Historical Note

Here we shall present a brief chronology of the appearance of the concepts discussed in this book. The development of mathematical ideas generally proceeds in such a way that some concepts gradually emerge from others. Therefore, it is generally impossible to fix accurately the appearance of some particular idea. We shall only point out the important milestones and, it goes without saying, shall do so only roughly. In particular, we shall limit our view to Western European mathematics.

The principal stimulus was, of course, the creation of analytic geometry by Fermat and Descartes in the seventeenth century. This made it possible to specify points (on the line, in the plane, and in three-dimensional space) using numbers (one, two, or three), to specify curves and surfaces by equations, and to classify them according to the algebraic nature of their equations. In this regard, linear transformations were used frequently, especially by Euler, in the eighteenth century.

Determinants (particularly as a symbolic apparatus for finding solutions of systems of n linear equations in n unknowns) were considered by Leibniz in the seventeenth century (even if only in a private letter) and in detail by Gabriel Cramer in the eighteenth. It is of interest that they were constructed on the basis of the rule of “general expansion” of the determinant, that is, on the basis of the most complex (among those that we considered in Chap. 2) way of defining them. This definition was discovered “empirically,” that is, conjectured on the basis of the formulas for the solution of systems of linear equations in two and three unknowns. The broadest use of determinants occurred in the nineteenth century, especially in the work of Cauchy and Jacobi.

The concept of “multidimensionality,” that is, the passage from one, two, and three coordinates to an arbitrary number, was stimulated by the development of mechanics, where one considered systems with an arbitrary number of degrees of freedom. The idea of extending geometric intuition and concepts to this case was developed systematically by Cayley and Grassmann in the nineteenth century. At the same time, it became clear that one must study quadrics in spaces of arbitrary dimension (Jacobi and Sylvester in the nineteenth century). In fact, this question had already been considered by Euler.

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The study of concepts defined by a set of abstract axioms (groups, rings, algebras, fields) began as early as the nineteenth century in the work of Hamilton and Cayley, but it reached its full flowering in the twentieth century, chiefly in the schools of Emmy Noether and Emil Artin.

The concept of a projective space was first investigated by Desargues and Pascal in the seventeenth century, but systematic work in this direction began only in the nineteenth century, beginning with the work of Poncelet.

The axiomatic definition of vector spaces and Euclidean spaces as given in this book broke finally with the primacy of coordinates. It was first rigorously formulated almost simultaneously by Hermann Weyl and John von Neumann. Both came to this from work on questions in physics. Then two versions of quantum mechanics were created: the "wave mechanics" of Schrödinger and the "matrix mechanics" of Heisenberg. It was necessary to work out that in some sense, they were "one and the same."

Both mathematicians developed an axiomatic theory of Euclidean spaces and vector spaces and showed that quantum-mechanical theories are connected with two isomorphic spaces. However, the difference between those theories and what we presented in this book lies in the fact that they worked with infinite-dimensional spaces. In any case, for finite-dimensional spaces, there appeared an invariant (that is, independent of the choice of coordinates) theory that by now has become universally accepted.

The introduction of the axiomatic approach in geometry was discussed in sufficient detail in Chap. 11, devoted to the hyperbolic geometry of Lobachevsky. Such studies began at the end of the nineteenth century, but their definitive influence in mathematics dates from the beginning of the twentieth century. The central figure here was Hilbert. For example, he contributed to the application of geometric intuition to many problems in analysis.

References

We recall first those books that were in vogue when the lectures on which this book is based were given. Many of these books have been reprinted, and we have tried to provide information on the latest available version.[1]

1. I.M. Gelfand, *Lectures on Linear Algebra* (Dover, New York, 1989)
2. A.G. Kurosh, *Linear Equations from a Course of Higher Algebra* (Oregon State University Press, Corvallis, 1969)
3. F.R. Gantmacher, *The Theory of Matrices* (American Mathematical Society, Chelsea, 1959)
4. A.I. Malcev, *Foundations of Linear Algebra* (Freeman, New York, 1963)
5. P.R. Halmos, *Finite-Dimensional Vector Spaces* (Springer, New York, 1974)
6. G.E. Shilov, *Mathematical Analysis: A Special Course* (Pergamon, Elmsford, 1965)
7. O. Schreier, E. Sperner, *Introduction to Modern Algebra and Matrix Theory*, 2nd edn. (Dover, New York, 2011)
8. O. Schreier, E. Sperner, *Einführung in die analytische Geometrie und Algebra* (Teubner, Leipzig, 1931)

The book by Shilov is of particular interest for its large number of analytic applications. The following books could also be recommended. However, the conciseness of their presentation and abstract approach put them far beyond the capacity of the average student.

9. B.L. Van der Waerden, *Algebra* (Springer, New York, 2003)
10. N. Bourbaki, *Algebra I* (Springer, Berlin, 1998)
11. N. Bourbaki, *Algebra II* (Springer, Berlin, 2003)

Since the lectures on which this book is based were given, so many books on the subject have appeared that we give here only a small sample.

[1] *Translator's note:* Wherever possible, English-language versions have been given. Some of these were written originally in English, while others are translations from original Russian or German sources.

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12. E.B. Vinberg, *A Course in Algebra* (American Mathematical Society, Providence, 2003)
13. A.I. Kostrikin, Yu.I. Manin, *Linear Algebra and Geometry* (CRC Press, Boca Raton, 1989)
14. A.I. Kostrikin, *Exercises in Algebra: A Collection of Exercises* (CRC Press, Boca Raton, 1996)
15. S. Lang, *Algebra* (Springer, 1992)
16. M.M. Postnikov, *Lectures in Geometry: Semester 2* (Mir, Moscow, 1982)
17. D.K. Faddeev, *Lectures on Algebra* (Lan, St. Petersburg, 2005) (in Russian)
18. R.A. Horn, C.R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge 1990)

With regard to applications to mechanics, see the book *The Theory of Matrices* by Gantmacher mentioned above as well as the following.

19. F.R. Gantmacher, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems* (American Mathematical Society, Providence, 2002)

Relationships with differential geometry, which we briefly touched on in this course, are described, for example, in the following.

20. A.S. Mishchenko, A.T. Fomenko, *A Course of Differential Geometry and Topology* (Mir, Moscow, 1988)

In presenting Lobachevsky's hyperbolic geometry, we have followed for the most part the following brochure.

21. B.N. Delone, *Elementary Proof of the Consistency of Hyperbolic Geometry* (State Technical Press, Moscow, 1956) (in Russian)

All the results concerning the foundations of geometry whose proofs we omitted are contained in the following books.

22. N.Yu. Netsvetaev, A.D. Alexandrov, *Geometry* (Nauka, Fizmatlit, Moscow, 1990)
23. N.V. Efimov, *Higher Geometry* (Mir, Moscow, 1980)

Facts about analytic geometry that were briefly mentioned in this course, such as the connection with the theory of quadrics, can be found in the following books.

24. P. Dandelin, *Mémoire sur l'hyperboloïde de révolution, et sur les hexagones de Pascal et de M. Brianchon. Nouveaux mémoires de l'Académie Royale des Sciences et Belles-Lettres de Bruxelles*, T. III (1826), pp. 3–16
25. B.N. Delone, D.A. Raikov, *Analytic Geometry* (State Technical Press, Moscow-Leningrad, 1949) (in Russian)
26. P.S. Alexandrov, *Lectures in Analytic Geometry* (Nauka, Fizmatlit, Moscow, 1968) (in Russian)
27. D. Hilbert, S. Cohn-Vossen, *Geometry and the Imagination* (AMS, Chelsea, 1999)
28. A.P. Veselov, E.V. Troitsky, *Lectures in Analytic Geometry* (Lan, St. Petersburg, 2003) (in Russian)

Connections between the hyperbolic geometry of Lobachevsky and other branches of projective geometry are well described in the following book.

29. F. Klein, *Nicht-Euklidische Geometrie* (Göttingen, 1893). Reprinted by AMS, Chelsea, 2000

In connection with representation theory, the following book is to be recommended.

30. J.-P. Serre, *Linear Representations of Finite Groups* (Springer, Berlin, 1977)

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