Appendix

Bessel Function Generation by Chebyshev Polynomial Methods

One possible approach to generating the Bessel functions required in Sect. 3.6 to 3.10 is to evaluate the series expansions which serve as their definitions. From the point of view of numerical evaluation this is undesirable owing to the required fractional exponentiation for functions of fractional order. Furthermore it is difficult, with this method, to obtain an estimate of the error made by truncating the series at a particular term. A more efficient technique is to employ a Chebyshev polynomial expansion of the desired function in the form

$$f(x) = \sum_{r=0}^{\infty}{}' a_r T_r(x) \qquad \text{(A1)[1]}$$

where the Chebyshev polynomial, $T_r(x)$, defined by

$$T_r(x) = \cos(r\cos^{-1} x), \quad -1 \leq x \leq 1$$

is subject to the recurrence relation

$$T_{r+1}(x) - 2xT_r(x) + T_{r-1}(x) = 0. \qquad \text{(A2)}$$

An approximation $g(x)$, to a function $f(x)$, can be produced by truncating the expansion such that

$$g(x) = \sum_{r=0}^{n}{}' a_r T_r(x) \qquad \text{(A3)}$$

where the error incurred by truncation has an upper bound given simply by

$$\varepsilon_n(x) = \sum_{r=n+1}^{\infty} |a_r|.$$

Owing to the rapid convergence of the series, a desired accuracy (and thus the value of n) can be chosen simply by inspection of the available range of expansion coefficients. As a result of this feature, and the fact that tables of Chebyshev polynomial expansion coefficients are readily available, this approach is an attractive practical one. Clenshaw and Picken [1] have produced tables of Chebyshev expansion coefficient for Bessel functions of fractional order. The coefficients are given to the 20th decimal place.

[1] The prime on the summation indicates that the leading term in the series is halved.

The more usual method of carrying out a Chebyshev expansion is to make use of the recurrence relation in Eq. (A2). Since $T_0(x) = 1$ and $T_1(x) = x$ the higher polynomials are generated quite readily and may be incorporated into an algorithm for machine evaluation. A disadvantage with this approach lies in the number of arithmetic operations necessary to evaluate a given number of terms in the series. If computer time rationing is not a major consideration, the accompanying advantages of flexibility and accuracy offset slow computation. The technique is flexible in that a change in order of the approximation simply requires the addition of more terms to the expansion whilst accuracy is guaranteed by the properties of Chebyshev expansions.

The alternative to direct evaluation of the Chebyshev polynomials involves the rearrangement of a given Chebyshev series into a power series. This power series may then be evaluated by the well known technique of nested multiplication, leading to rapid computation. A requirement of this method is that the power series expansion coefficients must be generated from the Chebyshev coefficients. Whilst this in itself is particularly simple, it does represent a loss of flexibility since a change in the order of the approximation to a function requires a new set of power series coefficients. Another limitation is the possible ill-determination of the power series coefficients, thus leading to some uncertainty as to the accuracy of the technique. This is discussed in Clenshaw [2].

The accuracy of the Bessel function values generated by Chebyshev methods can be checked against the 10 decimal place tables produced by the National Bureau of Standards [3, 4].

References for Appendix

1. Clenshaw, C. W.; Picken, S. M.: Chebyshev series for Bessel functions of fractional order. N.P.L. Mathematical tables, Vol. 8. London: H. M. Stat. Office 1966
2. Clenshaw, C. W.: Chebyshev series for mathematical functions. N.P.L. Mathematical tables, Vol. 5 London: H. M. Stat. Office 1962
3. Nat. Bur. Stand. Computation Laboratory: Tables of Bessel functions of fractional order, Vol. 1. New York: Columbia U.P. 1948
4. Nat. Bur. Stand. Computation Laboratory: Tables of Bessel functions of fractional order, Vol. 2. New York: Columbia U.P. 1948

Subject Index