Answers to problems

Chapter II

Exercise 1. Transformation by systematic obliteration

The process $P_k(a)$ has independent random intervals. Each interval obeys $\gamma(k\,d)$.

Exercise 2. Process of the "midway points"

We put

$$M_n M_{n+1} = M_n$$

and

$$A_n A_{n+1} = V_n,$$

then

$$\begin{cases} v_n = \frac{1}{2}(M_n + M_{n+1}), \\ v_{n+1} = \frac{1}{2}(M_{n+1} + M_{n+2}). \end{cases}$$

These relations show that v_n obeys $\gamma(2, 2d)$.

Two consecutive intervals are correlated.

The correlation coefficient between v_n and v_{n+1} is

$$\varrho_{n,\,n+1} = -\tfrac{1}{2}.$$

Two intervals such as v_n and v_{n+k} (with $k \geqq 2$) are mutually independent.

Exercise 3. Transformation by random blotting out

The process obtained is a uniform Poisson one. Its density is $p\,d$.

Exercise 4. Transformation by random displacings

4.1. $A_1 B_1 = U + V' + V''$ where U; V' and V'' are independent random variables.

Let $\varphi_{(z)}$ be the characteristic function of V, then $\overline{\varphi}_{(z)}$ is the characteristic function of $(-V)$ and the c.f. of $\overline{A_1 B_1}$ is

$$\Phi_{(z)} = \frac{\varphi_{(z)}\overline{\varphi}_{(z)}}{1 - i z} = \frac{|\varphi_{(z)}|^2}{1 - i z}.$$

$\overline{A_1 B_1}$ obeys the γ_1 distribution if and only if V is almost certainly a constant.

If V obeys the γ_1 law, we have:

$$\Phi_{(z)} = [(1 - i z)(1 + z^2)]^{-1}.$$

4.2. (*a*) Let dP be the probability that an event (A) occurs on $(t, t + dt)$

$$dP = \int_{v=-\infty}^{+\infty} dt\, g(v)\, dv = dt.$$

Moreover the numbers of events (A) occurring on two disjoint intervals are mutually independent; consequently $\{A_n\}$ is a uniform Poisson process. Its density is 1.

4.2. (*b*) Suppose now that V depends on $s = OM$, then

$$dP = \int_{v=0}^{\infty} dt\, g(s - v, v)\, dv,$$

$$= a(t)\, dt \quad \text{where} \quad a(t) = \int g(t - v, v)\, dv$$

and $\{A_n\}$ is a non-uniform Poisson process.

Exercise 5

Put

$$t_n = O M_n, \quad t_{n+1} = O M_{n+1}, \quad t = OA,$$

is known, but t_n and t_{n+1} are random variables.

5.1. $(t - t_n)$ and $(t_{n+1} - t)$ are mutually independent, each obeys γ_1. So $(t_{n+1} - t_n) = U$ obeys the γ_2-distribution $(e^{-cu}\, c\, u\, c\, du)$.

5.2. If $t > 0$ and $t_n > 0$, then $V = t - t_n$ obeys the following truncated γ_1 distribution

$$\frac{e^{-cv}\, c\, dv}{\int_{v=0}^{t} e^{-cv}\, c\, dv} \quad \text{where} \quad 0 < v < t.$$

Exercise 6

Let c be the parameter of a cluster process and $g(z)$ the characteristic function of the number of events in one cluster.

We have (see 22.1)

$$G_{(z)}^{(t)} = \exp\{c\, t(g(z) - 1)\}$$

and we must have

$$G_{(z)}^{(t)} = \exp\{m(z-1)\}.$$

So

$$g(z) = 1 - \frac{m}{ct} + \frac{m}{ct} z + \cdots,$$

$$1 - \frac{m}{ct} = \text{Prob}\{K = 0\},$$

$$\frac{m}{ct} = \text{Prob}\{K = 1\}.$$

These probabilities must not depend on t.

If

$$K \geqq 1 \Rightarrow m = ct \quad \text{and} \quad g(z) = z,$$

$$\text{Prob}\{K = 1\} = 1.$$

The process of events is a Poisson one. Its density is c. If $K \geqq 0 \Rightarrow m = a c t$

$$\text{Prob}\{K = 0\} = 1 - a, \qquad \text{Prob}\{K = 1\} = a.$$

The process of events is a Poisson one. Its density is $a c$.

Exercise 7

The random function $X(t)$ is

$$X(t) = (-1)^K$$

where K is the random number of events occurring in $[0, t]$.

We have

$$p_k = \text{Prob}\{K = k\} = e^{-t} \frac{t^k}{k!},$$

hence

$$\begin{cases} \text{Prob}\{X_{(t)} = 1\} = a_{(t)} = p_0 + p_2 + \cdots + p_{2n} + \cdots + = e^{-t} \dfrac{e^t + e^{-t}}{2}, \\ \text{Prob}\{X_{(t)} = -1\} = b_{(t)} = p_1 + p_3 + \cdots + p_{2n+1} + \cdots + = e^{-t} \dfrac{e^t - e^{-t}}{2}. \end{cases}$$

Finally

$$a_{(t)} = \tfrac{1}{2}[1 + e^{-2t}] \quad \text{and} \quad b_{(t)} = \tfrac{1}{2}[1 - e^{-2t}].$$

In order to study the pair $(X_t X_{t+h})$ we introduce Y_h by

$$Y_h = (-1)^m$$

where m is the number of events occurring on $(t, t+h)$.

Thus

$$X_{t+h} = X_t Y_h,$$

X_t and Y_h being mutually independent,

$$Y_h \quad \text{is} \begin{cases} +1 \text{ with probability } a_{(h)}, \\ -1 \text{ with probability } b_{(h)}. \end{cases}$$

The joint distribution of $X_t\, X_{t+h}$ is the following

X_t \ X_{t+h}	$+1$	-1
$+1$	$a_{(t)}\, a_{(h)}$	$a_{(t)}\, b_{(h)}$
-1	$b_{(t)}\, b_{(h)}$	$b_{(t)}\, a_{(h)}$

,

hence $E(X_t\, X_{t+h}) = a_h - b_h = e^{-2h}$ for $h > 0$.

Exercise 8

$Y_t = t - t_n$ obeys the γ_1-distribution: $e^{-y}\, dy$.
If $Y_t = y$, we have

$$Y_{t+h} = t + h - t_m.$$

If some events occur in $[t, t+h]$; the last occurring at t_m, otherwise:

$$Y_{t+h} = Y + h.$$

The probability distribution of $z = Y_{(t+h)}$ is the following: for

$$y < z < y + h$$

the density is $e^{-(z-y)}$ and

$$\text{Prob}\{z = y + h\} = e^{-h}.$$

Exercise 9. Moving average

$N(t)$ obeys the $P(d)$ Poisson distribution.
Suppose $0 < h < 1$ and put

$$n_1 = \text{number of events occurring in } [t - \tfrac{1}{2}, t - \tfrac{1}{2} + h],$$

$$n_2 = \text{number of events occurring in } [t - \tfrac{1}{2} + h, t + \tfrac{1}{2}],$$

$$n_3 = \text{number of events occurring in } [t + \tfrac{1}{2}, t + \tfrac{1}{2} + h].$$

n_1, n_2, n_3 are mutually independent; they obey Poisson-distributions the parameter being respectively $c\,h$; $c(1-h)$; $c\,h$.

Hence

$$N_{(t)} = n_1 + n_2,$$
$$N_{(t+h)} = n_2 + n_3$$

and this defines the joint distribution

$$\varrho_{(t,h)} = |1-h| \quad \text{for} \quad -1 < h < 1,$$
$$= 0 \qquad \text{for} \quad |h| > 1.$$

Chapter III

Exercise 10

Let $dN(u, t)$ be the number of saltus having a magnitude u (more precisely between u and $u + du$) and occurring in $[0, t]$.

$dN(u, t)$ obeys a Poisson distribution. The parameter is $a\,t\,\frac{e^{-\Theta u}}{u}\,du$. The corresponding second characteristic function is

$$(e^{iz} - 1)\,a\,t\,\frac{e^{-\Theta u}}{u}\,du.$$

The second characteristic function of the sum of these saltus is:

$$(e^{izu} - 1)\,a\,t\,\frac{e^{-\Theta u}}{u}\,du,$$

hence

$$\psi^{(t)}_{(z)} = \log E(e^{izX_t}) = \int_{u=0}^{\infty} (e^{izu} - 1)\,dt\,\frac{e^{-\Theta u}}{u}\,du,$$

$$\psi_{(z)} = a\,t \int \left(\sum_{n=1}^{\infty} \frac{(iz)^n}{n!}\,u^n \right) e^{-\Theta u}\,du,$$

$$= a\,t\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(i\frac{z}{\Theta}\right)^n\right] = a\,t \log\left(1 - \frac{iz}{\Theta}\right)^{-1}$$

and

$$\varphi_{(z)} = \left(1 - i\frac{z}{\Theta}\right)^{-at}.$$

Conclusion: $X_{(t)}$ obeys $\gamma(a\,t, 0)$. $X_{(t)}$ is a uniform gamma process.

Chapter IV

Exercise 11

11.1. For $n \geqq 4$ M^n has no zero: it is the *regular positive case.*

11.2. The characteristic equation is

$$(1 - s)\left(s^2 + \frac{s}{2} - \frac{1}{4}\right) = 0.$$

The spectrum is then:

$$s_1 = 1, \quad s_2 = -\frac{1}{4} + \frac{\sqrt{5}}{4}, \quad s_3 = -\frac{1}{4} - \frac{\sqrt{5}}{4}.$$

The corresponding eigen-columns are:

$$V_1 = \begin{vmatrix} \frac{2}{5} \\ \frac{2}{5} \\ \frac{1}{5} \end{vmatrix}, \quad V_2 = \begin{vmatrix} -\frac{1}{5} - \frac{\sqrt{5}}{5} \\ -\frac{1}{5} + \frac{\sqrt{5}}{5} \\ \frac{2}{5} \end{vmatrix}, \quad V_3 = \begin{vmatrix} -\frac{1}{5} + \frac{\sqrt{5}}{5} \\ -\frac{1}{5} - \frac{\sqrt{5}}{5} \\ \frac{2}{5} \end{vmatrix}.$$

The eigen-rows are

$$W_1' = [\quad 1 \qquad 1 \qquad 1],$$

$$W_2' = \left[-\frac{1}{4} - \frac{\sqrt{5}}{4} \quad -\frac{1}{4} + \frac{\sqrt{5}}{4} \quad 1\right],$$

$$W_3' = \left[-\frac{1}{4} + \frac{\sqrt{5}}{4} \quad -\frac{1}{4} - \frac{\sqrt{5}}{4} \quad 1\right].$$

It is easy to see that

$$M = S\,\Lambda\,S^{-1}$$

where

$$S = \begin{vmatrix} V_1 & V_2 & V_3 \end{vmatrix}, \qquad S^{-1} = \begin{matrix} \overline{W_1'} \\ \overline{W_2'} \\ \overline{W_3'} \end{matrix},$$

$$\Lambda = \begin{vmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{vmatrix}.$$

Exercise 12

12.1. The only stable law is

$$P = \begin{vmatrix} 2/6 \\ 1/6 \\ 2/6 \\ 1/6 \end{vmatrix}.$$

12.2. It is not the regular case but there is only one closed class, a periodic class; its period is 2.

The spectrum is $S = \{1; -1; \frac{1}{2}; -\frac{1}{2}\}$.

The sub-classes are $\{A, B\}$ and $\{C, D\}$.

12.3.

$$M^2 = \left|\begin{array}{cc|cc} 3/4 & 1/2 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 1/4 & 1/2 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 3/4 & 1/2 \\ & & 1/4 & 1/2 \end{array}\right| .$$

The sequence $\{E^{t+2n}\}_n$ has two closed classes.

The stable laws are

$$P_1 = \left|\begin{array}{c} 2/3 \\ 1/3 \\ 0 \\ 0 \end{array}\right|, \quad P_2 = \left|\begin{array}{c} 0 \\ 0 \\ 2/3 \\ 1/3 \end{array}\right|$$

and every $p\,P_1 + q\,P_2$ with $p \geqq 0,\ q \geqq 0,\ p + q = 1$.

12.4. If $E^0 = c$ the limit probability law of E^{2n} for $(n = \infty)$ is P_2, whereas the limit probability law of E^{2n+1} is P_1.

12,5. If

$$P_0 = \left|\begin{array}{c} 2/5 \\ 1/5 \\ 1/5 \\ 1/5 \end{array}\right|,$$

then

$$E_0 \in \{A, B\} \quad \text{with probability} \quad 3/5,$$

$$E_0 \in \{C, D\} \quad \text{with probability} \quad 2/5.$$

The limit probability law of E^{2n} for $n = \infty$ is then

$$\tfrac{3}{5} P_1 + \tfrac{2}{5} P_2 = \left|\begin{array}{c} 6/15 \\ 3/15 \\ 4/15 \\ 2/15 \end{array}\right| .$$

Exercise 13

13.1. We put

$$a_i^n = \mathrm{Prob}\{E^n = E_i\}$$

and

$$e_{t,i}^{n,j} = \mathrm{Prob}\{E^t = E_i / E^n = E_j\}$$

for $n > t$.

13.1 (*a*) We have

$$e_{0,i}^{1,j} = \frac{a_i^0\, p_{ij}}{a_j^1}$$

this gives for $i = 1, 2, 3$ the conditional law of E^0 for $E^1 = E_j$.

13.1 (*b*)

$$e_{1,i}^{2,j} = \frac{a_i^1 p_{ij}}{a_j^2}.$$

13.1 (*c*)

$$\text{Prob}\{E^0 = E_i/E^1 = E_j;\ E^2 = E_k\} = \frac{a_i^0 p_{ij} p_{jk}}{a_j^1 p_{jk}} = e_{0,i}^{1,j}.$$

The reverse sequence is a non-homogeneous Markov sequence.

13.2. We have

$$e_{0,i}^{n,j} = \frac{a_i^0 p_{ij}^n}{a_j^n}$$

and the limit for $n = \infty$ is

$$\frac{a_i^0 b_j}{b_j} = a_i^0.$$

13.3. The given sequence is now a stationary Markov one, then

$$e_{n,i}^{n+1,j} = \frac{b_i p_{ij}}{b_j},$$

it does not depend on n and

$$\text{Prob}\{E^n = E_i/E^{n+1} = E_j;\ E^{n+2} = E_k\} = \frac{b_i p_{ij} p_{jk}}{p_j p_{jk}} = \frac{b_i p_{ij}}{b_i}.$$

This last probability depends only on the state of the system at $t = n + 1$.

The reverse sequence is now a homogeneous Markov one. We have

$$e_{0,i}^{n,j} = \frac{b_i p_{ij}^n}{b_j} \quad \text{and} \quad \lim_{n=\infty} (e_{0,i}^{n,j}) = b_i.$$

Chapter V

Exercise 14

14.1. The canonical development of the sequence $\{X_t\}$ is the following:

$$X_2 = U_1 + U_2,$$

$$X_3 = U_2 + U_3, \qquad X_n = U_{n-1} + U_n.$$

14.2. $\{X_n X_{n+1} X_{n+2}\}$ is a Laplace random set. Its covariance matrix H is

$$H = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} \quad \text{and} \quad H^{-1} = \begin{vmatrix} 3/4 & -1/2 & 1/4 \\ -1/2 & 1 & -1/2 \\ 1/4 & -1/2 & 3/4 \end{vmatrix}.$$

The frequency function is

$$\frac{1}{2} \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{1}{2} V' H V\right) \quad \text{with} \quad V = \begin{vmatrix} X_n \\ X_{n+1} \\ X_{n+2} \end{vmatrix}.$$

The conditional probabilities (3) and (4) can be deduced from the preceding frequency function or from the corresponding characteristic function; but it is easier to deduce the canonical development of $(X_n\, X_{n+1}\, X_{n+2})$, the variables being placed in a suitable order.

14.3. Conditional distribution of X_{n+1} for $X_n = x_n$ and $X_{n+2} = x_{n+2}$. We write

$$\begin{cases} X_n = \sqrt{2}\, U_n, \\ X_{n+2} = \sqrt{2}\, U_{n+1}, \\ X_{n+1} = \dfrac{1}{\sqrt{2}} U_n + \dfrac{1}{\sqrt{2}} U_{n+1} + U_{n+2}. \end{cases}$$

Hence

$$X_{n+1} = \frac{x_n x_{n+2}}{2} + U_3 \quad \text{and} \quad \begin{cases} E(X_{n+1}) = \dfrac{x_n x_{n+2}}{2}, \\ \operatorname{Var}(X_{n+1}) = 1. \end{cases}$$

14.4. Conditional distribution of (X_n, X_{n+2}) for $X_{n+1} = x_{n+1}$. Here we write

$$\begin{cases} X_{n+1} = \sqrt{2}\, U_n, \\ X_n = \dfrac{1}{\sqrt{2}} U_n + \dfrac{3}{\sqrt{2}} U_{n+1}, \\ X_{n+2} = \dfrac{1}{\sqrt{2}} U_n - \dfrac{1}{\sqrt{6}} U_{n+2} + \dfrac{1}{\sqrt{3}} U_{n+2}. \end{cases}$$

Hence

$$\begin{cases} X_n = \dfrac{x_{n+1}}{2} + \dfrac{\sqrt{3}}{\sqrt{2}} U_{n+1}, \\ X_{n+2} = \dfrac{x_{n+1}}{2} U_{n+1} + \dfrac{1}{\sqrt{3}} U_{n+2}. \end{cases}$$

The covariance-matrix of (X_n, X_{n+2}) for given x_{n+1} is:

$$\begin{vmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{vmatrix}$$

and the correlation coefficient is $\varrho = \dfrac{-1}{\sqrt{3}}$.

Exercise 15

There exists a Laplace random set which has the covariance-matrix H.

Indeed, let $\{a_1, a_2, \ldots, a_k, \ldots\}$ be such that

$$\begin{cases} a_1^2 = 1 - r_1^2, \\ \cdots\cdots, \\ a_k^2 = r_{k-1}^2 - r_k^2 \end{cases}$$

and consider $X_1, X_2, \ldots, X_n$ defined by

$$\begin{cases} X_1 = U_1, \\ X_2 = \frac{r_1}{r_2} X_1 + \frac{a_2}{r_1} U_2, \\ \cdots\cdots\cdots\cdots, \\ X_n = \frac{r_n}{r_{n-1}} X_{n-1} + \frac{a_n}{r_{n-1}} U_n. \end{cases}$$

$U_1, U_2, \ldots, U_n$ are independent Laplace variables with zero mean and unit variance.

The covariance matrix of $(X_1, X_2, \ldots, X_n)$ is H. $(X_1, X_2, \ldots, X_n)$ is obviously a Markov sequence.

Chapter VI

Exercise 16

16.1. $\gamma(t)$ is a characteristic function (a Polya-type characteristic function) then it is a positive definite function and also a covariance function.

$g_k(t)$ is a convex discrete function; so it is a positive definite function and also the covariance of a discrete stationary process.

Besides, the fact that $\gamma(t)$ and $g_k(t)$ are covariance functions is obviously implied by the following problems.

16.2. Canonical development of $X_2(t)$ (see Ex. No. 14)

$$X_2\left(\frac{t}{2}\right) = \frac{1}{\sqrt{2}} U\left(\frac{t-1}{2}\right) + \frac{1}{\sqrt{2}} U\left(\frac{t}{2}\right).$$

16.3. Canonical development of $X_{2n}(t)$

$$X_{2n}(t) = \sum_{k=t-2n+1}^{t} U\left(\frac{k}{2n}\right).$$

16.4. We have:

$$\lim_{n=\infty}\left(g_{2n}(t)\right) = \gamma(t).$$

The process $X_2(t)$ can be indefinitly interpolated; the limit for $n = \infty$ is the process $Z(t)$ which has the covariance $\gamma(t)$

$$Z(t) = \int_{s=t-1}^{t} U(s)\sqrt{ds} \quad \text{and} \quad Y = \int_{0}^{t} U(s)\sqrt{ds},$$

hence

$$Z(t) = Y(t, t') - Y(t).$$

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