Appendix A
Nonexponential Asymptotic Solutions of Systems of Functional-Differential Equations

As was mentioned in the introduction, the methods presented for constructing particular solutions are valid for a wider class of objects than systems of ordinary differential equations, among the simplest of which are systems of differential equations with deviating arguments:

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}(t), \mathbf{x}(t+t_1), \dots, \mathbf{x}(t+t_s)\right), \ \mathbf{x} \in \mathbb{R}^n, \ t_1, \dots, t_s \in \mathbb{R}, \tag{A.1}$$

where $\mathbf{f} = \mathbf{f}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \dots, \mathbf{x}_{(s)}\right)$ is a smooth vector function of its arguments, where we will suppose that each of its components can be expanded in a Maclaurin series in the component vectors $\mathbf{x}_{(0)} = \mathbf{x}(t), \dots, \mathbf{x}_{(s)} = \mathbf{x}(t+t_s)$.

For an acquaintance with the theory of such systems we can recommend the monograph [47]. We suppose that $\mathbf{x}(t) \equiv \mathbf{0}$ is a trivial solution of the system (A.1), i.e. $\mathbf{f}(\mathbf{0}, \mathbf{0}, \dots, \mathbf{0}) = \mathbf{0}$. Then the question of the stability of such a solution arises entirely naturally. The Lyapunov theory of stability with some (occasionally very substantive) changes translates to systems with deviating arguments. We won't dwell on the details and refer to the monograph [47] just cited. N.N. Krasovskiy proved theorems about stability and instability with respect to the first approximation (see his original paper [123] or see [47]). These theorems were mainly proved by means of the *second* Lyapunov method, which is likewise used for the analysis of critical cases [165–168]. It should be noted right away that the analysis of critical cases for systems of equations with a deviating argument is a highly laborious problem and that stability criteria are practically never expressed in terms of the coefficients of the *original* system. Thus the classical ideas of Lyapunov's first method is rather rarely used in the theory of such systems. We show that these ideas are useful in obtaining necessary conditions for instability precisely in the critical cases. For systems of the type considered, an important theorem on the center manifold was proved by Yu.S. Osipov under additional assumptions [147], which allowed the connection of investigations of the critical cases of stability of systems of equations with deviating argument with the analysis of certain finite-dimensional systems of differential equations. We will consider "supercritical" cases, where all

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the roots of the first approximation system are zero and, consequently, reduction on the finite-dimensional center manifold is impossible.

We equip the space $(\mathbb{R}^n)^{s+1}$ of variables $\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}$ with a quasihomogeneous structure. As usual, let $\mathbf{G}$ be some matrix with real elements, whose eigenvalues have strictly *positive* real parts. We represent the group of quasihomogeneous dilations in the following form:

$$\mathbf{x}_{(0)} \mapsto \mu_{(0)}^{\mathbf{G}}\mathbf{x},\ \mathbf{x}_{(1)} \mapsto \mu_{(1)}^{\mathbf{G}}\mathbf{x}, \ldots, \mathbf{x}_{(s)} \mapsto \mu_{(s)}^{\mathbf{G}}\mathbf{x}.$$

Definition A.1. We say that the system of equations (A.1) is *quasihomogeneous* with respect to the quasihomogeneous structure generated by the matrix $\mathbf{G}$, and we denote its right side by $\mathbf{f} = \mathbf{f}_{\mathrm{q}}$ if, for any $\mathbf{x}_{(0)}$, $\mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}$ and any $\mu \in \mathbb{R}^+$, the following equality is satisfied:

$$\mathbf{f}_{\mathrm{q}}\left(\mu_{(0)}^{\mathbf{G}}\mathbf{x}, \mu_{(1)}^{\mathbf{G}}\mathbf{x}, \ldots, \mu_{(s)}^{\mathbf{G}}\mathbf{x}\right) = \mu^{\mathbf{G}+\mathbf{E}}\mathbf{f}_{\mathrm{q}}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}\right). \tag{A.2}$$

It should be noted that, although for quasihomogeneous systems of ordinary differential equations the quasihomogeneous structure itself gives rise to a particular solution of ray type, quasihomogeneous systems of equations with deviating arguments generally don't have such solutions.

Definition A.2. We call the system of equations (A.1) *semi-quasihomogeneous* with respect to the structure generated by the matrix $\mathbf{G}$ if its right side can be represented as a formal sum

$$\mathbf{f}_{\mathrm{q}}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}\right) = \sum_{m=0}^{\infty} \mathbf{f}_{\mathrm{q}+m}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}\right)$$

such that there exists a positive number β such that, for any

$$\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}$$

and for any $m = 0, 1, 2, \ldots$, the form $\mathbf{f}_{\mathrm{q}+m}$ satisfies the equality:

$$\mathbf{f}_{\mathrm{q}+m}\left(\mu_{(0)}^{\mathbf{G}}\mathbf{x}, \mu_{(1)}^{\mathbf{G}}\mathbf{x}, \ldots, \mu_{(s)}^{\mathbf{G}}\mathbf{x}\right) \mu^{\mathbf{G}+m\beta\mathbf{E}}\mathbf{f}_{\mathrm{q}+m}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}\right). \tag{A.3}$$

By virtue of the specific systems of equations of type (A.1), it makes sense only to consider positive semi-quasihomogeneous systems, so that everywhere in the sequel we will assume that $\beta > 0$.

The selection of quasihomogeneous truncations of systems of equations with a deviating argument can be realized with the aid of the already described technique of Newton manifolds, which we will amply employ.

We consider a model system of *ordinary* differential equations:

$$\dot{\mathbf{x}} = \mathbf{g}_q(\mathbf{x}), \tag{A.4}$$

where

$$\mathbf{g}_q(\mathbf{x}) = \mathbf{f}_q(\mathbf{x}, \dots, \mathbf{x}).$$

It turns out that, in the “supercritical” case, a deviating argument in fact has no effect on the *instability* of the system. Roughly speaking, if system (A.4) is unstable, so too will system (A.1) be unstable. More precisely, the following assertion holds, which generalizes a theorem in the article [64].

Theorem A.1. *Let the system (A.1) be semi-quasihomogeneous and suppose that all the following conditions hold:*

1. *There exist a vector* $\mathbf{x}_0^\gamma \in \mathbb{R}^n$, $\mathbf{x}_0^\gamma \neq \mathbf{0}$ *and a number* $\gamma = \pm 1$ *such that the following equality holds:*

$$-\gamma \mathbf{G}\mathbf{x}_0^\gamma = \mathbf{g}_q(\mathbf{x}_0^\gamma), \tag{A.5}$$

2. $$\operatorname{sign}\gamma = \operatorname{sign} t_j, \quad j = 1, \dots, s. \tag{A.6}$$

Then system (A.1) has a particular solution $\mathbf{x}(t) \to 0$ *as* $St \to \gamma \times \infty$.

Proof.

First step. Construction of a formal solution.

We will look for the solution in the customary form:

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}} \sum_{k=0}^{\infty} \mathbf{x}_k \left(\ln(\gamma t)\right) (\gamma t)^{-k\beta}. \tag{A.7}$$

For proving the existence of a formal solution of system (A.1) in the form (A.7), we apply Theorem 3.3.2. Let $t_j \neq 0$ be one of the numbers $t_1, \dots, t_s$. We consider the formal expansion of the vector function $\mathbf{x}_{(j)}(t) = \mathbf{x}(t + t_j)$ in powers of the quantity t_j:

$$\mathbf{x}_{(j)}(t) = \sum_{p=0}^{\infty} \frac{\mathbf{x}^{(p)}(t)}{p!} (t_j)^p.$$

From this expansion it is clear that from the formal point of view the system of equations (A.1) can be rewritten in the form of the following system of ordinary differential equations

$$\dot{\mathbf{x}} = \tilde{\mathbf{f}}\left(\mathbf{x}(t), \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t), \dots, \mathbf{x}^{(p)}(t), \dots\right), \tag{A.8}$$

whose right side contains an *unlimited* number of higher order derivatives.

System (A.8), implicit in (and generally unsolvable for) the higher derivatives, is positive semi-quasihomogeneous with respect to the structure given by the matrix $\mathbf{G}$ in the sense of Definition 3.3.5. Its quasihomogeneous truncation clearly coincides with the system of ordinary differential equations (A.4) (see Eqs. (A.2) and (A.3)) which, in view of (A.5), has a solution in the form of a quasihomogeneous ray. Therefore the existence of a formal particular solution of system (A.1) in the form (A.7) automatically follows from Theorem 3.3.2.

It should be noted that we did not use condition (A.6) for constructing a *formal* asymptotic solution for the system considered.

Second step. The proof of existence of an asymptotic solution for system (A.1), for which (A.7) is an asymptotic expansion.

We make a change of dependent and independent variables by the formulas:

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}}\mathbf{y}(\xi), \quad \xi = \varepsilon^{-1}(\gamma t)^{-\beta}, \quad 0 < \varepsilon \ll 1.$$

After this, system (A.1) assumes the form:

$$-\gamma\beta\xi\frac{d\mathbf{y}}{d\xi}(\xi) = \gamma\mathbf{G}\mathbf{y}(\xi) + \sum_{m=0}\varepsilon^m\xi^m\mathbf{f}_{q+m}\left(\mathbf{y}_{(0)}(\xi,\varepsilon),\ldots,\mathbf{y}_{(s)}(\xi,\varepsilon)\right), \tag{A.9}$$

where we have introduced the following notation:

$$\mathbf{y}_{(j)}(\xi,\varepsilon) = \left(1+\gamma t_j(\varepsilon\xi)^{1/\beta}\right)^{-\mathbf{G}}\mathbf{y}\left(\xi\left(1+\gamma t_j(\varepsilon\xi)^{1/\beta}\right)^{-\beta}\right), \\ j = 0,\ldots,s, \quad t_0 = 0.$$

Let $\mathbf{y}(\xi)$ be a continuous function on the interval $[0, 1]$. It is easy to see that, as $\varepsilon \to 0+$, the vector function $\mathbf{y}_{(j)}(\xi,\varepsilon)$ tends to $\mathbf{y}(\xi)$ uniformly on $[0, 1]$, for any $j = 0,\ldots,s$.

The rest of the proof is almost an exact repetition of the proof of Theorem 1.1.2 (second step), in connection with the application to system (A.9) of the implicit function theorem [94]. From this alone we get a clear sense of condition (A.6): for its satisfaction $\mathbf{y}_{(j)}(\xi,\varepsilon)$ belongs to $\mathbf{C}[0, 1]$ only if the vector function $\mathbf{y}(\xi)$ belongs to $\mathbf{C}[0, 1]$, along with its first derivative, and $\varepsilon > 0$ is sufficiently small.

The only essential difference in the proof of the given theorem from the proof of Theorem 1.1.2 consists of the following. Consider the linear operator

$$\mathbf{T}_{(j)}(\varepsilon)\colon \mathfrak{B}_{1,\Delta} \to \mathfrak{B}_{0,\Delta}, \quad j = 1,\ldots,s,$$

acting on vector functions $\mathbf{z} \in \mathfrak{B}_{1,\Delta}$ according to the rule:

$$\mathbf{T}_{(j)}(\varepsilon)(\mathbf{z})(\xi) = \\ = \mathbf{z}_{(j)}(\xi,\varepsilon) = \left(1+\gamma t_j(\varepsilon\xi)^{1/\beta}\right)^{-\mathbf{G}}\mathbf{z}\left(\xi\left(1+\gamma t_j(\varepsilon\xi)^{1/\beta}\right)^{-\beta}\right).$$

It should be kept in mind that $\mathbf{T}_{(j)}(\varepsilon)$ can be considered on a much larger space, i.e. as a linear operator mapping the space $\mathfrak{B}_{0,\Delta}$ into itself. However, $\mathbf{T}_{(j)}(\varepsilon)$ is a continuous operator only as a transformation from $\mathfrak{B}_{1,\Delta}$ into $\mathfrak{B}_{0,\Delta}$. For example, the proof presented doesn't hold for a system of equations of neutral type where the deviating arguments enter into the expression with the higher derivatives (regarding classical systems of equations with deviating argument, see the monograph [47]).

Thus we prove the continuity of $\mathbf{T}_{(j)}(\varepsilon)$ in ε. This proof is based on the mean value theorem and the fact that the positive constant β can always be taken to be less than unity:

$$\begin{aligned}
&\left\|\mathbf{T}_{(j)}(\varepsilon_1)-\mathbf{T}_{(j)}(\varepsilon_2)\mathbf{z}\right\|_{0,\Delta}=\\
&=\left\|\left(1+\gamma t_j(\varepsilon_1\xi)^{1/\beta}\right)^{-\mathbf{G}}\mathbf{z}\left(\xi\left(1+\gamma t_j(\varepsilon_1\xi)^{1/\beta}\right)^{-\beta}\right)-\right.\\
&\left.-\left(1+\gamma t_j(\varepsilon_2\xi)^{1/\beta}\right)^{-\mathbf{G}}\mathbf{z}\left(\xi\left(1+\gamma t_j(\varepsilon_2\xi)^{1/\beta}\right)^{-\beta}\right)\right\|_{0,\Delta}\le\\
&\le\sup\nolimits_{\xi\in[0,1],\varepsilon\in(0,\varepsilon_0)}\xi^{-\Delta+1/\beta}\gamma t_j\left\|-\mathbf{G}\left(1+\gamma t_j(\varepsilon\xi)^{1/\beta}\right)^{-(\mathbf{G}+\mathbf{E})}\times\right.\\
&\times\mathbf{z}\left(\xi\left(1+\gamma t_j(\varepsilon\xi)^{1/\beta}\right)^{-\beta}\right)+\left(1+\gamma t_j(\varepsilon\xi)^{1/\beta}\right)^{-(\mathbf{G}+(\beta+1)\mathbf{E})}\times\\
&\left.\times\left(1+\gamma t_j(\varepsilon\xi)^{1/\beta}-\beta\xi\right)\mathbf{z}'\left(\xi\left(1+\gamma t_j(\varepsilon\xi)^{1/\beta}\right)^{-\beta}\right)\right\|\times\\
&\times\left|\varepsilon_1^{1/\beta}-\varepsilon_2^{1/\beta}\right|\le C\|\mathbf{z}\|_{1,\Delta}\left|\varepsilon_1^{1/\beta}-\varepsilon_2^{1/\beta}\right|,
\end{aligned}$$

where $\mathbf{z}\in\mathfrak{B}_{1,\Delta}$ is an arbitrary vector function and the constant $C>0$ doesn't depend on $\mathbf{z}$.

The theorem is proved.

To illustrate the proof of the theorem, we consider a simple example.

Example A.1. We consider a system of two differential equations:

$$\dot{x}(t)=-\frac{1}{3}x^4(t),\quad \dot{y}(t)=-\frac{2}{3}x(t+t_1)y^2(t).$$

According to the terminology we have introduced, this system is quasihomogeneous with respect to the structure given by the matrix $\mathbf{G}=\operatorname{diag}(1/3,2/3)$ which, as has already been noted, doesn't guarantee the existence of a particular solution in the form of a quasihomogeneous ray. It is, nonetheless, rather easy to "guess" at an asymptotic solution as $t\to\pm\infty$ of the system considered:

$$x(t)=t^{-1/3},\quad y(t)=(t+t_1)^{-2/3}.$$

This asymptotic solution admits an expansion into a series of form (A.7)

$$x(t)=t^{-1/3},\quad y(t)=t^{-2/3}\left(1-\frac{2}{3}\frac{t_1}{t}+\frac{5}{9}\left(\frac{t_1}{t}\right)^2-\ldots\right).$$

The model system of ordinary differential equations

$$\dot{x} = -\frac{1}{3}x^4, \quad \dot{y}(t) = -\frac{2}{3}xy^2,$$

has a particular solution in the form of a ray,

$$x^{\pm}(t) = t^{-1/3}, \quad y^{\pm}(t) = t^{-2/3},$$

which represents the principal term of the asymptotic expansion of the particular solution we found for the full system.

Theorem A.1 has important applications in stability theory.

Theorem A.2. *Let the system of equations (A.1) of delay type* $(t_j < 0, j = 1, \ldots, s)$ *be semi-quasihomogeneous and let there exist a vector* $\mathbf{x}_0^- \in \mathbb{R}^n$, $\mathbf{x}_0^-$ *such that the equality*

$$\mathbf{G}\mathbf{x}_0^- = \mathbf{g}_q(\mathbf{x}_0^-).$$

holds. Then the trivial solution, $\mathbf{x}(t) \equiv \mathbf{0}$, *of (A.1) is unstable.*

Proof. This is immediate from the existence of a particular solution of the system (A.1): $\mathbf{x}(t) \to \mathbf{0}$ as $t \to -\infty$.

Remark A.1. From Theorem A.1 it follows that, by fulfilling the specified hypothesis, a system of *advanced* type (i.e. $t_j > 0,\ j = 1, \ldots, s$) has a particular solution that is smooth on some *positive* half-line $[T, +\infty)$. This is a rather remarkable fact in that the Cauchy problem, as is known, isn't ÜÜwell posedÝÝ for the advanced problem.

The series constructed in Example A.1 are clearly convergent for $|t| > |t_1|$. In the general case, the problem of determining the convergence of the series (A.7) is more complicated than for ordinary differential equations and we won't dwell on it at length. Evidently the fact that a method of Chap. 1, based on the implicit function theorem, was applied to the system under investigation attests in favor of convergence. On the other hand, for the proof of the existence of a formal solution, we use Theorem 3.3.2, which is the route we expect to follow in analyzing *singular* problems for which formal series typically diverge. However, we remark that, as in the case of ordinary equations, the presence of a *nontrivial* linear part leads to divergence of the asymptotic series. The formulation of the theorem on the existence of asymptotic solutions in this case would require a discussion of the theory of the center manifold for systems of functional-differential equations, so that we will not present a general result here, but rather confine ourselves to considering examples analogous to Example 3.1.1. Some general results can be found in the article [64].

Example A.2. We consider a system of two equations

$$\dot{x} = -x(t) + y(t + t_1), \quad \dot{y} = -y^2(t), \quad t_1 < 0,$$

the second of which has the obvious solution $y(t) = t^{-1}$. Upon substitution into the first equation we obtain

$$x(t) = e^{-t} \int_{-\infty}^{t} (s + t_1)^{-1} e^{s} \, ds.$$

This function clearly belongs to the space $\mathbf{C}^{\infty}(-\infty, t_1)$, tends to zero as $t \to -\infty$ and its expansion into a formal series may diverge everywhere (although it is Borel summable [75]).

$$x(t) = \sum_{k=1}^{\infty} (k-1)!(t + t_1)^{-k}.$$

This series can be converted into a power series in whole negative powers of t. We find the asymptotic of its coefficients:

$$x(t) = e^{-t_1} \int_{-\infty}^{t_1} (\sigma + t)^{-1} e^{\sigma} \, d\sigma = -e^{-t_1} \sum_{k=1}^{\infty} \Gamma(k, -t_1) t^{-k},$$

where

$$\Gamma(k, \tau) = \int_{\tau}^{-\infty} u^{k-1} e^{u} \, du$$

is the incomplete Euler gamma function [91].

It is easy to see that the following bound holds:

$$\Gamma(k, \tau) \geq (k-1)! - k^{-1} \tau^{k}.$$

An account of the above method extends, generally speaking, to a broader class of systems. It evidently applies to functional-differential equations of the Volterra type in their most general form (for an account of the theory of these equations see the monograph [73]). Here we won't, however, introduce such generality and merely show how to apply this circle of ideas to autonomous functional-differential equations of the retarded type with discrete and distributed time lags. Thus we consider a system of equations to the form

$$\begin{aligned} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t - t_1), \ldots, \mathbf{x}(t - t_s), \textstyle\int_{-\infty}^{t} \mathbf{L}_1(t-u)\mathbf{x}(u)\, du, \ldots, \\ \ldots, \textstyle\int_{-\infty}^{t} \mathbf{L}_r(t-u)\mathbf{x}(u)\, du) \end{aligned} \tag{A.10}$$

where $t_1 > 0$, $i = 1, \ldots, s$ (for a system of retarded type with discrete time lags we will ascribe a *minus* sign), the $\mathbf{L}_j(\sigma)$, $j = 1, \ldots, r$, are matrix functions that are continuous on the positive interval $(0, +\infty)$, and

$$\mathbf{f} = \mathbf{f}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}, \mathbf{x}_{[1]}, \ldots, \mathbf{x}_{[r]}\right)$$

is a smooth vector function in its arguments, where we have introduced the notation:

$$\mathbf{x}_{[1]} = \int_{-\infty}^{t} \mathbf{L}_1(t-u)\mathbf{x}(u)\,du, \ldots, \mathbf{x}_{[r]} = \int_{-\infty}^{t} \mathbf{L}_r(t-u)\mathbf{x}(u)\,du.$$

As before, let $\mathbf{x}(t) \equiv \mathbf{0}$ be the trivial solution of the system considered. Our goal will be to establish instability criteria of this solution in "supercritical" cases. If there are no terms with discrete lags $\mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(s)}$ on the right side of system (A.10), then (A.10) is a system of integro-differential equations of Volterra type (for the basic theory of these systems, see e.g. [35]). In noncritical cases an analogue of Lyapunov's first method has been worked out for systems of such form [162, 163].

We make additional assumptions regarding system (A.10). Usually we assume that the components of the matrices $\mathbf{L}_j(\sigma)$, $j = 1, \ldots, r$ are exponentially decreasing, i.e. that we have the inequalities

$$\|\mathbf{L}_j(\sigma)\| \le L_j e^{-a_j \sigma}, \quad \sigma \in (0, +\infty), \quad j = 1, \ldots, r.$$

We impose harsher conditions on the matrices $\mathbf{L}_j(\sigma)$, i.e. we suppose that they can be expanded in absolutely convergent series of the form:

$$\mathbf{L}_j(\sigma) = \sum_{l=1}^{\infty} e^{-a_{jl}\sigma} \left(\mathbf{M}_{jl} \cos(b_{jl}\sigma) + \mathbf{N}_{jl} \sin(b_{jl}\sigma) \right). \tag{A.11}$$

We introduce the notations:

$$\begin{aligned}
\mathbf{M}_{jl}^{(0)} &= \mathbf{M}_{jl}, \quad \mathbf{N}_{jl}^{(0)} = \mathbf{N}_{jl}, \\
\mathbf{M}_{jl}^{(p+1)} &= \frac{1}{a_{jl}^2 + b_{jl}^2} \left(a_{jl}\mathbf{M}_{jl}^{(p)} + b_{jl}\mathbf{N}_{jl}^{(p)} \right), \\
\mathbf{N}_{jl}^{(p+1)} &= \frac{1}{a_{jl}^2 + b_{jl}^2} \left(b_{jl}\mathbf{M}_{jl}^{(p)} - a_{jl}\mathbf{N}_{jl}^{(p)} \right),
\end{aligned}$$

for $p = 0, 1, \ldots$.

The following assumption, stronger than the assumption of absolute convergence for the series (A.11), consists of the matrix series

$$\sum_{l=1}^{\infty} \mathbf{M}_{jl}^{(p)}, \quad \sum_{l=1}^{\infty} \mathbf{N}_{jl}^{(p)}, \tag{A.12}$$

which is absolutely convergent to certain matrices $\mathbf{M}_j^{(p)}$, $\mathbf{N}_j^{(p)}$, $j = 1, \ldots, r$, $p = 0, 1, \ldots$

For a system of type (A.10) we introduce concepts of quasihomogeneity and semi-quasihomogeneity.

Definition A.3. We say that the system of equations (A.10) is *quasihomogeneous* with respect to the quasihomogeneous structure generated by the matrix $\mathbf{G}$ and will denote its right side by $\mathbf{f} = \mathbf{f}_\mathrm{q}$ if, for any system $\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \dots, \mathbf{x}_{(s)}, \mathbf{x}_{[1]}, \dots, \mathbf{x}_{[r]}$ and any $\mu \in \mathbb{R}^+$, we have the equality

$$\begin{aligned}
&\mathbf{f}_\mathrm{q}\left(\mu_{(0)}^{\mathbf{G}}\mathbf{x}, \mu_{(1)}^{\mathbf{G}}\mathbf{x}, \dots, \mu_{(s)}^{\mathbf{G}}\mathbf{x}, \mu_{[1]}^{\mathbf{G}}\mathbf{x}, \dots, \mu_{[r]}^{\mathbf{G}}\mathbf{x}\right) = \\
&= \mu^{\mathbf{G}+\mathbf{E}}\mathbf{f}_\mathrm{q}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \dots, \mathbf{x}_{(s)}, \mathbf{x}_{[1]}, \dots, \mathbf{x}_{[r]}\right).
\end{aligned} \tag{A.13}$$

Definition A.4. We say that the system of equations (A.10) is *semi-quasihomogeneous* with respect to the structure generated by the matrix $\mathbf{G}$ if its right side is represented as a formal sum

$$\mathbf{f}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \dots, \mathbf{x}_{(s)}\right) = \sum_{m=0}^{\infty} \mathbf{f}_{\mathrm{q}+m}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \dots, \mathbf{x}_{(s)}\right)$$

such that, for any $\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \dots, \mathbf{x}_{(s)}, \mathbf{x}_{[1]}, \dots, \mathbf{x}_{[r]}$ and $\mu \in \mathbb{R}^+$ for any m–th form $\mathbf{f}_{\mathrm{q}+m}$, we have the equality

$$\begin{aligned}
&\mathbf{f}_{\mathrm{q}+m}\left(\mu_{(0)}^{\mathbf{G}}\mathbf{x}, \mu_{(1)}^{\mathbf{G}}\mathbf{x}, \dots, \mu_{(s)}^{\mathbf{G}}\mathbf{x}, \mu_{[1]}^{\mathbf{G}}\mathbf{x}, \dots, \mu_{[r]}^{\mathbf{G}}\mathbf{x}\right) = \\
&= \mu^{\mathbf{G}+m\beta\mathbf{E}}\mathbf{f}_{\mathrm{q}+m}\left(\mathbf{x}_{(0)}, \mathbf{x}_{(1)}, \dots, \mathbf{x}_{(s)}, \mathbf{x}_{[1]}, \dots, \mathbf{x}_{[r]}\right),
\end{aligned} \tag{A.14}$$

where β is some positive number.

We consider a model system of ordinary differential equations:

$$\dot{\mathbf{x}} = \mathbf{g}_\mathrm{q}(\mathbf{x}), \tag{A.15}$$

where

$$\mathbf{g}_\mathrm{q}(\mathbf{x}) = \mathbf{f}_\mathrm{q}\left(\mathbf{x}, \dots, \mathbf{x}, \mathbf{M}_1^{(1)}\mathbf{x}, \dots, \mathbf{M}_r^{(1)}\mathbf{x}\right).$$

We will determine a connection between instability for the model system (A.15) and instability for the full system (A.10). However, instability for the full system bears a *formal* character that will be elaborated below.

Theorem A.3. *Let the system of equations (A.10) be semi-quasihomogeneous and suppose there exists a nonzero vector* $\mathbf{x}_0^-$ *such that we have the following equality:*

$$\mathbf{G}\mathbf{x}_0^- = g_\mathrm{q}(\mathbf{x}_0^-).$$

Then system (A.10) has a particular formal solution represented in the form of a series, each term of which converges to zero as $t \to -\infty$.

If the series referred to in the statement of Theorem A.3 converges, then system (A.10) will have a particular solution

$$\mathbf{x}(t) \to \mathbf{0} \ \text{ as } \ t \to -\infty,$$

which points to instability. We show below, in a concrete example, that this series can diverge. In the case considered, however, the theory of Kuznetsov [125, 126] is inapplicable, since this was developed only for systems of *ordinary* differential equations. We therefore speak only of *formal* instability.

Proof of Theorem A.3. The required particular formal solution will be sought in the usual form:

$$\mathbf{x}_{[j]}(t) = (-t)^{-\mathbf{G}} \sum_{k=0}^{\infty} \mathbf{x}\,(\ln(-t))\,(-t)^{-k\beta}. \tag{A.16}$$

Here we can again use Theorem 3.3.2. Applying integration by parts an infinite number of times to the vector functions $\mathbf{x}(t), \ldots, \mathbf{x}_{[r]}(t)$, we obtain the formal expansion

$$\mathbf{x}_{[j]}(t) = \sum_{p=0}^{\infty} (-1)^p \mathbf{M}_j^{(p+1)} \mathbf{x}^{(p)}(t)$$

for each $j = 1, \ldots, r$. Here we use the absolute convergence of the series (A.12).

Such a representation of the quantities $\mathbf{x}_{[1]}, \ldots, \mathbf{x}_{[r]}$ again converts the system being investigated into a system of ordinary differential equations of type (A.8). From the equalities (A.13) and (A.14), it follows that this system (implicit with respect to higher derivatives!) is semi-quasihomogeneous with respect to the structure given by the matrix $\mathbf{G}$ (see Definition 3.3.5), where its quasihomogeneous truncation coincides with system (A.15) which has, by the hypothesis of the theorem, a particular solution in the form of a ray tending to the point $\mathbf{x} = \mathbf{0}$ as $t \to -\infty$. Application of Theorem 3.3.2 allows us to assert that system (A.10) has a formal particular solution in the form of series (A.16).

The theorem is proved.

Example A.3. We consider the system of equations:

$$\dot{x} = x^2(t)\left(1 + \int_{-\infty}^{t} e^{-(t-u)} y(u)\,du\right), \quad \dot{y}(t) = y^2(t).$$

According to the definitions introduced above, this system is semi-(quasi)-homogeneous with respect to the structure generated by the identity matrix $\mathbf{E}$. The corresponding model system

$$\dot{x} = x^2, \quad \dot{y} = y^2$$

has the obvious asymptotic solution $x^-(t) = y^-(t) = -\dfrac{1}{t}$.

The second equation of the original system is easily integrated. Its asymptotic solution has the simple form:

$$y(t) = -t^{-1}.$$

But then the result of an integral transformation of the function $y(t)$,

$$\int_{-\infty}^{t} e^{-(t-u)} y(u)\, du = \sum_{k=1}^{\infty} (k-1)!t^{-k},$$

is represented as an everywhere divergent series.

We denote by $\phi(t)$ the formal expansion of the following integral:

$$\phi(t) = \int_{-\infty}^{t} \left(\int_{-\infty}^{u} e^{-(u-v)} v^{-1}\, dv - u^{-1} \right) du = -\sum_{k=2}^{\infty} (k-2)!t^{-k+1}.$$

Now we can find an expansion of a formal asymptotic solution for the first equation of the system considered:

$$x(t) = -t^{-1} \left(1 + t^{-1} \ln(-t) + t^{-1}\phi(t)\right)^{-1} =$$

$$= (-t)^{-1} \left(1 - (-t)^{-1} \ln(-t) + \sum_{k=2}^{\infty} (-1)^{k-1} (k-2)!(-t)^{-k} \right)^{-1}.$$

Having converted the last expression to an expansion in powers of $-t$, we obtain a series of type (A.16) which will, of course, diverge. As is to be expected, the principal term of this expansion will coincide with $x^{-}(t)$.

We consider some applications of the theory we have constructed.

Example A.4. In (1.63) of the first chapter we considered a so-called logistical system of ordinary differential equations, (1.64), describing processes of mutual interaction. We should, however, keep in mind that the impact upon the numbers of individuals in populations by the birthrates of species proceeds with a certain time lag, as reflected in the following mathematical model, described by a system of functional differential equations containing discrete and continuous time lags [180, 189]:

$$\dot{N}^i(t) =$$
$$= N^i(t) \left(k_i + b_i^{-1} \sum_{p=1}^{n} \left(a^i_p N^p(t - t_{ip}) + \int_{-\infty}^{t} f^i_p(t-u) N^p(u)\, du \right) \right), \qquad \text{(A.17)}$$

where $i = 1, \ldots, n$.

Here the $t_{ip} > 0, i, p = 1, \ldots, n$ are discrete lag arguments, and the matrix

$$\mathbf{F}(\sigma) = \left(f_p^i(\sigma)\right)_{i,p=1}^n$$

possesses the very same properties as the matrices $\mathbf{L}_1(\sigma), \ldots, \mathbf{L}_r(\sigma)$ that figure in the statement of Theorem A.3

We likewise introduce the following notation:

$$\tilde{\mathbf{F}} = \left(\tilde{f}_p^i\right)_{i,p=1}^n = \left(\int_0^\infty f_p^i(\sigma)\, d\sigma\right)_{i,p=1}^n .$$

We will find sufficient conditions for the instability of the trivial solution of system (A.17) for the "supercritical" case, where the values of all the "Malthusian birthrates" k_i, $i = 1, \ldots, n$, i.e. the average number of individuals in populations when left to themselves, equal zero. Thus the object of our investigations will be the system

$$\begin{gathered} \dot{N}^i(t) = \\ = b_i^{-1} N^i(t) \sum_{p=1}^n \left(a_p^i N^p(t - t_{ip}) + \int_{-\infty}^t f_p^i(t-u) N^p(u)\, du\right), \\ i = 1, \ldots, n. \end{gathered} \tag{A.18}$$

This system (A.18) of functional-differential equations of retarded type is (quasi)homogeneous, according to the definitions introduced. The corresponding model system of ordinary differential equations has the following appearance:

$$\dot{N}^i = b_i^{-1} N^i \sum_{p=1}^n \left(a_p^i + \tilde{f}_p^i\right) N^p, \quad i = 1, \ldots, n. \tag{A.19}$$

System (A.19) has an increasing solution in the form of a linear ray,

$$\mathbf{N}^-(t) = (-t)_0^{-1} \mathbf{N}_0^-, \quad \mathbf{N} = (N^1, \ldots, N^n),$$

provided the system, when solved with respect to $\mathbf{N}_0^- = (N_0^{-1}, \ldots, N_0^{-n})$, is a linear system of algebraic equations

$$\sum_{p=1}^n \left(a_p^i + \tilde{f}_p^i\right) N_0^{-p} = b_i .$$

Here, of course, we need only look for the *positive* solution of this linear system, since the population sizes must be positive.

The asymptotic particular solution as $t \to -\infty$ of the truncated system (A.19) generates a formal solution of the full system (A.18) of the form

$$\mathbf{N}(t) = \sum_{k=1}^{\infty} \mathbf{N}_k \left(\ln(-t)\right) (-t)^k .$$

In the case of convergence of this series, the existence of a similar solution would indicate instability. If a continuous time lag is absent, then by Theorem A.2 we will have Lyapunov instability. This instability has an explosive character: populations which find themselves on the verge of extinction at some initial moment begin to grow by a roughly exponential law, whereby the number of individuals in them theoretically can become infinite in finite time. This interesting effect, where populations "pull themselves up by their hair" (like baron Münchhausen) from the "morass of extinction", was discussed previously in the article [64] for systems with discrete time lags.

Appendix B
Arithmetic Properties of the Eigenvalues of the Kovalevsky Matrix and Conditions for the Nonintegrability of Semi-quasihomogeneous Systems of Ordinary Differential Equations

In the introduction we mentioned, in passing, a series of papers dedicated to problems of integrability of equations of motion and to the chaotic behavior of trajectories of nonlinear dynamical systems from the point of view of bifurcation, in the complex time plane, of particular solutions with generalized power asymptotic. It was noted that there are no rigorous results that allow us to treat properties such as the integrability of these solutions. In this appendix we state a series of results [65] concerning the nonintegrability of semi-quasihomogeneous differential equations obtained by using the proven existence of solutions with generalized power asymptotic. This should be regarded as a zero-th order theory, since it only contains properties of quasihomogeneous truncations and principal terms of asymptotic expansions of solutions. The results obtained constitute an extension of those in a paper by H. Yoshida [197], where it was first noted that the eigenvalues of the Kovalevsky matrix of an integrable quasihomogeneous system of equations must satisfy certain resonance relationships.

Here we introduce a different treatment of this idea, based on different considerations. In order to better understand the essence of this phenomenon, we consider the following simplified situation. Let there be given a system of differential equations with analytic right side:

$$\dot{\mathbf{u}} = \mathbf{h}(\mathbf{u}), \quad \mathbf{u} \in \mathbb{C}^{\mathbf{n}}, \tag{B.1}$$

for which the origin $\mathbf{u} = \mathbf{0}$ is a critical point ($\mathbf{h}(\mathbf{0}) = \mathbf{0}$).

Let $\mathbf{A} = \mathbf{dh}(\mathbf{0})$ be the Jacobian matrix of the vector field $\mathbf{h}(\mathbf{u})$, computed at the critical point. For simplicity, we further assume that the matrix $\mathbf{A}$ is diagonalizable and that the coordinates $(u^1, \dots, u^n)$ are chosen so that $\mathbf{A}$ already has diagonal form: $\mathbf{A} = \mathrm{diag}\,(\boldsymbol{\lambda}_\mathbf{1}, \dots, \boldsymbol{\lambda}_\mathbf{n})$.

The basic idea amounts to the observation that, if system (B.1) has an analytic integral, then the eigenvalues of the matrix $\mathbf{A}$ must satisfy certain resonance conditions. More precisely, we have

Lemma B.1. *Suppose* $\det \mathbf{A} \neq \mathbf{0}$ *and that, for any choice of nonnegative integers* $k_1, \dots, k_n$*,* $k_j \in \mathbb{N} \cup \{0\}$*, where* $\sum_{j=1}^{n} k_j \geq 1$,

V.V. Kozlov and S.D. Furta, *Asymptotic Solutions of Strongly Nonlinear Systems of Differential Equations*, Springer Monographs in Mathematics,
DOI 10.1007/978-3-642-33817-5,

$$\sum_{j=1}^{n} k_j \lambda_j \neq 0. \tag{B.2}$$

Then the system (B.1) has no integral that is infinitely differentiable in the neighborhood of $\mathbf{u} = \mathbf{0}$*, i.e. which admits a nontrivial formal Maclaurin expansion.*

Proof. Suppose there exists a smooth function $\phi(\mathbf{u})$ that admits expansion into a nontrivial formal Maclaurin series and is an integral of system (B.1). This function must satisfy the following first order partial differential equation:

$$\langle d\phi(\mathbf{u}), \mathbf{h}(\mathbf{u})\rangle = 0, \tag{B.3}$$

where the symbol $\langle\cdot,\cdot\rangle$ now denotes the Hermite scalar product on $\mathbb{C}^n$.

Without loss of generality we may suppose that $\phi(\mathbf{0}) = 0$. Consider the expansion of $\phi(\mathbf{u})$ into a series in the neighborhood of $\mathbf{u} = \mathbf{0}$:

$$\phi(\mathbf{u}) = \sum_{\mathbf{s}=\mathbf{1}}^{\infty} \phi_{(\mathrm{s})}(\mathbf{u}),$$

where the $\phi_{(s)}(\mathbf{u})$, $s = 1, 2, \ldots,$ are homogeneous polynomials in $\mathbf{u}$ of degree s.

We consider first the form of the expansion of the integral $\phi(\mathbf{u})$:

$$\phi_{(1)}(\mathbf{u}) = \langle \mathbf{b}, \mathbf{u}\rangle,$$

where $\mathbf{b} \in \mathbb{C}^{\mathbf{n}}$ is a constant vector. If we equate the terms of (B.3) that are linear in $\mathbf{u}$, we obtain the equality

$$\langle \mathbf{b}, \mathbf{A}\mathbf{u}\rangle = \mathbf{0}, \tag{B.4}$$

which must be satisfied for any $\mathbf{u} \in \mathbb{C}^{\mathbf{n}}$.

Then it follows from (B.4) that the vector $\mathbf{b}$ is an eigenvector of the matrix $\mathbf{A}^*$ (the symbol $(\ )^*$ denotes Hermite conjugation) with zero eigenvalue, which contradicts the condition $\det \mathbf{A} \neq \mathbf{0}$. Therefore $\mathbf{b} = \mathbf{0}$.

Suppose now that we have proved that $\phi_{(1)} \equiv \ldots \equiv \phi_{(s-1)} \equiv 0$. Then from (B.3) it follows that

$$\left\langle d\phi_{(s)}(\mathbf{u}), \mathbf{A}\mathbf{u}\right\rangle = 0. \tag{B.5}$$

We denote by $D^{(s)}\phi(\mathbf{v}, \mathbf{u})$ the s–th differential of the function ϕ at the point $\mathbf{v}$, computed at the vector $\mathbf{u}$.

On the basis of (B.5) we can make an important observation: the first nontrivial form in the expansion of the integral of system (B.1),

$$\phi_{(s)}(\mathbf{u}) = \mathbf{D}^{(\mathbf{s})}\phi(\mathbf{0}, \mathbf{u}),$$

is an integral of the linear system

$$\dot{\mathbf{u}} = \mathbf{A}\mathbf{u}.$$

Equation (B.5) can be rewritten in the following form:

$$\sum_{j=1}^{n} \lambda_j \frac{\partial \phi_{(s)}}{\partial u^j}(\mathbf{u}) u^j = 0. \tag{B.6}$$

We rewrite the homogeneous polynomials $\phi_{(s)}(\mathbf{u})$ in the form of a sum of elementary monomials:

$$\phi_{(s)}(u^1, \ldots, u^n) = \sum_{k_1+\ldots+k_n=s} \phi_{k_1\ldots k_n} (u^1)^{k_1} \ldots (u^n)^{k_n}.$$

Then (B.6) assumes the form:

$$\sum_{k_1+\ldots+k_n=s} (k_1\lambda_1 + \ldots + k_n\lambda_n)\, \phi_{k_1\ldots k_n} (u^1)^{k_1} \ldots (u^n)^{k_n} = 0, \tag{B.7}$$

and it follows from equality (B.7) that, if only the coefficient $\phi_{k_1\ldots k_n} \neq$ is nonzero, then

$$k_1\lambda_1 + \ldots + k_n\lambda_n = 0,$$

which contradicts condition (B.2).

The lemma is proved.

Remark B.1. The explicit requirement that $\det \mathbf{A} \neq \mathbf{0}$ can in fact be dropped, since condition (B.2) on the absence of resonances itself contains this requirement.

We now consider a more complicated situation. Let there be given a semi-quasihomogeneous system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}.) \tag{B.8}$$

Here it will be more convenient for us to use the traditional definitions of quasihomogeneity and semi-quasihomogeneity 1.1.2 and 1.1.4, whose introduction was based on the Newton manifold technique. Then the elements of the diagonal matrix $\mathbf{G} = \alpha\mathbf{S}$, $\alpha = \dfrac{1}{q-1}$, $\mathbf{S} = \mathrm{diag}\,(\mathbf{s_1}, \ldots, \mathbf{s_n})$, $s_j \in \mathbb{N} \cup \{0\}$, $j = 1, \ldots, n$, $q \in \mathbb{N}$, $q \neq 1$, that determine the quasihomogeneous structure, are rational and nonnegative.

We will consider two different problems:

(a) If system (B.8) is positive semi-quasihomogeneous, we investigate for the system the question of existence of smooth integrals, represented in the form of nontrivial Maclaurin series

$$F(x) = F(x^1, \ldots, x^n) = \sum_{k_1\geq 0,\ldots,k_n\geq 0}^{\infty} F_{k_1\ldots k_n} (x^1)^{k_1} \ldots (x^n)^{k_n}; \tag{B.9}$$

(b) If system (B.8) is negative semi-quasihomogeneous, then we will look for polynomial integrals, i.e. integrals for which expansion (B.9) contains only a finite number of terms.

We have:

Lemma B.2. *If the semi-quasihomogeneous system (B.8) has a nontrivial integral of the form (B.9), polynomial in the case of positive semi-quasihomogeneity, then the corresponding truncation of the system has a quasihomogeneous integral.*

Proof. We consider first case (a). The function $F(\mathbf{x})$ can be assumed to be positive semi-quasihomogeneous and re-expanded in a series of quasihomogeneous form, conforming to the quasihomogeneous structure with matrix $\mathbf{S}$:

$$F(\mathbf{x}) = \sum_{m=0}^{\infty} F_{N+m}(\mathbf{x}).$$

After the substitution $\mathbf{x} \mapsto \lambda^{\mathbf{S}}\mathbf{x}$ this function takes the form

$$F(\lambda^{\mathbf{S}}\mathbf{x}) = \lambda^{N} \sum_{m=0} \lambda^{m} F_{N+m}(\mathbf{x}).$$

Regardless of the sign of semi-quasihomogeneity the system of equations (B.8), after the substitution $\mathbf{x} \mapsto \mu^{\mathbf{G}}\mathbf{x}$, $t \mapsto \mu^{-1}t$, is written in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mu) = \mathbf{f}_{\mathrm{q}}(\mathbf{x}) + \sum_{m=1} \mu^{\beta m}\mathbf{f}_{\mathrm{q}+\chi \mathbf{m}}(\mathbf{x}). \tag{B.10}$$

Here $\mu = \lambda^{\mathrm{q}-1}$, $\beta = \pm\alpha$, $\chi = \operatorname{sign}\beta$.
In case (a), as $\mu \to 0$, Eq. (B.10) is transformed into the truncated system

$$\dot{\mathbf{x}} = \mathbf{f}_{\mathrm{q}}(\mathbf{x}). \tag{B.11}$$

System (B.10) clearly has, for any $\mu \in (0, +\infty)$, the integral

$$F(\mathbf{x},\mu) = F_N(\mathbf{x}) + \sum_{m=1} \mu^{\beta m} F_{N+\chi m}(\mathbf{x}), \tag{B.12}$$

(in case (a), $\chi = +1$) as $\mu \to 0+$, passing to the quasihomogeneous function

$$F_N(\mathbf{x}), \tag{B.13}$$

which in its turn is an integral of the truncated system (B.11).

Case (b) is treated analogously. The polynomial integral $F(\mathbf{x})$ decomposes into a *finite* sum of quasihomogeneous form, generated by the structure with matrix $\mathbf{G}$. Therefore this function may be taken to be negative semi-quasihomogeneous, whence the relation

$$F(\lambda^{\mathbf{S}}\mathbf{x}) = \lambda^N \sum_{m=0}^{M} \lambda^{-m} F_{N-m}(\mathbf{x}), \quad N \geq M.$$

In case (b), system (B.10) also has an integral of type (B.12), whose expansion in powers of μ^β, $\beta = -\alpha$ in fact contains only a finite number of terms. When $\mu \to +\infty$, system (B.10) passes to the truncated system (B.11), and the integral (B.12) to the integral (B.13) of the truncated system.

The lemma is proved.

We now consider more closely the truncated system (B.11), which we suppose has a particular solution in the form of a (possibly complex) quasihomogeneous ray

$$\mathbf{x}_{(0)}(t) = t^{-\mathbf{G}}\mathbf{x_0}, \tag{B.14}$$

where $\mathbf{x_0} \in \mathbb{C}^\mathbf{n}$ is a constant nonzero vector satisfying the equation

$$\mathbf{G}\mathbf{x_0} + \mathbf{f}_\mathrm{q}(\mathbf{x_0}) = 0.$$

It is clear that, if $F_N(\mathbf{x})$ is an integral of system (B.11), then $F_N(\mathbf{x_0}) = 0$.

We compute the Kovalevsky matrix for the linear system

$$\mathbf{K} = \mathbf{G} + d\mathbf{f}_\mathrm{q}(\mathbf{x_0}).$$

Let $\rho_1, \ldots, \rho_n$ be the roots of the characteristic equation

$$\det(\mathbf{K} - \rho\mathbf{E}) = \mathbf{0}.$$

For simplicity we will hereafter always assume that the Kovalevsky matrix reduces to diagonal form.

We have:

Lemma B.3. *Suppose that, for any choice of the numbers* $(k_1 \ldots k_n)$, $k_j \in \mathbb{N} \cup \{0\}$, $\sum_{j=1}^n k_j \geq 1$, *the inequality*

$$\sum_{j=1}^n k_j \rho_j \neq 0. \tag{B.15}$$

is satisfied. Then any quasihomogeneous integral of the truncated system (B.11) is trivial.

Proof. Suppose system (B.11) has a nontrivial quasihomogeneous integral (B.13). After the change of variables

$$\mathbf{x} = t^{-\mathbf{G}}(\mathbf{x_0} + \mathbf{u})$$

the system of equations (B.11) is transformed to the system

$$t\dot{\mathbf{u}} = \mathbf{K}\mathbf{u} + \hat{\mathbf{f}}(\mathbf{u}), \tag{B.16}$$

where $\mathbf{K}$ is the Kovalevsky matrix and $\hat{\mathbf{f}}(\mathbf{u})$ (with $\hat{\mathbf{f}}(\mathbf{0}) = \mathbf{0}$, $d\hat{\mathbf{f}}(\mathbf{0}) = \mathbf{0}$) is the vector field computed by the formula

$$\hat{\mathbf{f}}(\mathbf{u}) = \mathbf{f}_{\mathrm{q}}(\mathbf{x_0} + \mathbf{u}) - \mathbf{f}_{\mathrm{q}}(\mathbf{x_0}) - d\mathbf{f}_{\mathrm{q}}(\mathbf{x_0})\mathbf{u}.$$

If we make a logarithmic change of time $\tau = \ln t$, then system (B.16) transforms into the system

$$\mathbf{u}' = \mathbf{K}\mathbf{u} + \hat{\mathbf{f}}(\mathbf{u}), \tag{B.17}$$

where the prime signifies differentiation with respect to "logarithmic time" τ of the analogous system (B.1) that was considered at the beginning of this appendix.

However, the originally *autonomous* integral (B.13) of system (B.11) is transformed into a *time-dependent* integral of system (B.17):

$$F_N(\mathbf{x}) = e^{-\alpha N\tau} F_N(\mathbf{x_0} + \mathbf{u}).$$

We make an additional substitution $u^0 = e^{-\alpha\tau}$ and consider the extended system of equations

$$u^{0'} = -\alpha u^0, \quad \mathbf{u}' = \mathbf{K}\mathbf{u} + \hat{\mathbf{f}}(\mathbf{u}). \tag{B.18}$$

If the quasihomogeneous system (B.11) has a nontrivial quasihomogeneous integral (B.13), then the autonomous extended system (B.18) has a nontrivial *autonomous* integral

$$\phi(u^0, \mathbf{u}) = (u^0)^N F_N(\mathbf{x_0} + \mathbf{u}).$$

Then, by virtue of Lemma B.1, there exists a set of nonnegative integers $(k_0^*, k_1^*, \ldots, k_n^*)$, $\sum_{j=0}^n k_j^* \geq 1$, such that

$$-\alpha N k_0^* + \sum_{j=1}^n k_j^* \rho_j = 0. \tag{B.19}$$

According to Lemma 1.1.2, $\rho = -1$ is always an eigenvalue of the Kovalevsky matrix $\mathbf{K}$, so that without loss of generality we may assume that the first eigenvalue ρ_1 of this matrix equals -1. We rewrite the equality (B.19) in the form

$$-1(N k_0^* + (q-1)k_1^*) + (q-1)\sum_{j=2}^n k_j^* \rho_j = 0.$$

However, according to the hypothesis of the lemma (cf. (B.15)), this equation be cannot be satisfied.

The lemma is proved.

In proving Lemma B.3 based on Lemma B.1, we implicitly used a fact that we will be need in the sequel.

Remark B.2. Suppose the truncation of system (B.8) has a nontrivial quasihomogeneous integral $F_N(\mathbf{x})$. Then the linear system

$$\mathbf{u^0}' = -\alpha\mathbf{u^0}, \quad \mathbf{u}' = \mathbf{K}\mathbf{u} \tag{B.20}$$

has a homogeneous integral of the form

$$\phi_{(s+N)}(u^0, \mathbf{u}) = (u^0)^N D^{(s)} F_N(\mathbf{x_0}, \mathbf{u}),$$

where $s \geq 1$ is a natural number such that

$$D^{(p)} F_N(\mathbf{x_0}, \mathbf{u}) \equiv 0, \quad p = 1, \dots, s, \quad D^{(s)} F_N(\mathbf{x_0}, \mathbf{u}) \not\equiv 0.$$

The proof of the following theorem on nonintegrability consists of a succession of applications of Lemmas B.2 and B.3.

Theorem B.1. *Suppose that system (B.8) has a infinitely differentiable right side and is semi-quasihomogeneous with respect to some structure given by the diagonal matrix* $\mathbf{G}$ *with rational nonnegative elements. Let* $\rho_1, \dots, \rho_n$ *be the eigenvalues (possibly complex) of the diagonalizable Kovalevsky matrix, computed for one of the solutions of the truncated system (B.11). Suppose, for any choice of numbers* $(k_1, \dots, k_n)$, $k_j \in \mathbb{N} \cup \{0\}$, $\sum_{j=1}^n k_j \geq 1$, *that we have the inequality (B.15). Then system (B.8) has no semi-quasihomogeneous integral. If system (B.8) is positive semi-quasihomogeneous, then (B.8) likewise doesn't have any infinitely differentiable integral that is represented as a nontrivial formal Maclaurin series (B.9).*

A much more complicated and interesting integrability problem arises in the case where system (B.8) has a nontrivial integral. The question usually arises in the following way: do their exist other integrals in the given class, functionally independent from those that are known?

We first present the following result, stated and proved in the paper [197].

Lemma B.4. *We suppose that the quasihomogeneous system (B.11) has a quasihomogeneous integral (B.13) that is nondegenerate for any particular solution of this system of quasihomogeneous ray type* $\mathbf{x_{(0)}}(\mathbf{t}) = \mathbf{t}^{-\mathbf{G}}\mathbf{x_0}$, *i.e.* $dF_N(\mathbf{x_0}) \neq \mathbf{0}$. *Let* $Q = \alpha N$ *be a rational quasihomogeneous "degree" of this integral:* $F_N(\mu^{\mathbf{G}}\mathbf{x}) = \mu^Q F_N(\mathbf{x})$. *Then* Q *is one of the eigenvalues of the Kovalevsky matrix.*

Proof. We introduce the notation:

$$\mathbf{b} = dF_N(\mathbf{x_0}).$$

Then the auxiliary system (B.18) has an integral, which can be expanded in a series of homogeneous forms in the variables $(u^0, \mathbf{u})$:

$$\phi(u^0, \mathbf{u}) = (u^0)^N F_N(\mathbf{x_0} + \mathbf{u}) = \sum_{s=1}^{\infty} \phi_{(s)}(u^0, \mathbf{u}),$$

where

$$\phi_{(1)}(u^0, \mathbf{u}) \equiv \ldots \equiv \phi_{(N)}(u^0, \mathbf{u}) \equiv 0, \quad \phi_{(N+1)}(\mathbf{u^0}, \mathbf{u}) \equiv (u^0)^N \langle \mathbf{b}, \mathbf{u} \rangle.$$

According to Remark B.2, the form $\phi_{(N+1)}(u^0, \mathbf{u})$ is an integral of the linear system (B.20), from which it immediately follows that, for any $\mathbf{u} \in \mathbb{C}^{\mathbf{n}}$,

$$\langle \mathbf{b}, (\mathbf{K} - \alpha N \mathbf{E})\mathbf{u} \rangle = 0.$$

Since the matrix $\mathbf{K}$ is diagonalizable by assumption, it follows from the last relation that the vector $\mathbf{b}$ is its eigenvector with eigenvalue Q.

The lemma is proved.

Above we stipulated that $\rho_1 = -1$ is the *first* eigenvalue of the Kovalevsky matrix, with eigenvector $\mathbf{p} = \mathbf{f}_{\mathrm{q}}(\mathbf{x_0})$. Without loss of generality we may suppose that the vector $\mathbf{p}$ is parallel to the *first* basis vector $\mathbf{e_1} = (\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0})$. Using Lemma B.4, we may assume that $\rho_N = Q$ is the *last* eigenvalue of the Kovalevsky matrix and that $\mathbf{b}$ is parallel to the *last* basis vector $\mathbf{e_n} = (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{1})$.

Thus one of the resonance relationships satisfied by the eigenvalues of the Kovalevsky matrix has, in this case, the form

$$-k_1 + Qk_n = 0, \quad k_1 = r, \quad k_n = p,$$

where $Q = p/r$, $\gcd(p, r) = 1$.

We have the following result on the nonintegrability of a system of type (B.8), based on the arithmetic properties of the Kovalevsky matrix.

Theorem B.2. *Suppose that the system (B.8), with infinitely differentiable right side, is semi-quasihomogeneous with respect to some structure with diagonal matrix* $\mathbf{G}$ *with nonnegative rational elements, and that it has a nontrivial integral* $F(\mathbf{x})$ *that is polynomial in the case of a negative semi-quasihomogeneous system, or infinitely differentiable and represented as a formal nontrivial Maclaurin series in the case of a positive semi-quasihomogeneous system. Let the quasihomogeneous truncation of this integral* $F_N(\mathbf{x})$ *be nondegenerate for one of the solutions of the truncated system (B.11) in the form of a ray. Suppose likewise that the Kovalevsky matrix* $\mathbf{K}$*, computed* *at this solution, is diagonalizable and that its first* $n-1$ *eigenvalues* $\rho_1, \dots, \rho_{n-1}$ *are such that for any choice of (possibly negative) integers* $k_1, \dots, k_{n-1}$, $k_j \in \mathbb{Z}$, $\sum_{j=1}^{n-1} |k_j| \neq 0$, *the inequality*

$$\sum_{j=1}^{n-1} k_j \rho_j \neq 0. \tag{B.21}$$

is satisfied. Then any other integral $G(\mathbf{x})$ *of system (B.8), polynomial in the negative semi-quasihomogeneous case and represented in the form of a formal power series in the positive semi-quasihomogeneous case, is functionally dependent on* $F(\mathbf{x})$*, i.e. there exists a smooth function* $H: \mathbb{R} \to \mathbb{R}$ *such that*

$$G(\mathbf{x}) = H(F(\mathbf{x})). \tag{B.22}$$

The proof depends on several auxiliary assertions.

Lemma B.5. *Suppose that system (B.8) has a smooth integral* $F(\mathbf{x})$ *and that any nontrivial quasihomogeneous integral* $G_K(\mathbf{x})$ $(G_K(\lambda^{\mathbf{S}}\mathbf{x}) = \lambda^K G_K(\mathbf{x}))$ *of the truncated system (B.11) is a function of the truncation* $F_N(\mathbf{x})$ *of the integral* $F(\mathbf{x})$*, i.e.*

$$G_K(\mathbf{x}) = \Phi(\mathbf{F_N}(\mathbf{x})) \tag{B.23}$$

(The function Φ: $R \to R$ generally might be just an elementary monomial.) Then any nontrivial integral $G(x)$ of system (B.8) is a smooth function of $F(x)$ (see (B.22)).

Proof. Making the change of variables $\varepsilon = \mu^\beta$ in (B.10), we obtain

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\varepsilon}) = \sum_{\mathbf{m=0}}^{\infty} \boldsymbol{\varepsilon}^{\mathbf{m}} \mathbf{f}_{\mathrm{q}+\boldsymbol{\chi}\mathbf{m}}(\mathbf{x}). \tag{B.24}$$

As $\varepsilon \to 0$, system (B.24) is transformed into the truncated system (B.11). System (B.24) has the integral form

$$F(\mathbf{x},\varepsilon) = \sum_{m=0}^{\infty} \varepsilon^m \mathbf{F}_{\mathbf{N}+\boldsymbol{\chi}\mathbf{m}}(\mathbf{x}), \tag{B.25}$$

which, as $\varepsilon \to 0$, is transformed into an integral of the truncated system (B.11).

If (B.8) has an additional integral $G(\mathbf{x})$, then (B.24) likewise has an additional integral of type analogous to (B.25):

$$G(\mathbf{x},\varepsilon) = \sum_{m=0}^{\infty} \varepsilon^m G_{\mathbf{K}+\boldsymbol{\chi}\mathbf{m}}(\mathbf{x}), \tag{B.26}$$

As $\varepsilon \to 0$, the integral (B.26) passes to the truncation of $G_K(\mathbf{x})$, which is an integral of system (B.11).

The function in the variables $\mathbf{x},\boldsymbol{\varepsilon}$,

$$G^{(1)}(\mathbf{x},\varepsilon) = G^{(0)}(\mathbf{x},\varepsilon) - \Phi^{(0)}\left(F(\mathbf{x},\varepsilon)\right),$$

where we have introduced the notations $G^{(0)}(\mathbf{x},\varepsilon) = G(\mathbf{x},\varepsilon)$, $\Phi^{(0)}(F) = \Phi(F)$, is likewise an integral of (B.24) and, by (B.23) and has a first order of smallness in ε.

We consider the following function:

$$G^{(1)}_{K_1}(\mathbf{x}) = \lim_{\varepsilon\to 0} \varepsilon^{-1} G^{(1)}(\mathbf{x},\varepsilon).$$

Since the integrals (B.25) and (B.26) are power series, whose coefficients are quasihomogeneous functions, $G^{(1)}_{K_1}(\mathbf{x})$ is likewise a quasihomogeneous function of some degree K_1, where $G^{(1)}_{K_1}(\lambda^{\mathbf{S}}\mathbf{x}) = \lambda^{K_1} G^{(1)}_{K_1}(\mathbf{x})$, and is an integral of (B.11). But then

$$G^{(1)}_{K_1}(\mathbf{x}) = \Phi^{(1)}\left(F_N(\mathbf{x})\right),$$

where $\Phi^{(1)}\colon \mathbb{R} \to \mathbb{R}$ is some smooth function.

The function

$$G^{(2)}(\mathbf{x},\varepsilon) = G^{(1)}(\mathbf{x},\varepsilon) - \Phi^{(1)}\left(F(\mathbf{x},\varepsilon)\right)$$

is an integral of (B.24) and has a second order degree of smallness in ε.

But then

$$G^{(2)}_{K_1}(\mathbf{x}) = \lim_{\varepsilon\to 0} \varepsilon^{-2} G^{(2)}(\mathbf{x},\varepsilon)$$

will be a quasihomogeneous integral of the truncated system (B.11), from which it follows that

$$G^{(2)}_{K_2}(\mathbf{x}) = \Phi^{(2)}\left(F_N(\mathbf{x})\right),$$

for some smooth function $\Phi^{(2)}\colon \mathbb{R} \to \mathbb{R}$.

Repeating this process ad infinitum we obtain that

$$G(x,\varepsilon) = \sum_{m=0}^{\infty} \Phi^{(m)}\left(F(\mathbf{x})\right).$$

But this indicates that there exists a function $H\colon \mathbb{R} \to \mathbb{R}$, such that (B.22) holds.

The lemma is proved.

If system (B.8) has an additional integral $G(\mathbf{x})$ then, according to Remark B.2, the linear system (B.20) has two homogeneous integrals

$$\phi_{(N+1)}(u^0,\mathbf{u}) = (u^0)^N \langle \mathbf{b},\mathbf{u}\rangle, \quad \psi_{(s+K)}(u^0,\mathbf{u}) = (u^0)^K \varphi_{(s)}(u),$$
$$\varphi_{(s)}(\mathbf{u}) = D^{(s)} G_K(x_0,\mathbf{u}).$$

As has already been noted, we may assume that the first basis vector $\mathbf{e_1} = (\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0})$ is an eigenvector of the Kovalevsky matrix $\mathbf{K}$ with eigenvalue $\rho_1 = -1$. With this assumption we have:

Lemma B.6. *The integral $\psi_{(s+K)}(u^0, \mathbf{u})$ of the linear system (B.20) does not depend on the variable u^1.*

Proof. Consider the quasihomogeneous integral $G_K(\mathbf{x})$ of the truncated system (B.11), and let $\dot{G}_K(\mathbf{x})$ denote its total time derivative using system (B.11):

$$\dot{G}_K(x) = \left\langle dG_K(\mathbf{x}), \mathbf{f}_{\mathrm{q}}(\mathbf{x}) \right\rangle.$$

By the assumptions made above respecting the smoothness of the vector field $\mathbf{f}(\mathbf{x})$ and the properties of the matrix $\mathbf{G}$ determining the quasihomogeneous structure, $\dot{G}_K(\mathbf{x})$ is a polynomial in $\mathbf{x}$ that is *identically* equal to zero. Therefore,—by the Leibniz rule—for any natural number j,

$$D^{(j)}\dot{G}_K(\mathbf{x_0}, \mathbf{u}) = \sum_{i=0}^{j} C_j^i \left\langle D^{(j-1)}(d\dot{G}_K)(x_0, u), D^{(j)}\mathbf{f}_{\mathrm{q}}(\mathbf{x_0}, \mathbf{u}) \right\rangle \equiv 0,$$

where $C_j^i = \dfrac{j!}{i!(j-i)!}$.

Since the differential operators D and d clearly commute, by taking $j = s$, where s is the smallest integer for which $D^{(s)}G_K(\mathbf{x_0}, \mathbf{u}) \not\equiv 0$, we obtain

$$D^{(s)}\dot{G}_K(\mathbf{x_0}, \mathbf{u}) = \left\langle d\left(D^{(s)}\dot{G}_K(\mathbf{x_0}, \mathbf{u})\right), \mathbf{f}_{\mathrm{q}}(\mathbf{x_0}) \right\rangle = \left\langle d\varphi_{(\mathbf{s})}(u), \mathbf{p} \right\rangle \equiv 0.$$

From this, taking into account that $\mathbf{p} = p\mathbf{e_1} = (p, 0, \ldots, 0)$, we obtain

$$\frac{\partial \varphi_{(s)}}{\partial u^1}(\mathbf{u}) \equiv 0,$$

and the lemma is proved.

As discussed above, we may suppose that $\mathbf{b} = (0, \ldots, 0, b)$ is an eigenvector of the Kovalevsky matrix $\mathbf{K}$ with eigenvalue $\rho_n = Q$. We have

Lemma B.7. *Let the quasihomogeneous integral $F_N(\mathbf{x})$ of the truncated system (B.11) be nonsingular for the solution of this system (B.14), i.e. $dF_N(\mathbf{x_0}) \neq \mathbf{0}$. If this system has a quasihomogeneous integral $G_K(\mathbf{x})$ that is functionally independent of $F_N(\mathbf{x})$, then the linear system (B.20) has a nontrivial homogeneous integral $\tilde{\psi}_{l+R}(u^0, \mathbf{u})$, independent of u^1 and u^n, of the following form:*

$$\tilde{\psi}_{l+R}(u^0, \mathbf{u}) = (u^0)^R \tilde{\varphi}_{(l)}(u^2, \ldots, u^{n-1}), \tag{B.27}$$

where R is some (possibly negative) integer, and $\tilde{\varphi}_{(l)}$ is a homogeneous form in the variables $u^2, \ldots, u^n$ of positive integral degree l.

Proof. We rewrite the integral $\psi_{l+R}(u^0, \mathbf{u})$ in the following way:

$$\psi_{s+K}(u^0, \mathbf{u}) = (u^{\mathbf{0}})^{\mathbf{K}} \sum_{\mathbf{j=0}}^{\mathbf{s}} \tilde{\varphi}_{(s-j)}(u^2, \ldots, u^{n-1})(u^n)^j, \tag{B.28}$$

where $\tilde{\varphi}_{(s-j)}$ is a homogeneous form of degree $s - j$ in its arguments.

The integral of system (B.20) has the form:

$$\phi_{(N+1)}(u^0, \mathbf{u}) = (u^0)^N \langle \mathbf{b}, \mathbf{u} \rangle = b(u^0)^N u^n \equiv \mathbf{c} = \text{const}.$$

Therefore we can rewrite (B.28) in the form

$$\psi(u^0, \mathbf{u}, c) = \sum_{j=0}^{s} (u^0)^{K-jN} \tilde{\varphi}_{(s-j)}(u^2, \ldots, u^{n-1}) \left(\frac{c}{b}\right)^j.$$

Since the constant c is arbitrary, each coefficient

$$\tilde{\psi}_{(s+K-(N+1)j)}(u^0, \mathbf{u}) = (\mathbf{u^0})^{\mathbf{K-jN}} \tilde{\varphi}_{(s-j)}(u^2, \ldots, u^{n-1})$$

of $\left(\frac{c}{b}\right)^j$ is an integral of system (B.20) having the required form.

However, a theoretically possible situation is where all the homogeneous polynomials $\tilde{\varphi}_{(s-j)}$, $j = 0, \ldots, s-1$ are identically equal to zero, i.e. the integral (B.28) of system (B.20) has the form

$$\psi_{(s+K)}(u^0, \mathbf{u}) = a_s (u^0)^K (u^n)^s.$$

Since $Q = \alpha N$ is the last eigenvalue of the Kovalevsky matrix, it follows immediately that

$$K = sN.$$

Consider another integral of the truncated system (B.11):

$$G_K^*(\mathbf{x}) = G_K(\mathbf{x}) = \frac{a_s}{s!} \left(b^{-1} F_N(\mathbf{x})\right)^s,$$

which likewise is a quasihomogeneous function of degree K.

The Taylor series expansion of $G_K^*(\mathbf{x})$ in the neighborhood of $\mathbf{x} = \mathbf{x_0}$ begins with terms of order at least $s + 1$, so that the linear system (B.20) has a homogeneous integral of the form

$$\psi^*_{(s_1+K)}(u^0, \mathbf{u}) = (u^0)^K \varphi^*_{(s_1)}(\mathbf{u}),$$

$$\varphi^*_{(s_1)}(\mathbf{u}) = D^{(s_1)} G_K^*(\mathbf{x}_0)\mathbf{u}, \quad s < s_1 < +\infty.$$

With this integral we can perform the procedure described above. If this integral likewise has the form

$$\psi^*_{(s_1+K)}(u^0, \mathbf{u}) = a_{s_1}(u^0)^K (u^n)^{s_1},$$

then

$$K = s_1 N,$$

which contradicts the condition $s_1 > s$.

But if $s_1 = +\infty$, then $G^*_K(\mathbf{x}) \equiv 0$. This means that $G_K(\mathbf{x})$ is the function $F_N(\mathbf{x})$ and we have obtained a contradiction.

The lemma is proved.

We continue the proof of Theorem B.2. If the truncated system (B.11) has a quasihomogeneous integral $G_K(\mathbf{x})$ that is functionally independent of $F_N(\mathbf{x})$, then according to Lemmas B.6 and B.7 the linear system (B.20) possesses a nontrivial integral of the form (B.27). Then the functions $\tilde{\varphi}_{(l)}(u^2, \ldots, u^{n-1})$ must satisfy the following first order partial differential equation

$$\begin{aligned} &-\alpha R\tilde{\varphi}_{(l)}(\mathbf{u}) + \langle d\tilde{\varphi}_{(l)}(\mathbf{u}), \mathbf{K}\mathbf{u}\rangle = \\ &= -\alpha R\tilde{\varphi}_{(l)}(\mathbf{u}) + \sum\nolimits_{j=2}^{n-1} \rho_j u^j \frac{\partial \tilde{\varphi}_{(l)}}{\partial u^j}(u) = 0. \end{aligned} \tag{B.29}$$

Writing, as before, the homogeneous function $\tilde{\varphi}_{(l)}(u^2, \ldots, u^{n-1})$ as a sum of elementary monomials

$$\tilde{\varphi}_{(l)}(u^2, \ldots, u^{n-1}) = \sum_{k_2+\ldots+k_{n-1}=l} \tilde{\varphi}_{k_2\ldots k_{n-1}} (u^2)^{k_2} \ldots (u^{n-1})^{k_{n-1}},$$

from Eq. (B.29) we obtain that any nonzero coefficient $\tilde{\varphi}_{k_2\ldots k_{n-1}}$ in the above expansion must satisfy the resonance relation

$$-R + (q-1)\sum_{j=2}^{n-1} k_j \rho_j = 0,$$

which can be interpreted as a condition of the form (B.21), since $\rho_1 = -1$.

We need only apply Lemma B.5 to complete the proof.

The theorem is proved.

Unfortunately, the result obtained in not applicable to Hamiltonian systems, which are the most frequently encountered systems in mechanics. As shown in the paper [197], Hamiltonian systems necessarily have resonances of a special type between *pairs* of eigenvalues of the Kovalevsky matrix. On the other hand, it follows from results from the paper [109] that these resonances are not so much consequences of the presence of integrals—Hamiltonian functions—as they are consequences of the presence of invariant measures.

We apply our results to a concrete system of equations.

Example B.1. We consider the two-dimensional logistical system [189]

$$\dot{x} = x(a + cx + dy), \quad \dot{y} = y(b + ex + fy) \tag{B.30}$$

and find arithmetical conditions that must be satisfied by the coefficients of (B.30) so that the system considered not have smooth integrals represented in the form of nontrivial formal Maclaurin series in the neighborhood of the origin $x = y = 0$.

Of course, this autonomous two-dimensional system with analytic right sides cannot display chaotic behavior and must be integrable in the defined sense. However, under certain conditions, the integrals of this system will in some sense be "bad".

First of all, as follows from Lemma B.1, system (B.30) doesn't have integrals that can be written as formal power series in x, y if, for any $k_1, k_2 \in \mathbb{N} \cup \{0\}$, $k_1 + k_2 \geq 1$, the equation

$$k_1 a + k_2 b \neq 0,$$

holds. This may be rewritten as

$$-\frac{a}{b} \notin \mathbb{Q}^+.$$

On the other hand, system (B.30) is negative semi-quasihomogeneous. Its truncation

$$\dot{x} = x(cx + dy), \quad \dot{y} = y(ex + fy) \tag{B.31}$$

is a homogeneous system of order 2.

The truncated system (B.31) has an affine solution in the form of a ray

$$x_{(0)}(t) = \frac{x_0}{t}, \quad x_{(0)}(t) = \frac{y_0}{t}, \quad x_0 = \frac{d - f}{cf - ed}, \quad y_0 = \frac{e - c}{cf - ed},$$

provided that $cf - ed \neq 0$.

It is not difficult to compute the eigenvalues of the Kovalevsky matrix:

$$\rho_1 = -1, \quad \rho_2 = \rho = \frac{(d - f)(e - c)}{cf - ed}.$$

Consequently, according to Theorem B.1, system (B.30) has no polynomial integrals if

$$k_1 \neq k_2 \rho$$

for any $k_1, k_2 \in \mathbb{N} \cup \{0\}$, $k_1 + k_2 \geq 1$, i.e. if

$$\frac{(d - f)(e - c)}{cf - ed} \notin \mathbb{Q}^+.$$

Example B.2. We consider a system of equations describing the perturbed Oregonator [186]:

$$\begin{aligned} \dot{x} &= a(y - xy + x - \varepsilon xz - bx^2), \\ \dot{y} &= a^{-1}(-y - xy + cz), \\ \dot{z} &= d(x - \varepsilon xz - z). \end{aligned} \tag{B.32}$$

This system of equations describes a certain hypothetical chemical reaction of the Belousov-Zhabotinsky type, where the variables x, y, z denote concentrations of reagents and the constants a, b, c, d, ε are reaction coefficients, where $\varepsilon > 0$ is sufficiently small. Chronologically, the unperturbed case ($\varepsilon = 0$) was the first considered [50]. It turns out, even for very small values of ε, that the behavior of the system differs sharply from that of the unperturbed system. It was discovered that chaotic behavior is possible in system (B.32).

We will find arithmetic relations which, when satisfied, would guarantee that system (B.32) does not have polynomial integrals or even integrals that can be represented as formal Maclaurin series.

Explicit formulae for the eigenvalues of the matrix for the linear approximating system are quite unwieldy, so that Lemma B.1 isn't of much use as a means for establishing nonintegrability.

We apply Theorem B.1. System (B.32) is negative semi-quasihomogeneous. Its truncation

$$\dot{x} = -ax(y + bx), \quad \dot{y} = -a^{-1}xy, \quad \dot{z} = dx(1 - \varepsilon z) \tag{B.33}$$

is a quasihomogeneous system of order $q = 2$ with indices $s_x = s_y = 1$, $s_z = 0$.

System (B.33) has a real particular solution of quasihomogeneous ray type:

$$\begin{aligned} x_{(0)}(t) = \tfrac{x_0}{t}, \quad y_{(0)}(t) = \tfrac{y_0}{t}, \quad z_{(0)}(t) = z_0, \\ x_0 = a, \quad y_0 = a^{-1} - ab, \quad z_0 = \varepsilon^{-1}. \end{aligned}$$

The eigenvalues of the Kovalevsky matrix are:

$$\rho_1 = -1, \quad \rho_2 = 1 - a^2 b, \quad \rho_3 = -ad\varepsilon.$$

By Theorem B.1, the system of equations (B.32) doesn't have polynomial integrals when

$$-k_1 + (1 - a^2 b)k_2 - ad\varepsilon k_3 \neq 0,$$

where $k_1, k_2, k_3 \in \mathbb{N} \cup \{0\}$, $k_1 + k_2 + k_3 \geq 1$.

Example B.3. By way of an example that illustrates Theorem B.2, we consider the problem of the integrability of the Euler-Poincaré equations over a Lie algebra [152], which has been studied in various fields of mathematical physics. In the occidental literature these equations likewise are often called the Poincaré -Arnold equations in connection with the paper [4] of V.I. Arnold. They are natural generalizations of the famous Euler equations from dynamics, which describe the inertial motion of a solid body from a point at rest and can be written in the following way:

$$\dot{m}_l = \sum_{i,j=1}^{n} C_{il}^{j} \omega^i m_j, \quad l = 1, \dots, n. \tag{B.34}$$

The constants $\{C_{il}^{j}\}_{i,j,l=1}^{n}$ are so-called structure constants of some n–dimensional Lie algebra $\mathfrak{g}$, and $\boldsymbol{\omega} = (\omega^1, \dots, \omega^n)$ is the "angular velocity" vector that is connected with the momentum covector $\mathbf{m} = (m_1, \dots, m_n)$ by

$$m_j = \sum_{i=1}^{n} I_{ij} \omega^i,$$

where $\{I_{ij}\}_{i,j=1}^{n}$ is a positive definite symmetric tensor of type $(0,2)$, analogous to the energy tensor for an ordinary solid body.

In the usual situation of the motion of a solid body, we have $\mathfrak{g} = \mathfrak{so}(3)$.

Equation (B.34) always have an "energy integral":

$$T = \frac{1}{2} \sum_{i,j=1}^{n} I_{ij} \omega^i \omega^j. \tag{B.35}$$

As usual, the question arises as to whether Eq. (B.34) has additional integrals that are functionally independent of (B.35). A good bit of research has been devoted to this and related problems (see e.g. the monographs [51, 112, 185] along with the literature cited there). The publications [107, 112] study the closely related question of the existence of an integral invariant (an invariant measure with an infinitely smooth density function) for the equations (B.34). Exhaustive answers to the question posed have been given in the cited papers in the case of low dimensions. It turns out that, for $n = 3$, only the *solvable* Lie algebras provide us examples of Euler-Poincaré systems without integral invariants. It is entirely possible that the solvability of the algebra $\mathfrak{g}$ will turn out to be a general obstacle to integrability.

Thus we limit ourselves to the case of solvable algebras with $n = 3$.

According to [89] there exists a canonical basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ such that the only nonzero structure constants C_{il}^{j} are

$$C_{13}^{1} = -C_{31}^{1} = \alpha, \quad C_{13}^{2} = C_{31}^{2} = \beta,$$
$$C_{23}^{1} = -C_{32}^{1} = \gamma, \quad C_{23}^{2} = C_{32}^{2} = \delta.$$

We likewise impose a nondegeneracy condition of the structure constants of the algebra $\mathfrak{g}$:

$$\alpha\delta - \beta\gamma \neq 0.$$

So as to keep the "angular velocity" in analogy with the dynamics of a solid body, we use the notation $\boldsymbol{\omega} = (p, q, r)$. For the "inertial tensor" $\mathbf{I}$ and its inverse $\mathbf{I}^{-1}$ we introduce the following notation:

$$\mathbf{I} = \begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix}, \quad \mathbf{I}^{-1} = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$$

We let $\mathbf{m} = (x, y, z)$ denote the vector of "moments of momentum". Then the corresponding Euler-Poincaré equations assume the form:

$$\begin{aligned} \dot{x} &= -r(\alpha x + \beta y), \\ \dot{y} &= -r(\gamma x + \delta y), \\ \dot{z} &= p(\alpha x + \beta y) + q(\gamma x + \delta y), \end{aligned} \tag{B.36}$$

where $p = ax + dy + ez$, $q = dx + by + fz$, $r = ex + fy + cz$.

The integral of energy can be written in the following way:

$$\begin{aligned} T &= \tfrac{1}{2}(xp + yq + zr) \\ &= \tfrac{1}{2}(Ap^2 + Bq^2 + Cr^2 + 2Dpq + 2Epr + 2Fqr) \\ &= \tfrac{1}{2}(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz). \end{aligned} \tag{B.37}$$

The system (B.36) is homogeneous with a quadratic right side, so that we may regard it as quasihomogeneous and semi-quasihomogeneous. The integral (B.37) is, of course, nondegenerate for any vector (x_0, y_0, z_0).

We can show rather easily that Eq. (B.36) has a linear particular solution in the form of a ray:

$$x_{(0)}(t) = \frac{x_0}{t}, \quad y_{(0)}(t) = \frac{y_0}{t}, \quad z_{(0)}(t) = \frac{z_0}{t},$$

where (x_0, y_0, z_0) are certain (in general complex) numbers and $|x_0|^2 + |y_0|^2 + |z_0|^2 \neq 0$. However, in order to compute the eigenvalues of the Kovalevsky matrix, we need to use the following technical device. The matrix $\mathbf{K}$ in general has the form:

$$\mathbf{K} = \begin{pmatrix} 1 - eX - \alpha w & -fX - \beta w & -cX \\ -eY - \gamma w & 1 - fY - \delta w & -cY \\ aX + dY + \alpha u + \gamma v & dX + bY + \beta u + \delta v & 1 + eX + fY \end{pmatrix},$$

where we have introduced the notations

$$u = ax_0 + dy_0 + ez_0, \quad v = dx_0 + by_0 + fz_0, \quad w = ex_0 + fy_0 + cz_0,$$
$$X = \alpha x_0 + \beta y_0, \quad Y = \gamma x_0 + \delta y_0.$$

Since Eq. (B.36) has a quadratic energy integral (B.37), we have a priori knowledge of *two* roots of the characteristic polynomial $\Delta_K(\rho)$:

$$\rho_1 = -1, \quad \rho_3 = 2.$$

Consequently the quadratic trinomial

$$P(\rho) = \rho^2 - \rho - 2$$

divides the polynomial $\Delta_K(\rho)$.

Looking at the matrix **K**, we can compute the remainder upon division of $\Delta_K(\rho)$ by $P(\rho)$:

$$Q(\rho) = \rho - 2 + (\alpha + \delta)w.$$

Consequently

$$\rho = \rho(w) = 2 - (\alpha + \delta)w,$$

so that, in determining ρ_2, it suffices to compute the quantity w.

For this we need to pay attention to the fact that the equation for determining x_0, y_0 "almost" has the closed form:

$$(1 - \alpha w)x_0 - \beta w y_0 = 0, \quad -\gamma w x_0 + (1 - \delta w)y_0 = 0.$$

This linear homogeneous system has a nontrivial solution if and only if its determinant equals zero. From this we get

$$w = \frac{1}{2}\frac{\alpha + \delta \pm \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}}{\alpha\delta - \beta\gamma}. \tag{B.38}$$

If

$$\rho(w) \in \mathbb{Q},$$

where the quantity w is given by the equalities (B.38), then the eigenvalues of the Kovalevsky matrix don't satisfy resonance relations of type (B.24) and the Euler-Poincaré equations (B.36) don't have an integral that is representable as a nontrivial formal power series in x, y, z or in p, q, r, and that is functionally independent of (B.37).

We look at two interesting particular cases. Let $\alpha = \delta = 0$. Then

$$\rho_2 = 2,$$

and (B.36) becomes suspect for integrability.

In fact, in this case there is an obvious additional first integral:

$$G = \frac{1}{2}(\gamma x^2 - \beta y^2).$$

It is interesting to note that if $\beta\gamma < 0$, then, as shown in [107, 112] the system considered has an integral invariant (the Lie group that generates the Lie algebra $\mathfrak{g}$ is unimodular). If, conversely, $\beta\gamma > 0$, then there is no integral invariant but there exists an invariant measure, whose density function has the desired high (but finite) order of smoothness.

Now let $\beta = \gamma = 0$. Then

$$\rho_2 = 1 - \frac{\alpha}{\delta} \quad \text{or} \quad 1 - \frac{\delta}{\alpha},$$

and the condition for integrability assumes the form

$$\frac{\alpha}{\delta} \notin \mathbb{Q}.$$

If $a\delta > 0$, then it follows from [107, 112] that the Euler-Poincaré system (B.36) does not possess an invariant measure even with summable density.

In the paper [88] the Euler-Poincaré equation is studied on a multidimensional solvable Lie algebra. It is shown that, independent of the structure of the inertia tensor for a typical non-nilpotent solvable Lie algebra, the Euler-Poincaré equation branches in the plane of complex time.

In conclusion we will briefly discuss the conditions for nonintegrability obtained in Theorems B.1 and B.2 and compare them with Yosida's criterion [197].

According to Yoshida's criterion, if at least one of the eigenvalues of the Kovalevsky matrix is irrational, then the quasihomogeneous system of differential equations is *algebraically nonintegrable*. This assertion admits a complex-analytic interpretation in the spirit of a paper of S.L. Ziglin [202]. The basic idea of that paper is that the branching of solutions in the complex plane impedes integrability.

Consider a Fuchsian system of linear differential equations, obtained from system (B.16) by linearization in the neighborhood of $\mathbf{u} = \mathbf{0}$:

$$t\dot{\mathbf{u}} = \mathbf{K}\mathbf{u}. \tag{B.39}$$

The monodromy operator, acting on the space of solutions of this system, has the matrix

$$\mathbf{M} = \exp(2\pi i \mathbf{K})$$

which, if the hypothesis of Yoshida's criterion is satisfied, cannot be a rational root of the identity matrix $\mathbf{E}$.

This means that system (B.39) has a particular solution with an unbounded Riemann surface. But if either of the inequalities (B.15) or (B.21) is satisfied, then the space of solutions of system (B.39) has a still more complicated structure.

As follows from Lemma B.4, any $n-1$ quasihomogeneous integrals of the truncated system (B.11) are functionally dependent at the point $\mathbf{x} = \mathbf{x_0}$ if the hypothesis of Yoshida's criterion is satisfied. It is, however, unclear—as is noted in the paper [69]—whether this domain of dependency can be extended. The hypothesis of Theorems B.1 and B.2 impose substantially harsher requirements on the properties of the eigenvalues of the Kovalevsky matrix, which allow us to formulate a stronger assertion about nonintegrability.

Literature

Translator's Note: English translations are cited when available and *Mathematical Reviews* and/or *Zentralblatt für Mathematik* references are given for untranslated books and articles, when available, e.g. MR 01234567 and/or Zbl 0123.45678. In transcription of most Russian words and proper names, the BGN/PCGN system is followed. Exceptions occur when a name has an accepted spelling in English (e.g. *Kovalevsky*) or when another spelling is used in a publication (e.g. *Arnold* as in [6]). The name *Bohl* has the Russian form *Bol'* in [23], as well as the Latin form in [22].

The language of publication is English except as otherwise noted or as is obvious from the title (for French, German and Italian titles). For the original version of translated *books*, only the place and year of publication and the name of the publisher is noted, e.g. "Publication source: Moscow: Editorial URSS (2002)". For translated *articles*, the exact journal reference of the original is given, but not the original title.

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