Appendix A: Heuristic Algorithms

In this book, you will hear about or even directly encounter a number of solution algorithms. All of these algorithms fall into two broad categories: *exact algorithms* (sometimes also somewhat misleadingly referred to as *optimal algorithms*) and *heuristic methods* usually simply called *heuristics*. Exact algorithms have the obvious advantage of providing the best possible solution there is, given the user-defined constraints, whereas heuristics do not. Some heuristics do have error bounds, some actually proven, while others are empirical, i.e., they state that a certain heuristic usually (typically on average) finds solutions that have a certain quality. On the other hand, there is computing speed. Some models are such that it takes an exact algorithm exceedingly long to find the optimal solution. Is this relevant? Well, it depends. If the task at hand is to, say, locate a landfill for millions of dollars, you will not care if it takes a laptop 2 or 3 weeks to run, so that it can find a solution that may potentially save hundreds of thousands of dollars. There are limits to this argument, of course: if it takes years or even longer to find a solution, most problems have either solved themselves or have become irrelevant by that time. So, this is not acceptable.

In order to make the case for heuristics, consider the situation of automated guided vehicles (*AGV*s). Suppose you have a number of individual work stations on the shop floor, each of which processes a given piece from a semi-finished product to the finished good. For that purpose, it will be necessary to move pieces, on which work has been finished at one station to the next station. Note that it is not necessarily the case that all goods are running through the same sequence of jobs, not all tasks have to be performed at all work stations, and different levels of customization are possible. The movement of goods may be accomplished by automated guided vehicles that receive a message from a workstation whenever a piece is ready for pickup, and the machine knows where the piece has to go next. However, temporarily there may be more pieces to be transported than the machine can handle, so that it will put the tasks in a list. Whenever it is ready it will work on that list, depending on where it is at any point in time, how far it is to the destination for that particular transportation job, how many work stations will be idle if they have to wait for the next job, and many other considerations. It is apparent that solutions to that problem have to be found in real time, i.e., immediately. This is where heuristic algorithms come in.

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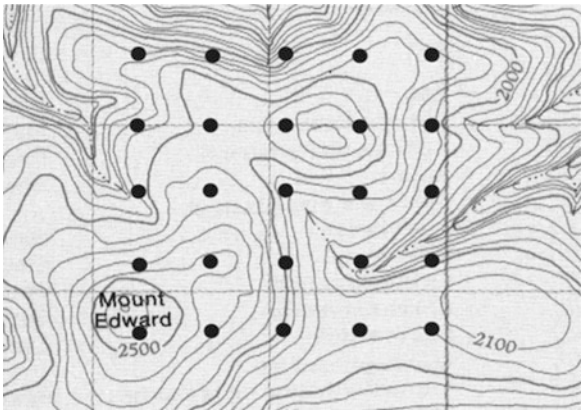

Fig. A.1 Map for the lost hiker example

Heuristic algorithms typically have two phases. The first phase is the *construction phase*, in which a solution is established. This phase starts with nothing, and by the end of the phase, we have a solution. Phase 1 should be followed by Phase 2, which is an *improvement phase*, in which the method uses simple modifications of the present solution that improve the quality of the solution. The best known heuristics are the *Greedy Method* (a construction method), and the *Swap* (or *pairwise exchange*) method, which is an improvement method.

In order to illustrate the Greedy technique, consider the following example. Suppose that a hiker is lost in the woods. In order to be visible from the air for a rescue mission, he decides to climb as high as possible. Since he has neither a map, nor an idea of where he really is (and it is very foggy, so that visibility is very limited, making his vision myopic), he can only determine the shape of the land in close vicinity. At any point, he can examine the terrain to his North, Northwest, West, Southwest, South, Southeast, East, and Northeast. If there are higher points in any of these directions, he will go into that direction that features the highest nearby point (i.e., the best possible improvement, hence the name "Greedy"). If the points in all eight directions are lower than where he is right now, he will conclude that he is standing on top of a hill and stay there, assuming that he has arrived.

As an example, consider the situation shown in Fig. A.1. In order to simplify matters, we have determined the altitudes of the points shown as big black dots and have them displayed again in Fig. A.2.

Suppose now that the hiker is presently at the Northeasterly point in Fig. A.2 at an altitude of 2,160 ft. There are only three surrounding points, those to the West, the Southwest, and the South, with altitudes of 2,210, 2,230, and 2,050, respectively. The highest of these is the point to the Southwest, which is then where the hiker walks to. This point now has eight neighbors, the highest of which is located directly to the West of the hiker's present position at an altitude of 2,340. Consequently, the hiker relocates to this point. Repeating the procedure, the hiker notices that all points in the vicinity of his present location are lower. So, he concludes that he is standing

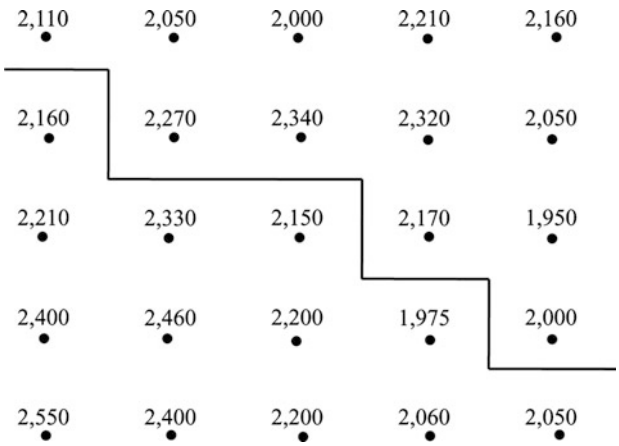

Fig. A.2 Altitudes (in feet) for the lost hiker example

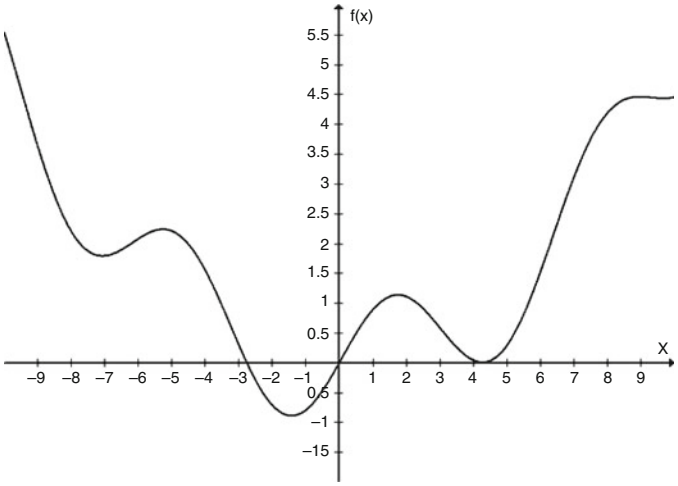

Fig. A.3 Function with three local minima

on top of a hill. From the topographic map we know that he is, but we also know that this is not the highest hill in the area of interest.

Formally, a point all of whose neighbors are lower (higher), is called a *local maximum* (*local minimum*). If a local maximum (local minimum) is also the highest (lowest) point overall, it is referred to as a *global maximum* (*global minimum*). As an illustration, consider the function $y = \sin x + 0.05x^2$. For values of x between -10 and $+10$, the function is shown in Fig. A.3.

The function has three minima on the domain shown, they are at $x = -7.0689$ (with $y = 1.7911$), at $x = -1.4755$ (with $y = -.8879$), and at $x = 4.2711$ (with $y = .0079$). (Myopic) heuristics (as well as derivatives) will readily find

local optima, but not necessarily global optima. The reason for this is apparent: if any myopic method arrives at, say, the rightmost of the three local minima in Fig. A.3, how is the method to know whether or not there are better points located to the left? If a graph such as this were available, it is easy to see, but typically it is not, as almost all problems are multidimensional, and thus cannot be graphed in three or fewer dimensions.

The hiker's plight described above might now be considered his own problem and of little general interest, if it were not for the fact that each point on the map may represent a course of action (determined by the values of its coordinates), and the contour lines may represent their profit. So the search for the highest hill has now become a search for the point of maximal profit, which is of considerable general interest.

To make the problem even more interesting, had the hiker been in the extreme Southeast and had used the Greedy method for his progress, he would actually have ended up at Mount Edward, the overall highest point in the region of interest. As a matter of fact, the line that divides Fig. A.2 indicates what is usually referred to as *catchment areas*: if the hiker starts at any point to the Northeast of the line, he ends up at the local optimum at altitude 2,340 ft, if he starts at any point to the Southwest of the line, he ends up on Mount Edward at the global maximum. A similar situation applies to the function in Fig. A.3. Starting a Greedy minimization method any point to the left of $x = -5.2671$ (a local maximum), will end up at the local minimum at $x = -7.0689$. Starting at any point between $x = -5.2671$ and $x = 1.7463$ will end at the global minimum at $x = -1.4276$, and starting anywhere to the right of $x = 1.7463$ will lead to the local minimum at $x = 4.2711$.

This immediately suggests a technique that is called a *multistart method*. The idea is simply to apply a Greedy technique starting at a number of different points, compare the results, and choose the best (highest or lowest, depending on whether a minimum or a maximum is sought).

Next we will discuss an improvement method. The Swap method is easy to describe. Given a solution that has been obtained "somehow," it takes two components and exchanges them. Depending on the specific application, this may mean exchange their sequence, their inclusion/exclusion status, or whatever the problem commands. The method then computes the change of the value of the objective function. If the value has improved (i.e., has increased in case of a maximization problem or decreased in case of a minimization problem), the modified solution becomes the new starting point and the previous solution is discarded. This step is repeated until no further Swap step can improve the solution any further.

As an example, consider the map in Fig. A.4 and assume that the task at hand is to determine the shortest route from Memphis, Tennessee, to Reno, Nevada. The traveler has outlined the tour in layers, so that a drive from a city in one layer to a city in the next is about a day's drive.

The distances are as follows: From Memphis to Omaha, Wichita, Oklahoma City, and Dallas we have 650, 555, 460, and 460 miles, respectively. From Omaha to Denver and Albuquerque there are 540 and 860 miles. The distances from

Fig. A.4 Map of the U.S. for the shortest route example

Wichita to Denver and Albuquerque are 510 and 585 miles, from Oklahoma City to Denver and Albuquerque are 630 and 550 miles, and from Dallas to Denver and Albuquerque are 780 and 640 miles, respectively. The distances from Denver to Salt Lake City and Phoenix are 505 and 835 miles, while from Albuquerque, there are 610 and 460 miles to Salt Lake City and Phoenix, respectively. Finally, the distance from Salt Lake City to Reno is 525 miles and from Phoenix to Reno there are 735 miles.

In order to obtain some solution, we use the Greedy algorithm, starting at the origin of our trip in Memphis. From here, Greedy will choose the nearest neighbor, which is either Dallas or Oklahoma City, both 460 miles away. We arbitrarily choose Dallas. The nearest neighbor of Dallas is Albuquerque, which is 640 miles away (as opposed to Denver, which is 780 miles from Dallas). From Albuquerque, we take the closest connection to Phoenix (460 miles), and from there we have no choice but take the last long trip to Reno (735 miles). The trip leads us on the route Memphis – Dallas – Albuquerque – Phoenix – Reno, and its total length is 2,295 miles.

At this point we start to swap. One possibility is to swap Phoenix and Salt Lake City. This means that we have to add the connections from Albuquerque to Salt Lake City (610 miles) and from Salt Lake City to Reno (525 miles), and subtract the connections from Albuquerque to Phoenix (460 miles) and from Phoenix to Reno (735 miles), for net savings of 60 miles. This is an improvement, and so our new route is Memphis – Dallas – Albuquerque – Salt Lake City – Reno, and its total length is 2,235 miles.

We can now use the new solution and try other Swap moves. For instance, we could attempt to swap Albuquerque and Denver. The net change of such a swap move is +35 miles, so we will not make this change. Another possibility is to swap Dallas and Oklahoma City. The net change is −90 miles, so we make the change and obtain the route Memphis – Oklahoma City – Albuquerque – Salt Lake City – Reno, whose length is 2,145 miles.

At this time we may examine again the pair Albuquerque and Denver. This swap did not improve the solution the last time we tried it, but since then the solution has changed. In fact, swapping the two cities at this point results in a net change of $+630 + 505 - 550 - 610 = -25$, so that the change is made. This results in the new route Memphis – Oklahoma City – Denver – Salt Lake City – Reno, which is 2,120 miles long.

We may now try to exchange Oklahoma City and Wichita, which leads to a net change of $+555 + 510 - 460 - 630 = -25$, for another reduction in terms of the total distance, which is now 2,095 miles. The route leads from Memphis – Oklahoma City – Denver – Salt Lake City – Reno. At this point we may try to further reduce the length of the tour, which is no longer possible with swap moves. As a matter of fact (unbeknownst to us when we are just using heuristics), the tour is actually optimal.

Our final example of a heuristic method deals with a much-studied field called *bin packing*. We have an unspecified number of bins, all of which are of the same prespecified length. We also have a number of rods that are to be placed into the bins. The problem is one-dimensional, in that the bins and the rods have the same height and width, so that only the length of the bins and the rods that are placed into them will decide whether or not they actually fit. For instance, if the bin is 20 ft and there are one 6 ft rod, one 3 ft rod, and one 5 ft rod, then these three rods will occupy 14 ft of the bin and leave 6 ft unoccupied. The task at hand is now to put the existing rods into the smallest number of bins possible.

Despite its apparent simplicity, the problem has been proven to be very difficult from a computational point of view. A Greedy-like heuristic is the so-called *First Fit* (*FF*) *Algorithm*. In order to implement the method, we first assume that a sufficiently large number of bins is available. These bins are numbered 1, 2, ... The First Fit Algorithm that can be described as follows:

FF Algorithm: Put the next rod into the bin with the smallest number into which it will fit.

As an example, suppose that all bins are 19 ft long. In addition, we have six 11 ft rods, six 6 ft rods, and twelve 4 ft rods. In this type of situation, we can actually compute a very simple bound for the number of bins that will be needed. Here, the total length of the rods is $6(11) + 6(6) + 12(4) = 150$ ft. Given that each bin is 19 ft long, we will need at least $150/19 \cong 7.89$ bins. Since the number of bins must be integer, we will need at least eight bins. This also means that if we were to find a solution to the problem that requires eight bins, this solution must be optimal.

Apply now the First Fit Algorithm. Assigning the rods in order of their lengths (i.e., the 11 ft rods first, then the 6 ft rods, and finally the 4 ft rods), we notice that

only a single 11 ft rod fits into each bin. This means that we have to put each of the 11 ft rods into one bin each, so that we now have dealt with all 11 ft rods and have used parts of six bins. Next, we assign the six 6 ft rods. Since each of them fits into one of the already partially used bins, we now have six bins with one 11 ft and one 6 ft rod each, leaving 2 ft of free space in each of the six bins. This is not sufficient for any of the remaining 4 ft bins, so that we have to use additional bins. We can place four 4 ft rods in each bin, leaving an empty (and unusable for us) space of 3 ft each, which requires another three bins. We now have assigned all rods to the bins. This solution requires a total of nine bins. There is no apparent pairwise exchange (swap) step able to improve the solution.

On the other hand, if we were to put one 11 ft and two 4 ft rods into each of six bins, this would leave no empty space at all. The remaining six 6 ft rods can be put into two bins. Having again assigned all rods, this solution requires only eight bins and, given the bound computed above, must be optimal.

A variety of other heuristics exists for this problem. An excellent (albeit difficult) pertinent reference is the book by Garey and Johnson (1979). An interesting extension of the problem makes available the different rods over time. This is reminiscent of ready times in machine scheduling. The solution obtained in such a case will be no better than the one found in the case in which all rods are available at the beginning of the process (simply because the problem that makes rods only available over time is more restrictive than the problem discussed here).

The efficiency of the heuristics and their performance may be evaluated in different ways. An obvious evaluation is to use simulation (see Chap. 13 of this book). This will enable users to specify an average or expected *error bound* of the algorithm. For instance, in the above example the heuristic method uses nine instead of the optimal eight bins, i.e., 12.5 % more than optimal.

However, in some cases it is possible to determine theoretical error bounds, i.e., bounds that cannot be violated. For instance, it has been shown that the solution found by the First Fit Heuristic cannot be worse than about 70 % higher than the optimal solution. One problem associated with the theoretical bounds is that while they represent a reliable, provable property, they tend to be very high.

Many other heuristics have been presented in the literature. Many improvement algorithms are *neighborhood searches*, whose main distinguishing feature is that they start with a given solution and search for better solutions in the neighborhood of the present solution. The Swap Method described above belongs to this class. Another very successful heuristic in this class is *tabu search*. The idea of this method is to get out of a local optimum by temporarily allowing the current solution to deteriorate. In order to avoid cycling between solutions that are better and those that are worse, a list of prohibited moves (a tabu list) is set up that is updated as the algorithm progresses. This procedure allows to "get over the hump" from the present point to other solutions that are hopefully better than the best solution known at this point. For example, in Fig. A.3, if the best known minimum is $x = -7.0689$, we may allow worse solutions (i.e., those with higher functional values) in our move to the left. This may allow us to find the global minimum at $x = -1.4755$.

Other techniques are based on observations made in the technical or the natural world. Examples are *simulated annealing*, a technique modeled after the way molten metal cools. Similar to tabu search, it allows moves to solutions worse than the best presently known solution. Such moves are allowed with a certain probability that decreases during the course of the algorithm. The formulas ensure that the probability to accept a move to a very bad solution is very small. Other methods follow some behavioral patterns of ant colonies or bees.

Appendix B: Vectors and Matrices

This appendix is intended to provide the reader with some basic refresher regarding some basic operations that involve matrices and vectors and the solution of systems of simultaneous linear equations. It is not designed to replace a text, but as a mere quick reference for some material used in this book.

Definition B1. An $[m \times n]$-dimensional matrix

$$\mathbf{A} = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is a two-dimensional array of elements a_{ij} arranged in m horizontal rows and n vertical columns, so that the element a_{ij} is positioned in row i and column j. If $m = n$, the matrix is said to be *square*, if $m = 1$, it is called a *row vector*, if $n = 1$, it is a *column vector*, and if $m = n = 1$, it is a *scalar*.

It is common practice to denote scalars by italicized letters, vectors by lowercase bold letters, and matrices by capitalized letters in boldface.

Definition B2. Given an $[m \times n]$-dimensional matrix $\mathbf{A}$ and an $[n \times p]$-dimensional matrix $\mathbf{B}$, the product $\mathbf{C} = \mathbf{AB}$ is an $[m \times p]$-dimensional matrix $\mathbf{C} = (c_{ij})$, such that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}, \text{ for } i = 1, \ldots, m \text{ and } j = 1, \ldots, n.$$

Example B1

With $\mathbf{a} = [3, \sqrt{2}, -6]$ and

$$\mathbf{B} = \begin{bmatrix} 0 & 5 & \pi \\ 7 & -6 & 2 \\ \sqrt{3} & -11 & 1 \end{bmatrix},$$

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we obtain $\mathbf{aB} = \left[7\sqrt{2} - 6\sqrt{3},\ 81 - 6\sqrt{2},\ 3\pi + 2\sqrt{2} - 6\right]$, and

$$\mathbf{B}^2 = \mathbf{BB} = \begin{bmatrix} 35 + \pi\sqrt{3} & -11\pi - 30 & 10 + \pi \\ -42 + 2\sqrt{3} & 49 & -10 + 7\pi \\ -77 + \sqrt{3} & 55 + 5\sqrt{3} & -21 + \pi\sqrt{3} \end{bmatrix}.$$

Definition B3. Given an $[m \times n]$-dimensional matrix $\mathbf{A}$, the *transpose* $\mathbf{A}^T = (a_{ij}^T)$ is the $[n \times m]$-dimensional matrix with $a_{ij}^T = a_{ji}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Example B2

Use the vector $\mathbf{a}$ and the matrix $\mathbf{B}$ of Example B1, we then obtain

$$\mathbf{a}^T = \begin{bmatrix} 3 \\ \sqrt{2} \\ -6 \end{bmatrix} \text{ and } \mathbf{B}^T = \begin{bmatrix} 0 & 7 & \sqrt{3} \\ 5 & -6 & -11 \\ \pi & 2 & 1 \end{bmatrix}.$$

Appendix C: Systems of Simultaneous Linear Equations

Definition C1. A mathematical relation is written as

LHS R RHS,

where *LHS* denotes the left-hand side, *R* is the relation, and *RHS* is the right-hand side of the relation.

Typically (but not necessarily), *LHS* is a function $f(x_1, x_2, \ldots, x_n)$ of n variables $x_1, x_2, \ldots, x_n$, *R* is a relation of type $<, \leq, = \geq, >$, or $\neq$, and *RHS* is a scalar.

Definition C2. A relation $f(x_1, x_2, \ldots, x_n)\ R\ b$ is said to be *linear*, if the function f can be written as $f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$. We will refer to $f(x_1, x_2, \ldots, x_n)$ as *LHS*, while b is the *RHS*.

Example C1

The relation $2x_1 - .7x_2 + \sqrt{11}x_3 \leq 59$ is linear, whereas the relations $2\sqrt{x_1} - 0.7x_2 + \sqrt{11}x_3 \leq 59$, $2x_1 - 0.7x_1x_2 + \sqrt{11}x_3 \leq 59$, and $2x_1 - 0.7x_2 + \sqrt{11}x_3^3 \leq 59$ are not, due to the appearance of the square root $\sqrt{x_1}$, the product x_1x_2, and the cubic x_3^3, respectively.

We will first deal with the case when the relation *R* is an equation. Assume now that we have a system of m linear equations with n unknowns, and that we want to find a solution, i.e., an assignment of values to the variables, that satisfies all equations. It is now possible to prove the following

Theorem C1

▶ Consider a system of *m* simultaneous linear equations in *n* variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\qquad & \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

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The system has either no solution, exactly one solution, or an infinite number of solutions.

Example C2

Consider the system

$$2x_1 + 3x_2 = 7$$
$$4x_1 + 6x_2 = 10.$$

It is known that if we simultaneously multiply right-hand side and left-hand side of an equation, we do not change its content. Multiplying the first equation by 2, we obtain $4x_1 + 6x_2 = 14$. Now the left-hand side of this equation and that of the second equation are equal, but its right-hand side differs, indicating that there is an inherent contradiction in the system. Thus it is no surprise that the system has no solution.

On the other hand, consider the system

$$2x_1 + 3x_2 = 7$$
$$4x_1 + 6x_2 = 14.$$

Multiplying both sides of the first equation by 2 results in the second equation. In other words, the two relations have exactly the same informational content. This means that we really have only a single equation, and since we need one equation to specify the value of each unknown. Hence this system has an infinite number of solutions x_1 and $x_2 = \frac{1}{3}(7 - 2x_1)$.

Consider now the case that has exactly one solution. Here, we are not concerned with conditions in which a system has exactly one solution, but our focus is on how to actually obtain such a solution, given that it exists. There are many different versions of the *Gaussian elimination technique* (named after the German mathematician Carl Friedrich Gauss, 1777–1855). In order to illustrate the technique, consider the system of simultaneous linear equations

$$\begin{aligned} 2x_1 + 3x_2 - 5x_3 &= 1 && (I)\\ x_1 - 2x_2 + 4x_3 &= -3 && (II)\\ 4x_1 + x_2 + 6x_3 &= 2 && (III) \end{aligned}$$

The idea is to first eliminate one variable, say x_3, from all equations but one. The system has then one equation in all (here: three) variables, whereas two equations include only the remaining variables (here: x_1 and x_2). Among these remaining variables, we now choose another variable to be eliminated (here: x_2), and the procedure is repeated. In the end, we have a single equation in just one variable, which is then replaced by its value in all of the remaining equations, resulting in another system, in which again one of the equations is just a function of a single variable, which is replaced by its value everywhere, and so forth, until the system is solved.

Applying this idea to our example, we fist eliminate the variable x_3 from Eq. *II* by multiplying (*I*) by 4 and multiplying (*II*) by 5, adding them, and then replacing Eq. *II* by $4 \times (I) + 5 \times (II)$. The revised system can then be written as

$$\begin{aligned} 2x_1 + 3x_2 - 5x_3 &= 1 && (I) \\ 13x_1 + 2x_2 &= -11 && (II') = 4\times(I) + 5\times(II) \\ 4x_1 + x_2 + 6x_3 &= 2 && (III) \end{aligned}$$

Next, we eliminate x_3 from Eq. *III* and replace Eq. *III* by $6 \times (I) + 5 \times (III)$. This results in the system

$$\begin{aligned} 2x_1 + 3x_2 - 5x_3 &= 1 && (I) \\ 13x_1 + 2x_2 &= -11 && (II') \\ 32x_1 + 23x_2 &= 16 && (III') = 6\times(I) + 5\times(III) \end{aligned}$$

Since the Eqs. *II*′ and *III*′ now contain only the two variables x_1 and x_2, we can eliminate x_2 from (*III*′) by replacing (*III*′) by $23 \times (II') - 2 \times (III')$. This process results in

$$\begin{aligned} 2x_1 + 3x_2 - 5x_3 &= 1 && (I) \\ 13x_1 + 2x_2 &= -11 && (II') \\ 235x_1 &= -285 && (III'') = 23\times(II') - 2\times(III'). \end{aligned}$$

We say that the system is now in triangular form, due to the pattern of coefficients on the left hand side. This allows us to obtain the values of the variables in a recursive procedure. First we can determine the value of x_1 from Eq. *III*″. Clearly, $x_1 = -\frac{285}{235} = -\frac{57}{47}$. Inserting the value of x_1 into Eq. *II*′ allows us to solve for the variable x_2. In particular, we have $13(-\frac{57}{47}) + 2x_2 = -11$ results in $x_2 = \frac{112}{47}$. Finally, inserting the values of x_1 and x_2 into Eq. *I*, we can solve for x_3. The relation reads $2(-\frac{57}{47}) + 3(\frac{112}{47}) - 5x_3 = 1$, which results in $x_3 = \frac{35}{47}$. The system has now been completely solved. The solution is $[x_1, x_2, x_3] = [-\frac{57}{47}, \frac{112}{47}, \frac{35}{47}] \cong [-1.2128, 2.3830, 0.7447]$. Inserting the values of the unknowns into the original Eqs. *I*, *II*, and *III*, we can verify that the solution is indeed correct. In analogous fashion, we can determine the solution of any system of linear equations. For further details, see any pertinent introductory text on linear algebra or the short summaries in Eiselt and Sandblom (2007) or (2004).

Appendix D: Probability and Statistics

While most chapters in this book deal with deterministic models, some include probabilistic models, in which concepts of probability are needed. This appendix is intended to briefly cover those probabilistic concepts needed in this book. It is by no means intended to replace a thorough knowledge of statistics.

Definition D1. The *probability* of an event (or outcome) of a random experiment is a number p between 0 and 1, which measures the likelihood that the event will occur.

A value of p close to 0 indicates that the event is unlikely to happen, while a value of p close to 1 means that the event is very likely. We can interpret the probability as the proportion of times that the even or outcome will occur, if the experiment is repeated a large number of times.

Suppose now that the events are *mutually exclusive* (i.e., the events are distinct and do not overlap) and *collectively exhaustive* (meaning that exactly one of them will occur). We then obtain the following result.

Theorem D1

If $p_1, p_2, \ldots, p_n$ denote the probabilities of the mutually exclusive and collectively exhaustive outcomes of an experiment, then $p_1 + p_2 + \ldots + p_n = 1$.

Example D1

Consider an experiment that involves tossing a fair coin three times, suppose that each outcome is either head or tail, standing on edge does not occur. Denote H for "head" and T for tail, the outcome "first tail, then head, then tail" is written as THT. Then there are eight possible outcomes: HHH, HHT, HTH, HTT, THH, THT, TTH, and TTT. Given a fair coin, all outcomes are equally likely, and since $p_1 + p_2 + \ldots + p_8 = 1$, we obtain the result that each outcome has a probability of $P(HHH) = P(HHT) = \ldots = P(TTT) = \frac{1}{8}$.

The probability of a *composite event* that consists of several outcomes is the sum of probabilities of the outcome of the event. For instance, the event "obtain exactly one tail in three flips of a fair coin" refers to the event $\{HHT, HTH, THH\}$, so that the probability of such an event is $P(\{HHT, HTH, THH\}) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$. The *event space* is defined as the set of all possible events of an experiment.

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Definition D2. A *random variable X* is a function defined on the event space of an experiment.

Example D2

Given the above coin tossing experiment, let X denote the number of heads that come up in the three tosses. There are four possibilities: $X = 0$ (which occurs only in the event $\{TTT\}$, so that the probability of this event is $P(X = 0) = \frac{1}{8}$), $X = 1$ (which happens if one of the events $\{HTT, THT, TTH\}$ occurs, so that the probability $P(X = 1) = \frac{3}{8}$), $X = 2$ (an event that occurs if one of $\{HHT, HTH, THH\}$ occurs, so that $P(X = 2) = \frac{3}{8}$), and finally $X = 3$ (which occurs only if $\{HHH\}$ happens, so that $P(X = 3) = \frac{1}{8}$). Again, the sums of all possible events add up to 1.

In general, we distinguish between two different types of random variables, *discrete* and *continuous random variables*.

Definition D3. A random variable X is called *discrete*, if it can assume only one of (countably many) values $a_1, a_2, \ldots$. The function $P(a_j) = P(X = a_j) = p(a_j)$ is called the *discrete probability distribution* (function) of X. The function $F(a_j) = P(X \leq a_j)$ is called the c*umulative probability distribution* (function) of X.

Example D3

Again, define the random variable X as the number of heads in three tosses of a fair coin. Table D.1 shows the probability distribution function for this event.

We note that $F(a_j)$ is an increasing function of a_j that eventually reaches the value of 1 for the largest value of a_j (in case of finitely many outcomes), and that converges towards 1 for infinitely many outcomes a_j. Since P and F are probabilities, we must have $0 \leq P(a_j) \leq 1$ and $0 \leq F(a_j) \leq 1$ for all events a_j.

Definition D4. A discrete random variable X with a probability distribution function $p(x) = P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}$ $x = 0, 1, 2, \ldots$ is called *Poisson-distributed* with parameter λ. In case a random variable follows this distribution, we will write $X \sim Po(\lambda)$, where *Po* stands for Poisson.

The Poisson distribution is named in honor of Siméon Denis Poisson, a French mathematician, 1781–1840. The distribution is of major importance in queuing (Chap. 13 of this book) and will be extensively used in that context.

Example D4

If the random variable $X \sim P_o(\lambda = 1.3)$, then $P(X = 2) = p(2) = e^{-1.3}\frac{(1.3)^2}{2!} = 0.2303$, and $F(X \leq 1) = p(0) + p(1) = 0.6268$.

Table D.1 Probability distribution for the coin toss example

X	P	F
0	$\frac{1}{8}$	$\frac{1}{8}$
1	$\frac{3}{8}$	$\frac{4}{8}$
2	$\frac{3}{8}$	$\frac{7}{8}$
3	$\frac{1}{8}$	$\frac{8}{8}$

Definition D5. A random variable X is called *continuous*, if there exists a function $f(x)$, such that the cumulative distribution function $F(x)$ for X can be written as $F(x) = \int_{-\infty}^{t} f(t)dt$. The function $f(x)$ is called the (*probability*) *density function* of X.

There are two different continuous distributions that are used in this book, the *exponential* and the *normal distribution*.

Definition D6. A continuous random variable X with the density function $f(x) = \lambda e^{-\lambda x}$ with $x \geq 0$ is said to *exponentially distributed* with parameter λ (a positive constant). If X is exponentially distributed, we write $X \sim Ex(\lambda)$.

It is not difficult to demonstrate that the cumulative distribution function of an exponentially distributed variable with parameter λ is $F(x) = 1 - e^{-\lambda x}$.

Definition D7. A continuous random variable X with density function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is said to follow a *normal distribution* (or, alternatively, *Gaussian distribution*) with parameters μ and $\sigma > 0$. In such a case, we write $X \sim N(\mu, \sigma)$. The distribution $N(0, 1)$, i.e., the normal distribution with $\mu = 0$ and $\sigma = 1$ is called the *standard normal distribution* and is usually denoted by Z. Its density function is called the *normal curve* (or *bell curve*), given by the function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-1/2x^2}$.

Definition D8. The *expected value* (or *mean* or *expectation*) $E(X)$ or μ of a discrete random variable X is given by $E(X) = \mu = a_1 P(X = a_1) + a_2 P(X = a_2) + \ldots$, meaning we multiply each outcome by its associated probability and add them up. For a continuous random variable X, the expected value is given by $E(X) = \mu = \int_{-\infty}^{\infty} tf(t)dt$.

There are discrete and continuous random variables for which there exists no expected value.

Example D5

For the random variable X of Examples $D2$ and $D3$ where X denotes the number of heads in three tosses of a fair coin, we obtain the expected value $E(X) = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = 1\text{1/2}.$

Theorem D2

If $X \sim Po(\lambda)$, then $E(X) = \lambda$, if $X \sim Ex(\lambda)$, then $E(X) = \frac{1}{\lambda}$, and if $X \sim N(\mu, \sigma)$, then $E(X) = \mu$.

Definition D9. The *variance* $V(X) = \sigma^2$ of a random variable X with mean μ is defined as $E((X - \mu)^2)$. We call $\sigma = \sqrt{V(X)}$ the *standard deviation* of X. The expression $\sigma_\mu = \sigma/\mu$ is referred to as the *coefficient of variation* of X.

There are discrete and continuous variables for which the mean exists, but the variance does not. One can show that $V(X) = E(X^2) - \mu^2$ and that $V(X) \geq 0$ for all random variables X, as long as $E(X^2)$, $E(X)$, and $V(X)$ exist.

Example D6

For the random variable X of Examples $D2$ and $D3$ where X denotes the number of heads in three tosses of a fair coin, we obtain the variance $V(X) = E(X^2) - \mu^2 = 3 - 2.25 = 0.75$. The standard deviation $\sigma = \sqrt{V(X)} = \sqrt{.75} \approx .8660$. The coefficient of variation $\sigma_\mu = \sqrt{.75}/1.5 \approx 0.5774$.

Theorem D3

If the random variable $X \sim P_o(\lambda)$, then $V(X) = \lambda$, if $X \sim Ex(\lambda)$, then $V(X) = \frac{1}{\lambda^2}$, and if $X \sim N(\mu, \sigma)$, then $V(X) = \sigma^2$.

Theorem D4

If the random variable $X \sim N(\mu, \sigma)$, then $Z = [(X - \mu)/\sigma] \sim N(0,\ 1)$.

Theorem D5

For any random variable X and for any given numbers $a < b$, we have $P(a < X \leq b) = F(b) - F(a)$. For any continuous random variable X, $P(a \leq X \leq b) = P(a \leq X < b) = P(a < X < b) = F(b) - F(a)$ as well as $P(X = a) = P(X = b) = 0$.

Example D7

Let the random variable $X \sim N(2.3,\ 0.90)$ and compute the probability $P(1.7 \leq X \leq 2.6)$. We find that $P(1.7 \leq X \leq 2.6) = P\left(\frac{1.7-2.3}{0.9} \leq \frac{X-2.3}{0.9} \leq \frac{2.6-2.3}{0.9}\right) = P(-0.6667 \leq Z \leq 0.3333) = F(0.3333) - F(-0.6667)$. The function $F(x) =$

$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-1/2t^2} dt$ is called area under the normal curve and it is tabulated at the end of this section for values $x \geq 0$. By virtue of symmetry of the normal curve, we find that $F(-x) = 1 - F(x)$, therefore $F(0.3333) - F(-0.6667) = F(0.3333) - 1 + F(0.6667) = 0.6306 + 1 - 0.7475 = 0.3781$ by reading the table and interpolating as necessary.

It has been observed in practice that empirical data that are bell-shaped and symmetric share some common properties. In particular, the *empirical rule* holds, according to which about 68 % of all observations lie within one standard deviation about the mean, about 95 % of the observations are within two standard deviations about the mean, and virtually all observations are within three standard deviations about the mean. It is worth mentioning that this is not a provable property, but an observed rule that often occurs in practice.

Making now stronger assumptions about the distribution in particular that it is not only bell-shaped and symmetric, but normal, we are able to confirm the assertion of the empirical rule. In particular, for a standard normal variable $Z \sim N(0, 1)$ we have $F(0) = 0.5$, $F(1) = 0.8413$, $F(2) = 0.9773$, and $F(3) = 0.9987$. Then $P(-1 \leq Z \leq 1) = 2(0.8413 - 0.5) = 0.6826$, $P(-2 \leq Z \leq 2) = 2(0.9773 - 0.5) = 0.9546$, and $P(-3 \leq Z \leq 3) = 2(0.9987 - 0.5) = 0.9974$.

We will now consider events A, B, C, ..., which are sets of outcomes of experiments.

Definition D10. Given the events A and B, the *union* $A \cup B$ is the event consisting of outcomes that are in A or in B or both. In contrast, the *intersection* $A \cap B$ is the event that consists of outcomes that are in both A and B.

Theorem D6 (The addition law for probabilities)
For any events A and B, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Example D8

Recall Example *D*4, in which $X \sim P_o(\lambda = 1.3)$. Furthermore, let $A = \{X = 0 \text{ or } 1\}$ and $B = \{1 \text{ or } 2\}$. Then $A \cup B = \{X = 0, 1, \text{or } 2\}$, while $A \cap B = \{X = 1\}$. We then find $P(A) = p(0) + p(1) = 0.6268, P(B) = p(1) + p(2) = 0.5846, P(A \cup B)$ $p(0) + p(1) + p(2) = 0.8571, P(A \cap B) = p(1) = 0.3543$. In accordance with the theorem, we find that $P(A) + P(B) - P(A \cap B) = 0.6268 + 0.5846 - 0.8571 = P(A \cup B)$.

Definition D11. The sets A and B are said to be *collectively exhaustive*, if their union $A \cup B$ includes all possible outcomes of an experiment. The two sets are called *mutually exclusive*, if their intersection is empty. The *complement* $\bar{A}$ of a set A (sometimes also written as $\neg A$) is the set of all possible outcomes not in A.

As far as probabilities are concerned, $P(\bar{A}) = 1 - P(A)$.

The addition low for probabilities can be generalized. For instance, for mutually exclusive sets $A_1, A_2, \ldots, A_m$, we obtain $P(A_1 \cup A_2 \cup ... \cup A_m) = P(A_1) + P(A_2) + ... + P(A_m)$. To establish a multiplication law for probabilities, we need the following

Definition D12. For any events A and B with $P(B) \neq 0$, the *conditional probability of A given B* is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example D9

Consider an experiment, in which the random variable X denotes that at most two heads come up in three tosses of a fair coin, i.e., $X \leq 2$, and define B as an event that sees at least one head in three tosses of a fair coin, i.e. $X \geq 1$. Then $A \cap B = \{X = 1 \text{ or } 2\}$, so that $P(A \cap B) = \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$ and $P(B) = \frac{3}{8} + \frac{3}{8} + \frac{1}{4} = \frac{7}{8}$. Therefore, $P(A|B) = \frac{3/4}{7/8} = \frac{6}{7} \approx 0.8571$.

Theorem D7 (The multiplication law for probabilities)
For any events A and B, $P(A \cap B) = P(A|B)\, P(B)$.

Note that if $P(B) = 0$, then $P(A|B)$ is not defined, but in this case the right-hand side is interpreted as being zero.

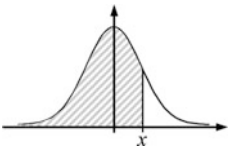
Definition D13. The events A and B are said to be *statistically independent*, if $P(A \cap B) = P(A)\, P(B)$.

Theorem D8 (Bayes's theorem)
Let the events $A_1, A_2, \ldots, A_m$ be mutually exclusive and collectively exhaustive. Then for any event B with $P(B) \neq 0$, we have

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + ... + P(B|A_m)P(A_m)}, i = 1, ..., m$$

The theorem of Bayes (Thomas Bayes, English clergyman, 1702–1761) will be used in Chap. 9 of this book. In the context of Bayes's theorem, the unconditional probabilities $P(A_i)$ are called *prior probabilities*, while the conditional probabilities $P(A_i|B)$ are referred to as *posterior probabilities*.

Area under the normal curve



x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
.00	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.10	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5754
.20	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.30	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.40	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.50	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.60	.7258	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7518	.7549
.70	.7580	.7612	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.80	.7881	.7910	.7939	.7967	.7996	.8023	.8051	.8079	.8106	.8133
.90	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9430	.9441
1.6	.9452	.9463	.9474	.9485	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9700	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9762	.9767
2.0	.9773	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9865	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9978	.9980	.9980	.9981
2.9	.9981	.9982	.9983	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.5	.999767									
4.0	.9999683									
4.5	.99999660									
5.0	.999999713									

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