\mu)r_0^2}{R^2 - r_0^2}\int_{r_0}^{R} r\varepsilon_T(r)dr\right]. \tag{A.15b}$$

Formulas for a compact plate are resulted at $r_0 = 0$. In the plate center, the limiting transition $r_0 \to 0$ leads to $u(0) = 0$, and

$$\sigma_r(0) = \sigma_\varphi(0) = E\left[\frac{1}{R^2}\int_{r_0}^{R} r\varepsilon_T(r)dr - 0.5\varepsilon_T(0)\right]. \quad \text{(A.16)}$$

A.2.1.3 Thermal Stresses in Plates

It follows from analysis of Eqs. (A.15a, b–A.16):

- At a *uniform temperature field* and $\varepsilon_T = \text{const}$ the displacements of a plate with free edges change linearly, i.e., $u(r) = \varepsilon_T r$, and thermal stresses do not arise. But if at least one of the plate edges is fixed the thermal stresses appear. So, as the outer edge of the plate is fixed, i.e., $u(R) = 0$, radial stresses at this radius are

$$\sigma_r(R) = -E\varepsilon_T \frac{R^2 - r_0^2}{(1-\mu)R^2 + (1+\mu)r_0^2}.$$

In this case the strains in a compact plate are zero at all radii and compressing thermal stresses

$$\sigma_r = \sigma_\varphi = -\frac{E\varepsilon_T}{1-\mu}$$

occur over the entire plate.

- *At a non-uniform temperature field* the stresses arise even if the edges of a plate are free.

Let the temperature field change step-wisely, so that the internal part of a plate I from $r = 0$ to $r = r_*$ is regularly heated upon $\Delta T > 0$ and peripheral part II—from $r = r_*$ to $r = b$—is regularly cooled upon the same size $\Delta T < 0$, so as $\varepsilon_T^I = \varepsilon_T > 0$ and $\varepsilon_T^{II} = -\varepsilon_T$.

Under free expansion of each of the parts there would be no stresses in them, and their radial displacements at the common radius $r = r_*$ would be identical in size and different in sign, $u^I(r_*) = \varepsilon_T r_* = -u^{II}(r_*)$, i.e., the plane parts would "crawl" against each other over a quantity $\Delta u = u^I(r_*) - u^{II}(r_*) = 2\varepsilon_T r_*$. At joint plane parts displacements, a radial stress $\sigma_r(r_*) = \sigma_{r*}$ appears between parts, which inhibits their crawling. It follows from previous equations that

$$\sigma_{r*} = -E\varepsilon_T\left[1 - (r_*/R)^2\right].$$

If a temperature of the internal plane part is higher than the peripheral one, compressing radial temperature stresses cover the entire plate (including colder peripheral part). At the border of the parts at temperature jump $2\Delta T$, circumferential stresses undergo discontinuity and have different signs within hot and cold parts. Under smooth radial temperature distribution there will be no circumference stress discontinuity.

A.2.2 Axisymmetric Bending of Plates

At axisymmetric bending of a thin circular plate stress state in all points is biaxial, normal stresses in the radial σ_r and circumferential σ_φ directions alter along the plate thickness δ but Eqs. (A.11a) and (A.11b) remain valid. Temperature may change along the plate radius and thickness, but thickness variation of the elasticity modulus can be neglected.

Radial displacements along a plate thickness $u(r,z)$ are defined on the basis of the condition of "a rigid normal" (Kirchhoff's hypothesis)

$$u(r,z) = u(r) - z\psi(r), \tag{A.17a}$$

where $u(r)$ is a displacement of a point at a median plane of the plate, z is a distance from it to a current point ($-0.5\delta \le z \le 0.5\delta$), $\psi(r)$ is a rotation angle of a normal to this plane at radius r.

For small displacements

$$\psi(r) = w'(r), \tag{A.17b}$$

where $w(r)$ is a plate deflection at a current radius.

At axisymmetric deformations Eq. (A.12), fair for any points, become

$$\varepsilon_r(r,z) = u'(r,z), \varepsilon_\varphi(r,z) = u(r,z)/r,$$

so

$$\varepsilon_r(r,z) = \varepsilon_r(r) - z\psi'(r), \varepsilon_\varphi(r,z) = \varepsilon_\varphi(r) - z\psi(r)/r, \tag{A.18}$$

at $\varepsilon_r(r) = u'(r)$ and $\varepsilon_\varphi(r) = u(r)/r$.

Using a linear statement, the problems of a plate bending and tension/compression can be considered irrespectively, accepting for the bend $u(r) = 0$. At uniaxial stress state, taking into account Eq. (A.11b) for $\tilde{\varepsilon} = \varepsilon_T$, the stresses will be defined as

$$\begin{aligned} \sigma_r &= -\frac{E}{1-\mu^2} z(\psi' + \mu\frac{\psi}{r}) - \frac{E}{1-\mu}\varepsilon_T z, \\ \sigma_\varphi &= -\frac{E}{1-\mu^2} z(\frac{\psi}{r} + \mu\psi') - \frac{E}{1-\mu}\varepsilon_T z, \end{aligned} \tag{A.19}$$

and "linear" bending moments at a radius r, will be

$$
\begin{aligned}
M_r &= -\int\limits_{-0.5\delta}^{0.5\delta} \sigma_r z dz = D(\psi' + \mu\psi/r) + M_T, \\
M_\varphi &= -\int\limits_{-0.5\delta}^{0.5\delta} \sigma_\varphi z dz = D(\psi/r + \mu\psi') + M_T
\end{aligned}
\tag{A.20a}
$$

where is $D = E\delta^3/12(1-\mu^2)$ and a "temperature" moment is

$$
M_T = \frac{E}{1-\mu} \int\limits_{-0.5\delta}^{0.5\delta} \varepsilon_T z dz. \tag{A.20b}
$$

Thermal expansion of a material leads to a plate bend only when the temperature changes over its thickness. In this case the moment $M_T \neq 0$.

The equilibrium equation of a plate element under the action of the bending moments in absence of lateral forces is

$$
\frac{d(rM_r)}{dr} - M_\varphi = 0. \tag{A.21}
$$

Having input Eq. (A.20a) into Eq. (A.21), we obtain the differential equation of plate bending presented in angles of rotation ψ

$$
\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} - \frac{\psi}{r^2} = \frac{1}{D}\frac{dM_T}{dr}, \tag{A.22}
$$

which formally coincides with Eq. (A.14a, b) for a plate tension/compression at $q_r = 0$.

Therefore the resolving equations of a plate deformations presented in angles of rotation and also formulas following from them, will have an identical appearance for both problems. Plate deflections $w(r)$ will be defined by integrating Eq. (A.17b).

The general calculation methods for bend of circular plates of variable thickness and various temperature distributions on radius and on thickness, including temperature-dependent material characteristics, were considered in a number of works, cited in Chaps. 2 and 3.

A.3 Non-uniformly Heated Truss

A problem of analysis of thermal stresses in a non-uniformly heated truss is presented in matrix term, generally used in FE codes.

A.3.1 Thermoelastic Deformation

A.3.1.1 Basic Set of Matrix Equations

Let us consider a truss consisting of two rods of a constant cross section A with a single common node 1 (Fig. A.3).

A force F is applied to this node on an angle β to a vertical axis, and a spring with rigidity k_{str} limits the node displacement along y direction. Nodes 2 and 3 at the other ends of the rods are fixed. Deformation of a cold rod I is assumed be elastic ($\varepsilon_I^p \equiv 0$), its length is equal to$l/\cos\alpha_I$, the elasticity modulus is E_I = const. The rod II of the length $l/\cos\alpha_{II}$ is heated up to temperature $T_{II} > T_I$, to which there corresponds the elasticity modulus $E_{II}(T) < E_I$ and temperature expansion $\varepsilon_T(T)$. Displacements are assumed be small, not affecting the system configuration.

Element strains ε_I and ε_{II} are expressed through the components u_x and u_y of the node 1 displacement by relations

$$\begin{aligned}\varepsilon_I &= \frac{\cos\ \alpha_I}{l}(u_x \cos\alpha_I + u_y \sin\alpha_I),\\ \varepsilon_{II} &= \frac{\cos\ \alpha_{II}}{l}(u_x \cos\alpha_{II} - u_y \sin\alpha_{II}).\end{aligned} \tag{A.23}$$

A constitution equation for a non-isothermal problem of a rod of a material with any characteristics is

$$\varepsilon(\sigma, T) = \frac{\sigma}{E(T)} + \varepsilon_T(T) + \varepsilon^{\circ}(\sigma, T), \tag{A.24a}$$

and vice versa

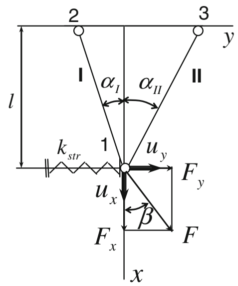

Fig. A.3 Scheme of a non-uniformly heated two-rod truss

$$\sigma = E(T)\,[\varepsilon(\sigma, T) - \varepsilon_T(T) - \varepsilon^{\circ}(\sigma, T)]. \tag{A.24b}$$

Here ε° is a nonelastic part of the strain.

Equations (A.23) and (A.24a, b) lead to

$$\begin{aligned} \sigma_I &= E_I \frac{\cos\alpha_I}{l}(u_x \cos\alpha_I + u_y \sin\alpha_I), \\ \sigma_{II} &= E_{II}\left[\frac{\cos\alpha_{II}}{l}(u_x \cos\alpha_{II} - u_y \sin\alpha_{II}) - \varepsilon_T - \varepsilon^{\circ}\right], \end{aligned} \tag{A.25a}$$

and external loads affecting the rods are

$$F_I = A\sigma_I,\;\; F_{II} = A\sigma_{II}. \tag{A.25b}$$

The equilibrium conditions of node 1

$$\begin{aligned} &F_x - F_I \cos\alpha_I - F_{II} \cos\alpha_{II} = 0, \\ &F_y - F_I \sin\alpha_I + F_{II} \sin\alpha_{II} - k_{str} u_y = 0 \end{aligned} \tag{A.26}$$

yield, accounting Eq. (A.25a, b),

$$\begin{aligned} K_{11}u_x + K_{12}u_y &= F_x + F_x^T + F_x^{\circ}, \\ K_{21}u_x + K_{22}u_y &= F_y + F_y^T + F_y^{\circ} \end{aligned} \tag{A.27}$$

where coefficients of the rigidity matrix $[K]$ are

$$\begin{aligned} K_{11} &= \frac{A}{l}(E_I \cos^3\alpha_I + E_{II} \cos^3\alpha_{II}), \\ K_{22} &= \frac{A}{l}(E_I \cos\alpha_I \sin^2\alpha_I + E_{II} \cos\alpha_{II} \sin^2\alpha_{II}) + k_{str}, \\ K_{12} = K_{21} &= \frac{A}{l}(E_I \cos^2\alpha_I \sin\alpha_I - E_{II} \cos^2\alpha_{II} \sin\alpha_{II}), \end{aligned} \tag{A.28a}$$

and thermal and nonelastic components of the right parts are

$$\begin{aligned} F_x^T &= E_{II} A \varepsilon_T \cos\alpha_{II}, \quad & F_y^T &= -E_{II} A \varepsilon_T \sin\alpha_{II}, \\ F_x^{\circ} &= E_{II} A \varepsilon^{\circ} \cos\alpha_{II}, \quad & F_y^{\circ} &= -E_{II} A \varepsilon^{\circ} \sin\alpha_{II}. \end{aligned} \tag{A.28b}$$

A matrix form of set of Eq. (A.27) is

$$[K]\{u\} = \{F\} + \{F^T\} + \{F^{\circ}\}. \tag{A.29}$$

After the displacements u_x and u_y are found from the solution of the equation set (A.27), all other variables of a problem in the elastic statement can be calculated.

For variable loading and heating regimes a rigidity matrix $[K(T)]$ changes against temperature according to change of the elasticity modulus $E(T)$.

A.3.1.2 Conditions of Thermal Stresses Arising

It follows from Eq. (A.27) that balance of node 1 in absence of the external loads ($F_x = F_y = 0$) and the spring ($k_{str} = 0$) is possible only for $\sigma_I = \sigma_{II} = 0$. In this case rod II may freely elongate and rotate on heating. Rod I turns together with rod II, so that their displacements at node 1 are identical but the length of rod I remains invariable. Thermal stresses in the system do not arise. In the presence of the spring ($k_{str} \neq 0$) the loads at node 1 differ from zero, and thermal stresses arise in both rods also at $F_x = F_y = 0$.

A.3.1.3 Variational Equations

In the FEM theory, matrix equations of type (A.29) are usually deduced from variational equation

$$\partial(W_{inn} - W_{out}) = 0,$$

where work variations of internal and external forces are

$$\partial W_{inn} = \int_V \{\sigma^T\}\{\partial\varepsilon\}dV,$$

$$\delta W_{out} = \int_V \{p^T\}\{\partial u\}dV + \int_S \{F^T\}\{\partial u\}dS.$$

Having expressed the strains ε_I, ε_{II} and the stresses σ_I, σ_{II} in elements of the considered system through the displacements u_x,u_y, and their variations, we come at $\partial u_x \neq 0$ and $\partial u_y \neq 0$ to the set of Eqs. (A.27)–(A.29).

A.3.2 Inelastic Deformations

A procedure of finite element modeling of any problem begins with elastic calculation. For the given truss system, this corresponds to $\varepsilon^\circ = 0$, $F_x^\circ = F_y^\circ = 0$. The "elastic" values of stresses $\sigma = \sigma^e$ and total strains $\varepsilon = \varepsilon^e$ are used for calculation of the first approach of nonelastic strains according to the corresponding theory of plasticity or creep. Values of additional deformations ε° are then refined by successive approximations.

Temperature field of a body, external forces, and loads from thermal expansion thus do not change.

For step-by-step procedure, Eq. (A.24b) is presented in increment terms

$$\Delta\sigma = E(T)[\Delta\varepsilon - \Delta\varepsilon_T(T) - \Delta\varepsilon^\circ(\sigma, T, \varepsilon, \ldots)], \tag{A.30}$$

where increment of the additional deformations $\Delta\varepsilon^\circ$ may depend on current values of stress, strain, temperature, and also parameters reflecting influence of the previous steps (see Chaps. 5 and 8).