



Appendix

Numerical Integration

A

A Numerical Integration

A differential equation of order n

$$y^{(n)}(t) = f[t, y(t), y'(t), \dots, y^{(n-1)}(t)] \quad (\text{A.1})$$

can be transformed with the help of the functions

$$\begin{aligned} z_1(t) &= y(t) , \\ z_2(t) &= y'(t) , \\ \dots &= \dots , \\ z_n(t) &= y^{(n-1)}(t) \end{aligned}$$

to a system of n differential equations of first order:

$$\begin{pmatrix} z_1' \\ z_2' \\ \dots \\ z_n' \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \dots \\ f[t, z_1, z_2, \dots, z_n] \end{pmatrix} \rightarrow \mathbf{z}' = \mathbf{f}[t, z_1, z_2, \dots, z_n]. \quad (\text{A.2})$$

To solve this system in the time interval $[t_s, t_e]$ we need the *initial conditions*

$$\{y(t_s), y'(t_s), \dots, y^{(n-1)}(t_s)\} \rightarrow \{z_{10}, z_{20}, \dots, z_{(n-1)0}\} ,$$

which are summarized in the column vector \mathbf{z}_0 .

```
given:  $\mathbf{z}_0, t_s, t_e, n, \mathbf{f}[t, \mathbf{z}]$ 
time increment:  $\Delta t = (t_e - t_s)/n$ 
loop over all time increments
FOR  $i$  FROM 0 TO  $n$  DO
     $t_i = t_s + i \Delta t$ 
     $\mathbf{z}_{i+1} = \mathbf{z}_i + \mathbf{f}[t_i, \mathbf{z}_i] \Delta t$ 
END DO
```

Fig. A.1 Euler's method

The numerical solution of (A.2) can be obtained with Euler's method. Its algorithmic procedure is summarized in Fig. A.1.

For the numerical solution of (A.2) with the fourth-order Runge-Kutta method, we have to perform four function evaluations for each of the n first-order differential equations. We assemble these additional function evaluations k_{jl} for $l = 1, 2, 3, 4$ within the column vectors

$$\mathbf{k}_1 = \begin{pmatrix} k_{11} \\ k_{21} \\ \dots \\ k_{n1} \end{pmatrix}, \quad \mathbf{k}_2 = \begin{pmatrix} k_{12} \\ k_{22} \\ \dots \\ k_{n2} \end{pmatrix}, \quad \mathbf{k}_3 = \begin{pmatrix} k_{13} \\ k_{23} \\ \dots \\ k_{n3} \end{pmatrix}, \quad \mathbf{k}_4 = \begin{pmatrix} k_{14} \\ k_{24} \\ \dots \\ k_{n4} \end{pmatrix}.$$

A compact program flow of this method is given in Fig. A.2.

```

given:   $z_0, t_s, t_e, n, \mathbf{f}[t, \mathbf{z}]$ 
time increment:   $\Delta t = (t_e - t_s)/n$ 
loop over all time increments
FOR  $i$  FROM 0 TO  $n$  DO
     $t_i = t_s + i \Delta t$ 
     $\mathbf{k}_1 = \mathbf{f}[t_i, \mathbf{z}_i]$ 
     $\mathbf{k}_2 = \mathbf{f}[t_i + \frac{\Delta t}{2}, \mathbf{z}_i + \mathbf{k}_1 \frac{\Delta t}{2}]$ 
     $\mathbf{k}_3 = \mathbf{f}[t_i + \frac{\Delta t}{2}, \mathbf{z}_i + \mathbf{k}_2 \frac{\Delta t}{2}]$ 
     $\mathbf{k}_4 = \mathbf{f}[t_i + \Delta t, \mathbf{z}_i + \mathbf{k}_3 \Delta t]$ 
     $\mathbf{z}_{i+1} = \mathbf{z}_i + \frac{\Delta t}{6}(\mathbf{k}_1 + 2 \mathbf{k}_2 + 2 \mathbf{k}_3 + \mathbf{k}_4)$ 
END DO

```

Fig. A.2 Fourth-order Runge-Kutta method

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