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Appendix A
Recalls on Sliding Modes Techniques

In this appendix we will recall some sliding modes principle, precisely the finite time convergence and the notion of equivalent vector.

Let us consider the following system input:

$$\dot{x} = F(x, t, u) \tag{A.1}$$

where $x \in \Re^n$ is the state, and $u \in \Re$ is the control vector.
For this system, we define the discontinuous control given by:

$$u_i(x, t) = \begin{cases} u^+(x, t) \text{ si } s(x) > 0 \\ u^-(x, t) \text{ si } s(x) < 0 \end{cases}$$

where $s(x) \in \Re$ is a function.
the closed loop system is then noted

$$\dot{x} = f(x, t) \tag{A.2}$$

If there exists a positive constant k such that the Lyapunov function defined by

$$v = \frac{s^2}{2}$$

verifies

$$\dot{v} \leq -k|s| = -k\sqrt(v)$$

then the sliding mode occurs (i.e $s(x) = 0$) after a finite time inteval. We will establish this using a comparison method.
In fact , the existence of such a constant k implies that there exists another constant μ such that

$$\begin{array}{l} v(t) \leq \rho(t),\ \dot{\rho} = \mu\sqrt(\rho),\ \rho(0) = v(0) \\ 0 < v(t) = \rho(t) = (v(0) - \mu t/2)^2, \\ v(t) = 0(s(t) = 0) \text{ for } t > t_1 = 2v(0)/\mu(\text{ because} v \geq 0) \end{array}$$

Another demonstration in [Kha92] (chapter 7) establishes that $t_1 leq|s(t = 0)|/k$ by integrating

$$\frac{1}{2}\frac{d}{dt}s^2 \leq -k|s|$$

Now, we interest to the dynamics of the system on the sliding surface. The system's motion on the sliding surface can give an interesting geometric problem interpretation as an average of the system's dynamics on both sides of the surface. Thus, by solving formally the equation $\dot{s} = 0$ for the control input, we obtain an expression for u called the equivalent control denoted by u_{eq}, which can be interpreted as the continuous control law that would maintain $\dot{s} = 0$.

This result is a consequent of the Fillipov theorem [Fil60]. The trajectories of the system (A.2) on the sliding surfaces are not defined as the control vector is also not defined on $s = 0$. Filippov [Fil60] defined a solution of (A.2) in terms of differential inclusions:

Definition A.1. (Solution of (A.2) in the sens of Filippov) The stae vector $x(t)$ défined on $[t_1, t_2]$ is a solution of (A.2) in the Filippov's sens, if $x(t)$ is absolutely continuous, and if for almost all $t \in [t_1, t_2]$,

$$\dot{x}(t) \in \bigcap_{\delta>0} \bigcap_{\mu N=0} \overline{conv}\, f(B(x,\delta) - N, t) \tag{A.3}$$

where $\overline{conv}$ designes the close convex envelope, $B(x, \delta)$ is the ball centered in x and of ray δ, μ is the Lebegue's measure. The notation,

$$\bigcap_{\mu N=0}$$

indicates the intersection of all the null measure sets.

So in the Filippov sense, the differential equation (A.2) is substituted by the differential inclusion (A.3).

• **The dynamics of the system on the sliding surface**

For sake of simplicity, we take the following notation:

$$S = \{x \in \Re^n \ : \ s(x) = 0\}$$

The surface S separates the state space into two parts S^+ $(s(x) > 0)$ and S^- $(s(x) < 0)$. We suppose that the functions $f^+(x, t)$ and $f^-(x, t)$ defined by

$$\lim_{s\to 0^+} f(x,t) = f^+(x,t)$$
$$\lim_{s\to 0^-} f(x,t) = f^-(x,t)$$

exist for all given t.

Let $f_0^+(x,t) =< \nabla s, f^+(x,t) >$ (resp. $f_0^-(x,t) =< \nabla s, f^-(x,t) >$) the projection of f^+ (resp. f^-) in the normal direction of the sliding surface S oriented to S^- (resp. S^+).
with these notations, we announce the Fillipov theorem

Theorem A.2. *Let $x(t)$ absolutely continuous such that $x(t) \in S$, verify $f_0^-(x,t) \geq 0$, $f_0^+(x,t) \leq 0$ and $f_0^-(x,t) - f_0^+(x,t) > 0$, then $x(t)$ is a solution solution of (A.2) (in the sens of the definition A.1), if and only if*

$$\dot{x}(t) = \alpha(t)f^+(x,t) + (1-\alpha(t))f^-(x,t) \quad \textit{whith} \tag{A.4}$$

$$\alpha(t) = \frac{f_0^-(x,t)}{f_0^-(x,t) - f_0^+(x,t)} \tag{A.5}$$

The right hand of the equation (A.4) is orthogonal to ∇s. In fact, we verify that $< \nabla s, \alpha f^+ + (\alpha - 1)f^- >= 0$.

Consequently the solution $x(t)$ remains on the surface S. The values of $f(x,t)$ in the neighborhood of S generate solutions which are constraint to slide on the surface S (see figures A.1 and A.2).

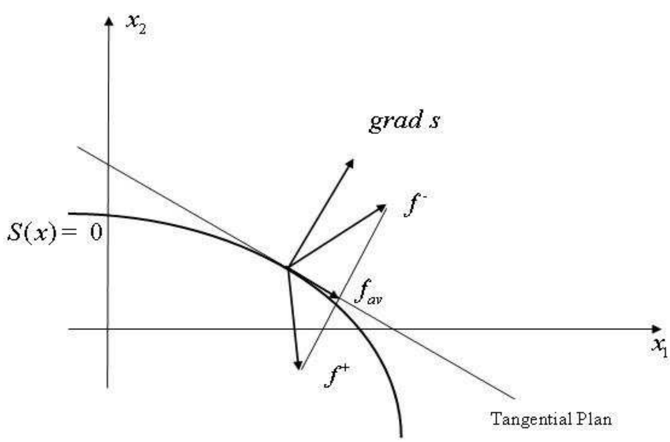

Fig. A.1 Filipppov definition of the sliding mode equations

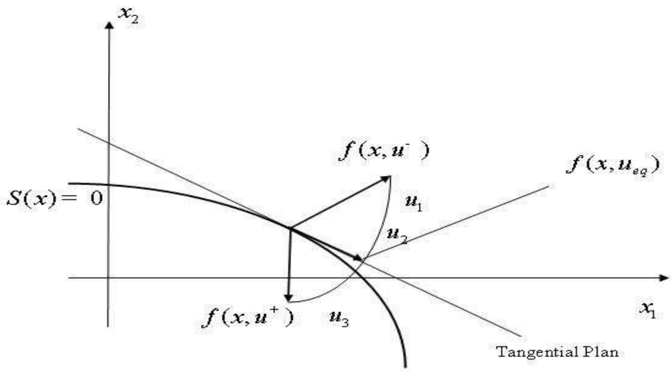

Fig. A.2 Equivalent control method definition of the sliding mode equations

Appendix B
Equivalent Control Concept

B.1 Motivation

When using SM control, one of the most interesting practical problems appearing is that of finding the trajectory of the state variables, so called, the sliding equations [Utk92].

A formal approach is that of solution of differential inclusions in the Filippov sense [Fil60]. However, a simpler way to study the effect of a discontinuous control acting on the system is the *equivalent control method (ECM)* which, in fact, for affine systems, it turns out to give the same results as studying differential inclusion in the Filippov sense. In this chapter a short description of the ECM is introduced.

B.2 Equivalent Control Method

Let us consider the system described by the following differential equation:

$$\dot{x}\left(t\right) = f\left(x, t\right) + B\left(x, t\right) u\left(t\right), \; t \geq t_0 \tag{B.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, and they represent the state vector and the control vector, respectively. Moreover, $f\left(x, t\right)$ and $B\left(x, t\right)$ are continuous vector and matrix functions, respectively, with respect to all the arguments. Here, u is to be designed as a discontinuous control to compel the trajectories of (B.1) to enter into the sliding manifold $S = \{x : s\left(x\right) = 0\}$ and to be maintained there for all the time forward. The function $s\left(x\right) \in \mathbb{R}^m$ is to be designed according to some specific requirements, we will called it sliding variable. Once the trajectories of (B.1) are into the manifold S, i.e. $s\left(x\right) = 0$, we say that (B.1) is on a sliding mode (SM). An u achieving the SM will be called sliding mode control.

Let us assume that $s(x) \equiv 0$, then its derivative would be also identical to zero. Thus, we have that

$$\dot{s}(x) = \frac{\partial s}{\partial x}\left[f(x,t) + B(x,t)\,u\right] = 0 \tag{B.2}$$

Assuming that $G(x) := \frac{\partial s}{\partial x}$ fulfills with the condition $\det G(x)\,B(x) \neq 0$. The function u taken from (B.2) is the so-called equivalent control, thus we have that,

$$u_{\text{eq}} = -\left[G(x)\,B(x,t)\right]^{-1}\left[G(x)\,f(x,t)\right] \tag{B.3}$$

What the EC method asserts is that the dynamics of (B.1) can be calculated by the substitution of u_{eq} in the place of u, i.e., on the sliding mode the system is governed by the following equations,

$$\dot{x}(t) = f(x,t) - B(x,t)\left[G(x)\,B(x,t)\right]^{-1}\left[G(x)\,f(x,t)\right] \tag{B.4}$$

Let us consider the following simple scalar example:

$$\dot{x}(t) = ax + bu + \gamma(t) \tag{B.5}$$

where a and $b \neq 0$ are real scalars and $\gamma(t)$ is a disturbance. Let say that we wish to constrain $x(t)$ to the origin in a finite time and in spite of the lack of knowledge of $\gamma(t)$. This can be achieved by selecting $u = -b^{-1}M(t)\operatorname{sign} x$ and $M(t) > |ax| + |\gamma(t)| + \epsilon$, for some arbitrarily small ϵ. By deriving $V = \frac{1}{2}|x|^2$ we get that

$$\begin{aligned}\dot{V} &= |x|\left(ax + bu + \gamma(t)\right) \leq -|x|\left(M(t) - |ax| - |\gamma(t)|\right)\\ &\leq -|x|\,\epsilon = -\sqrt{2}\epsilon\sqrt{V}\end{aligned}$$

By using the comparison principle, we obtain that

$$V(t) \leq \left(V(t_0) - \frac{\epsilon}{\sqrt{2}}(t - t_0)\right)^2 \quad \text{for all } t \geq t_0 \tag{B.6}$$

Since $V(t)$ is by definition a positive function, from (B.6) we can calculate an upper-estimation of the time t_s when $V(t)$ vanishes and consequently also $x(t)$ do it. Thence, we obtain that

$$t_s \leq \frac{\sqrt{2}}{\epsilon}V(t_0) + t_0$$

Thus in this example the EC is obtain from (B.5) when $\dot{x}$ and x are identical to zero, i.e. $u_{\text{eq}} = -b^{-1}\gamma(t)$. We immediately, notice that the disturbance $\gamma(t)$ might be estimated by means of the equivalent control, a way to do it will be given below.

Notice that with the control u being a signum function the right-hand side of (B.5) is not Lipschitz, therefore, we can not use the theory of differential equations. To overcome such a complexity, we can use the theory of differential inclusions treated extensively in [Fil60]. Thus, we can obtain a solution of (B.5) in the Filippov sense.

Nevertheless, the effects of real devices, let say small delays, uncertainties, hysteresis, digital computations, etc., always avoid to achieve the identity $s(x) \equiv 0$. And the trajectories are constraint to some region around the origin, i.e., $\|s(x)\| \leq \Delta$. That is why, that we can ask for the limit solution of (B.1) when Δ tends to zero. That solution is in fact the solution of (B.1) on the sliding mode and it will be found using the equivalent control method, which will be justified by means of Theorem B.1.

Let $\tilde{u}$ be a control for which we obtain the boundary layer $\|s(x)\| \leq \Delta$, we could say that $\tilde{u}$ is the *real control* with which we obtain a real sliding mode. Thus, the dynamic equations are,

$$\dot{x}(t) = f(x,t) + B(x,t)\tilde{u}(t) \tag{B.7}$$

Let us notate by x^* the state vector obtained using the EC method, i.e. the trajectories whose dynamics is governed by (B.4). Let us assume that the distance of any point in the set $S_r = \{x : \|s(x)\| \leq \Delta\}$ to the manifold S is estimated by the inequality

$$d(x,S) \leq P\Delta, \ for \ P > 0.$$

Such a number P always exists if all gradients of functions $s_i(x)$ are linearly independent and are lower bounded in the norm by some positive number. In fact the first condition follows from the assumption that $\det(GB) \neq 0$.

Theorem B.1. *Let us assume that the following 4 conditions are satisfied:*

1. *there is a solution $x(t)$ of system (B.7) which, on the interval $[0,T]$, fulfills the inequality $\|s(x)\| \leq \Delta$;*
2. *for the right-hand part of (B.4), rewritten using x^* as*

$$\dot{x}^*(t) = f(x^*,t) - B(x^*,t)\left[G(x^*)B(x^*,t)\right]^{-1}\left[G(x^*)f(x^*,t)\right], \tag{B.8}$$

   a Lipschitz constant exists;
3. *partial derivatives of the function $B(x,t)\left[G(x)B(x,t)\right]^{-1}$ with respect to all arguments exist and are bounded in every bounded domain, and*
4. *for the right-hand part (B.7) there exist positive numbers M and N such that*

$$\|f(x,t) + B(x,t)\tilde{u}\| \leq M + N\|x\|. \tag{B.9}$$

Then for any pair of solutions to eqs. (B.8) and (B.7) with their initial conditions satisfying

$$\|x(0) - x^*(0)\| \leq P\Delta$$

there exists a positive number H such that

$$\|x(t) - x^*(t)\| \leq H\Delta \text{ for all } t \in [0, T].$$

Proof. For (B.7) we will obtain the following derivative on time of $s(x)$,

$$\dot{s}(x) = G(x) f(x,t) + G(x) B(x,t) \tilde{u}(t) \tag{B.10}$$

since we have assumed that $\det(GB) \neq 0$, from (B.10) we obtain that

$$\tilde{u}(t) = [G(x) B(x,t)]^{-1} \dot{s}(x) - [G(x) B(x,t)]^{-1} G(x) f(x,t) \tag{B.11}$$

The substitution of $\tilde{u}(t)$ into (B.7) yields

$$\dot{x} = f - B[GB]^{-1} Gf + B[GB]^{-1} \dot{s} \tag{B.12}$$

Thus, we have that (B.8) and (B.12) differ from a term depending on $\dot{s}$. By integrating, x^* and x can be written by the following integral equations,

$$x^*(t) = x^*(0) + \int_0^t \left\{ f(x^*,\tau) - B(x^*,\tau)[G(x^*) B(x^*,\tau)]^{-1} [G(x^*) f(x^*,\tau)] \right\} d\tau, \tag{B.13}$$

$$x(t) = x(0) + \int_0^t \left\{ f(x,\tau) - B(x,\tau)[G(x) B(x,\tau)]^{-1} [G(x) f(x,\tau)] \right\} d\tau + \\ + \int_0^t B(x,\tau)[G(x) B(x,\tau)]^{-1} \dot{s}(x)\, d\tau \tag{B.14}$$

By integrating the last term of (B.14) by parts, and taking into account the hypothesis of the theorem, we can obtain the following estimation of the difference of the two solutions,

$$\|x(t) - x^*(t)\| \leq P\Delta + \int_0^t L\|x(\tau) - x^*(\tau)\| d\tau \\ + \left\| B(x,\tau)[G(x) B(x,\tau)]^{-1} s(x) \right\| \Big|_0^t \\ + \int_0^t \left\| \frac{d}{d\tau} B(x,\tau)[G(x) B(x,\tau)]^{-1} \right\| \|s(x)\| d\tau \tag{B.15}$$

By the assumption (B.9), we have that the norm of $x(t)$ is bounded in a interval $[0, T]$, indeed, since

$$\|x(t)\| \leq \|x(0)\| + MT + \int_0^t N \|x(\tau)\| \, d\tau.$$

According to the Bellman-Gronwall lemma (see, e.g. [Poz08]) the following inequality is satisfied,

$$\|x(t)\| \leq (\|x(0)\| + MT) e^{NT}, \text{ for all } t \in [0, T]. \tag{B.16}$$

Thus by the continuity of f and B, and taking into account hypothesis 3 of the theorem, the inequality (B.15) may be represented as follows,

$$\|x(t) - x^*(t)\| \leq Q\Delta + \int_0^t L \|x(\tau) - x^*(\tau)\| \, d\tau$$

where Q is a positive number. Using again the Bellman-Gronwall lemma, we obtain the inequality

$$\|x(t) - x^*(t)\| \leq Q\Delta e^{LT}$$

Taking $H = Qe^{LT}$, the theorem is proven.

Thus, from the theorem we have that $\lim_{\Delta \to 0} x(t) \to x^*(t)$ in a finite interval. This justifies the equivalent control method.

We have say the equivalent control method might be used for the estimation of the matched disturbances, as in the example where $u_{\text{eq}} = -\gamma$. Next, we will see how to estimate the function u_{eq} by means of a first-order low-pass filter. We will make use of the following lemma.

Lemma B.2. *Let the differential equation be as follows*

$$\tau \dot{z}(t) + z(t) = h(t) + H(t) \dot{s} \tag{B.17}$$

where τ is a scalar constant and z, h and s are m-dimensional function vectors. If the following assumptions are satisfied,

i) the functions $h(t)$ and $H(t)$, and their first order derivatives are bounded in magnitude by a certain number M and
ii) $\|s(t)\| \leq \Delta$, Δ being a constant positive value,

then, for any pair of positive numbers Δt and ε, there exists a number $\delta = \delta(\varepsilon, \Delta t, z(0))$ such that the following inequality is fulfilled

$$\|z(t) - h(t)\| \leq \varepsilon$$

provided that $0 < \tau \leq \delta$, $\Delta/\tau \leq \delta$ and $t \geq \Delta t$.

Proof. Let us write the solution of (B.17).

$$z(t) = e^{-t/\tau} z(0) + \frac{1}{\tau} \int_0^t e^{-(t-\sigma)/\tau} \left[h(\sigma) + H(\sigma) \dot{s}(\sigma) \right] d\sigma$$

By integrating by parts we obtain,

$$\begin{aligned} z(t) = {} & e^{-t/\tau} z(0) + h(t) - h(0) e^{-t/\tau} \\ & - \int_{0h}^t e^{-(t-\sigma)/\tau} \dot{h}(\sigma) d\sigma + H(t) \frac{s}{\tau} - H(0) e^{-t/\tau} \frac{s(0)}{\tau} \\ & - \frac{1}{\tau} \int_0^t e^{-(t-\sigma)/\tau} \left[\dot{H}(\sigma) + \frac{1}{\tau} H(\tau) \right] s(\sigma) d\sigma \end{aligned}$$

Then, by the assumptions (i) and (ii), we deduce the following inequality,

$$\| z(t) - h(t) \| \leq \| z(0) - h(0) \| e^{-t/\tau} + M\tau + \frac{2M\Delta}{\tau} + M\Delta + \frac{M\Delta}{\tau}$$

putting similar terms together yields

$$\| z(t) - h(t) \| \leq \| z(0) - h(0) \| e^{-t/\tau} + M(\tau + \Delta) + 3M\frac{\Delta}{\tau} \tag{B.18}$$

Therefore, it is easy to conclude from (B.18) that for any positive number Δt, the following identity is achieved,

$$\lim_{\substack{\tau \to 0 \\ \Delta/\tau \to 0}} z(t) = h(t) \text{ for all } t \geq \Delta t \tag{B.19}$$

Thus, the lemma is proven.

From (B.19), we see that Δ should be much smaller that τ in order to achieve a good estimation of $h(t)$ by means of $z(t)$. Furthermore, (B.18) gives us a more qualitative expression to measure the effect of τ on the estimation. That is, there we can see that if τ is too small then the term depending on the difference on the initial conditions could be considered negligible, i.e. $z(t)$ reaches rapidly a neighborhood around $h(t)$ of order $O(\tau + \Delta) + O\left(\frac{\Delta}{\tau}\right)$. In this case, if Δ is not much smaller than τ, then the neighborhood around $h(t)$ would be big. On the other hand if $\Delta << \tau$, but τ is not so small, then $z(t)$ would last some time before reaching a small neighborhood around $h(t)$. That is why, we can say that an 'ideal' case case is when $\Delta << \tau << 1$.

Thus, the filter designed as

$$\tau u_{\text{av}}(t) + u_{\text{av}}(t) = \tilde{u}(t) \tag{B.20}$$

can be used to estimate u_{eq}. Indeed, from (B.3) and (B.11), (B.20) takes the form

$$\tau u_{\text{av}}(t) + u_{\text{av}}(t) = u_{\text{eq}} + [G(x)\,B(x,t)]^{-1}\,\dot{s}(x) \tag{B.21}$$

Hence, by comparing (B.17) with (B.21), lemma implies that

$$\lim_{\substack{\tau \to 0 \\ \Delta/\tau \to 0}} u_{\text{av}} = u_{\text{eq}} \; for \; t \in (0, T] \tag{B.22}$$

provided that u_{eq} and $(GB)^{-1}$ are bounded and have bounded derivatives, which is fulfilled if conditions of Theorem B.1 are fulfilled.

Now, let us assume that Δ is known (which in general might be not true). In that case we could select $\tau = \Delta^{1/r}$ $(r > 1)$, implying that $\Delta/\tau = \Delta^{\frac{r-1}{r}}$. Thus, as Δ tends to zero, Δ/τ tends to zero also. Therefore, in that case, B.22 is still satisfied. For the same qualitative arguments given above, a good estimation of u_{eq} using u_{av} is obtained when it is satisfied that $\Delta << \tau << 1$. When r is close to 1 then τ is close to Δ; therefore, r near 1 is not a good selection. On the other hand, for $r >> 1$, τ is close to 1, then in that case r is not a good choice either. By selecting $r = 2$, we obtain, for Δ enough small, that $\Delta << \tau << 1$. Hence, by selecting $\tau = \Delta^{1/2}$ and provided that Δ is much smaller than 1, we obtain a good estimation of u_{eq}.

Appendix C
Vehicle Parameters Description

C.1 Vehicle Data

Parameter	Value
M	1296 Kg
r_{1i}	0.28 m
I_{ri}	0.9 $Kg.m^2$
l_o	-0.03 m
C_{ij}	50000 N/rad

Parameter	Value
M	1296 Kg
r_{1i}	0.28 m
I_{ri}	0.9 $Kg.m^2$
l_o	-0.03 m
C_{ij}	50000 N/rad

C.2 Friction Parameters Characteristics

Type of cover	c1	c2	c3
Asphalt, dry	1.2801	23.99	0.52
Asphalt, wet	0.857	33.822	0.347
Concrete, dry	1.1973	25.168	0.5373
Cobblestones, dry	1.3713	6.4565	0.6691
Cobblestones, wet	0.4004	33.7080	0.1204
Snow	0.1946	94.129	0.0646
Ice	0.05	306.39	0

Appendix D
Matrices Definitions

The matrices E_1, E_2, A_1 and A_2 have been used in the chapter IV in the section "Unknown Forces Estimation".

The matrices $E_1$and E_2 are defined as follows:

$$E_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

The matrices A_1 and A_2 are defined as follows:

$$A_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}$$