Solutions to Selected Problem

Chapter 2

Section 2.1

1. The collections (a), (c) are equal setes. The collection (b) is not a set since the element -3 appears twice.
2. If $x \in A - B$ then $x \in A$, $x \notin B$, i.e., $x \notin B - A$. 4. (b) There are 8 elements in S.
7. Let $x \in A - (B \cup C)$. Then, $x \in A$, $x \notin B \cup C$. Consequently $x \notin B$, $x \notin C$, i.e., $x \in A - B$ and finally $x \in (A - B) - C$. Thus, $A - (B \cup C) \subseteq (A - B) - C$ and Similarly $A - (B \cup C) \supseteq (A - B) - C$, which completes the proof.

Section 2.2

8. (a) By definition $(n+1)^2 = (n+1)(n+1)$ which implies

$$(n+1)^2 = n \cdot (n+1) + 1 \cdot (n+1) = (n^2 + n \cdot 1) + (1 \cdot n + 1 \cdot 1)$$

and finally leads to $(n+1)^2 = n^2 + 2n + 1$

9. The inequality $n^2 > n + 3$ certainly holds for $n = 3$ since $3^2 = 9 > 3 + 3 = 6$. Assume that it holds for some arbitrary n.

Section 2.3

3. Define $2^1 = 2$ and $2^{n+1} = 2^n \cdot 2$, $n \ge 1$. The inequality $2^n > n$ holds for $n = 1$ since $2 > 1$. If it holds for arbitrary n, i.e., if $2^n > n$, then

$$2^{n+1} = 2^n \cdot 2 = 2^n + 2^n > n + n \ge n + 1$$

7. $t = \dfrac{r+s}{2}$

9. (a) $n=1$: $(1+r)^n = 1+r = 1+nr$ (b) Assume $(1+r)^n \ge 1+nr$ for arbitrary n, Then, since $1+r>0$, we get $(1+r)^{n+1} = (1+r)^n(1+r) \ge (1+nr)(1+r)$, i.e.,

$$(1+r)^{n+1} \ge 1+nr+r+r^2 = 1+(n+1)r+r^2 \ge 1+(n+1)r$$

Section 2.4

1. Let k be a prime number and consider a rational number $x = \dfrac{m}{n}$ such that m,n have no common factor and $m^l = kn^l$. Consequently, k must divide both m and n (why?) which contradicts the given input.

16. Substitute $a=b=1$ in Newton's binomial theorem (Eq. (2.4.3)).

Section 2.5

6. We consider four cases: (a) $x+5 \ge 0, x-7 \ge 0, x+5 < x+7 \Rightarrow$ no solution.
(b) $x+5 \ge 0, x-7 < 0, x+5 < 7-x \Rightarrow -5 \le x < 1$. The other two cases are left for the reader.

7. The given inequality is equivalent, for all $a>0$, to $a^2+1 \ge 2a$ which holds since $a^2+1-2a = (a-1)^2 \ge 0$.

10. Clearly, $\sqrt[n]{1\cdot 2\cdot \ldots \cdot n} \le \dfrac{1+2+\ldots+n}{n} = \dfrac{n(n+1)}{2n} = \dfrac{n+1}{2}$. Hence, $n! \le \left(\dfrac{n+1}{2}\right)^n$.

Chapter 3

Section 3.1

4. The two straight lines $x=2$, $x=3$.

5. The particular part of the graph is the set $\{(2,2),(3,3),(3,7),(5,5),(5,7),(7,3),(7,5)\}$.

6. $S = \{(x,y) \mid -\infty < x < \infty, y > 0\}$.

Section 3.2

1. (a) The relation is a function with domain $D = \{x \mid x \ge 1\}$.

   (d) The relation has a domain $D = \{x \mid 0 \le x \le 1\}$ and it is not a function.

3. The domain is $D=\{x \mid x\neq 2,3\}$ with range $R=\{y \mid -\infty<y\leq -4\ ,\ y>0\}$

6. (b) $x=\dfrac{1+5y}{4-3y}$ (d) Show that the function does not have an inverse by obtaining a solution other than $x_1=x_2$ to $x_1^2+\dfrac{1}{x_1}=x_2^2+\dfrac{1}{x_2}$.

10. (a) strictly increasing 12. (a) $f\circ g=\dfrac{|x|\sqrt{x^2+2}}{1+x^2}$.

Section 3.3

2. $y=\sqrt{x}+x$ implies $(y-x)^2=x$ and $y^2-2xy+x^2-x=0$, i.e., y is an algebraic function
7. (a) $x=\sqrt{2}+\sqrt{3}$ implies $x^2=5+2\sqrt{6}$ and $(x^2-5)^2-24=0$, i.e. x is an Algebraic number.
12. The function is not defined for $x<0$, i.e., the domain is the single number $x=0$.
15. The function $f(x)g(x)$ is odd.

Section 3.4

2. For example choose $n_0=\left[\dfrac{23}{7\cdot 10^{-3}}\right]+1$. This is no guarantee that that the sequence converges to $\dfrac{1}{7}$ or to any limit.

4. The limit exists and equals $\dfrac{1}{1-q}$.

7. Clearly $\dfrac{1}{3}+\dfrac{1}{4}>\dfrac{1}{4}+\dfrac{1}{4}=\dfrac{1}{2}$, $\dfrac{1}{5}+\dfrac{1}{6}+\dfrac{1}{7}+\dfrac{1}{8}>\dfrac{1}{8}+\dfrac{1}{8}+\dfrac{1}{8}+\dfrac{1}{8}=\dfrac{1}{2}$ etc. Thus, a_n exceeds $\dfrac{k}{2}$ for all positive integer k as $n\to\infty$.

10. $a_n=1+\left(1-\dfrac{1}{2}\right)+\left(\dfrac{1}{2}-\dfrac{1}{3}\right)+\ldots+\left(\dfrac{1}{n-1}-\dfrac{1}{n}\right)=2-\dfrac{1}{n}$ which implies $\lim\limits_{n\to\infty}a_n=2$.

Section 3.5

2. $\lim\limits_{n\to\infty}(a_nb_nc_n)=\lim\limits_{n\to\infty}((a_nb_n)c_n)=\lim\limits_{n\to\infty}(a_nb_n)\lim\limits_{n\to\infty}c_n=(AB)C=ABC$.

7. Without loss of generality we assume $a \ge 0$. Let m be an integer such that $m = [a]+1$ which implies $m > a$. For an arbitrary $n > m$ we get

$$\frac{a^n}{n!} = \frac{a^{m-1}}{(m-1)!}\frac{a}{m}\frac{a}{m+1}\cdots\frac{a}{n} < \frac{a^{m-1}}{(m-1)!}\left(\frac{a}{m}\right)^{n-m+1} \to 0 \ as\ n \to \infty$$

11. $A = \frac{1+\sqrt{5}}{2}$ 12. Not necessarily: $\{a_n\} = \{0,1,0,1,\ldots\}$ and $\{b_n\} = \{1,0,1,0,\ldots\}$ are divergent while their sum clearly converges to 1.

13. For arbitrary r the sequence $a_n = r + \frac{\sqrt{2}}{n}$, $n \ge 1$ is a sequence of irrational numbers which converges to r.

14. No – use the example of problem 12.

Section 3.6

1. 0 3. (c) $\sup(S_3) = \frac{4}{3}$, $\inf(S_3) = -1$, $\overline{\lim}(S_3) = \underline{\lim}(S_3) = 1$.

4. (d) $\sup(S_4) = \frac{2}{3}$, $\inf(S_4) = -\frac{1}{2}$, $\overline{\lim}(S_4) = \underline{\lim}(S_4) = 0$.

9. The even and the odd terms separately, converge to 0. This is no guarantee for convergence. For example consider the sequence $\left\{1,0,1,\frac{1}{2},0,2,\frac{1}{3},0,3,\ldots\right\}$.

10. The sequence does not converge. $\sup(S) = 1.1$, $\inf(S) = 10^{-6}$, $\overline{\lim}(S) = \underline{\lim}(S) = 1$.

Section 3.7

1. Clearly $a_2 = 1+\sqrt{1} = 2 > 1 = a_1$. Using Assume $a_{n+1} > a_n$. We then get $a_{n+2} = 1+\sqrt{a_{n+1}} > 1+\sqrt{a_n} = a_{n+1}$ (obviously $a_n > 0$, $n \ge 1$ - why?).

3. All the sequence elements are obviously positive and $a_{n+1} = a_n + \sqrt{a_n} > a_n$. If the sequence converges to some positive A then $A = A + \sqrt{A}$ which leads to $A = 0$ and to contradiction.

9. $a_n = \frac{n}{1}\cdot\frac{n}{2}\cdots\frac{n}{n-1}\cdot\frac{n}{n} \ge \frac{n}{1} = n \to \infty$

Chapter 4

Section 4.1

2. For an arbitrary $x_0 > 0$ write

$$\sqrt{x} - \sqrt{x_0} = \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}}$$

from which we easily get that $f(x)$ has a limit at x_0. The case $x_0 = 0$ is straightforward. The function does not have a limit at $x = -1$ since it is not defined at the interval between -1 and 0.

4. The function has the limit 0 at $x = 0$.

Section 4.2

1. (b) The function is clearly continuous for $x > 0$ where $f(x) = x$ and for $x < 0$ where $f(x) = -x$. It is also continuous at $x = 0$: Indeed, given $\varepsilon > 0$ we look for $\delta > 0$ such that $|x - 0| < \delta$ implies $|f(x) - f(0)| < \varepsilon$ and the obvious choice is $\delta = \varepsilon$.

3. For any given x_0 and arbitrary $\varepsilon > 0$ choose $\delta = \frac{\varepsilon}{L}$.

Section 4.3

1. Clearly,
$f^n(x) - f^n(x_0) = [f^{n-1}(x) + f^{n-2}(x)f(x_0) + \ldots + f^{n-1}(x_0)][f(x) - f(x_0)]$.
Since $f(x)$ is continuous, it is bounded by some $M > 0$ which implies

$$\left|f^n(x) - f^n(x_0)\right| \le nM^{n-1}\left|f(x) - f(x_0)\right|$$

where nM^{n-1} is constant. Consequently, $x \to x_0$ implies $f^n(x) \to f^n(x_0)$ as well.

3. The function $g(x) = x^3 + \sin^2(x)$ is a sum of two functions which are continuous everywhere. Thus, it is continuous everywhere and so is $f(x)$ as a composite function.

4. (b) $x \neq 1$ (c) $-\infty < x < \infty$

Section 4.4

2. The inverse function is $f^{-1}(y)=\dfrac{1+\sqrt{8y-7}}{4}$ and it is continuous over $[2,7]$.
3. The inverse is $x=\sqrt{y^2-1}$ defined for $1<y<\infty$.
5. The function $y=\cos(x)$, $-\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2}$ does not have an inverse since it is not one to one. For example $\cos(\pi/4)=\cos(-\pi/4)$.

Section 4.5

1. The function is both continuous and uniformly continuous.
2. (a) The function is continuous. (b) It is not uniformly continuous.
   (c) The function in problem 1 has a limit as $x\to\infty$.
3. A function defined over a *closed interval* and satisfies a Lipschitz condition, is continuous over this interval and is therefore uniformly continuous there.
7. (a) The function satisfies a Lipschitz condition:

$$\left|x^3-y^3\right|=|x-y|\left|x^2+xy+y^2\right|\le 27|x-y|$$

Chapter 5

Section 5.1

1. The derivatives are $3x^2+2x$, 8.
5. The function is not continuous at any point and therefore derivative nowhere.
7. When two derivatives are identical the corresponding functions differ by a constant.
10. The function has derivative for $x>0$ and $f'(x)=\dfrac{1}{2\sqrt{x}}$.
12. At $x=0$ the function has a right derivative 0 and a left derivative 1. Thus it does not possess a derivative there.

Section 5.2

1. (c) $f'(x)=-3x^{-4}\sqrt{x}+\dfrac{1}{2\sqrt{x}}x^{-3}=-2.5x^{-4}\sqrt{x}$

   2. (a) $f'(x)=3(x^2+x)^2(2x+1)$

4. The function is one – to – one and consequently the inverse exists and

$$\left[f^{-1}(x)\right]' = \frac{1}{1+3x^2}$$

7. Solve $\left(x^3 + x^2\right)' = 3x^2 + 2x = 10$. The solutions are $\frac{-2 \pm \sqrt{124}}{6}$ and the positive one is within the interval $[1,2]$.

9. $\xi = \frac{38}{15}$.

Section 5.3

1. $\left[\sin^2(x) + \cos^2(x)\right]' = 2\sin(x)\cos(x) + 2\cos(x)\left[-\sin(x)\right] = 0$.

3. The left-hand side of the relation is always positive! (why?).

8. (a) $\cosh^2(x) - \sinh^2(x) = \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4} = 1$.

9. $\tanh'(x) = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}$ and similarly

   $\coth'(x) = -\frac{1}{\sinh^2(x)}$.

11. (a) The domain is all x for which $\cosh(x) > 0$, i.e., $-\infty < x < \infty$.

Section 5.4

1. (a) $y'' = -\sin(e^x)e^{2x} + e^x\cos(e^x)$. 2. The stationary points are $x = 0$, $x = n$.

4. The function is defined for all $x \neq 0$ and has a minimum at $x = \sqrt[3]{-\frac{1}{4}}$.

8. Clearly $e^x > 1$ for arbitrary $x > 0$. By the first mean-value theorem

$$e^x - 1 = e^x - e^0 = xe^{\xi}\ ,\ 0 < \xi < x$$

   and thus $e^x = 1 + xe^{\xi} > 1 + x$, $x > 0$.

Section 5.5

1. (a) $\frac{3}{4}$ 2. (b) -2 4. (b) 0.

Chapter 6

Section 6.1

1. (a) $\int_0^3 f(x)dx = 5$ (b) The integral still exists and its value unchanged.
4. No.

Section 6.2

1. (a) Differentiability (or continuity) (b) Continuity.

Section 6.3

2. Since $f(x)$ is continuous and $g(x)$ monotonic increasing and bounded by 0.001, all functions are integrable (Theorems 6.3.1, 6.3.5, 6.3.6).
5. Use Theorem 6.3.6.

Section 6.4

1. (a) 10 (c) $1+\ln(2)$ 2. (a) $x+\frac{x^3}{3}-x+\tan^{-1}(x)$.
5. (a) $\frac{7\pi}{12}$ (b) $\frac{(a+2)^8-3^8}{8}$ 6. (a) $\frac{1}{2}e^{(x^2)}$

Section 6.5

2. $\lambda=\frac{e^2-1}{2}$, $x_0=\ln\left(\frac{e^2-1}{2}\right)\approx 1.164$ and x_0 is unique.
3. $\lambda=\frac{2}{\pi}$, $x_0=\sin^{-1}\left(\frac{2}{\pi}\right)\approx 0.690$, 2.451
5. By Theorem 6.5.3: $\int_0^{\pi}\sin(x)\cos(x)dx=\cos(0)\int_0^{\xi}\sin(x)dx+\cos(\pi)\int_{\xi}^{\pi}\sin(x)dx$ for some $\xi: 0\le\xi\le\pi$. This implies $\cos(\xi)=0$, i.e., $\xi=\frac{\pi}{2}$.

Section 6.6

1. (b) $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$ 4. Substitute $u = 1 - x^2$.

7. (a) $-\ln(\cos(x))$ 10. The integral is $\frac{1}{a-b}\ln\left(\frac{x-a}{x-b}\right)$ if $a \neq b$. Otherwise it is $\frac{1}{a-x}$.

Section 6.7

2. The integrals (b) and (c) are divergent. 3. The integrals (a) and (b) are divergent.

Chapter 7

Section 7.1

1. Yes 2. No 4. (a) Converges (b) Converges (d) Diverges

Section 7.2

2. $\int_1^\infty \frac{dx}{x\sqrt{x}} = \left[\frac{-2}{\sqrt{x}}\right]_1^\infty = 2$ and the series converges.

5. $\frac{a_{n+1}}{a_n} = \frac{\left(\frac{1}{2}\right)^{n+1} + \left(\frac{2}{3}\right)^{n+1}}{\left(\frac{1}{2}\right)^n + \left(\frac{2}{3}\right)^n} = \frac{\frac{1}{2}\left(\frac{3}{4}\right)^n + \frac{2}{3}}{1 + \left(\frac{3}{4}\right)^n} \underset{n\to\infty}{\to} \frac{2}{3} < 1$

6. No, since $\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{n^2}} \to 1$ as $n \to \infty$.

Section 7.3

1. Clearly $\frac{1}{n\sqrt{n}} > \frac{1}{(n+1)\sqrt{n+1}}$ and $\lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0$, i.e., convergence follows by Theorem 7.3.1.

3. (a) The series is not absolutely convergent ($\int_2^\infty \frac{dx}{x\ln(x)}$ does not exist).

   (b) The series is not absolutely convergent ($\lim_{x\to 0}\frac{\sin(x)}{x}=1$ and $\sum\left(\frac{1}{2}\right)^n$ converges).

5. Before each negative term insert k (which varies) positive terms whose sum exceeds 1.

Section 7.4

2. Cauchy product is $1+\frac{3}{4}+\frac{35}{72}+\frac{11}{36}+\ldots$

Section 7.5

1. The radius of convergence is $R=1$. The first two derivatives are $f'(x)=\frac{1}{1-x}$ and $f''(x)=\frac{1}{(1-x)^2}$.

2. The Taylor series is $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ and the radius of convergence is $R=1$.

6. The Taylor series is $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ and it converges uniformly for $|x|\le 100$ since for an arbitrary $\varepsilon>0$ we can find n such that

$$\frac{e^{100}\cdot 100^{n+1}}{(n+1)!}<\varepsilon$$

   A similar argument holds for $|x|\le 200$ (see Example 7.5.7 and Eq. (10.2.8)).

Chapter 8

Section 8.1

1. (i) Clearly $I=\int_a^a f(x)dx=\int_0^a f(x+T)dx=\int_0^a f(x+T)d(x+T)$. By substituting $u=x+T$ we get $I=\int_T^{a+T} f(u)du=\int_T^{a+T} f(x)dx$.

2. $f(x) = 2(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) + \ldots)$

6. (d) $a_n = 0$, $n \ge 0$; $b_n = \frac{1}{\pi}\int_0^{\pi} \cos^2(x)\sin(nx)dx$, $n \ge 1$

Section 8.2

4. The function $f(x) = \sqrt{x}$, $0 < x < 2$ does not satisfy the requirements since $f'(x) = \frac{1}{2\sqrt{x}} \to \infty$ as $x \to 0$. The function $f(x) = \sqrt{x}$, $0.001 < x < 2$ satisfies both requirements of Theorem 8.2.1.

Section 8.3

1. (a) even (c) odd (d) neither
2. $f'(0) = \lim_{h \to 0} \frac{f(h) - f(-h)}{h} = \lim_{h \to 0} \frac{f(h) - f(h)}{h} = 0$

   $g(-x) = -g(x) \Rightarrow g(x) + g(-x) = 0 \Rightarrow 2g(0) = 0 \Rightarrow g(0) = 0$
3. Consider an even function $f(x)$. Then

$$\frac{f(x+h) - f(x)}{h} = -\frac{f(-x-h) - f(-x)}{-h}$$

   and if $h \to 0$ the left-hand side converges to $f'(x)$ while the right-hand side converges to $-f'(x)$.

Chapter 9

Section 9.1

1. We discuss only real solutions. **Step 1.** Calculate $A = b^2 - 4ac$. **Step 2.** If $A < 0$ there are no real solutions – stop; If $A = 0$ get the single solution $x_1 = \frac{-b}{2a}$ and stop; Otherwise get the two different solutions $x_1 = \frac{-b + \sqrt{A}}{2a}$, $x_2 = \frac{-b - \sqrt{A}}{2a}$ and stop.
5. 5 iterations.

Section 9.2

1. Consider the function

$$f(x)=\begin{cases} 2x\,,\ 0\le x<\dfrac{1}{2} \\ 1-x\,,\ \dfrac{1}{2}\le x\le 1 \end{cases}$$

   Clearly it does not satisfy either condition, yet $f\left(\frac{1}{2}\right)=\frac{1}{2}$.

5. There is an indication that $|f(s)|>1$.

Section 9.3

1. 5 iterations, $x_5=1.718771927$. 3. The Newton method provides the particular solution $s=0$ where $f'(s)=0$. This is not the case at the other solution $s=3$.

Section 9.4

2. The maximum error is bounded by $6\cdot 10^{-2}$.

Section 9.5

3. The quadratic least-squares approximation is $p_2(x)=x$ since the data is an exact straight line.
4. (b) The data is almost linear which indicates a small coefficient for x^2.
5. The least-squares approximation is $y=4.671x-2.143$.

Section 9.6

1. 3.116 2. The exact value is π and Eq. (9.6.4) provides $f''(\xi)=-1.006$. It is easily seen that that the second derivative $f''(x)=\sin(x)+x\cos(x)$ obtains this value somewhere between $\frac{\pi}{2}$ and π. 5. The approximated value is 8.3894 vs. the exact value 8.3891. 6. The integrand's fourth derivative is bounded by $6e^2$. Hence $\frac{6e^2\cdot 2\cdot h^4}{180}\le 10^{-8}$ which implies $h\le 0.0119$.

Chapter 10

Section 10.1

1. If $\sqrt{e} = \frac{m}{n}$ where m,n integers than $e = \frac{m^2}{n^2}$ is rational which contradicts Theorem 10.1.1. Similarly, if $\sqrt{\frac{e+1}{e-1}} = r$ for some rational r than $e = \frac{r+1}{r-1}$, i.e., rational.

Section 10.2

1. $\binom{2n}{n} \approx \frac{2^{2n}}{\sqrt{\pi n}}$, $n \to \infty$ 5. The approximate value is $\sqrt{\frac{81}{39}} \approx 1.446$. The exact value is $1.441\ldots$

Section 10.3.1

1. (b) Definition: the whole plane, discontinuity: both axes.
2. (a) $\frac{\partial f}{\partial x} = -\frac{1}{x^2} + \frac{x}{\sqrt{x^2+y^2}}$, $x \neq 0$, $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$, $(x.y) \neq (0,0)$

Section 10.3.2

1. $x_1 = x_2 = \frac{\sqrt{2}}{2}$, $x_1 = x_2 = -\frac{\sqrt{2}}{2}$ 3. (b) $x = \frac{ab^2}{a^2+b^2}$, $y = \frac{a^2 b}{a^2+b^2}$

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