

# References

1. Ackerberg, R.C., O'Malley, R.E.: Boundary layer problems exhibiting resonance. *Stud. Appl. Math.* **49**, 277–295 (1970)
2. Balser, W.: *Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations*. Springer, New York (2000)
3. Benoît, É., Callot, J.-L., Diener, F., Diener, M.: Chasse au canard. *Collect. Math.* **31**, 37–119 (1981)
4. Benoît, É., El Hamidi, A., Fruchard, A.: On combined asymptotic expansions in singular perturbations. *Electron. J. Diff. Equat.* **51**, 1–27 (2002)
5. Benoît, É., Fruchard, A., Schäfke, R., Wallet, G.: Solutions surstables des équations différentielles complexes lentes-rapides à point tournant. *Ann. Fac. Sci. Toulouse Math.* VII **4**, 627–658 (1998)
6. Benoît, É., Fruchard, A., Schäfke, R., Wallet, G.: Overstability: Toward a global study. *C. R. Acad. Sci. Paris I* **326**, 873–878 (1998)
7. Bonckaert, P., De Maesschalck, P., Gevrey normal forms of vector fields with one zero eigenvalue. *J. Math. Anal. Appl.* **344**, 301–321 (2008)
8. Canalis-Durand, M., Mozo-Fernandez, J., Schäfke, R.: Monomial summability and doubly singular differential equations. *J. Differ. Equat.* **233**, 485–511 (2007)
9. Canalis-Durand, M., Ramis, J.-P., Schäfke, R., Sibuya, Y.: Gevrey solutions of singularly perturbed differential equations. *J. Reine Angew. Math.* **518**, 95–129 (2000)
10. De Maesschalck, P.: On maximum bifurcation delay in real planar singularly perturbed vector fields. *Nonlinear Anal.* **68**, 547–576 (2008)
11. De Maesschalck, P.: Ackerberg-O'Malley resonance in boundary value problems with a turning point of any order. *Commun. Pure Appl. Anal.* **6**, 311–333 (2007)
12. De Maesschalck, P.: Gevrey properties of real planar singularly perturbed systems. *J. Differ. Equat.* **238**, 338–365 (2007)
13. De Maesschalck, P., Dumortier, F.: Canard solutions at non-generic turning points. *Trans. Am. Math. Soc.* **358**, 2291–2334 (2006)
14. Diener, M.: Regularizing microscopes and rivers. *SIAM J. Math. Anal.* **25**, 148–173 (1994)
15. Dorodnitsyn, A.A.: Asymptotic solution of the Van der Pol equation. *Priklad. Mat. Mekh.* **11**, 313–328 (1947) (in russian)
16. Dumortier, F., Roussarie, R.: Canard cycles and center manifolds. *Memoir. Am. Math. Soc.* **577**, 1996
17. Eckhaus, W.: *Asymptotic analysis of singular perturbations. Studies in Mathematics and its Applications*, vol. 9. North-Holland, Amsterdam (1979)
18. Erdélyi, A.: Singular perturbations of boundary value problems involving ordinary differential equations. *J. Soc. Indust. Appl. Math.* **11**, 105–116 (1963)

19. Fenichel, N.: Geometric singular perturbation theory for ordinary differential equations. *J. Differ. Equat.* **31**, 53–98 (1979)
20. Forget, T.: Points tournants dégénérés, Thèse de Doctorat, Université de La Rochelle, 2007
21. Forget, T.: Solutions canards en des points tournants dégénérés. *Ann. Fac. Sci. Toulouse Math.* **16**, 799–816 (2007)
22. Forget, T.: Asymptotic study of planar canard solutions. *Bull. Belg. Math. Soc. Simon Stevin* **15**, 809–824 (2008)
23. Fraenkel, L.E.: On the method of matched asymptotic expansions. *Proc. Cambridge Philos. Soc.* **65**, 209–284 (1969)
24. Fruchard, A., Schäfke, R.: Exceptional complex solutions of the forced Van der Pol equation. *Funkcialaj Ekvacioj* **42**, 201–223 (1999)
25. Fruchard, A., Schäfke, R.: Overstability and resonance. *Ann. Inst. Fourier Grenoble* **53**, 227–264 (2003)
26. Fruchard, A., Schäfke, R.: A survey of some results on overstability and bifurcation delay. *Discrete Cont. Dyn. Syst. S* **2**, 931–965 (2009)
27. Fruchard, A., Schäfke, R.: De nouveaux développements asymptotiques combinés pour la perturbation singulière. *Actes du Colloque à la mémoire d’Emmanuel Isambert Publications Univ. Paris* **13**, 125–161 (2012)
28. Fruchard, A., Schäfke, R.: Composite asymptotic expansions and turning points of singularly perturbed ordinary differential equations. *C. R. Math. Acad. Sci.* **348**, 1273–1277 (2010)
29. Gautheron, V., Isambert, E.: Finitely differentiable ducks and finite expansions. In: Benoît, E. (ed.) *Dynamic Bifurcations*, *Lect. Notes Math.*, vol. 1493, pp. 40–56. Springer, New York (1991)
30. Van Gils, S., Krupa, M., Szmolyan, P.: Asymptotic expansions using blow-up *Z. Angew. Math. Phys.* **56**, 369–397 (2005)
31. Hek, G.: Geometric singular perturbation theory in biological practice. *J. Math. Biol.* **60**, 347–386 (2010)
32. Isambert, E.: Nonsmooth ducks and regular perturbations of rivers, I and II. *J. Math. Anal. Appl.* **200**, 14–33 and 289–306 (1996)
33. Jones, C.K.R.T.: Geometric singular perturbation theory. In: *Dynamical Systems*, *Lect. Notes Math.*, vol. 1609. Springer, New York (1995)
34. Kaplun, S., Lagerstrom, P.A.: Asymptotic expansions of Navier-Stokes solutions for small Reynolds numbers. *J. Math. Mech.* **6**, 585–593 (1957)
35. Kevorkian, J., Cole, J.D.: *Perturbation methods in applied mathematics*. Applied Mathematical Sciences, vol. 34. Springer, New York (1981)
36. Kopell, N.: A geometric approach to boundary layer problems exhibiting resonance. *SIAM J. Appl. Math.* **37**, 436–458 (1979)
37. Kopell, N.: The singularly perturbed turning-point problem: a geometric approach. In: *Singular perturbations and asymptotics*, *Proc. Adv. Sem., Math. Res. Center, University of Wisconsin, Madison, Wisconsin* (1980) 173–190
38. Krupa, M., Szmolyan, P.: Relaxation oscillation and canard explosion. *J. Differ. Equat.* **174**, 312–368 (2001)
39. Krupa, M., Szmolyan, P.: Extending geometric singular perturbation theory to nonhyperbolic points-fold and canard points in two dimensions. *SIAM J. Math. Anal.*, **33**, 286–314 (2001)
40. Lagerstrom, P.A.: *Matched asymptotic expansions: ideas and techniques*. Applied Mathematical Sciences, vol. 76. Springer, New York (1988)
41. Lobry, C.: *Dynamic bifurcations*. In: Benoît, E. (ed.) *Dynamic Bifurcations*, *Lect. Notes Math.*, vol. 1493, pp. 1–13. Springer, New York (1991)
42. Matzinger, É.: Étude d’équations différentielles ordinaires singulièrement perturbées au voisinage d’un point tournant. Thesis, Preprint IRMA 2000/53, Strasbourg (2000)
43. Matzinger, É.: Étude des solutions surstables de l’équation de Van der Pol. *Ann. Fac. Sci. Toulouse* **10**, 713–744 (2001)
44. Matzinger, É.: Asymptotic behaviour of solutions near a turning point: the example of the brusselator equation. *J. Differ. Equat.* **220**, 478–510 (2006)

45. Mischenko, E.F., Rozov, N.Ch.: *Differential Equations with Small Parameters and Relaxation Oscillations*. Plenum Press, New York and London (1980)
46. O'Malley, R.E.: *Singular perturbation methods for ordinary differential equations*. Applied Mathematical Sciences, vol. 89. Springer, New York (1991)
47. Panazzolo, D.: On the existence of canard solutions. *Publ. Mat.* **44**, 503–592 (2000)
48. Ramis, J.-P.: Dévissage Gevrey. *Astérisque* **59–60**, 173–204 (1978)
49. Ramis, J.-P.: Les séries  $k$ -sommables et leurs applications. In: *Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory*, Lect. Notes Physics, vol. 126, pp. 178–199. Springer, New York (1980)
50. Sibuya, Y.: Gevrey property of formal solutions in a parameter. In: *Asymptotic and computational analysis* (Winnipeg, MB, 1989). *Lecture Notes in Pure and Appl. Math.*, vol. 124, pp. 393–401. Dekker, New York (1990)
51. Sibuya, Y.: *Linear differential equations in the complex domain*. Problems of Analytic Continuation. Am. Math. Soc., Providence (RI) (1990)
52. Sibuya, Y.: Uniform simplification in a full neighborhood of a transition point. *Memoi. Am. Math. Soc.* **149** (1974)
53. Sibuya, Y.: A theorem concerning uniform simplification at a transition point and a problem of resonance. *SIAM J. Math. Anal.* **12**(5), 653–668 (1981)
54. Skinner, L.A.: *Singular Perturbation Theory*. Springer, New York (2011)
55. Skinner, L.A.: Uniform solution of boundary layer problems exhibiting resonance. *SIAM J. Appl. Math.* **47**, 225–231 (1987)
56. Skinner, L.A.: Matched expansion solutions of the first-order turning point problem. *SIAM J. Math. Anal.* **25**, 1402–1411 (1994)
57. Skinner, L.A.: A class of singularly perturbed singular Volterra integral equations. *Asymptot. Anal.* **22**, 113–127 (2000)
58. Szmolyan, P., Wechselberger, M.: Canards in  $R^3$ . *J. Differ. Equat.* **177**, 419–453 (2001)
59. Vasil'eva, A.B., Butuzov, V.F.: *Asymptotic Expansions of the Solutions of Singularly Perturbed Equations*. Izdat. "Nauka", Moscow (1973) (in Russian)
60. Wallet, G.: Surstabilité pour une équation différentielle analytique en dimension un. *Ann. Inst. Fourier* **40**, 557–595 (1990)
61. Wasow, W.: *Asymptotic Expansions for Ordinary Differential Equations*. Interscience, New York (1965)
62. Wasow, W.: *Linear Turning Point Theory*. Springer, New York (1985)

# Index

## Symbols

$A(\sigma, \varrho)$ , annulus 59  
 $A(r, \infty)$ , infinite annulus 18  
 $D(0, r)$ , the disk of center 0 and radius  $r$  17  
 $R_d$ , landscape function 94  
 $S(\alpha, \beta, r)$ , the sector of vertex 0, radius  $r$  and angles  $\alpha, \beta$  17  
 $S_{l,l+1}$ , intersection of consecutive sectors 67  
 $U^+$ , special function 5  
 $U^-$ , special function 4  
 $U_j^-$ , special function 12  
 $V(\alpha, \beta, r, \mu)$ , quasi-sector 18  
 $V^j$ , quasi-sector for  $X$  59  
 $\widetilde{V}_l^{j,j+1}(\eta)$ , intersection of consecutive quasi-sectors in  $x$  65  
 $V_l^j(\eta)$ , quasi-sector for  $x$  60  
 $\mathcal{A}(\eta_0, r_0, \mu)$ , domain covered by quasi-sectors 60  
 $\Delta$ , differential operator 25  
 $\Delta_2$ , difference operator 78, 96, 106  
 $\mathcal{G}(V)$ , the space of bounded holomorphic functions in  $V$  18  
 $\widetilde{\mathcal{H}}$ , space of (unbounded) holomorphic functions 90  
 $\mathcal{H}(r_0)$ , the space of bounded holomorphic functions in  $D(0, r_0)$  19  
**S**, shift operator 4, 19  
**T**, shift operator 18  
**D**, usual differential operator 4  
**I**, canonical inclusion 20  
 $\mathbb{C}^*$ , the nonzero complex numbers 17  
 $\widetilde{\mathbb{C}}$ , the Riemann surface of the logarithm 17  
 $\mathbb{N}$ , the set of natural numbers, including 0 17  
 $\widetilde{\mathcal{C}}(V)$ , space of (unbounded) composite formal series 90

$\widehat{\mathcal{C}}(r_0, V)$ , space of composite formal series w.r.t.  $V$  and  $D(0, r_0)$  20  
 $\ell$ , kind of logarithm 27  
 $\widehat{R}$ , series of residues 27, 140  
 $\sim$ , asymptotic expansion 18  
 $\sim_{\frac{1}{p}}$ , Gevrey asymptotic expansion 47  
 $y_l^{\text{ext}}$ , function of  $x$  and  $\eta$  on a good covering 66  
 $y_l^{\text{int}}$ , function of  $x$  and  $\eta$  on a good covering 66  
 $g_n^j$ , function of  $X$  71  
 $y_l^j$ , function of  $x$  and  $\eta$  on a good covering 66

## A

Ackerberg-O'Malley resonance 136  
   local  $\mathcal{C}^\infty$ -resonant solution 136  
 Ackerberg, R. C. 119, 139  
 Asymptotic expansion  
   at infinity 18  
   inner expansion vi, 29, 30, 33  
   matched expansion vi, 29, 30, 33  
   outer expansion vi, 29, 30, 33  
   in the sense of Poincaré 3, 18

## B

Balsler, Werner 55  
 Benoît, Éric vi, 21, 82  
 Borel-Ritt theorem  
   for CASES 28  
   classical 28

- Borel-Ritt-Gevrey theorem  
 for CASEs 55  
 classical 55  
 Borel transform 56, 85, 100, 110  
 Boundary layer v  
 Bounded 2, 9  
 Butuzov, V. F. v, 21, 82
- C**
- Callot, Jean-Louis 139  
 Canalis-Durand, Mireille vii, 23, 43, 48, 82, 145  
 Canard solution or duck  
 $\mathcal{C}^m$  canard 121  
 $\mathcal{C}^\infty$  canard 120  
 angular canard 132  
 formal canard 136  
 global canard 105, 120  
 local canard 120  
 non-smooth canard 128, 132  
 Cauchy-Heine formula 65, 66  
 Cole, J. D. vi, 118  
 Composite asymptotic expansion, CASE v, 5, 22  
 classical composite formal series 21  
 composite formal series 19  
 composition of CASEs 23  
 convergent CASE 29, 134  
 differentiation of CASEs 26  
 fast part of a CASE 19, 23  
 generalized CASE 28  
 integration of CASEs 27  
 slow part of a CASE 19, 23
- D**
- De Maesschalck, Peter 119, 120, 139, 146  
 $\delta$ -descending 94  
 Diener, Marc 119, 127, 128  
 Dorodnitsyn, A. A. 118  
 Dumortier, Freddy 120, 152
- E**
- Eckhaus, Wiktor 33, 151  
 El Hamidi, Abdallah vi, 21, 82  
 Elimination of the time v, 118
- Erdélyi, A. 151  
 Exponential decay 21  
 Exponentially small vii
- F**
- Fast variable 4  
 Fenichel, Niel 152  
 Flat 21, 53  
 in the strong sense 53  
 in the weak sense 53  
 Forget, Thomas 12, 15, 121  
 Fraenkel, L. E. 151  
 Fruchard, Augustin vi, 17, 21, 82, 84, 119, 120, 123, 132, 139, 145
- G**
- Gautheron, Véronique 31, 119, 132  
 Geometric singular perturbation 152  
 Gevrey CASE 43, 50  
 composition of CASEs 76  
 consistent 43  
 differentiation of CASEs 48  
 integration of CASEs 48, 52  
 order 43  
 simultaneous integration of CASEs 78  
 type 43, 44, 105  
 Good covering vii  
 consistent 60, 63  
 resolution 61
- I**
- Isambert, Emmanuel 31, 119, 128, 129, 132, 134
- K**
- Kaplun, S. 151  
 Kevorkian, J. vi, 118
- L**
- Lagerstrom, P. A. 151  
 Landscape 94, 112  
 Lobry, Claude 9

**M**

Matzinger, Éric 116  
 Monomial  
   expansion 23, 153  
   Gevrey expansion 48, 153  
   summability 48  
 Mountain 10, 13, 99, 100, 103, 112  
 Mozo, Jorge 23, 48

**O**

O'Malley, Robert E. 118, 119, 139  
 Overstability 1, 3, 10, 13

**Q**

Quasi-linear equation 89, 94, 105

**R**

Ramis, Jean-Pierre vii, 43, 68, 82, 145  
 Ramis-Sibuya theorem  
   for CASEs 63  
   classical 68  
 Regular point v, 81  
   attracting vi  
 Roussarie, Robert 152

**S**

Schäfke, Reinhard vii, 17, 23, 43, 48, 82, 84,  
 119, 120, 123, 132, 139, 145

Sector 17  
   annulus 18, 23, 28, 29, 48, 59, 60, 79,  
     153  
   infinite quasi-sector 18  
   quasi-sector 18  
 Series of residues 27  
 Sibuya, Yasutaka vii, 43, 68, 82, 145, 152  
 Skinner, Lindsay A. vi, 29, 151  
 Slow function 81  
 Slow set v, 81  
 Slow variable 4

**T**

Truncated Laplace transform 56, 85, 100,  
 110  
 Turning point v, 1, 81, 106

**U**

Union Jack equation 127

**V**

Valley 10, 99, 100, 103, 112  
 Van der Pol equation 118  
 Vasil'eva, Adelaida Borisovna v, 21, 82

**W**

Wallet, Guy 1, 3, 10, 82  
 Wasow, Wolfgang 30, 151  
 Watson lemma 54

Edited by J.-M. Morel, B. Teissier; P.K. Maini

**Editorial Policy** (for the publication of monographs)

1. Lecture Notes aim to report new developments in all areas of mathematics and their applications - quickly, informally and at a high level. Mathematical texts analysing new developments in modelling and numerical simulation are welcome.

Monograph manuscripts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. They may be based on specialised lecture courses. Furthermore, the manuscripts should provide sufficient motivation, examples and applications. This clearly distinguishes Lecture Notes from journal articles or technical reports which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this “lecture notes” character. For similar reasons it is unusual for doctoral theses to be accepted for the Lecture Notes series, though habilitation theses may be appropriate.

2. Manuscripts should be submitted either online at [www.editorialmanager.com/lnm](http://www.editorialmanager.com/lnm) to Springer’s mathematics editorial in Heidelberg, or to one of the series editors. In general, manuscripts will be sent out to 2 external referees for evaluation. If a decision cannot yet be reached on the basis of the first 2 reports, further referees may be contacted: The author will be informed of this. A final decision to publish can be made only on the basis of the complete manuscript, however a refereeing process leading to a preliminary decision can be based on a pre-final or incomplete manuscript. The strict minimum amount of material that will be considered should include a detailed outline describing the planned contents of each chapter, a bibliography and several sample chapters. Authors should be aware that incomplete or insufficiently close to final manuscripts almost always result in longer refereeing times and nevertheless unclear referees’ recommendations, making further refereeing of a final draft necessary. Authors should also be aware that parallel submission of their manuscript to another publisher while under consideration for LNM will in general lead to immediate rejection.
3. Manuscripts should in general be submitted in English. Final manuscripts should contain at least 100 pages of mathematical text and should always include

- a table of contents;
- an informative introduction, with adequate motivation and perhaps some historical remarks: it should be accessible to a reader not intimately familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

For evaluation purposes, manuscripts may be submitted in print or electronic form (print form is still preferred by most referees), in the latter case preferably as pdf- or zipped psfiles. Lecture Notes volumes are, as a rule, printed digitally from the authors’ files. To ensure best results, authors are asked to use the LaTeX2e style files available from Springer’s web-server at:

<ftp://ftp.springer.de/pub/tex/latex/svmonot1/> (for monographs) and  
<ftp://ftp.springer.de/pub/tex/latex/svmult1/> (for summer schools/tutorials).

Additional technical instructions, if necessary, are available on request from [lnm@springer.com](mailto:lnm@springer.com).

4. Careful preparation of the manuscripts will help keep production time short besides ensuring satisfactory appearance of the finished book in print and online. After acceptance of the manuscript authors will be asked to prepare the final LaTeX source files and also the corresponding dvi-, pdf- or zipped ps-file. The LaTeX source files are essential for producing the full-text online version of the book (see <http://www.springerlink.com/openurl.asp?genre=journal&issn=0075-8434> for the existing online volumes of LNM). The actual production of a Lecture Notes volume takes approximately 12 weeks.
5. Authors receive a total of 50 free copies of their volume, but no royalties. They are entitled to a discount of 33.3 % on the price of Springer books purchased for their personal use, if ordering directly from Springer.
6. Commitment to publish is made by letter of intent rather than by signing a formal contract. Springer-Verlag secures the copyright for each volume. Authors are free to reuse material contained in their LNM volumes in later publications: a brief written (or e-mail) request for formal permission is sufficient.

**Addresses:**

Professor J.-M. Morel, CMLA,  
École Normale Supérieure de Cachan,  
61 Avenue du Président Wilson, 94235 Cachan Cedex, France  
E-mail: [morel@cmla.ens-cachan.fr](mailto:morel@cmla.ens-cachan.fr)

Professor B. Teissier, Institut Mathématique de Jussieu,  
UMR 7586 du CNRS, Équipe “Géométrie et Dynamique”,  
175 rue du Chevaleret  
75013 Paris, France  
E-mail: [teissier@math.jussieu.fr](mailto:teissier@math.jussieu.fr)

*For the “Mathematical Biosciences Subseries” of LNM:*

Professor P. K. Maini, Center for Mathematical Biology,  
Mathematical Institute, 24-29 St Giles,  
Oxford OX1 3LP, UK  
E-mail : [maini@maths.ox.ac.uk](mailto:maini@maths.ox.ac.uk)

Springer, Mathematics Editorial, Tiergartenstr. 17,  
69121 Heidelberg, Germany,  
Tel.: +49 (6221) 4876-8259

Fax: +49 (6221) 4876-8259  
E-mail: [lnm@springer.com](mailto:lnm@springer.com)