Appendix A
Coordinate Systems

The coordinate systems used in this book are depicted in Fig. A.1. The spherical coordinates (r, α, β) are related to the Cartesian coordinates $[x, y, z]^T$ by (Weisstein 2002)

$$r = \sqrt{x^2 + y^2 + z^2} \tag{A.1a}$$

$$\alpha = \arctan\left(\frac{y}{x}\right) \tag{A.1b}$$

$$\beta = \arccos\left(\frac{z}{r}\right), \tag{A.1c}$$

where $r \in [0, \infty)$, $\alpha \in [0, 2\pi)$, and $\beta \in [0, \pi]$, and the inverse tangent must be suitably defined to take the correct quadrant of (x, y) into account (Weisstein 2002).

The Cartesian coordinates $[x, y, z]^T$ and $[k_x, k_y, k_z]^T$ are related to the spherical coordinates (r, α, β) and (k, θ, ϕ) by

$$x = r \cos\alpha \sin\beta \tag{A.2a}$$

$$y = r \sin\alpha \sin\beta \tag{A.2b}$$

$$z = r \cos\beta. \tag{A.2c}$$

and

$$k_x = k \cos\theta \sin\phi \tag{A.3a}$$

$$k_y = k \sin\theta \sin\phi \tag{A.3b}$$

$$k_z = k \cos\phi \tag{A.3c}$$

respectively.

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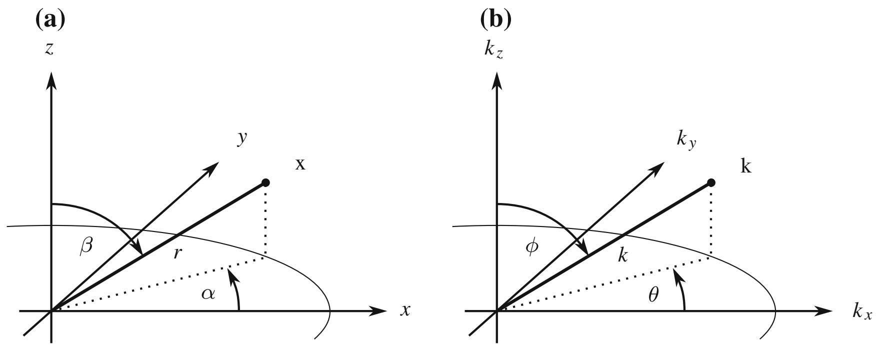

Fig. A.1 The coordinate systems used in this paper. **a** Spatial domain. **b** Wavenumber dom

The angles α and θ are termed *azimuth*, β and ϕ are termed *spherical polar angle*, or *zenith angle*, or *colatitude*.

Appendix B
Definition of the Fourier Transform

The temporal Fourier transform used in this work is defined as (Bracewell 2000)

$$S(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} s(\mathbf{x}, t) e^{-i\omega t} dt. \tag{B.1}$$

The inverse temporal Fourier transform is therefore

$$s(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\mathbf{x}, \omega) e^{i\omega t} d\omega. \tag{B.2}$$

The spatial Fourier transform is defined as

$$\tilde{S}(k_x, y, z, \omega) = \int_{-\infty}^{\infty} S(\mathbf{x}, \omega) e^{ik_x x} dx \tag{B.3}$$

exemplarily for the x-dimension. The corresponding inverse spatial Fourier transform is

$$S(\mathbf{x}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{S}(k_x, y, z, \omega) e^{-ik_x x} dk_x. \tag{B.4}$$

Note that reversed exponents are used in the spatial Fourier transform compared to the temporal one. The motivation for this choice is outlined in Sect. 2.2.6.

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Appendix C
Fourier Transforms of Selected Quantities

C.1 Fourier Transforms of a Plane Wave

A monochromatic plane wave with radian frequency ω_{pw} and wave vector $\mathbf{k}_{\text{pw}}$ is given by (Williams 1999)

$$s(\mathbf{x}, t) = e^{-i\mathbf{k}_{\text{pw}}^T \mathbf{x}} \cdot e^{i\omega_{\text{pw}} t} \tag{C.1a}$$

$$= e^{i\frac{\omega_{\text{pw}}}{c}\left(ct - \mathbf{n}_{\text{pw}}^T \mathbf{x}\right)}, \tag{C.1b}$$

with $\mathbf{n}_{\text{pw}}$ denoting the unit length vector pointing in the same direction like $\mathbf{k}_{\text{pw}}$, i.e. in propagation direction of the plane wave. The term in brackets in (C.1b) is termed the *Hesse normal form* of a plane propagating in direction $\mathbf{n}_{\text{pw}}$ with speed c (Weisstein 2002).

Recall that

$$\mathbf{k}_{\text{pw}}^T = [k_{\text{pw,x}} k_{\text{pw,y}} k_{\text{pw,z}}] \tag{C.2}$$

$$= k_{\text{pw}} \cdot [\cos\theta_{\text{pw}} \sin\phi_{\text{pw}} \sin\theta_{\text{pw}} \sin\phi_{\text{pw}} \cos\phi_{\text{pw}}] \tag{C.3}$$

with $(\theta_{\text{pw}}, \phi_{\text{pw}})$ being the propagation direction of the plane wave in spherical coordinates.

The Fourier transform of $s(\mathbf{x}, t)$ with respect to t yields (Girod et al. 2001)

$$S(\mathbf{x}, \omega) = e^{-i\mathbf{k}_{\text{pw}}^T \mathbf{x}} \cdot 2\pi\delta(\omega - \omega_{\text{pw}}). \tag{C.4}$$

A further Fourier transform with respect to x yields

$$\tilde{S}(k_x, y, z, \omega) = 2\pi\delta(k_x - k_{\text{pw},x})\, e^{-ik_{\text{pw},y} y}\, e^{-ik_{\text{pw},z} z} \cdot 2\pi\delta(\omega - \omega_{\text{pw}}), \tag{C.5}$$

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a further Fourier transform with respect to z yields

$$\tilde{S}(k_x, y, k_z, \omega) = 4\pi^2 \delta(k_x - k_{\text{pw},x})\, e^{-ik_{\text{pw},y} y} \delta(k_z - k_{\text{pw},z}) \cdot 2\pi\, \delta(\omega - \omega_{\text{pw}}), \quad \text{(C.6)}$$

and finally a further Fourier transform with respect to y yields

$$\tilde{S}(\mathbf{k}, \omega) = 8\pi^3 \delta(\mathbf{k} - \mathbf{k}_{\text{pw}}) \cdot 2\pi\, \delta(\omega - \omega_{\text{pw}}). \quad \text{(C.7)}$$

C.2 Fourier Transforms of the Free-Field Green's Function

The three-dimensional free-field Green's function $g_0(\mathbf{x}, t)$ for excitation at the coordinate origin is given in time domain by (Williams 1999)

$$g_0(\mathbf{x}, t) = \frac{1}{4\pi} \frac{\delta\left(t - \frac{r}{c}\right)}{r}. \quad \text{(C.8)}$$

Applying a Fourier transform with respect to t to (C.8) yields

$$G_0(\mathbf{x}, \omega) = \frac{1}{4\pi} \frac{e^{-i\frac{\omega}{c} r}}{r}. \quad \text{(C.9)}$$

The Fourier transform with respect to x is calculated by applying Euler's formula (Weisstein 2002) and using (Gradshteyn and Ryzhik 2000, Eqs. (3.876-1) and (3.876-2); Morse and Feshbach 1953, p. 1323). It is given by

$$\tilde{G}_0(k_x, y, z, \omega) = \begin{cases} -\frac{i}{4} H_0^{(2)}\left(\sqrt{\left(\frac{\omega}{c}\right)^2 - k_x{}^2}\sqrt{y^2 + z^2}\right) & \text{for } 0 \leq |k_x| < \left|\frac{\omega}{c}\right| \\ \frac{1}{2\pi} K_0\left(\sqrt{k_x{}^2 - \left(\frac{\omega}{c}\right)^2}\sqrt{y^2 + z^2}\right) & \text{for } 0 < \left|\frac{\omega}{c}\right| < |k_x|\,. \end{cases} \quad \text{(C.10)}$$

$H_0^{(2)}(\cdot)$ denotes the zero-th order Hankel function of second kind, $K_0(\cdot)$ the zero-th order modified Bessel function of second kind (Williams 1999). A further Fourier transform with respect to z is yielded using (Gradshteyn and Ryzhik 2000, Eqs. (6.677-3)–(6.677-5)). It is given by

$$\tilde{G}_0(k_x, y, k_z, \omega) = \begin{cases} -\frac{i}{2} \frac{e^{-i\sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2 - k_z^2} \cdot y}}{\sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2 - k_z^2}} & \text{for } 0 \leq \sqrt{k_x^2 + k_z^2} < \left|\frac{\omega}{c}\right| \\ \frac{1}{2} \frac{e^{-\sqrt{k_x^2 + k_z^2 - \left(\frac{\omega}{c}\right)^2} \cdot y}}{\sqrt{k_x^2 + k_z^2 - \left(\frac{\omega}{c}\right)^2}} & \text{for } 0 < \left|\frac{\omega}{c}\right| < \sqrt{k_x^2 + k_z^2}. \end{cases} \quad \text{(C.11)}$$

Note that (C.11) is only valid for $y > 0$ (Gradshteyn and Ryzhik 2000).

Finally, $\tilde{G}_0(\mathbf{k}, \omega)$ is yielding using (Gradshteyn and Ryzhik 2000, Eq. (3.893-2)). It is given by

$$\tilde{G}_0(\mathbf{k}, \omega) = \tilde{G}(k, \omega) = \frac{1}{k^2 - \left(\frac{\omega}{c}\right)^2}. \quad \text{(C.12)}$$

Appendix D
Convolution Theorems

D.1 Fourier Series Domain

A representation of the Fourier series expansion coefficients $\mathring{H}_m(r, \beta, \omega)$ of a function $H(\mathbf{x}, \omega)$ which is given by a multiplication of two functions $F(\mathbf{x}, \omega)$ and $G(\mathbf{x}, \omega)$ as

$$H(\mathbf{x}, \omega) = F(\mathbf{x}, \omega) \cdot G(\mathbf{x}, \omega) \tag{D.1}$$

in terms of the Fourier series expansion coefficients $\mathring{F}_m(r, \beta, \omega)$ and $\mathring{G}_m(r, \beta, \omega)$ of $F(\mathbf{x}, \omega)$ and $G(\mathbf{x}, \omega)$ respectively is derived in this section. Applying (2.36) yields

$$\begin{aligned}
\mathring{H}_m(r, \beta, \omega) &= \frac{1}{2\pi} \int_0^{2\pi} F(\mathbf{x}, \omega) G(\mathbf{x}, \omega)\, e^{-im\alpha}\, d\alpha \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{m_1=-\infty}^{\infty} \mathring{F}_{m_1}(r, \beta, \omega)\, e^{im_1\alpha} \\
&\quad \times \sum_{m_2=-\infty}^{\infty} \mathring{G}_{m_2}(r, \beta, \omega) e^{im_2\alpha}\, e^{-im\alpha}\, d\alpha \\
&= \frac{1}{2\pi} \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \mathring{F}_{m_1}(r, \beta, \omega) \mathring{G}_{m_2}(r, \beta, \omega) \\
&\quad \times \int_0^{2\pi} e^{i(m_1+m_2-m)\alpha}\, d\alpha.
\end{aligned} \tag{D.2}$$

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The integral in (D.2) vanishes unless $m_1 + m_2 - m = 0$ or $m_2 = m - m_1$ respectively. In these cases it equals 2π so that finally (Girod et al. 2001)

$$\begin{aligned}\mathring{H}_m(r,\beta,\omega) &= \sum_{m_1=-\infty}^{\infty} \mathring{F}_{m_1}(r,\beta,\omega)\mathring{G}_{m-m_1}(r,\beta,\omega) \\ &= \mathring{F}_m(r,\beta,\omega) *_m \mathring{G}_m(r,\beta,\omega),\end{aligned} \tag{D.3}$$

which represents a *convolution theorem* for the Fourier series expansion.

D.2 Spherical Harmonics Domain

The procedure outlined in Sect. D.1 is adapted here in order to obtain a representation of the coefficients $\mathring{H}_n^m(r,\omega)$ of a function $H(\mathbf{x},\omega)$ which is given by a multiplication of two functions $F(\mathbf{x},\omega)$ and $G(\mathbf{x},\omega)$ as

$$H(\mathbf{x},\omega) = F(\mathbf{x},\omega) \cdot G(\mathbf{x},\omega) \tag{D.4}$$

in terms of the coefficients $\mathring{F}_n^m(r,\omega)$ and $\mathring{G}_n^m(r,\omega)$ of $F(\mathbf{x},\omega)$ and $G(\mathbf{x},\omega)$ respectively. Applying (2.33) yields

$$\begin{aligned}\mathring{H}_n^m(r,\omega) &= \int_0^{2\pi}\int_0^{\pi} F(\mathbf{x},\omega)G(\mathbf{x},\omega)Y_n^{-m}(\alpha,\beta)\sin\beta\, d\beta\, d\alpha \\ &= \int_0^{2\pi}\int_0^{\pi} \sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1} \mathring{F}_{n_1}^{m_1}(r,\omega)Y_{n_1}^{m_1}(\alpha,\beta) \sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2} \mathring{G}_{n_2}^{m_2}(r,\omega) \\ &\qquad \times Y_{n_2}^{m_2}(\alpha,\beta)Y_n^{-m}(\alpha,\beta)\sin\beta\, d\beta\, d\alpha \\ &= \sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}\sum_{n_2=0}^{\infty}\sum_{m_2=-n_2}^{n_2} \mathring{F}_{n_1}^{m_1}(r,\omega)\mathring{G}_{n_2}^{m_2}(r,\omega) \\ &\qquad \times \underbrace{\int_0^{2\pi}\int_0^{\pi} Y_{n_1}^{m_1}(\alpha,\beta)Y_{n_2}^{m_2}(\alpha,\beta)Y_n^{-m}(\alpha,\beta)\sin\beta\, d\beta\, d\alpha\,.}_{=\gamma_{n_1,n_2,n}^{m_1,m_2,m}}\end{aligned} \tag{D.5}$$

Integrals like the one in (D.5) often appear in problems in quantum mechanics and their properties are well investigated (Arfken and Weber 2005). The result is a real number and these integrals are also referred to as *Gaunt coefficients* $\gamma_{n_1,n_2,n}^{m_1,m_2,m}$ (Sébilleau 1998). The integral form of $\gamma_{n_1,n_2,n}^{m_1,m_2,m}$ as given in (D.5) is inconvenient for evaluation since it can not be solved analytically. More convenient is the representation (Gumerov and Duraiswami 2004, Eq. (3.2.28), p. 99)

$$\gamma_{n_1,n_2,n}^{m_1,m_2,m} = \frac{1}{4\pi}\sqrt{\frac{(2n_1+1)(2n_2+1)(2n+1)}{4\pi}}\ \mathcal{E}\begin{pmatrix} m_1 & m_2 & -m \\ n_1 & n_2 & n \end{pmatrix}. \tag{D.6}$$

The *E-symbol* $\mathcal{E}(\cdot)$ is defined as (Gumerov and Duraiswami 2004, Eq. (3.2.27), p. 99)

$$\mathcal{E}\begin{pmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} = 4\pi\,\varepsilon_{m_1}\varepsilon_{m_2}\varepsilon_{m_3}\begin{pmatrix} n_1 & n_2 & n_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} n_1 & n_2 & n_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{D.7}$$

with

$$\varepsilon_m = i^{m+|m|} = \begin{cases} (-1)^m & \forall m \geq 0 \\ 1 & \forall m \leq 0 \end{cases} \tag{D.8}$$

and $\begin{pmatrix} \cdots \\ \cdots \end{pmatrix}$ denoting the *Wigner 3j-Symbol*. The Wigner 3j-Symbol is defined in (Weisstein 2002). The MATLAB simulations presented in this book employ the script provided by (Kraus 2008).

The E-symbol and thus the Gaunt coefficients $\gamma_{n_1,n_2,n}^{m_1,m_2,m}$ satisfy the following selection rules:

1. $m_2 = m - m_1$.
2. $|n - n_2| \leq n_1 \leq n + n_2$ (*triangle inequalities* or *triangle rule* (Weisstein 2002)).
3. $n + n_1 + n_2$ is even or zero.

If these rules are not satisfied then $\gamma_{n_1,n_2,n}^{m_1,m_2,m} = 0$. Actually, it can be shown that $\gamma_{n_1,n_2,n}^{m_1,m_2,m}$ vanishes in more cases than stated above (Gjellestad 1955; Gumerov and Duraiswami 2004). In order to retain notational clarity the selection rules are only occasionally explicitly considered.

Reformulating (D.5) by explicitly considering rule 1 reads then (Arfken and Weber 2005; Shirdhonkar and Jacobs 2005)

$$\mathring{H}_n^m(r,\omega) = \sum_{n_1=0}^{\infty}\sum_{m_1=-n_1}^{n_1}\sum_{n_2=0}^{\infty}\mathring{F}_{n_1}^{m_1}(r,\omega)\mathring{G}_{n_2}^{m-m_1}(r,\omega)\gamma_{n_1,n_2,n}^{m_1,m-m_1,m} \tag{D.9}$$

$$= \mathring{F}_n^m(r,\omega) *_n^m \mathring{G}_n^m(r,\omega). \tag{D.10}$$

Equation (D.9) constitutes a *convolution theorem* for the spherical harmonics expansion.

Appendix E
Miscellaneous Mathematical Considerations

E.1 Translation of Spherical Harmonics Expansions

Assume the coefficients $\breve{S}'^{m'}_{n',e}(\omega)$ represent an exterior sound field $S(\mathbf{x}, \omega)$ with respect to a local coordinate system with origin at $\Delta\mathbf{x}$, which can be transformed into the global coordinate system by a simple translation as depicted in Fig. E.1. Then $S(\mathbf{x}', \omega)$ can be described as (refer to (2.32b))

$$S(\mathbf{x}', \omega) = \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \breve{S}'^{m'}_{n',e}(\omega) h^{(2)}_{n'}\left(\frac{\omega}{c} r'\right) Y^{m'}_{n'}(\beta', \alpha') \tag{E.1}$$

with respect to the local coordinate system. Note that $\mathbf{x}' = \mathbf{x}'(\mathbf{x}) = \mathbf{x} + \Delta\mathbf{x}$.

It is now desired to describe $S(\mathbf{x}, \omega)$ by means of a spherical harmonics expansion around the origin of the global coordinate system. This *translation* of the coordinate system is described below.

Assuming that the origin of the global coordinate system is located in the exterior domain with respect to the local coordinate system, then it must be possible to expand the term $h^{(2)}_{n'}\left(\omega/c\, r'\right) Y^{m'}_{n'}(\beta', \alpha')$ with respect to the global coordinate system as (Gumerov and Duraiswami 2004, Sect. 3.2)

$$h^{(2)}_{n'}\left(\frac{\omega}{c} r'\right) Y^{m'}_{n'}(\beta', \alpha') = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-1)^{n+n'} (E|I)^{m\,m'}_{n\,n'}(\Delta\mathbf{x}, \omega)\, j_n\left(\frac{\omega}{c} r\right) Y^{m}_{n}(\beta, \alpha), \tag{E.2}$$

since this term constitutes a solution to the wave equation. The notation $(E|I)$ indicates that the translation represents a change from an exterior expansion to an interior expansion (Williams 1999; Gumerov and Duraiswami 2004). The factor $(-1)^{n+n'}$ arises since the *translation coefficients* $(E|I)$ are defined in (Gumerov and

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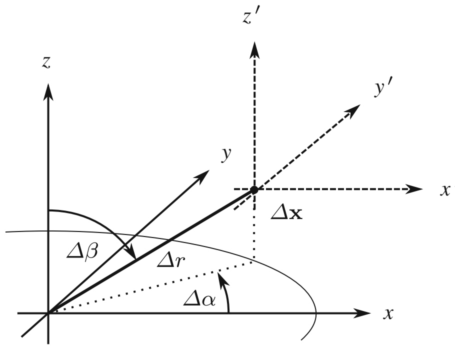

Fig. E.1 Illustration of the local coordinate system employed in (E.1)

Duraiswami 2004) for translation in opposite direction. Refer also to (ibidem, Eq. (3.2.54), p. 103).

Inserting (E.2) in (E.1) and re-ordering of the sums reveals the general form of $\breve{S}_n^m(\omega)$ as

$$S(\mathbf{x},\omega) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \times \underbrace{\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} \breve{S}'^{m'}_{n',e}(\omega)(-1)^{n+n'}\,(E|I)^{m\,m'}_{n\,n'}\,(\Delta\mathbf{x},\omega)}_{=\breve{S}_n^m(\omega)}\, j_n\left(\frac{\omega}{c}r\right) Y_n^m(\beta,\alpha). \tag{E.3}$$

Applying (2.33) to (E.2) yields an integral representation for the translation coefficients $(E|I)$ as (Gumerov and Duraiswami 2004, Eq. (3.2.12), p. 96)

$$(E|I)^{m\,m'}_{n\,n'}\,(\Delta\mathbf{x},\omega) = \frac{(-1)^{n+n'}}{j_n\left(\frac{\omega}{c}r\right)} \int_0^{2\pi}\int_0^{\pi} h^{(2)}_{n'}\left(\frac{\omega}{c}r'\right) Y^{m'}_{n'}(\beta',\alpha')\, Y_n^{-m}(\beta,\alpha)\sin\beta\, d\beta\, d\alpha, \quad \forall r < \Delta r. \tag{E.4}$$

Equation (E.4) is not practical since it can not be evaluated analytically. However, other representations of the translation coefficients $(E|I)$ are available which are somewhat more convenient. Several alternatives are discussed in (Gumerov and Duraiswami 2004). For convenience, only the most compact representation given in (Gumerov and Duraiswami 2004, Eqs. (3.2.30), (3.2.36); p. 100, 101) is stated here. It reads

$$(E|I)_{n\,n'}^{m\,m'}(\Delta\mathbf{x},\omega) = \sum_{n''=|n'-n|}^{n'+n} i^{n+n''-n'} \sqrt{\frac{(2n+1)(2n'+1)(2n''+1)}{4\pi}} \times \mathcal{E}\begin{pmatrix} m' & -m & m-m' \\ n' & n & n'' \end{pmatrix} h_{n''}^{(2)}\left(\frac{\omega}{c}\Delta r\right) Y_{n''}^{m'-m}(\Delta\beta,\Delta\alpha). \tag{E.5}$$

$\mathcal{E}(\cdot)$ is defined in (D.7).

Similar considerations like above yield the translation coefficients $(E|E)$ and $(I|I)$ for exterior-to-exterior and interior-to-interior translation respectively as (Gumerov and Duraiswami 2004, Eqs. (3.2.18), (3.2.46); pp. 97, 102)

$$(E|E)_{n\,n'}^{m\,m'}(\Delta\mathbf{x},\omega) = (I|I)_{n\,n'}^{m\,m'}(\Delta\mathbf{x},\omega) = \sum_{n''=|n'-n|}^{n'+n} i^{n+n''-n'} \sqrt{\frac{(2n+1)(2n'+1)(2n''+1)}{4\pi}} \times \mathcal{E}\begin{pmatrix} m' & -m & m-m' \\ n' & n & n'' \end{pmatrix} j_{n''}\left(\frac{\omega}{c}\Delta r\right) Y_{n''}^{m'-m}(\Delta\beta,\Delta\alpha). \tag{E.6}$$

Note that every second addend in the summations in (E.5) and (E.6) is zero. This is not explicitly indicated to retain notational clarity.

Equation (E.5) and (E.6) do not represent the most efficient translation operators. However, they are employed in this book since they are the most compact expressions. Refer to (Gumerov and Duraiswami 2004) for alternatives.

E.2 Rotation of Spherical Harmonics Expansions

Rotation of a spherical harmonics expansion along the azimuth α is achieved by replacing $\breve{S}_n^m(\omega)$ with $\breve{S}_n^m(\omega)e^{-im\alpha_{\text{rot}}}$, whereby α_{rot} denotes the rotation angle (Gumerov and Duraiswami 2004, Eq. (3.3.31), p. 127). Other types of rotation are more complicated and are not relevant in the context of this book. The reader is referred to (Gumerov and Duraiswami 2004) for an extensive treatment of rotation of spherical harmonics expansions.

E.3 Recursion Formulae for Exterior-to-Interior Sectorial Translation

As outlined in Sect. 3.5.3, the sectorial translation coefficients $(E|I)_{|m|n'}^{m\,m'}(\Delta\mathbf{x},\omega)$ can be computed using (Gumerov and Duraiswami 2004, Eq. (3.2.79), p. 109)

$$b_{-m}^{m}\,(E|I)_{n',\,|m|}^{m',\,m}(\Delta\mathbf{x},\omega) = b_{n'}^{m'}\,(E|I)_{n'-1,\,|m+1|}^{m'+1,\,m+1}(\Delta\mathbf{x},\omega) - b_{n'+1}^{-m'-1}\,(E|I)_{n'+1,\,|m+1|}^{m'+1,\,m+1}(\Delta\mathbf{x},\omega), \tag{E.7}$$

for $m \leq 0$ and (Gumerov and Duraiswami 2004, Eq. (3.2.78), p. 108)

$$\begin{aligned} & b_m^{-m} \, (E|I)_{n',m}^{m',m} \, (\Delta\mathbf{x}, \omega) \\ & = b_{n'}^{-m'} \, (E|I)_{n'-1,m-1}^{m'-1,m-1} \, (\Delta\mathbf{x}, \omega) - b_{n'+1}^{m'-1} \, (E|I)_{n'+1,m-1}^{m'-1,m-1} \, (\Delta\mathbf{x}, \omega), \end{aligned} \tag{E.8}$$

for $m \geq 0$ with (Gumerov and Duraiswami, 2004, Eq. (2.2.10), p. 68)

$$b_n^m = \begin{cases} \sqrt{\frac{(n-m-1)(n-m)}{(2n-1)(2n+1)}} & \text{for } 0 \leq m \leq n \\ -\sqrt{\frac{(n-m-1)(n-m)}{(2n-1)(2n+1)}} & \text{for } -n \leq m < 0 \\ 0 & \text{for } |m| > n. \end{cases} \tag{E.9}$$

E.4 Derivation of the Relations Between the Signature Function and Various Other Presentations

E.4.1 From Signature Function to Spherical Harmonics Expansion

Assume a plane wave propagating in direction (ϕ, θ). Its spherical harmonics expansion is given by (2.38) as

$$\begin{aligned} S(\mathbf{x}, \omega) &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} 4\pi i^{-n} Y_n^{-m}(\phi, \theta) \, j_n\left(\frac{\omega}{c} r\right) Y_n^m(\beta, \alpha) \\ &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \underbrace{4\pi i^{-n} Y_n^{-m}(\beta, \alpha) \, j_n\left(\frac{\omega}{c} r\right)}_{=\mathring{S}_n^m(r,\omega)} Y_n^m(\phi, \theta) \end{aligned} \tag{E.10}$$

Inserting (E.10) into (2.45) yields (Gumerov and Duraiswami 2004)

$$\begin{aligned} S(\mathbf{x}, \omega) =& \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \bar{S}(\phi, \theta, \omega) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} 4\pi i^{-n} Y_n^{-m}(\phi, \theta) \\ & \times j_n\left(\frac{\omega}{c} r\right) Y_n^m(\beta, \alpha) \sin\phi \, d\phi \, d\theta \\ =& \sum_{n=0}^{\infty} \sum_{m=-n}^{n} j_n\left(\frac{\omega}{c} r\right) Y_n^m(\beta, \alpha) \\ & \underbrace{\times \, i^{-n} \int_0^{2\pi} \int_0^{\pi} \bar{S}(\phi, \theta, \omega) Y_n^{-m}(\phi, \theta) \sin\phi \, d\phi \, d\theta,}_{\breve{S}_n^m(\omega)} \end{aligned} \tag{E.11}$$

so that

$$\breve{S}_n^m(\omega) = i^{-n} \int_0^{2\pi} \int_0^{\pi} \bar{S}(\phi, \theta, \omega) Y_n^{-m}(\phi, \theta) \sin\phi \, d\phi \, d\theta. \tag{E.12}$$

E.4.2 From Spherical Harmonics Expansion to Signature Function

Inserting (E.10) into (2.33) yields (Gumerov and Duraiswami 2004)

$$4\pi i^{-n} Y_n^{-m}(\beta, \alpha) j_n\left(\frac{\omega}{c} r\right) = \int_0^{2\pi} \int_0^{\pi} e^{-i\mathbf{k}^T\mathbf{x}} Y_n^{-m}(\phi, \theta) \sin\phi \, d\phi \, d\theta$$

$$Y_n^m(\beta, \alpha) j_n\left(\frac{\omega}{c} r\right) = \frac{i^n}{4\pi} \int_0^{2\pi} \int_0^{\pi} e^{-i\mathbf{k}^T\mathbf{x}} Y_n^m(\phi, \theta) \sin\phi \, d\phi \, d\theta \tag{E.13}$$

Composing then $S(\mathbf{x}, \omega)$ via (2.32a) yields

$$S(\mathbf{x}, \omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \breve{S}_n^m(\omega) \frac{i^n}{4\pi} \int_0^{2\pi} \int_0^{\pi} e^{-i\mathbf{k}^T\mathbf{x}} Y_n^m(\phi, \theta) \sin\phi \, d\phi \, d\theta \tag{E.14}$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \underbrace{\sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n \breve{S}_n^m(\omega) Y_n^m(\phi, \theta)}_{=\bar{S}(\phi,\theta,\omega)} e^{-i\mathbf{k}^T\mathbf{x}} \sin\phi \, d\phi \, d\theta, \tag{E.15}$$

so that the signature function $\bar{S}(\phi, \theta, \omega)$ is given by

$$\bar{S}(\phi, \theta, \omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n \breve{S}_n^m(\omega) Y_n^m(\phi, \theta). \tag{E.16}$$

E.4.3 From Time-Frequency Domain to Signature Function

Recall (E.16).

$$\bar{S}(\phi, \theta, \omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n \breve{S}_n^m(\omega) Y_n^m(\phi, \theta).$$

Using (2.33),

$$\begin{aligned}\bar{S}(\phi,\theta,\omega) &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{i^n}{j_n\left(\frac{\omega}{c}r\right)}\oint_{S_u} S(\mathbf{x},\omega)Y_n^{-m}(\phi,\theta)dS_u Y_n^m(\phi,\theta)\\ &= \sum_{n=0}^{\infty}\frac{i^n}{j_n\left(\frac{\omega}{c}r\right)}\oint_{S_u} S(\mathbf{x},\omega)\sum_{m=-n}^{n} Y_n^{-m}(\phi,\theta)Y_n^m(\phi,\theta)dS_u\\ &= \sum_{n=0}^{\infty}\frac{i^n(2n+1)}{4\pi j_n\left(\frac{\omega}{c}r\right)}\oint_{S_u} S(\mathbf{x},\omega)P_n\left(\frac{\mathbf{x}}{r}\bullet\frac{\mathbf{k}}{k}\right)dS_u \end{aligned} \tag{E.17}$$

$\oint_{S_u}(\cdot)dS_u$ denotes integration over the unit sphere (such as in (2.33)). In the last equality, the addition theorem for spherical harmonics (2.29) was exploited.

E.5 The Stationary Phase Approximation Applied to the Rayleigh I Integral

The objective of this section is approximating the Rayleigh I integral (3.91) in the horizontal plane. Consider the integral over z_0 in (3.91) assuming that the driving function $D(\mathbf{x}_0,\omega)$ is independent of z_0, thus (Berkhout et al. 1993)

$$\int_{-\infty}^{\infty}\frac{1}{4\pi}\left.\frac{e^{-i\frac{\omega}{c}|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|}\right|_{z=0} dz_0. \tag{E.18}$$

Such an integral can be approximated by the stationary phase approximation (Williams 1999). The latter provides an approximative solution to integrals of the form

$$I=\int_{-\infty}^{\infty} f(z_0)\, e^{i\zeta(z_0)}\, dz_0 \tag{E.19}$$

which is given by

$$I\approx\sqrt{\frac{2\pi i}{\zeta''(z_\mathrm{p})}}\, f(z_\mathrm{p})\, e^{i\zeta(z_\mathrm{p})}. \tag{E.20}$$

$\zeta''(z_0)$ denotes the second derivative of $\zeta(z_0)$ with respect to z_0. z_p denotes the *stationary phase point* which corresponds to the zero of $\zeta'(z_0)$.

In the present case ($z=0$),

$$f(z_0) = \frac{1}{4\pi} \cdot \frac{1}{\sqrt{(x-x_0)^2 + y^2 + z_0^2}}, \tag{E.21}$$

$$\zeta(z_0) = -\frac{\omega}{c}\sqrt{(x-x_0)^2 + y^2 + z_0^2}, \tag{E.22}$$

$$\zeta'(z_0) = -\frac{\omega}{c}\frac{1}{\sqrt{(x-x_0)^2 + y^2 + z_0^2}} z_0. \tag{E.23}$$

Thus $z_\text{p} = 0$.

$$\zeta''(z_0) = -\frac{\omega}{c}\frac{1}{\sqrt{(x-x_0)^2 + y^2 + z_0^2}} + \frac{\omega}{c}\frac{z_0^2}{\left((x-x_0)^2 + y^2 + z_0^2\right)^{\frac{3}{2}}} \tag{E.24}$$

so that (Berkhout et al. 1993)

$$\zeta''(z_\text{p}) = -\frac{\omega}{c}\frac{1}{\sqrt{(x-x_0)^2 + y^2}}. \tag{E.25}$$

Inserting above results into (E.20) and the result in (3.91) yields the 2.5-dimensional approximation of the Rayleigh I integral (3.92).

E.6 Derivation of (4.14) and (4.15)

As indicated in (2.33), the spherical harmonics transform $\mathring{\Phi}_{n_1}^{m_1}(L)$ of the Gauß sampling grid can be determined via

$$\mathring{\Phi}_{n_1}^{m_1}(L) = \int_0^{2\pi}\int_0^{\pi} \Phi(\alpha, \beta, L) Y_{n_1}^{-m_1}(\beta, \alpha) \sin\beta \, d\beta \, d\alpha. \tag{E.26}$$

The integrals in (E.26) can be solved independently as

$$\int_0^{2\pi} \sum_{l_1=0}^{2L-1} \delta\left(\alpha - \frac{2\pi l_1}{2L}\right) e^{-im_1\alpha} d\alpha = \sum_{l_1=0}^{2L-1} e^{-im_1 2\pi \frac{l_1}{2L}} \tag{E.27}$$

$$= \begin{cases} 2L & \forall m_1 = \mu 2L, \mu \in \mathbb{Z} \\ 0 & \text{elsewhere} \end{cases}, \tag{E.28}$$

and

$$\int_0^\pi \sum_{l_2=0}^{L-1} w_{l_2} \delta\left(\beta - \beta_{l_2}\right) P_{n_1}^{|m_1|}(\cos\beta) \sin\beta \, d\beta$$
$$= \sum_{l_2=0}^{L-1} w_{l_2} P_{n_1}^{|m_1|}\left(\cos\beta_{l_2}\right) \sin\beta_{l_2}. \quad \text{(E.29)}$$

From the parity properties of the sampling locations β_{l_2}, the associated Legendre functions, and the sine function in (E.29), it can be deduced that the result equals zero for $m_1 + n_1$ being odd.

The spherical harmonics expansion coefficients $\mathring{\Phi}_{n_1}^{m_1}(L)$ of the sampling grid are finally given by

$$\mathring{\Phi}_{n_1}^{m_1}(L) = \begin{cases} \frac{\pi(-1)^{m_1}}{L} \sqrt{\frac{2n_1+1}{4\pi} \frac{(n_1-|m_1|)!}{(n_1+|m_1|)!}} \sum_{l_2=0}^{L-1} w_{l_2} P_{n_1}^{|m_1|}\left(\cos\beta_{l_2}\right) \sin\beta_{l_2} & \forall m_1 = \mu 2L \\ 0 & \text{elsewhere.} \end{cases} \quad \text{(E.30)}$$

Introducing (E.30) into (4.13), changing the order of summations, and considering selection rule 2 from Appendix D.2 yields

$$\mathring{D}_{n,\mathrm{S}}^{m}(R,\omega) = \sum_{\mu=-\infty}^{\infty} \sum_{n_2=|m-\mu 2L|}^{\infty} \mathring{D}_{n_2}^{m-\mu 2L}(R,\omega) \Upsilon_{n_2,n}^{\mu,m}(L), \quad \text{(E.31)}$$

with

$$\Upsilon_{n_2,n}^{\mu,m}(L) = \sum_{n_1=|n-n_2|}^{n+n_2} \mathring{\Phi}_{n_1}^{\mu 2L}(L) \gamma_{n_1,n_2,n}^{\mu 2L, m-\mu 2L, m}. \quad \text{(E.32)}$$

E.7 Derivation of (5.31) and (5.32)

Equation (5.31) can be simplified via the substitution $u = \cos\beta$ as

$$\Psi_n^m = \int_{-1}^{1} P_n^{|m|}(u) du. \quad \text{(E.33)}$$

From the parity relation (Arfken and Weber 2005)

$$P_n^{|m|}(-u) = (-1)^{n+|m|} P_n^{|m|}(u) \quad \text{(E.34)}$$

it can be deduced that the integral in (E.33) vanishes for $n + |m|$ being odd. Furthermore,

$$\int_{-1}^{1} P_n^{|m|}(u)du = 2\int_{0}^{1} P_n^{|m|}(u)du \quad \forall n + |m| \text{ even}. \tag{E.35}$$

The solution to the integral on the right hand side of (E.35) is given in (Gradshteyn and Ryzhik 2000, 7.126-2) so that Ψ_n^m is finally given by

$$\begin{aligned}\Psi_n^m =& \frac{\pi\, 2^{-2|m|}\, \Gamma(1+|m|+n)}{\Gamma\left(\frac{1}{2}+\frac{|m|}{2}\right)\Gamma\left(\frac{3}{2}+\frac{|m|}{2}\right)\Gamma(1-|m|+n)} \\ &\times {}_3F_2\left(\frac{|m|+n+1}{2}, \frac{|m|-n}{2}, \frac{|m|}{2}+1; |m|+1, \frac{|m|+3}{2}; 1\right) \\ &\qquad \forall n+|m| \text{ even},\end{aligned} \tag{E.36}$$

and $\Psi_n^m = 0$ elsewhere. $\Gamma(\cdot)$ denotes the gamma function and ${}_3F_2(\cdot)$ the generalized hypergeometric function (Arfken and Weber 2005).

$\chi^m(\eta)$ given by (5.32) can be determined to be

$$\chi^m(\eta) = \sum_{l=0}^{2\eta-1} (-1)^l \times \begin{cases} \alpha_{l+1} - \alpha_l & \forall m = 0 \\ \frac{i}{m}\left(e^{-im\alpha_{l+1}} - e^{-im\alpha_l}\right) & \forall m \neq 0 \end{cases}. \tag{E.37}$$

E.8 Projection of a Sound Field onto the Horizontal Planes

Assume an $(N-1)$-th order sound field $S(\mathbf{x}, \omega)$ that is described by the signature function $\bar{S}(\phi, \theta, \omega)$ as

$$S(\mathbf{x}, \omega) = \frac{1}{4\pi}\int_{0}^{2\pi}\int_{0}^{\pi} \bar{S}(\phi, \theta, \omega)e^{-i\mathbf{k}^T\mathbf{x}} \sin\phi \, d\phi \, d\theta. \tag{E.38}$$

The projection $S_{\text{proj}}\left(\mathbf{x}\big|_{z=0}, \omega\right)$ of $S(\mathbf{x}, \omega)$ onto the horizontal plane is given by

$$S_{\text{proj}}\left(\mathbf{x}\big|_{z=0}, \omega\right) = \frac{1}{4\pi}\int_{0}^{2\pi}\underbrace{\int_{0}^{\pi} \bar{S}(\phi, \theta, \omega) \sin\phi \, d\phi}_{=\bar{S}_{\text{proj}}(\theta,\omega)} \; e^{-ikr\cos(\theta-\alpha)}d\theta. \tag{E.39}$$

In the following, the integral over ϕ in (E.39) is evaluated in order to derive the signature function $\bar{S}_{\text{proj}}(\theta, \omega)$ of the projected sound field.

Exploiting (E.16) yields

$$\bar{S}_{\text{proj}}(\theta, \omega) = \sum_{n=0}^{N-1} \sum_{m=-n}^{n} i^n \breve{S}_n^m(\omega)(-1)^m e^{im\theta} \sqrt{\frac{(2n+1)}{4\pi} \frac{(n-|m|)!}{(n+|m|)!}} \times \int_0^{\pi} P_n^{|m|}(\cos\phi) \sin\phi \, d\phi. \tag{E.40}$$

The integral in (E.40) can be simplified via the substitution $u = \cos\phi$ as

$$\int_0^{\pi} P_n^{|m|}(\cos\phi) \sin\phi \, d\phi = \int_{-1}^{1} P_n^{|m|}(u) \, du. \tag{E.41}$$

From the parity relation (2.20) it can be deduced that the integral in (E.41) vanishes for $n + |m|$ being odd. Furthermore,

$$\int_{-1}^{1} P_n^{|m|}(u) du = 2 \int_0^1 P_n^{|m|}(u) \, du \quad \forall \, n + |m| \text{ even} . \tag{E.42}$$

The solution to the integral on the right hand side of (E.42) is given in (Gradshteyn and Ryzhik 2000, 7.126-2) so that the projected signature function $\bar{S}_{\text{proj}}(\theta, \omega)$ is finally given by

$$\bar{S}_{\text{proj}}(\theta, \omega) = \sum_{n=0}^{N-1} \sum_{m'=0}^{n} \Psi_n^{2m'-n} \, i^n \breve{S}_n^{2m'-n}(\omega) \, e^{i(2m'-n)\theta}, \tag{E.43}$$

whereby Ψ_n^l is a real number given by

$$\Psi_n^l = \pi \sqrt{\frac{(2n+1)}{4\pi} \frac{(n-|l|)!}{(n+|l|)!}} \frac{2^{-2|l|} \, \Gamma(1+|l|+n)}{\Gamma\left(\frac{1}{2}+\frac{|l|}{2}\right) \Gamma\left(\frac{3}{2}+\frac{|l|}{2}\right) \Gamma(1-|l|+n)} \times {}_3F_2\left(\frac{|l|+n+1}{2}, \frac{|l|-n}{2}, \frac{|l|}{2}+1; |l|+1, \frac{|l|+3}{2}; 1\right). \tag{E.44}$$

$\Gamma(\cdot)$ denotes the gamma function and ${}_3F_2(\cdot)$ the generalized hypergeometric function (Arfken and Weber 2005).

E.9 Integration Over Plane Wave Driving Signals

The objective is finding the solution to

$$D(\mathbf{x}, \omega) = -\frac{i}{4\pi}\frac{\omega}{c}\int_{\alpha_\mathrm{n}-\frac{\pi}{2}}^{\alpha_\mathrm{n}+\frac{\pi}{2}} \bar{S}_\mathrm{proj}(\theta, \omega)\cos\left(\theta - \alpha_\mathrm{n}\right) e^{-ikr\cos(\theta-\alpha)} d\theta. \tag{E.45}$$

$\bar{S}_\mathrm{proj}(\cdot)$ is expressed by (E.43) and Euler's identity (Weisstein 2002) is applied to the cosine factor to yield

$$\begin{aligned} D(\mathbf{x}, \omega) = &- \frac{i}{4}\frac{\omega}{c}\sum_{n=0}^{N-1}\sum_{m=-n}^{n} \Psi_n^m\, i^n\, \breve{S}_n^m(\omega) \\ &\times \left(\frac{e^{-i\alpha_\mathrm{n}}}{2\pi}\int_0^{2\pi} w\left(\alpha_\mathrm{n}, \theta\right) e^{-ikr\cos(\theta-\alpha)} e^{-i(-1-m)\theta} d\theta \right. \\ &\left. + \frac{e^{i\alpha_\mathrm{n}}}{2\pi}\int_0^{2\pi} w\left(\alpha_\mathrm{n}, \theta\right) e^{-ikr\cos(\theta-\alpha)} e^{-i(1-m)\theta} d\theta \right), \end{aligned} \tag{E.46}$$

whereby $w\left(\alpha_\mathrm{n}, \theta\right)$ denotes a rectangular window given by

$$w\left(\alpha_\mathrm{n}, \theta\right) = \begin{cases} 1 & \text{for } \alpha_\mathrm{n} - \frac{\pi}{2} \le \theta \le \alpha_\mathrm{n} + \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases}. \tag{E.47}$$

The integrals in (E.46) yield the $(-m-1)$-th and $(m-1)$-th Fourier series expansion coefficients of the window function multiplied with the exponential describing a plane wave (Williams 1999). As stated by (D.3), the Fourier series expansion coefficients of a multiplication of two functions $u(\theta)$ and $v(\theta)$ is given by a discrete convolution of the Fourier series expansion coefficients $\mathring{u}_m$ and $\mathring{v}_m$ of $u(\theta)$ and $v(\theta)$ respectively as

$$\frac{1}{2\pi}\int_0^{2\pi} u(\theta)v(\theta)\, e^{-im\theta} d\theta = \sum_{l=-\infty}^{\infty} \mathring{u}_l\, \mathring{v}_{m-l}. \tag{E.48}$$

The Fourier expansion coefficients $\mathring{w}_m\left(\alpha_\mathrm{n}\right)$ of the window function $w\left(\alpha_\mathrm{n}, \theta\right)$ can be determined to be

$$\mathring{w}_m\left(\alpha_\mathrm{n}\right) = \begin{cases} \frac{1}{2} & \text{for } m = 0 \\ i\,\frac{e^{-im\alpha_\mathrm{n}}}{2\pi m}\left(i^{-m} - i^m\right) & \text{for } m \neq 0 \end{cases}. \tag{E.49}$$

The Fourier expansion coefficients of the plane wave can be deduced from the Jacobi-Anger expansion (Weisstein 2002) as $i^{-m} J_m\left(\frac{\omega}{c}r\right) e^{-im\alpha}$, whereby $J_m(\cdot)$ denotes the m-th order Bessel function (Arfken and Weber 2005).

The driving signal $D(\mathbf{x}, \omega)$ is thus finally given by

$$D(\mathbf{x}, \omega) = \sum_{n=0}^{N-1} \sum_{m=-n}^{n} \Psi_n^m \, i^n \breve{S}_n^m(\omega) \, \Lambda_m(\mathbf{x}, \omega), \tag{E.50}$$

with

$$\begin{aligned} \Lambda_m(\mathbf{x}, \omega) = & -\frac{i}{4}\frac{\omega}{c} \sum_{l=-\infty}^{\infty} i^{-l} J_l\left(\frac{\omega}{c}r\right) e^{-il\alpha} \\ & \times \left(e^{-i\alpha_\mathrm{n}} \mathring{w}_{-1-m-l}(\alpha_\mathrm{n}) + e^{i\alpha_\mathrm{n}} \mathring{w}_{1-m-l}(\alpha_\mathrm{n}) \right). \end{aligned} \tag{E.51}$$

E.10 Derivation of the Gradient of a Convolution of Two Functions With Respect to Time

Evaluating the expression

$$\frac{\partial}{\partial \mathbf{n}} \left(u(\mathbf{x}, t) *_t v(\mathbf{x}, t) \right) \tag{E.52}$$

is sought after. A Fourier transform with respect to t is applied to (E.52) and the product rule for derivatives is applied, which yields (Weisstein 2002; Girod et al. 2001)

$$\frac{\partial}{\partial \mathbf{n}} \left(U(\mathbf{x}, \omega) \cdot V(\mathbf{x}, \omega) \right) = U(\mathbf{x}, \omega) \frac{\partial}{\partial \mathbf{n}} V(\mathbf{x}, \omega) + V(\mathbf{x}, \omega) \frac{\partial}{\partial \mathbf{n}} U(\mathbf{x}, \omega). \tag{E.53}$$

An inverse Fourier transform applied to the right hand side of (E.53) yields the desired result, which is given by

$$\frac{\partial}{\partial \mathbf{n}} \left(u(\mathbf{x}, t) *_t v(\mathbf{x}, t) \right) = u(\mathbf{x}, t) *_t \frac{\partial}{\partial \mathbf{n}} v(\mathbf{x}, t) + v(\mathbf{x}, t) *_t \frac{\partial}{\partial \mathbf{n}} u(\mathbf{x}, t). \tag{E.54}$$

E.11 The Components of (5.80)

The gradients of the monopole component of (5.80) have been derived in (5.65) and (5.66).

In the following, the directional gradient of $\bar{s}'(\cdot)$ via its spherical harmonics representation (Eq. (E.16)) is derived since this representation is the most general one. If the signature function $\bar{s}(\cdot)$ is known analytically, then the gradient can be applied to the latter directly.

The directional gradient of $\bar{s}'(\cdot)$ expressed in spherical harmonics is given by

$$\frac{\partial}{\partial \mathbf{n}} \bar{s}'\left(\tilde{\alpha}, \tilde{\beta}, t\right) = -\frac{c}{8\pi} \operatorname{sign}(t) *_t \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n \breve{s}_n^m(t) \frac{\partial}{\partial \mathbf{n}} Y_n^m\left(\tilde{\beta}, \tilde{\alpha}\right), \tag{E.55}$$

with

$$\begin{aligned} \frac{\partial}{\partial \mathbf{n}} Y_n^m\left(\tilde{\beta}, \tilde{\alpha}\right) =& (-1)^m \sqrt{\frac{(2n+1)}{4\pi} \frac{(n-|m|)!}{(n+|m|)!}} \\ & \times \left(P_n^{|m|}\left(\cos\tilde{\beta}\right) \frac{\partial}{\partial \mathbf{n}} e^{im\tilde{\alpha}} + e^{im\tilde{\alpha}} \frac{\partial}{\partial \mathbf{n}} P_n^{|m|}\left(\cos\tilde{\beta}\right)\right). \end{aligned} \tag{E.56}$$

Finally,

$$\begin{aligned} \frac{\partial}{\partial x} e^{im\tilde{\alpha}} = & -\frac{imy}{(x - x_\mathrm{s}(t-\tau))^2 + y^2} \\ & \times \left(1 + \frac{M}{1-M^2}\left(M + \frac{x - x_\mathrm{s}(t)}{\Delta(\mathbf{x}, t)}\right)\right) e^{im\tilde{\alpha}}, \end{aligned} \tag{E.57}$$

$$\frac{\partial}{\partial y} e^{im\tilde{\alpha}} = \frac{im\,(x - x_\mathrm{s}(t-\tau))}{(x - x_\mathrm{s}(t-\tau))^2 + y^2} \times \left(1 - \frac{My^2}{\Delta(\mathbf{x}, t)\,(x - x_\mathrm{s}(t-\tau))}\right) e^{im\tilde{\alpha}}, \tag{E.58}$$

$$\frac{\partial}{\partial z} e^{im\tilde{\alpha}} = -\frac{imMyz}{\Delta(\mathbf{x}, t)\left((x - x_\mathrm{s}(t-\tau))^2 + y^2\right)} e^{im\tilde{\alpha}}, \tag{E.59}$$

$$\begin{aligned} \frac{\partial}{\partial x} P_n^{|m|}\left(\cos\tilde{\beta}\right) = & -\left(P_n^{|m|}\left(\cos\tilde{\beta}\right)\right)' \frac{z\,(x - x_\mathrm{s}(t-\tau))}{\tilde{r}^3} \\ & \times \left(1 + \frac{M}{1-M^2}\left(M + \frac{x - x_\mathrm{s}(t)}{\Delta(\mathbf{x}, t)}\right)\right), \end{aligned} \tag{E.60}$$

$$\frac{\partial}{\partial y} P_n^{|m|}\left(\cos\tilde{\beta}\right) = -\left(P_n^{|m|}\left(\cos\tilde{\beta}\right)\right)' \frac{zy}{\tilde{r}^3}\left(1 + M\frac{x - x_\mathrm{s}(t-\tau)}{\Delta(\mathbf{x}, t)}\right), \tag{E.61}$$

$$\begin{aligned} \frac{\partial}{\partial z} P_n^{|m|}\left(\cos\tilde{\beta}\right) = & \left(P_n^{|m|}\left(\cos\tilde{\beta}\right)\right)' \\ & \times \left(\frac{1}{\tilde{r}} - \frac{z^2}{\tilde{r}^3}\left(1 + M\frac{x - x_\mathrm{s}(t-\tau)}{\Delta(\mathbf{x}, t)}\right)\right). \end{aligned} \tag{E.62}$$

The derivative of $P_n^{|m|}(\cdot)$ with respect to the argument is given by (2.21)

References

Arfken, G., & Weber, H. (2005). *Mathematical methods for physicists.* San Diego: Elsevier Academic Press.

Berkhout, A. J., de Vries, D., & Vogel, P. (1993). Acoustic control by wave field synthesis. *JASA, 93*(5), 2764–2778.

Bracewell, R.N. (2000). *The Fourier transform and its applications.* Singapore: Mcgraw-Hill.

Girod, B., Rabenstein, R., & Stenger, A. (2001). *Signals and Systems.* New York: Wiley.

Gjellestad, G. (1995, November). Note on the definite integral over products of three legendre functions. *PNAS, 41*, 954–956.

Gradshteyn, I. S., & Ryzhik, I. M. (2000). *Table of integrals, series, and products.* San Diego: Academic.

Gumerov, N. A., & Duraiswami, R. (2004). *Fast multipole methods for the Helmholtz equation in three dimensions.* Amsterdam: Elsevier.

Kraus, K. (2008). Wigner3j symbol. Retrieved January 10, 2010, from http://www.mathworks.com/matlabcentral/fileexchange.

Morse, P. M., & Feshbach, H. (1953). *Methods of theoretical physics.* Minneapolis: Feshbach Publishing, LLC.

Sébilleau, D. (1998). On the computation of the integrated products of three spherical harmonics. *Journal of Physics A: Mathematical and General, 31*, 7157–7168.

Shirdhonkar, S., & Jacobs, D. (2005, October). Non-negative lighting and specular object recognition. In *IEEE International Conference on Computer Vision* (Vol. 2, pp. 1323–1330).

Weisstein, E. W. (2002). *CRC concise encyclopedia of mathematics.* London: Chapman and Hall/CRC.

Williams, E. G. (1999). *Fourier acoustics: Sound radiation and nearfield acoustic holography.* London: Academic.

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