

# Answers to problems

## Chapter II

### Exercise 1. Transformation by systematic obliteration

The process  $P_k(a)$  has independent random intervals. Each interval obeys  $\gamma(kd)$ .

### Exercise 2. Process of the "midway points"

We put

$$M_n M_{n+1} = M_n$$

and

$$A_n A_{n+1} = V_n,$$

then

$$\begin{cases} v_n = \frac{1}{2}(M_n + M_{n+1}), \\ v_{n+1} = \frac{1}{2}(M_{n+1} + M_{n+2}). \end{cases}$$

These relations show that  $v_n$  obeys  $\gamma(2, 2d)$ .

Two consecutive intervals are correlated.

The correlation coefficient between  $v_n$  and  $v_{n+1}$  is

$$\rho_{n, n+1} = -\frac{1}{2}.$$

Two intervals such as  $v_n$  and  $v_{n+k}$  (with  $k \geq 2$ ) are mutually independent.

### Exercise 3. Transformation by random blotting out

The process obtained is a uniform Poisson one. Its density is  $p d$ .

### Exercise 4. Transformation by random displacings

4.1.  $A_1 B_1 = U + V' + V''$  where  $U$ ;  $V'$  and  $V''$  are independent random variables.

Let  $\varphi_{(z)}$  be the characteristic function of  $V$ , then  $\bar{\varphi}_{(z)}$  is the characteristic function of  $(-V)$  and the c.f. of  $A_1 B_1$  is

$$\Phi_{(z)} = \frac{\varphi_{(z)} \bar{\varphi}_{(z)}}{1 - iz} = \frac{|\varphi_{(z)}|^2}{1 - iz}.$$

$\overline{A_1 B_1}$  obeys the  $\gamma_1$  distribution if and only if  $V$  is almost certainly a constant.

If  $V$  obeys the  $\gamma_1$  law, we have:

$$\Phi_{(z)} = [(1 - iz)(1 + z^2)]^{-1}.$$

**4.2. (a)** Let  $dP$  be the probability that an event ( $A$ ) occurs on  $(t, t + dt)$

$$dP = \int_{v=-\infty}^{+\infty} dt g(v) dv = dt.$$

Moreover the numbers of events ( $A$ ) occurring on two disjoint intervals are mutually independent; consequently  $\{A_n\}$  is a uniform Poisson process. Its density is 1.

**4.2. (b)** Suppose now that  $V$  depends on  $s = OM$ , then

$$\begin{aligned} dP &= \int_{v=0}^{\infty} dt g(s - v, v) dv, \\ &= a(t) dt \quad \text{where} \quad a(t) = \int g(t - v, v) dv \end{aligned}$$

and  $\{A_n\}$  is a non-uniform Poisson process.

### Exercise 5

Put

$$t_n = OM_n, \quad t_{n+1} = OM_{n+1}, \quad t = OA,$$

is known, but  $t_n$  and  $t_{n+1}$  are random variables.

**5.1.**  $(t - t_n)$  and  $(t_{n+1} - t)$  are mutually independent, each obeys  $\gamma_1$ . So  $(t_{n+1} - t_n) = U$  obeys the  $\gamma_2$ -distribution ( $e^{-cu} c u c du$ ).

**5.2.** If  $t > 0$  and  $t_n > 0$ , then  $V = t - t_n$  obeys the following truncated  $\gamma_1$  distribution

$$\frac{e^{-cv} c dv}{\int_{v=0}^t e^{-cv} c dv} \quad \text{where} \quad 0 < v < t.$$

### Exercise 6

Let  $c$  be the parameter of a cluster process and  $g(z)$  the characteristic function of the number of events in one cluster.

We have (see 22.1)

$$G_{(z)}^{(t)} = \exp\{ct(g(z) - 1)\}$$

and we must have

$$G_{(z)}^{(t)} = \exp\{m(z-1)\}.$$

So

$$g(z) = 1 - \frac{m}{ct} + \frac{m}{ct}z + \dots,$$

$$1 - \frac{m}{ct} = \text{Prob}\{K = 0\},$$

$$\frac{m}{ct} = \text{Prob}\{K = 1\}.$$

These probabilities must not depend on  $t$ .

If

$$K \geq 1 \Rightarrow m = ct \quad \text{and} \quad g(z) = z,$$

$$\text{Prob}\{K = 1\} = 1.$$

The process of events is a Poisson one. Its density is  $c$ . If  $K \geq 0 \Rightarrow m = act$

$$\text{Prob}\{K = 0\} = 1 - a, \quad \text{Prob}\{K = 1\} = a.$$

The process of events is a Poisson one. Its density is  $ac$ .

### Exercise 7

The random function  $X(t)$  is

$$X(t) = (-1)^K$$

where  $K$  is the random number of events occurring in  $[0, t]$ .

We have

$$\text{hence} \quad p_k = \text{Prob}\{K = k\} = e^{-t} \frac{t^k}{k!},$$

$$\left\{ \begin{array}{l} \text{Prob}\{X_{(t)} = 1\} = a_{(t)} = p_0 + p_2 + \dots + p_{2n} + \dots = e^{-t} \frac{e^t + e^{-t}}{2}, \\ \text{Prob}\{X_{(t)} = -1\} = b_{(t)} = p_1 + p_3 + \dots + p_{2n+1} + \dots = e^{-t} \frac{e^t - e^{-t}}{2}. \end{array} \right.$$

Finally

$$a_{(t)} = \frac{1}{2} [1 + e^{-2t}] \quad \text{and} \quad b_{(t)} = \frac{1}{2} [1 - e^{-2t}].$$

In order to study the pair  $(X_t, X_{t+h})$  we introduce  $Y_h$  by

$$Y_h = (-1)^m$$

where  $m$  is the number of events occurring on  $(t, t+h)$ .

Thus

$$X_{t+h} = X_t Y_h,$$

$X_t$  and  $Y_h$  being mutually independent,

$$Y_h \text{ is } \begin{cases} \nearrow +1 & \text{with probability } a_{(h)}, \\ \searrow -1 & \text{with probability } b_{(h)}. \end{cases}$$

The joint distribution of  $X_t X_{t+h}$  is the following

$$X_t \left\{ \begin{array}{c} \overbrace{\begin{array}{cc} & X_{t+h} \\ & +1 \quad -1 \end{array}} \\ +1 \quad \left[ \begin{array}{c|c} a_{(t)} a_{(h)} & a_{(t)} b_{(h)} \\ \hline b_{(t)} b_{(h)} & b_{(t)} a_{(h)} \end{array} \right. \\ -1 \end{array} \right.$$

hence  $E(X_t X_{t+h}) = a_h - b_h = e^{-2h}$  for  $h > 0$ .

**Exercise 8**

$Y_t = t - t_n$  obeys the  $\gamma_1$ -distribution:  $e^{-y} dy$ .

If  $Y_t = y$ , we have

$$Y_{t+h} = t + h - t_m.$$

If some events occur in  $[t, t + h]$ ; the last occurring at  $t_m$ , otherwise:

$$Y_{t+h} = Y + h.$$

The probability distribution of  $z = Y_{(t+h)}$  is the following: for

$$y < z < y + h$$

the density is  $e^{-(z-y)}$  and

$$\text{Prob}\{z = y + h\} = e^{-h}.$$

**Exercise 9. Moving average**

$N(t)$  obeys the  $P(d)$  Poisson distribution.

Suppose  $0 < h < 1$  and put

$$n_1 = \text{number of events occurring in } [t - \frac{1}{2}, t - \frac{1}{2} + h],$$

$$n_2 = \text{number of events occurring in } [t - \frac{1}{2} + h, t + \frac{1}{2}],$$

$$n_3 = \text{number of events occurring in } [t + \frac{1}{2}, t + \frac{1}{2} + h].$$

$n_1, n_2, n_3$  are mutually independent; they obey Poisson-distributions the parameter being respectively  $ch; c(1 - h); ch$ .

Hence

$$N_{(t)} = n_1 + n_2,$$

$$N_{(t+h)} = n_2 + n_3$$

and this defines the joint distribution

$$\begin{aligned} \varrho_{(t, h)} &= |1 - h| \quad \text{for } -1 < h < 1, \\ &= 0 \quad \text{for } |h| > 1. \end{aligned}$$

### Chapter III

#### Exercise 10

Let  $dN(u, t)$  be the number of saltus having a magnitude  $u$  (more precisely between  $u$  and  $u + du$ ) and occurring in  $[0, t]$ .

$dN(u, t)$  obeys a Poisson distribution. The parameter is  $a t \frac{e^{-\Theta u}}{u} du$ . The corresponding second characteristic function is

$$(e^{iz} - 1) a t \frac{e^{-\Theta u}}{u} du.$$

The second characteristic function of the sum of these saltus is:

$$(e^{iz} - 1) a t \frac{e^{-\Theta u}}{u} du,$$

hence

$$\psi_{(z)}^{(t)} = \log E(e^{izX_t}) = \int_{u=0}^{\infty} (e^{iz} - 1) dt \frac{e^{-\Theta u}}{u} du,$$

$$\begin{aligned} \psi_{(z)} &= a t \int \left( \sum_{n=1}^{\infty} \frac{(iz)^n}{n!} u^n \right) e^{-\Theta u} du, \\ &= a t \left[ \sum_{n=1}^{\infty} \frac{1}{n} \left( i \frac{z}{\Theta} \right)^n \right] = a t \log \left( 1 - \frac{iz}{\Theta} \right)^{-1} \end{aligned}$$

and

$$\varphi_{(z)} = \left( 1 - i \frac{z}{\Theta} \right)^{-at}.$$

*Conclusion:*  $X_{(t)}$  obeys  $\gamma(at, 0)$ .  $X_{(t)}$  is a uniform gamma process.

### Chapter IV

#### Exercise 11

**11.1.** For  $n \geq 4$   $M^n$  has no zero: it is the *regular positive case*.

**11.2.** The characteristic equation is

$$(1 - s) \left( s^2 + \frac{s}{2} - \frac{1}{4} \right) = 0.$$

The spectrum is then:

$$s_1 = 1, \quad s_2 = -\frac{1}{4} + \frac{\sqrt{5}}{4}, \quad s_3 = -\frac{1}{4} - \frac{\sqrt{5}}{4}.$$

The corresponding eigen-columns are:

$$V_1 = \begin{vmatrix} \frac{2}{5} \\ \frac{2}{5} \\ \frac{1}{5} \end{vmatrix}, \quad V_2 = \begin{vmatrix} -\frac{1}{5} - \frac{\sqrt{5}}{5} \\ -\frac{1}{5} + \frac{\sqrt{5}}{5} \\ \frac{2}{5} \end{vmatrix}, \quad V_3 = \begin{vmatrix} -\frac{1}{5} + \frac{\sqrt{5}}{5} \\ -\frac{1}{5} - \frac{\sqrt{5}}{5} \\ \frac{2}{5} \end{vmatrix}.$$

The eigen-rows are

$$W'_1 = [ \quad 1 \quad \quad 1 \quad \quad 1 ],$$

$$W'_2 = \left[ -\frac{1}{4} - \frac{\sqrt{5}}{4} \quad -\frac{1}{4} + \frac{\sqrt{5}}{4} \quad 1 \right],$$

$$W'_3 = \left[ -\frac{1}{4} + \frac{\sqrt{5}}{4} \quad -\frac{1}{4} - \frac{\sqrt{5}}{4} \quad 1 \right].$$

It is easy to see that

$$M = S \Lambda S^{-1}$$

where

$$S = \left| \begin{array}{c|c|c} V_1 & V_2 & V_3 \end{array} \right|, \quad S^{-1} = \begin{array}{c} \overline{W'_1} \\ \overline{W'_2} \\ \overline{W'_3} \end{array},$$

$$\Lambda = \begin{vmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{vmatrix}.$$

### Exercise 12

12.1. The only stable law is

$$P = \begin{vmatrix} 2/6 \\ 1/6 \\ 2/6 \\ 1/6 \end{vmatrix}.$$

12.2. It is not the regular case but there is only one closed class, a periodic class; its period is 2.

The spectrum is  $S = \{1; -1; \frac{1}{2}; -\frac{1}{2}\}$ .

The sub-classes are  $\{A, B\}$  and  $\{C, D\}$ .

12.3.

$$M^2 = \left| \begin{array}{cc|cc} 3/4 & 1/2 & & \\ 1/4 & 1/2 & 0 & \\ \hline & 0 & 3/4 & 1/2 \\ & & 1/4 & 1/2 \end{array} \right|.$$

The sequence  $\{E^{t+2n}\}_n$  has two closed classes.

The stable laws are

$$P_1 = \left| \begin{array}{c} 2/3 \\ 1/3 \\ 0 \\ 0 \end{array} \right|, \quad P_2 = \left| \begin{array}{c} 0 \\ 0 \\ 2/3 \\ 1/3 \end{array} \right|$$

and every  $p P_1 + q P_2$  with  $p \geq 0$ ,  $q \geq 0$ ,  $p + q = 1$ .

12.4. If  $E^0 = c$  the limit probability law of  $E^{2n}$  for  $(n = \infty)$  is  $P_2$ , whereas the limit probability law of  $E^{2n+1}$  is  $P_1$ .

12.5. If

$$P_0 = \left| \begin{array}{c} 2/5 \\ 1/5 \\ 1/5 \\ 1/5 \end{array} \right|,$$

then

$E_0 \in \{A, B\}$  with probability  $3/5$ ,

$E_0 \in \{C, D\}$  with probability  $2/5$ .

The limit probability law of  $E^{2n}$  for  $n = \infty$  is then

$$\frac{3}{5} P_1 + \frac{2}{5} P_2 = \left| \begin{array}{c} 6/15 \\ 3/15 \\ 4/15 \\ 2/15 \end{array} \right|.$$

### Exercise 13

13.1. We put

$$a_i^n = \text{Prob}\{E^n = E_i\}$$

and

$$e_{i,i}^{n,j} = \text{Prob}\{E^t = E_i | E^n = E_j\}$$

for  $n > t$ .

13.1 (a) We have

$$e_{0,i}^{1,j} = \frac{a_i^0 p_{ij}}{a_j^1}$$

this gives for  $i = 1, 2, 3$  the conditional law of  $E^0$  for  $E^1 = E_j$ .

13.1 (b)

$$e_{1,i}^{2,j} = \frac{a_i^1 p_{ii}}{a_j^2}.$$

13.1 (c)

$$\text{Prob}\{E^0 = E_i | E^1 = E_j; E^2 = E_k\} = \frac{a_i^0 p_{ij} p_{jk}}{a_j^1 p_{jk}} = e_{0,i}^{1,j}.$$

The reverse sequence is a non-homogeneous Markov sequence.

13.2. We have

$$e_{0,i}^{n,j} = \frac{a_i^0 p_{ij}^n}{a_j^n}$$

and the limit for  $n = \infty$  is

$$\frac{a_i^0 b_j}{b_j} = a_i^0.$$

13.3. The given sequence is now a stationary Markov one, then

$$e_{n,i}^{n+1,j} = \frac{b_i p_{ij}}{b_j},$$

it does not depend on  $n$  and

$$\text{Prob}\{E^n = E_i | E^{n+1} = E_j; E^{n+2} = E_k\} = \frac{b_i p_{ij} p_{jk}}{p_j p_{jk}} = \frac{b_i p_{ij}}{b_j}.$$

This last probability depends only on the state of the system at  $t = n + 1$ .

The reverse sequence is now a homogeneous Markov one. We have

$$e_{0,i}^{n,j} = \frac{b_i p_{ij}^n}{b_j} \quad \text{and} \quad \lim_{n \rightarrow \infty} (e_{0,i}^{n,j}) = b_i.$$

## Chapter V

### Exercise 14

14.1. The canonical development of the sequence  $\{X_t\}$  is the following:

$$X_2 = U_1 + U_2,$$

$$X_3 = U_2 + U_3, \quad X_n = U_{n-1} + U_n.$$

14.2.  $\{X_n, X_{n+1}, X_{n+2}\}$  is a Laplace random set. Its covariance matrix  $H$  is

$$H = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} \quad \text{and} \quad H^{-1} = \begin{vmatrix} 3/4 & -1/2 & 1/4 \\ -1/2 & 1 & -1/2 \\ 1/4 & -1/2 & 3/4 \end{vmatrix}.$$



The frequency function is

$$\frac{1}{2} \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{1}{2} V' H V\right) \quad \text{with} \quad V = \begin{vmatrix} X_n \\ X_{n+1} \\ X_{n+2} \end{vmatrix}.$$

The conditional probabilities (3) and (4) can be deduced from the preceding frequency function or from the corresponding characteristic function; but it is easier to deduce the canonical development of  $(X_n X_{n+1} X_{n+2})$ , the variables being placed in a suitable order.

**14.3.** Conditional distribution of  $X_{n+1}$  for  $X_n = x_n$  and  $X_{n+2} = x_{n+2}$ . We write

$$\begin{cases} X_n = \sqrt{2} U_n, \\ X_{n+2} = \sqrt{2} U_{n+1}, \\ X_{n+1} = \frac{1}{\sqrt{2}} U_n + \frac{1}{\sqrt{2}} U_{n+1} + U_{n+2}. \end{cases}$$

Hence

$$X_{n+1} = \frac{x_n x_{n+2}}{2} + U_3 \quad \text{and} \quad \begin{cases} E(X_{n+1}) = \frac{x_n x_{n+2}}{2}, \\ \text{Var}(X_{n+1}) = 1. \end{cases}$$

**14.4.** Conditional distribution of  $(X_n, X_{n+2})$  for  $X_{n+1} = x_{n+1}$ . Here we write

$$\begin{cases} X_{n+1} = \sqrt{2} U_n, \\ X_n = \frac{1}{\sqrt{2}} U_n + \frac{3}{\sqrt{2}} U_{n+1}, \\ X_{n+2} = \frac{1}{\sqrt{2}} U_n - \frac{1}{\sqrt{6}} U_{n+2} + \frac{1}{\sqrt{3}} U_{n+2}. \end{cases}$$

Hence

$$\begin{cases} X_n = \frac{x_{n+1}}{2} + \frac{\sqrt{3}}{\sqrt{2}} U_{n+1}, \\ X_{n+2} = \frac{x_{n+1}}{2} U_{n+1} + \frac{1}{\sqrt{3}} U_{n+2}. \end{cases}$$

The covariance-matrix of  $(X_n, X_{n+2})$  for given  $x_{n+1}$  is:

$$\begin{vmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{vmatrix}$$

and the correlation coefficient is  $\rho = \frac{-1}{\sqrt{3}}$ .

### Exercise 15

There exists a Laplace random set which has the covariance-matrix  $H$ .

Indeed, let  $\{a_1, a_2, \dots, a_k, \dots\}$  be such that

$$\begin{cases} a_1^2 = 1 - r_1^2, \\ \dots \dots \dots, \\ a_k^2 = r_{k-1}^2 - r_k^2 \end{cases}$$

and consider  $X_1, X_2, \dots, X_n$  defined by

$$\begin{cases} X_1 = U_1, \\ X_2 = \frac{r_1}{r_2} X_1 + \frac{a_2}{r_1} U_2, \\ \dots \dots \dots, \\ X_n = \frac{r_{n-1}}{r_n} X_{n-1} + \frac{a_n}{r_{n-1}} U_n. \end{cases}$$

$U_1, U_2, \dots, U_n$  are independent Laplace variables with zero mean and unit variance.

The covariance matrix of  $(X_1, X_2, \dots, X_n)$  is  $H$ .  $(X_1, X_2, \dots, X_n)$  is obviously a Markov sequence.

Chapter VI

Exercise 16

**16.1.**  $\gamma(t)$  is a characteristic function (a Polya-type characteristic function) then it is a positive definite function and also a covariance function.

$g_k(t)$  is a convex discrete function; so it is a positive definite function and also the covariance of a discrete stationary process.

Besides, the fact that  $\gamma(t)$  and  $g_k(t)$  are covariance functions is obviously implied by the following problems.

**16.2.** Canonical development of  $X_2(t)$  (see Ex. No. 14)

$$X_2\left(\frac{t}{2}\right) = \frac{1}{\sqrt{2}} U\left(\frac{t-1}{2}\right) + \frac{1}{\sqrt{2}} U\left(\frac{t}{2}\right).$$

**16.3.** Canonical development of  $X_{2n}(t)$

$$X_{2n}(t) = \sum_{k=t-2n+1}^t U\left(\frac{k}{2n}\right).$$

**16.4.** We have:

$$\lim_{n \rightarrow \infty} (g_{2n}(t)) = \gamma(t).$$

The process  $X_2(t)$  can be indefinitely interpolated; the limit for  $n = \infty$  is the process  $Z(t)$  which has the covariance  $\gamma(t)$

$$Z(t) = \int_{s=t-1}^t U(s) \sqrt{ds} \quad \text{and} \quad Y = \int_0^t U(s) \sqrt{ds},$$

hence

$$Z(t) = Y(t, t') - Y(t).$$

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