Appendix A

A.1 Mathematical Preliminaries

A.1.1 Sets

Let $\mathcal{X}$ and $\mathcal{Y}$ be two arbitrary sets. Then

$$\begin{aligned}
&\mathcal{X} \subseteq \mathcal{Y} \iff (x \in \mathcal{X} \implies x \in \mathcal{Y}),\\
&\mathcal{X} \subset \mathcal{Y} \iff (\mathcal{X} \subseteq \mathcal{Y} \quad \text{and} \quad \mathcal{X} \neq \mathcal{Y}),\\
&\mathcal{X} \setminus \mathcal{Y} = \{x \in \mathcal{X} : x \notin \mathcal{Y}\}\,.
\end{aligned}$$

A.1.2 Spaces

Let $\mathcal{V}$ be a vector space. Elements of $\mathcal{V}$ are denoted by bold letters, e.g., $\boldsymbol{x} \in \mathcal{V}$. In addition, functions that map into a vector space are also denoted by bold letters, e.g., $\boldsymbol{f} : \mathcal{X} \to \mathcal{V}$, where $\mathcal{X}$ is arbitrary.

The set of real numbers is denoted by $\mathbb{R}$. Moreover,

$$\begin{aligned}
\mathbb{R}_+ &= \{x \in \mathbb{R} : x \geq 0\}\,,\\
\mathbb{R}_{++} &= \{x \in \mathbb{R} : x > 0\}\,.
\end{aligned}$$

The set of complex numbers is denoted by $\mathbb{C}$.

The Cartesian product of two sets $\mathcal{X}$ and $\mathcal{Y}$ is denoted by $\mathcal{X} \times \mathcal{Y}$, and $\mathcal{K}^N$ is the N-fold Cartesian product of $\mathcal{K}$. An exception is made if $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}\}$: $\mathbb{R}^K$ denotes the K-fold Cartesian product of $\mathbb{R}$, equipped with the usual vector space structure, an inner product

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$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{k=1}^{K} x_k y_k,$$

the induced norm

$$\|\boldsymbol{x}\|_2 = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle},$$

the metric

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_2,$$

and the standard topology induced by the metric d. Likewise, $\mathbb{C}^K$ is the K-fold Cartesian product of $\mathbb{C}$, equipped with the usual vector space structure, inner product, norm, and metric.

The vector $\mathbf{1} \in \mathbb{R}^K$ is the vector of all-ones, and $\mathbf{0} \in \mathbb{R}^K$ is the vector of all-zeros. Given $\boldsymbol{x} \in \mathbb{R}^K$ and $\eta \in \mathbb{R}_{++}$, an open ball $\mathcal{B}_\eta(\boldsymbol{x})$ is defined by

$$\mathcal{B}_\eta(\boldsymbol{x}) = \left\{ \boldsymbol{y} \in \mathbb{R}^K : d(\boldsymbol{x}, \boldsymbol{y}) < \eta \right\}.$$

The interior of a set $\mathcal{X} \subseteq \mathbb{R}^K$ is

$$\operatorname{int}(\mathcal{X}) = \left\{ \boldsymbol{x} \in \mathcal{X} : \exists \eta > 0 : \mathcal{B}_\eta(\boldsymbol{x}) \subset \mathcal{X} \right\}.$$

The boundary of a set $\mathcal{X} \subseteq \mathbb{R}^K$ is

$$\operatorname{bd}(\mathcal{X}) = \mathcal{X} \setminus \operatorname{int}(\mathcal{X}).$$

For $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}\}$, the vector space of matrices with M rows and N columns and entries in $\mathcal{K}$ is denoted by $\mathcal{K}^{M \times N}$. Matrices with $M, N > 1$ are frequently denoted by uppercase letters, although this convention is not strictly adhered to. The matrix $\boldsymbol{X}^{\mathrm{T}}$ is the transpose of $\boldsymbol{X}$, $\boldsymbol{X}^{\mathrm{H}}$ is the conjugate transpose of $\boldsymbol{X}$, $\operatorname{rank}(\boldsymbol{X})$ is the rank of $\boldsymbol{X}$, and $\operatorname{tr}(\boldsymbol{X})$ denotes the trace of $\boldsymbol{X}$. The matrix $\boldsymbol{X}^{-1}$ is the inverse of $\boldsymbol{X}$, provided it exists. The identity matrix is $\boldsymbol{I}$, and $\boldsymbol{e}_n \in \mathbb{R}^{M \times 1}$ is the n-th column of $\boldsymbol{I} \in \mathbb{R}^{M \times N}$. Range and nullspace of a matrix $\boldsymbol{A} \in \mathbb{R}^{M \times N}$ are defined as follows:

$$\operatorname{range}(\boldsymbol{A}) = \left\{ \boldsymbol{A}\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^{N \times 1} \right\},$$

$$\operatorname{null}(\boldsymbol{A}) = \left\{ \boldsymbol{x} \in \mathbb{R}^{N \times 1} : \boldsymbol{A}\boldsymbol{x} = \mathbf{0} \right\}.$$

By equipping $\mathbb{R}^{M \times N}$ with the inner product

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \operatorname{tr}\left(\boldsymbol{X}^{\mathrm{T}} \boldsymbol{Y} \right)$$

and the metric induced by this inner product, $\mathbb{R}^{M \times N}$ and $\mathbb{R}^{MN}$ are isometric. Accordingly, no distinction is made between $\mathbb{R}^{K \times 1}$ and $\mathbb{R}^K$, and the inner product of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^K$ is written as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \operatorname{tr}\left(\boldsymbol{x}^{\mathrm{T}} \boldsymbol{y} \right) = \boldsymbol{x}^{\mathrm{T}} \boldsymbol{y}.$$

For two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^K$, the element-wise product $\boldsymbol{x}\boldsymbol{y} \in \mathbb{R}^K$ is defined by

$$\boldsymbol{z} = \boldsymbol{x}\boldsymbol{y} \iff z_k = x_k y_k, \forall k.$$

If $\mathcal{X}$ and $\mathcal{Y}$ are subsets of a vector space $\mathcal{V}$, the sum of $\mathcal{X}$ and $\mathcal{Y}$ is defined as follows:

$$\mathcal{X} + \mathcal{Y} = \{\boldsymbol{x} + \boldsymbol{y} : (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y})\}.$$

Moreover, if $\mathcal{V}$ is a real vector space and $\alpha \in \mathbb{R}$, then

$$\alpha\mathcal{X} = \{\alpha\boldsymbol{x} : \boldsymbol{x} \in \mathcal{X}\}.$$

A.1.3 Functions

Let $f : \mathcal{X} \to \mathcal{Y}$ be a function. The image of a set $\mathcal{A} \subseteq \mathcal{X}$ under f is given by

$$f(\mathcal{A}) = \{f(x) : x \in \mathcal{A}\}.$$

The preimage of a set $\mathcal{C} \subseteq \mathcal{Y}$ under f is

$$f^{-1}(\mathcal{C}) = \{x \in \mathcal{X} : f(x) \in \mathcal{C}\}.$$

If f is a bijection, its inverse function is denoted by f^{-1}. Accordingly, for $c \in \mathcal{Y}$, $f^{-1}(c)$ is the value of the inverse function of f at c, while $f^{-1}(\{c\})$ is the preimage of $\{c\}$ under f.

Given a set $\mathcal{X}$, the identity function on $\mathcal{X}$ is the function $\mathrm{id} : \mathcal{X} \to \mathcal{X}$,

$$\mathrm{id}(x) = x.$$

The restriction of a function $f : \mathcal{X} \to \mathcal{Y}$ to a set $\mathcal{W} \subseteq \mathcal{X}$ is the function $f|_{\mathcal{W}} : \mathcal{W} \to \mathcal{Y}$,

$$f|_{\mathcal{W}}(x) = f(x).$$

Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{W} \to \mathcal{Z}$ be two functions. Provided that $f(\mathcal{X}) \subseteq \mathcal{W}$, the composite function $g \circ f$ is the function $g \circ f : \mathcal{X} \to \mathcal{Z}$,

$$(g \circ f)(x) = g(f(x)).$$

The function (f, g) is the function defined by $(f, g) : \mathcal{X} \times \mathcal{W} \to \mathcal{Y} \times \mathcal{Z}$,

$$(f, g)(x, w) = (f(x), g(w)).$$

The operator ∇ yields the gradient field of a differentiable function $f : \mathbb{R}^K \to \mathbb{R}$, i.e., $\nabla f(\boldsymbol{x})$ is the gradient of f at $\boldsymbol{x}$. Likewise, applying ∇ to a function

$\boldsymbol{f} : \mathbb{R}^K \to \mathbb{R}^N$ with differentiable component functions f_n yields a matrix $\nabla \boldsymbol{f}(\boldsymbol{x})$, with the n-th column corresponding to the gradient of the n-th component function at $\boldsymbol{x}$. Equivalently, $\nabla \boldsymbol{f}(\boldsymbol{x})$ is the transpose of the Jacobian of $\boldsymbol{f}$ at $\boldsymbol{x}$. A function $\boldsymbol{f} : \mathbb{R}^K \to \mathbb{R}^N$ whose component functions f_n are r- times continuously differentiable is said to be a C^r function.

Given a function $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}, (x, y) \mapsto f(x, y)$, a function $f(\cdot, y)$ is defined by

$$f(\cdot, y) : \mathcal{X} \to \mathcal{Z}, x \mapsto f(x, y), \tag{A.1}$$

where it is assumed that $y \in \mathcal{Y}$. Moreover, if $\mathcal{Z} \subseteq \mathbb{R}$, then $\min_{y \in \mathcal{Y}} f(\cdot, y)$ is the function defined by the point-wise minimum

$$\left(\min_{y \in \mathcal{Y}} f(\cdot, y) \right)(x) = \min_{y \in \mathcal{Y}} f(x, y), \tag{A.2}$$

where it is assumed that the minimum on the right-hand side of (A.2) is well-defined for all $x \in \mathcal{X}$.

For $\boldsymbol{x} \in \mathbb{R}^K$, $(\boldsymbol{x})^+$ is the vector in $\mathbb{R}^K_+$ obtained by setting the negative entries in $\boldsymbol{x}$ to zero.

A.1.4 Order Relations, Monotonicity, and Pareto Optimality

The following definitions of order relations between vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$, with $N > 1$, are used:

$$\begin{aligned} \boldsymbol{x} \geq \boldsymbol{y} &\iff \forall k : x_k \geq y_k, \\ \boldsymbol{x} > \boldsymbol{y} &\iff \boldsymbol{x} \geq \boldsymbol{y}, \exists k : x_k > y_k, \\ \boldsymbol{x} \gg \boldsymbol{y} &\iff \forall k : x_k > y_k. \end{aligned}$$

Order relations $\leq, <, \ll$ are defined in the same manner.

A function $\boldsymbol{f} : \mathcal{X} \to \mathbb{R}^N$, with $\mathcal{X} \subseteq \mathbb{R}^M$, is said to be *increasing* if

$$\boldsymbol{x} \in \mathcal{X} \text{ and } \boldsymbol{x}' \geq \boldsymbol{x} \implies \boldsymbol{x}' \in \mathcal{X} \text{ and } \boldsymbol{f}(\boldsymbol{x}') \geq \boldsymbol{f}(\boldsymbol{x}).$$

Likewise, $\boldsymbol{f}$ is said to be *strictly increasing* if it is increasing and

$$\boldsymbol{x}' > \boldsymbol{x} \implies \boldsymbol{f}(\boldsymbol{x}') > \boldsymbol{f}(\boldsymbol{x}).$$

Finally, $\boldsymbol{f}$ is said to be *strongly increasing* if it is increasing and

$$\boldsymbol{x}' \gg \boldsymbol{x} \implies \boldsymbol{f}(\boldsymbol{x}') \gg \boldsymbol{f}(\boldsymbol{x}).$$

For a set $\mathcal{S} \subset \mathbb{R}^K$, the set of Pareto optimal points of $\mathcal{S}$ is defined by

$$\begin{aligned}\text{par}(\mathcal{S}) &= \left\{\boldsymbol{x} \in \mathcal{S} : (\{\boldsymbol{x}\} + \mathbb{R}^K_+) \cap \mathcal{S} = \{\boldsymbol{x}\}\right\} \\ &= \left\{\boldsymbol{x} \in \mathcal{S} : \nexists \boldsymbol{x}' \in \mathcal{S} : \boldsymbol{x}' > \boldsymbol{x}\right\}.\end{aligned}$$

Likewise, the set of weakly Pareto optimal points of $\mathcal{S}$ is given by

$$\begin{aligned}\text{wpar}(\mathcal{S}) &= \left\{\boldsymbol{x} \in \mathcal{S} : (\{\boldsymbol{x}\} + \mathbb{R}^K_{++}) \cap \mathcal{S} = \emptyset\right\} \\ &= \left\{\boldsymbol{x} \in \mathcal{S} : \nexists \boldsymbol{x}' \in \mathcal{S} : \boldsymbol{x}' \gg \boldsymbol{x}\right\}.\end{aligned}$$

A.2 Elementary Topology

This appendix primarily summarizes basic topological definitions and results that are used throughout this work. Accordingly, in most cases proofs are omitted. Proofs and further details can be found in any textbook on elementary topology, such as [85]. Familiarity with the following basic concepts is assumed: topological space, topological subspace, metric space, and natural topology in a metric space.

The minimal topological structure considered in this work is that of a metric space. Accordingly, the following definitions and propositions are formulated using metric spaces.

Definition A.2.1 (Continuous Function). Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is continuous if and only if the preimage of any open subset of $\mathcal{Y}$ is an open subset of $\mathcal{X}$.

Definition A.2.2 (Homeomorphism). An invertible function f is a *homeomorphism* if both f and f^{-1} are continuous.

Proposition A.2.1. *Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a continuous function between two metric spaces $\mathcal{X}$ and $\mathcal{Y}$, and $\mathcal{Z} \subset \mathcal{Y}$ is closed. Then the preimage $f^{-1}(\mathcal{Z})$ is closed.*

Proposition A.2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous function between two metric spaces $\mathcal{X}$ and $\mathcal{Y}$. If $\mathcal{X}$ is compact, then $f(\mathcal{X})$ is compact.*

Proposition A.2.3. *Suppose $\mathcal{X}$ is a metric space, $\mathcal{F} \subset \mathcal{X}$ is compact, and $\mathcal{G} \subset \mathcal{X}$ is closed. Then $\mathcal{F} \cap \mathcal{G}$ is compact.*

Theorem A.2.4 (Weierstrass). *Let $\mathcal{X}$ be a nonempty compact set, and let $f : \mathcal{X} \to \mathbb{R}$ be a continuous function. Then there exist $\bar{x}, \underline{x} \in \mathcal{X}$ such that*

$$f(\bar{x}) = \sup\{f(x) : x \in \mathcal{X}\}, \tag{A.3}$$

$$f(\underline{x}) = \inf\{f(x) : x \in \mathcal{X}\}. \tag{A.4}$$

Proposition A.2.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces, with $\mathcal{X}$ nonempty and compact, and let $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be continuous. Then the function $g : \mathcal{Y} \to \mathbb{R}$, with*

$$g(\boldsymbol{y}) = \min_{\boldsymbol{x} \in \mathcal{X}} f(\boldsymbol{x}, \boldsymbol{y})$$

is well-defined and continuous.

Proof. The function f is continuous, thus the function $f(\cdot, \boldsymbol{y})$ is continuous, for any $\boldsymbol{y} \in \mathcal{Y}$. Moreover, $\mathcal{X}$ is compact. Consequently, the minimum $g(\boldsymbol{y})$ is attained on $\mathcal{X}$, thus g is well-defined.

Likewise, the function $f(\boldsymbol{x}, \cdot)$ is continuous for any $\boldsymbol{x} \in \mathcal{X}$. The function g corresponds to the point-wise minimum over the set of continuous functions $\{f(\boldsymbol{x}, \cdot)\}_{\boldsymbol{x} \in \mathcal{X}}$. Thus, g is continuous. □

Corollary A.2.6. *Let $\mathcal{Y}$ be a metric space, and let $\boldsymbol{h} : \mathcal{Y} \to \mathbb{R}^K$ be continuous. Then the function $g : \mathcal{Y} \to \mathbb{R}$, with*

$$g(\boldsymbol{y}) = \min_{k \in \{1,\ldots,K\}} h_k(\boldsymbol{y})$$

is continuous.

Proof. The result follows from Proposition A.2.5 by letting

$$\mathcal{X} = \{1, \ldots, K\},$$
$$f(x, \boldsymbol{y}) = h_x(\boldsymbol{y}),$$

and observing that $\mathcal{X}$ is compact and $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is continuous. □

Theorem A.2.7 (Heine-Borel). *Let $\mathcal{X} \subset \mathbb{R}^N$. Then the following statements are equivalent:*

(1) $\mathcal{X}$ is closed and bounded.
(2) $\mathcal{X}$ is compact.

A.3 Pareto Sets

Proposition A.3.1. *Let $\mathcal{X}$ denote a nonempty subset of $\mathbb{R}^K$. Then*

$$\operatorname{wpar}(\mathcal{X}) \subseteq \operatorname{bd}(\mathcal{X}). \tag{A.5}$$

Proof. Let $\boldsymbol{x} \in \operatorname{wpar}(\mathcal{X})$. Assume $\boldsymbol{x} \notin \operatorname{bd}(\mathcal{X})$. Then there exists $\eta > 0$ such that

$$\mathcal{B}_\eta(\boldsymbol{x}) \subset \mathcal{X},$$

which contradicts $\boldsymbol{x} \in \operatorname{wpar}(\mathcal{X})$. □

Proposition A.3.2. *Let $\mathcal{X}$ denote a nonempty compact subset of $\mathbb{R}_+^K$. Then*

$$\operatorname{par}(\mathcal{X}) \neq \emptyset. \tag{A.6}$$

Proof. Define

$$\bar{\mathcal{X}} = \operatorname*{argmax}_{\boldsymbol{x} \in \mathcal{X}} \|\boldsymbol{x}\|_2 .$$

The set $\mathcal{X}$ is compact and nonempty, and $\|\cdot\|_2$ is continuous, thus $\bar{\mathcal{X}}$ is well-defined and non-empty. Let $\boldsymbol{x}' \in \bar{\mathcal{X}}$. Assume $\boldsymbol{x}' \notin \operatorname{par}(\mathcal{X})$. Then there exists $\boldsymbol{v} \in \mathbb{R}_+^K \setminus \{\boldsymbol{0}\}$ such that $\boldsymbol{x}' + \boldsymbol{v} \in \mathcal{X}$. But

$$\left\|\boldsymbol{x}' + \boldsymbol{v}\right\|_2 = \sqrt{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} + 2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{v} + \boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}} > \left\|\boldsymbol{x}'\right\|_2 ,$$

where the inequality follows from the fact that $\boldsymbol{x}, \boldsymbol{v} \in \mathbb{R}_+^K$ and $\boldsymbol{v} \neq \boldsymbol{0}$. This contradicts $\boldsymbol{x}' \in \bar{\mathcal{X}}$, thus

$$\operatorname{par}(\mathcal{X}) \supseteq \bar{\mathcal{X}} \supset \emptyset.$$

□

Corollary A.3.3. *Let $\mathcal{X}$ denote a nonempty compact subset of $\mathbb{R}_+^K$. Then*

$$\operatorname{wpar}(\mathcal{X}) \neq \emptyset. \tag{A.7}$$

Proposition A.3.4. *Let $\mathcal{X}$ denote a compact subset of $\mathbb{R}^K$. Then* $\operatorname{wpar}(\mathcal{X})$ *is compact.*

Proof. Let $\left\{\boldsymbol{x}^{(n)}\right\}$ denote a sequence with $\boldsymbol{x}^{(n)} \in \operatorname{wpar}(\mathcal{X})$ and

$$\lim_{n \to \infty} \boldsymbol{x}^{(n)} = \boldsymbol{x}.$$

By definition, $\operatorname{wpar}(\mathcal{X}) \subseteq \mathcal{X}$, thus $\boldsymbol{x}^{(n)} \in \mathcal{X}$. The set $\mathcal{X}$ is closed, hence $\boldsymbol{x} \in \mathcal{X}$. Assume that $\boldsymbol{x} \notin \operatorname{wpar}(\mathcal{X})$. Then there exists $\boldsymbol{x}' \in \mathcal{X}$ such that

$$\boldsymbol{x}' \gg \boldsymbol{x}. \tag{A.8}$$

Let

$$\delta = \min_k x_k' - x_k .$$

Equation (A.8) implies that $\delta > 0$. It follows that

$$\boldsymbol{x}' \gg \boldsymbol{y}, \forall \boldsymbol{y} \in \mathcal{B}_\delta(\boldsymbol{x}).$$

As a result, $\mathcal{B}_\delta(\boldsymbol{x})$ does not contain any weak Pareto optimal points, which contradicts the fact that $\boldsymbol{x}$ is a limit point of a sequence of weak Pareto optimal points. Thus, $\boldsymbol{x} \in \operatorname{wpar}(\mathcal{X})$, implying that $\operatorname{wpar}(\mathcal{X})$ is closed. As a subset of a bounded set, $\operatorname{wpar}(\mathcal{X})$ is bounded. Theorem A.2.7 yields the result. □

Proposition A.3.5. *Let $\mathcal{X} \subset \mathbb{R}_+^K$ denote a compact set and let $\phi : \mathbb{R} \to \mathbb{R}^K$ denote a continuous function. Moreover, assume that there exist $\underline{\gamma}$ and $\overline{\gamma}$ such that $\phi(\underline{\gamma}) \in \mathcal{X}$ and ϕ restricted to $\{\overline{\gamma}\} + \mathbb{R}_+$ is strictly increasing. Then the set*

$$\mathcal{G} = \{\gamma \in \mathbb{R} : \phi(\gamma) \in \mathcal{X}\}$$

is closed, nonempty, and bounded from above. Furthermore,

$$\phi(\gamma^*) \in \mathrm{bd}\,(\mathcal{X}),$$

where

$$\gamma^* = \max\{\gamma \in \mathbb{R} : \phi(\gamma) \in \mathcal{X}\}.$$

Proof. The function ϕ is continuous and the set $\mathcal{X}$ is closed, thus the reverse image $\mathcal{G} = \phi^{-1}(\mathcal{X})$ is closed. Obviously, $\left\{\underline{\gamma}\right\} \subseteq \mathcal{G}$. From ϕ restricted to $\overline{\gamma} + \mathbb{R}_+$ being strictly increasing it follows that

$$\gamma' > \gamma \implies \left\|\phi(\gamma')\right\|_2 > \|\phi(\gamma)\|_2, \forall \gamma \geq \overline{\gamma}. \tag{A.9}$$

The set $\mathcal{X}$ is bounded, thus (A.9) implies that $\mathcal{G}$ is bounded from above. Assume that $\phi(\gamma^*) \notin \mathrm{bd}\,(\mathcal{X})$. This implies there exists $\eta > 0$ such that

$$\mathcal{B}_\eta(\phi(\gamma^*)) \in \mathcal{X},$$

which by continuity of ϕ imlies there exists $\epsilon > 0$ such that

$$\phi(\gamma^* + \epsilon) \in \mathcal{X},$$

which contradicts the definition of γ^*. □

Proposition A.3.6. *Let $\mathcal{X}$ denote a nonempty compact subset of $\mathbb{R}_+^K$. Let $\boldsymbol{x} \in \mathcal{X}$. Then there exists $\boldsymbol{x}^* \in \mathrm{par}\,(\mathcal{X})$ with*

$$\boldsymbol{x}^* \geq \boldsymbol{x}.$$

Proof. Let $\boldsymbol{x}^{(0)} = \boldsymbol{x}$. Define a sequence $\left\{\boldsymbol{x}^{(n)}\right\}_{n=0}^{\infty}$ as follows: If $\boldsymbol{x}^{(n)} \in \mathrm{par}\,(\mathcal{X})$, set

$$\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)}. \tag{A.10}$$

If $\boldsymbol{x}^{(n)} \notin \mathrm{par}\,(\mathcal{X})$, there exists $\boldsymbol{x}' \in \mathcal{X}$ such that $\boldsymbol{x}' > \boldsymbol{x}^{(n)}$. Let

$$\boldsymbol{v} = \frac{\boldsymbol{x}' - \boldsymbol{x}^{(n)}}{\left\|\boldsymbol{x}' - \boldsymbol{x}^{(n)}\right\|_2},$$

and define

$$\gamma^{(n)} = \max_{\gamma \in \mathbb{R}} \gamma \quad \text{s.t.} \quad \boldsymbol{x}^{(n)} + \gamma \boldsymbol{v} \in \mathcal{X}.$$

Set

$$\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} + \gamma^{(n)} \boldsymbol{v}. \tag{A.11}$$

By Proposition A.3.5 $\gamma^{(n)}$ is well-defined and $\boldsymbol{x}^{(n+1)} \in \mathcal{X}$. Moreover, $\boldsymbol{x}' \in \mathcal{X}$ and $\boldsymbol{x}' > \boldsymbol{x}^{(n)}$ implies that

$$\gamma^{(n)} \geq \left\| \boldsymbol{x}' - \boldsymbol{x}^{(n)} \right\|_2 > 0.$$

From (A.10) and (A.11), it follows that

$$\boldsymbol{x}^{(n+1)} \geq \boldsymbol{x}^{(n)}. \tag{A.12}$$

This implies

$$\left\| \boldsymbol{x}^{(n+k)} - \boldsymbol{x}^{(n)} \right\|_2 \geq \left\| \boldsymbol{x}^{(n+1)} - \boldsymbol{x}^{(n)} \right\|_2, \forall k \geq 1. \tag{A.13}$$

The set $\mathcal{X}$ is compact, thus the sequence $\left\{\boldsymbol{x}^{(n)}\right\}_{n=0}^{\infty}$ has a convergent subsequence with limit point $\boldsymbol{x}^* \in \mathcal{X}$. By (A.12),

$$\boldsymbol{x}^* \geq \boldsymbol{x}^{(0)} = \boldsymbol{x}.$$

Now, if $\boldsymbol{x}^{(n)} \notin \text{par}\,(\mathcal{X})$,

$$\left\| \boldsymbol{x}^{(n+1)} - \boldsymbol{x}^{(n)} \right\|_2 = \gamma^{(n)} > 0.$$

Thus,

$$\left\| \boldsymbol{x}^{(n+k)} - \boldsymbol{x}^{(n)} \right\|_2 \geq \gamma^{(n)}, \forall k \geq 1.$$

As a consequence, no point in $\mathcal{X} \setminus \text{par}\,(\mathcal{X})$ can be a limit point, implying that $\boldsymbol{x}^* \in \text{par}\,(\mathcal{X})$. □

A.4 Comprehensive Sets

Definition A.4.1 (Comprehensive Set). A subset $\mathcal{X}$ of $\mathbb{R}_+^K$ is comprehensive if

$$\boldsymbol{x} \in \mathcal{X}, \boldsymbol{x}' \in \mathbb{R}_+^K, \boldsymbol{x}' \leq \boldsymbol{x} \implies \boldsymbol{x}' \in \mathcal{X}.$$

Corollary A.4.1. *Let $\mathcal{X}$ be a nonempty comprehensive set. Then $\mathbf{0} \in \mathcal{X}$.*

Definition A.4.2 (Comprehensive Hull). The comprehensive hull of a set $\mathcal{X} \subseteq \mathbb{R}_+^K$ is the set

$$\text{comp}\,(\mathcal{X}) = \bigcup_{\boldsymbol{x} \in \mathcal{X}} \left\{ \boldsymbol{y} \in \mathbb{R}_+^K : \boldsymbol{y} \leq \boldsymbol{x} \right\}.$$

Corollary A.4.2.

$$\operatorname{comp}(\mathcal{X} \cup \mathcal{Y}) = \operatorname{comp}(\mathcal{X}) \cup \operatorname{comp}(\mathcal{Y}). \tag{A.14}$$

Proposition A.4.3. *Let $\mathcal{X}$ denote a comprehensive subset of $\mathbb{R}_+^K$ and let $\phi : \mathbb{R} \to \mathbb{R}_+^K$ denote a continuous increasing function. Moreover, assume that there exists a pair $(\gamma', \boldsymbol{s}) \in \mathbb{R} \times \mathcal{X}$ such that $\phi(\gamma') \ll \boldsymbol{s}$. Then there exists $\epsilon > 0$ such that*

$$\phi(\gamma' + \epsilon) \in \mathcal{X}.$$

Proof. Let

$$\mathcal{K}_+(\boldsymbol{x}) = \{\boldsymbol{x}\} + \mathbb{R}_+^K.$$

Due to the fact that ϕ is increasing,

$$\phi(\gamma' + \epsilon) \in \mathcal{K}_+\left(\phi(\gamma')\right), \forall \epsilon \geq 0. \tag{A.15}$$

By continuity of ϕ, for all $\eta > 0$ there exists $\epsilon > 0$ such that

$$\phi(\gamma' + \epsilon) \in \mathcal{B}_\eta(\phi(\gamma')). \tag{A.16}$$

Combining (A.15) and (A.16), it follows that for all $\eta > 0$ there exists $\epsilon > 0$ such that

$$\phi(\gamma' + \epsilon) \in \mathcal{B}_\eta(\phi(\gamma')) \cap \mathcal{K}_+\left(\phi(\gamma')\right). \tag{A.17}$$

From the comprehensiveness of $\mathcal{X}$, $\boldsymbol{s} \in \mathcal{X}$, $\phi(\gamma') \in \mathbb{R}_+^K$, and $\phi(\gamma') \leq \boldsymbol{s}$, it follows that

$$\left\{\boldsymbol{r} \in \mathbb{R}_+^K : \phi(\gamma') \leq \boldsymbol{r} \leq \boldsymbol{s}\right\} \subseteq \mathcal{X}.$$

Moreover, $\phi(\gamma') \ll \boldsymbol{s}$ implies that there exists $\eta > 0$ such that

$$\mathcal{B}_\eta(\phi(\gamma')) \cap \mathcal{K}_+\left(\phi(\gamma')\right) \subseteq \mathcal{X}. \tag{A.18}$$

Combined with (A.17), (A.18) yields the result. □

Note that the existence of a pair $(\gamma', \boldsymbol{s}) \in \mathbb{R} \times \mathcal{X}$ with $\phi(\gamma') \ll \boldsymbol{s}$ does not imply $\mathcal{B}_\eta(\phi(\gamma')) \subset \mathcal{X}$. As an example, take $\phi(\gamma') = \mathbf{0}$ and $\boldsymbol{s} = \mathbf{1}$. Then $\mathcal{B}_1(\phi(\gamma')) \cap \mathcal{K}_+(\phi(\gamma')) \subset \mathcal{X}$, while $\mathcal{B}_1(\phi(\gamma')) \not\subset \mathcal{X}$. In other words, the property that ϕ is increasing is needed to establish the result.

Proposition A.4.4. *Let $\mathcal{X} \subset \mathbb{R}_+^K$ denote a compact comprehensive set and let $\phi : \mathbb{R} \to \mathbb{R}^K$ denote a continuous function. Moreover, assume that there exist $\underline{\gamma}$ and $\overline{\gamma}$ such that $\phi(\underline{\gamma}) \in \mathcal{X}$ and ϕ restricted to $\overline{\gamma} + \mathbb{R}_+$ is strictly increasing. Then*

$$\phi(\gamma^*) \in \operatorname{wpar}(\mathcal{X}),$$

where

$$\gamma^* = \max\{\gamma \in \mathbb{R} : \phi(\gamma) \in \mathcal{X}\}.$$

Proof. From Proposition A.3.5 it follows that γ^* is well-defined and $\phi(\gamma^*) \in \mathcal{X}$.

Assume that $\phi(\gamma^*) \notin \mathrm{wpar}(\mathcal{X})$. By definition of $\mathrm{wpar}(\mathcal{X})$, this implies that there exists $\boldsymbol{s} \in \mathcal{X}$ with $\boldsymbol{s} \gg \phi(\gamma^*)$. By Proposition A.4.3, this implies that there exists $\epsilon > 0$ such that $\phi(\gamma^* + \epsilon) \in \mathcal{X}$, which contradicts the definition of γ^*. □

Proposition A.4.5. *Let $\mathcal{X}$ be a compact comprehensive set. Then*

$$\mathcal{X} = \mathrm{comp}(\mathrm{wpar}(\mathcal{X})). \tag{A.19}$$

Proof. If $\mathcal{X}$ is empty, the equation trivially holds. Assume $\mathcal{X}$ is nonempty. Then $\mathcal{X}$ is a nonempty compact subset of $\mathbb{R}_+^K$, thus, by Corollary A.3.3, $\mathrm{wpar}(\mathcal{X})$ is nonempty.

The relation $\mathrm{comp}(\mathrm{wpar}(\mathcal{X})) \subseteq \mathcal{X}$ follows from $\mathrm{wpar}(\mathcal{X}) \subseteq \mathcal{X}$, thus $\mathrm{comp}(\mathrm{wpar}(\mathcal{X})) \subseteq \mathrm{comp}(\mathcal{X}) = \mathcal{X}$.

To show $\mathcal{X} \subseteq \mathrm{comp}(\mathrm{wpar}(\mathcal{X}))$, let $\boldsymbol{x} \in \mathcal{X}$. If $\boldsymbol{x} = \boldsymbol{0}$, then $\boldsymbol{x} \in \mathrm{comp}(\mathrm{wpar}(\mathcal{X}))$, as $\mathrm{comp}(\mathrm{wpar}(\mathcal{X}))$ is a nonempty comprehensive set. If $\boldsymbol{x} \neq \boldsymbol{0}$, then by Proposition A.4.4 $\gamma^*\boldsymbol{x} \in \mathrm{wpar}(\mathcal{X})$, with γ^* as defined in Proposition A.4.4. By definition of γ^*, $\boldsymbol{x} \leq \gamma^*\boldsymbol{x}$, thus $\boldsymbol{x} \in \mathrm{comp}(\mathrm{wpar}(\mathcal{X}))$. □

Proposition A.4.6. *Let $\mathcal{X} \subset \mathbb{R}_+^K$ be comprehensive. Then*

$$\mathrm{par}(\mathcal{X} \cap \mathcal{Z}_{\mathcal{D}}) \subseteq \mathrm{wpar}(\mathcal{X}),$$

where $\mathcal{D} \subset \{1, \ldots, K\}$ and

$$\mathcal{Z}_{\mathcal{D}} = \left\{\boldsymbol{x} \in \mathbb{R}_+^K : x_k = 0, k \in \mathcal{D}\right\} \cap \left\{\boldsymbol{x} \in \mathbb{R}_+^K : x_k > 0, k \in \{1, \ldots, K\} \setminus \mathcal{D}\right\}.$$

Proof. Let $\boldsymbol{x} \in \mathrm{par}(\mathcal{X} \cap \mathcal{Z}_{\mathcal{D}})$. Assume that $\boldsymbol{x} \notin \mathrm{wpar}(\mathcal{X})$. This implies there exists $\boldsymbol{y} \in \mathcal{X}$ such that $\boldsymbol{y} \gg \boldsymbol{x}$. Let

$$z_k = \begin{cases} 0, k \in \mathcal{D}, \\ y_k, k \notin \mathcal{D}. \end{cases}$$

By comprehensiveness of $\mathcal{X}$, $\boldsymbol{z} \in \mathcal{X}$. But $\boldsymbol{z} \in \mathcal{Z}_{\mathcal{D}}$ and $\boldsymbol{z} > \boldsymbol{y}$, which contradicts $\boldsymbol{x} \in \mathrm{par}(\mathcal{X} \cap \mathcal{Z}_{\mathcal{D}})$. □

A.5 Convex Analysis

Let $\mathcal{V}$ denote a real or complex vector space.

Definition A.5.1. A set $\mathcal{C} \subseteq \mathcal{V}$ is convex if

$$\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in \mathcal{C}, \ \forall(\boldsymbol{x}, \boldsymbol{y}, \alpha) \in \mathcal{C} \times \mathcal{C} \times (0, 1).$$

Definition A.5.2. Let $\mathcal{C} \subseteq \mathcal{V}$ be convex. A function $f : \mathcal{C} \to \mathbb{R}^M$, with $M \geq 1$, is concave if

$$\boldsymbol{f}\left(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}\right) \geq \alpha \boldsymbol{f}(\boldsymbol{x}) + (1-\alpha)\boldsymbol{f}(\boldsymbol{y}), \ \forall(\boldsymbol{x}, \boldsymbol{y}, \alpha) \in \mathcal{C} \times \mathcal{C} \times (0,1).$$

Proposition A.5.1. *Let $\mathcal{C} \subseteq \mathbb{R}^N$ be convex. A concave function $\boldsymbol{f} : \mathcal{C} \to \mathbb{R}^M$ is continuous.*

Proof. By concavity of $\boldsymbol{f} = (f_1, \ldots, f_M)$, the component functions $f_m : \mathbb{R}^N \to \mathbb{R}$ are concave. A concave function $f_m : \mathcal{C} \to \mathbb{R}$ is continuous [28], thus $\boldsymbol{f}$ is continuous. □

Proposition A.5.2. *Let $\boldsymbol{g} : \mathcal{C} \to \mathbb{R}^M$ and $\boldsymbol{h} : \mathcal{D} \to \mathbb{R}^N$ be concave functions. Moreover, let $\boldsymbol{h}$ be increasing and $\boldsymbol{g}(\mathcal{C}) \subseteq \mathcal{D}$. Then the composite function $\boldsymbol{h} \circ \boldsymbol{g}$ is concave.*

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$ and $\alpha \in (0,1)$. Due to the concavity of $\boldsymbol{g}$,

$$\boldsymbol{g}\left(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}\right) \geq \alpha \boldsymbol{g}(\boldsymbol{x}) + (1-\alpha)\boldsymbol{g}(\boldsymbol{y}). \tag{A.20}$$

Hence,

$$\begin{aligned}\boldsymbol{h}\left(\boldsymbol{g}\left(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}\right)\right) &\geq \boldsymbol{h}\left(\alpha \boldsymbol{g}(\boldsymbol{x}) + (1-\alpha)\boldsymbol{g}(\boldsymbol{y})\right) \\ &\geq \alpha \boldsymbol{h}(\boldsymbol{g}(\boldsymbol{x})) + (1-\alpha)\boldsymbol{h}(\boldsymbol{g}(\boldsymbol{y})),\end{aligned}$$

where the first inequality follows from (A.20) and the fact that $\boldsymbol{h}$ is increasing, while the second inequality follows from the concavity of $\boldsymbol{h}$. □

Definition A.5.3 (Convex Hull). Let $\mathcal{X} \subseteq \mathcal{V}$. The convex hull of $\mathcal{X}$ is the smallest convex set containing $\mathcal{X}$,

$$\operatorname{conv}(\mathcal{X}) = \cap \left\{\mathcal{S} \subseteq \mathcal{V} : \mathcal{S} \text{ is convex and } \mathcal{X} \subseteq \mathcal{S}\right\}.$$

Theorem A.5.3. *Let $\mathcal{X} \subset \mathbb{R}^K$ be compact. Then* $\operatorname{conv}(\mathcal{X})$ *is compact.*

Proof. [22, Theorem 1.4.3]. □

Proposition A.5.4. *Let $\mathcal{X} \subset \mathbb{R}^K$ be comprehensive. Then $\mathcal{C} =$* $\operatorname{conv}(\mathcal{X})$ *is comprehensive.*

Proof. By Carathéodory's Theorem [22], each point in $\mathcal{C}$ can be written as a convex combination of at most $K+1$ points in $\mathcal{X}$. Accordingly, for any $\boldsymbol{s} \in \mathcal{C}$ there exist coefficients $b_k \geq 0$ and points $\boldsymbol{s}^k \in \mathcal{X}$ such that $\sum_{k=1}^{K+1} b_k = 1$ and

$$\boldsymbol{s} = \sum_{k=1}^{K+1} b_k \boldsymbol{s}^k.$$

Let $\boldsymbol{s} \in \mathcal{C}$ and $\mathbf{0} \leq \boldsymbol{s}' \leq \boldsymbol{s}$. There exists $\boldsymbol{a} \in \mathbb{R}_+^K$ with $\boldsymbol{a} \leq \mathbf{1}$ such that $\boldsymbol{s}' = \boldsymbol{a}\boldsymbol{s}$. Hence,

$$\boldsymbol{s}' = \sum_{k=1}^{K+1} b_k \boldsymbol{a}\boldsymbol{s}^k.$$

By comprehensiveness of $\mathcal{X}$, $\boldsymbol{a}\boldsymbol{s}^k \in \mathcal{X}$, thus $\boldsymbol{s}' \in \mathcal{C}$. □

Proposition A.5.5. *Let $\boldsymbol{x} \in \mathrm{bd}\,(\mathcal{C})$, where $\mathcal{C}$ is a nonempty compact convex subset of $\mathbb{R}^K$. Then there exists a hyperplane supporting $\mathcal{C}$ at $\boldsymbol{x}$, i.e.,*

$$\exists \boldsymbol{\lambda} \in \mathbb{R}^K \setminus \{\mathbf{0}\} : \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{x} = \max_{\boldsymbol{y} \in \mathcal{C}} \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{y}.$$

Proof. [22, Chap. III, Lemma 4.2.1]. □

Proposition A.5.6. *Let $\boldsymbol{x} \in \mathrm{wpar}\,(\mathcal{C})$, where $\mathcal{C}$ is a nonempty compact convex subset of $\mathbb{R}_+^K$. Then*

$$\exists \boldsymbol{\lambda} \in \mathbb{R}_+^K \setminus \{\mathbf{0}\} : \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{x} = \max_{\boldsymbol{y} \in \mathcal{C}} \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{y}.$$

Proof. Define a set $\mathcal{A} = \mathrm{comp}\,(\mathcal{C} + \{\mathbf{1}\})$. The set $\mathcal{A}$ is convex and comprehensive. Let $\boldsymbol{x} \in \mathrm{wpar}\,(\mathcal{C})$. Define $\boldsymbol{v} = \boldsymbol{x} + \mathbf{1}$. Clearly, $\boldsymbol{v} \in \mathbb{R}_{++}^K$. Moreover, $\boldsymbol{v} \in \mathrm{wpar}\,(\mathcal{A})$, thus $\boldsymbol{v} \in \mathrm{bd}\,(\mathcal{A})$. By Proposition A.5.5 there exists $\boldsymbol{\lambda} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$ such that

$$\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{v} = \max_{\boldsymbol{y} \in \mathcal{A}} \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{y}. \tag{A.21}$$

Define $\boldsymbol{z} \in \mathbb{R}^K$ as follows:

$$z_k = \begin{cases} v_k, \lambda_k \geq 0, \\ 0, \lambda_k < 0. \end{cases} \tag{A.22}$$

Observe that $\mathbf{0} \leq \boldsymbol{z} \leq \boldsymbol{v}$, $\boldsymbol{v} \in \mathcal{A}$, and $\mathcal{A}$ is comprehensive, hence $\boldsymbol{z} \in \mathcal{A}$. Assume that there exists k such that $\lambda_k < 0$. Then $\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{v} < \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{z}$, which contradicts (A.21). Thus, $\boldsymbol{\lambda} \in \mathbb{R}_+^K \setminus \{\mathbf{0}\}$. Moreover,

$$\begin{aligned} \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{1} &= \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{v} \\ &= \max_{\boldsymbol{y} \in \mathcal{A}} \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{y} \\ &\geq \max_{\boldsymbol{y} \in (\mathcal{C}+\{\mathbf{1}\})} \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{y} \\ &= \max_{\boldsymbol{y} \in \mathcal{C}} \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{1}. \end{aligned}$$

Subtracting $\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{1}$ on both sides and noting that $\boldsymbol{x} \in \mathcal{C}$ yields the result. □

Proposition A.5.7. *Let $\mathcal{X}$ denote a compact subset of $\mathbb{R}^K$. Any $\boldsymbol{x} \in \mathrm{bd}\,(\mathrm{conv}\,(\mathcal{X}))$ can be represented as a convex combination of at most K elements of $\mathcal{X}$.*

Proof. [22, Chap. III, Proposition 4.2.3]. □

Proposition A.5.8. *Let $\mathcal{C}$ denote a convex compact set. Let $\boldsymbol{s}$ be a convex combination of W elements of $\mathcal{C}$, i.e.,*

$$\boldsymbol{s} = \sum_{w=1}^{W} \alpha_w \boldsymbol{s}^{(w)}, \tag{A.23}$$

with $\boldsymbol{s}^{(w)} \in \mathcal{C}$, $\alpha_w > 0$, and $\sum_{w=1}^{W} \alpha_w = 1$. Moreover, assume that $\boldsymbol{s}$ maximizes $\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{y}$ over $\mathcal{C}$, i.e.,

$$\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s} = \max_{\boldsymbol{y}\in\mathcal{C}} \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{y}. \tag{A.24}$$

Then

$$\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s} = \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s}^{(w)}, \forall w = 1, \ldots, W.$$

Proof. First note that $\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s} < \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s}^{(w)}$ contradicts (A.24). Next, assume that there exist $q, r \in \{1, \ldots, W\}$ such that $\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s}^{(q)} < \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s}^{(r)}$. Let

$$\boldsymbol{s}' = \sum_{\substack{w=1\\ w\neq q}}^{W} \alpha_w \boldsymbol{s}^{(w)} + \alpha_q \boldsymbol{s}^{(r)}.$$

Then

$$\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s} < \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s}'.$$

This again contradicts (A.24), as, by convexity of $\mathcal{C}$, $\boldsymbol{s}' \in \mathcal{C}$. Thus,

$$\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s}^{(q)} = \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s}^{(r)}$$

and

$$\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s} = \sum_{w=1}^{W} \alpha_w \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s}^{(r)} = \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s}^{(r)} \sum_{w=1}^{W} \alpha_w = \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{s}^{(r)}.$$ □

Corollary A.5.9. *Let $\mathcal{R}$ be a nonempty compact subset of $\mathbb{R}_+^K$. Let $\mathcal{C} = \mathrm{conv}\,(\mathcal{R})$ and $\boldsymbol{\lambda} \in \mathbb{R}_+^K$. Then*

$$\max_{\boldsymbol{r}\in\mathcal{R}} \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{r} = \max_{\boldsymbol{r}\in\mathcal{C}} \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{r},$$

$$\operatorname*{argmax}_{\boldsymbol{r}\in\mathcal{R}} \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{r} \subseteq \operatorname*{argmax}_{\boldsymbol{r}\in\mathcal{C}} \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{r}.$$

Proposition A.5.10. *Let $\mathcal{R} \subset \mathbb{R}_+^K$ be nonconvex, compact, and comprehensive. Let $\mathcal{C} = \operatorname{conv}(\mathcal{R})$. Then there exists a weakly Pareto optimal point of $\mathcal{C}$ that is not in $\mathcal{R}$, i.e.,*

$$\operatorname{wpar}(\mathcal{C}) \setminus (\operatorname{wpar}(\mathcal{C}) \cap \mathcal{R}) \neq \emptyset.$$

Proof. By nonconvexity of $\mathcal{R}$, $\mathcal{R} \subset \mathcal{C}$. The set $\mathcal{C}$ is compact and comprehensive, thus, by Proposition A.4.5, $\mathcal{C} = \operatorname{comp}(\operatorname{wpar}(\mathcal{C}))$. Using the equality

$$\operatorname{wpar}(\mathcal{C}) = (\operatorname{wpar}(\mathcal{C}) \cap \mathcal{R}) \cup (\operatorname{wpar}(\mathcal{C}) \setminus (\operatorname{wpar}(\mathcal{C}) \cap \mathcal{R})) \tag{A.25}$$

yields

$$\mathcal{C} = \operatorname{comp}(\operatorname{wpar}(\mathcal{C}) \cap \mathcal{R}) \cup \operatorname{comp}(\operatorname{wpar}(\mathcal{C}) \setminus (\operatorname{wpar}(\mathcal{C}) \cap \mathcal{R}))\,. \tag{A.26}$$

But $\operatorname{comp}(\operatorname{wpar}(\mathcal{C}) \cap \mathcal{R}) \subseteq \operatorname{comp}(\mathcal{R}) = \mathcal{R}$. As a result,

$$\mathcal{R} \subset \mathcal{C} \subseteq \mathcal{R} \cup \operatorname{comp}(\operatorname{wpar}(\mathcal{C}) \setminus (\operatorname{wpar}(\mathcal{C}) \cap \mathcal{R}))\,. \tag{A.27}$$

Thus, $\operatorname{wpar}(\mathcal{C}) \setminus (\operatorname{wpar}(\mathcal{C}) \cap \mathcal{R}) \neq \emptyset$. □

A.6 The Set of Extended Real Numbers

The set of extended real numbers is obtained by adding the elements "$-\infty$" and "∞" to the set of real numbers:

$$\tilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}\,.$$

The following rules apply:

$$\begin{aligned}
-\infty \leq x \leq \infty,\ \forall x \in \mathbb{R} \cup \{-\infty\} \cup \{\infty\}\,,\\
-1 \cdot (-\infty) = \infty,\\
x + (-\infty) = -\infty,\ \forall x \in \mathbb{R} \cup \{-\infty\}\,,\\
x + \infty = \infty,\ \forall x \in \mathbb{R} \cup \{\infty\}\,,\\
x \cdot (-\infty) = -\infty,\ \forall x \in \mathbb{R}_{++} \cup \{\infty\}\,,\\
x \cdot \infty = \infty,\ \forall x \in \mathbb{R}_{++} \cup \{\infty\}\,.
\end{aligned}$$

Using the set of extended real numbers, it is possible to assign a value to the infimum and supremum of an empty set:

$$\inf \emptyset = \infty,$$
$$\sup \emptyset = -\infty.$$

A.7 Optimality Conditions

Definition A.7.1. Let $\mathcal{X}$ be a nonempty set, and $f : \mathcal{X} \to \mathbb{R}$ a function. Let $\boldsymbol{x}^* \in \mathcal{X}$. If

$$f(\boldsymbol{x}^*) \geq f(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathcal{X}$$

then $f(\boldsymbol{x}^*)$ is the *global maximum* in $f(\mathcal{X})$, and $\boldsymbol{x}^*$ is a *global maximizer* of f in $\mathcal{X}$.

Definition A.7.2. Let $\mathcal{X}$ be a nonempty set, and $f : \mathcal{X} \to \mathbb{R}$ a function. Let $\boldsymbol{x}^* \in \mathcal{X}$. If

$$f(\boldsymbol{x}^*) + \epsilon \geq f(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathcal{X}$$

then $f(\boldsymbol{x}^*)$ is a *global ϵ-maximum* in $f(\mathcal{X})$, and $\boldsymbol{x}^*$ is a *global ϵ-maximizer* of f in $\mathcal{X}$.

Definition A.7.3. Let $\mathcal{D}$ be a metric space, $f : \mathcal{D} \to \mathbb{R}$, and $\mathcal{X}$ a nonempty subset of $\mathcal{D}$. Let $\boldsymbol{x}^* \in \mathcal{X}$. If there exists $\epsilon > 0$ such that

$$f(\boldsymbol{x}^*) \geq f(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathcal{B}_\epsilon(\boldsymbol{x}^*) \cap \mathcal{X}$$

then $f(\boldsymbol{x}^*)$ is a *local maximum* in $f(\mathcal{X})$ and $\boldsymbol{x}^*$ a *local maximizer* of f in $\mathcal{X}$.

Definition A.7.4. Let $\mathcal{X}$ be a subset of $\mathbb{R}^K$. A vector $\boldsymbol{v} \in \mathbb{R}^K$ is a *tangent direction* of $\mathcal{X}$ at $\boldsymbol{x}$ if there exists a sequence $\{\boldsymbol{x}^{(n)}\}$, with $\boldsymbol{x}^{(n)} \in \mathcal{X}$, and a sequence $\{t^{(n)}\}$, $t^{(n)} \in \mathbb{R}_{++}$, such that

$$\lim_{n\to\infty} \boldsymbol{x}^{(n)} = \boldsymbol{x}, \tag{A.28}$$

$$\lim_{n\to\infty} t^{(n)} = 0, \tag{A.29}$$

$$\lim_{n\to\infty} \frac{\boldsymbol{x}^{(n)} - \boldsymbol{x}}{t^{(n)}} = \boldsymbol{v}. \tag{A.30}$$

The set of all tangent directions of $\mathcal{X}$ at $\boldsymbol{x}$ is the *tangent cone* of $\mathcal{X}$ at $\boldsymbol{x}$, denoted by $\mathcal{T}_\mathcal{X}(\boldsymbol{x})$.

Corollary A.7.1. *Let $\mathcal{S} = \mathcal{X} \cap \mathcal{Y}$. Then*

$$\mathcal{T}_\mathcal{S}(\boldsymbol{x}) \subseteq \mathcal{T}_\mathcal{X}(\boldsymbol{x}) \cap \mathcal{T}_\mathcal{Y}(\boldsymbol{x}).$$

Proposition A.7.2. *Let $\mathcal{X} \subset \mathbb{R}^K$. Then the vectors in the tangent cone of the weak Pareto set of $\mathcal{X}$ at $\boldsymbol{x}$ are neither strictly positive nor strictly negative:*

$$\mathcal{T}_{\mathrm{wpar}(\mathcal{X})}(\boldsymbol{x}) \subset \mathbb{R}^K \setminus (\mathbb{R}^K_{++} \cup -\mathbb{R}^K_{++}), \forall \boldsymbol{x} \in \mathbb{R}^K.$$

Proof. Assume that $\mathcal{T}_{\mathrm{wpar}(\mathcal{X})}(\boldsymbol{x})$ is nonempty, as otherwise the result is trivial. Let $\boldsymbol{v} \in \mathbb{R}^K$ be a tangent direction of wpar $(\mathcal{X})$ at $\boldsymbol{x}$. Then there exists a sequence $\{\boldsymbol{x}^{(n)}\}$, with $\boldsymbol{x}^{(n)} \in$ wpar $(\mathcal{X})$, and a sequence $\{t^{(n)}\}$, $t^{(n)} \in \mathbb{R}_{++}$, such that

$$\lim_{n\to\infty} \boldsymbol{x}^{(n)} = \boldsymbol{x},$$
$$\lim_{n\to\infty} t^{(n)} = 0,$$
$$\lim_{n\to\infty} \frac{\boldsymbol{x}^{(n)} - \boldsymbol{x}}{t^{(n)}} = \boldsymbol{v}.$$

The assumption $\boldsymbol{v} \in \mathbb{R}^K_{++}$ implies that there exists n' such that $\boldsymbol{x}^{(n)} \gg \boldsymbol{x}, \forall n \geq n'$. This contradicts $\boldsymbol{x} \in$ wpar $(\mathcal{X})$. Likewise, the assumption $\boldsymbol{v} \in -\mathbb{R}^K_{++}$ implies that there exists n' such that $\boldsymbol{x}^{(n)} \ll \boldsymbol{x}, \forall n \geq n'$. This contradicts $\boldsymbol{x}^{(n)} \in$ wpar $(\mathcal{X})$. □

Theorem A.7.3. *Let $\mathcal{X}$ be a nonempty subset of $\mathbb{R}^K$, $f : \mathcal{D} \to \mathbb{R}$, with $\mathcal{X} \subseteq \mathcal{D}$. Assume that f is differentiable at $\boldsymbol{x}^*$. If $\boldsymbol{x}^*$ is a local maximizer of f in $\mathcal{X}$ then*

$$\nabla f(\boldsymbol{x}^*)^{\mathrm{T}} \boldsymbol{v} \leq 0, \forall \boldsymbol{v} \in \mathcal{T}_{\mathcal{X}}(\boldsymbol{x}^*). \tag{A.31}$$

Proof. [25, Theorem 5.1.2]. □

Definition A.7.5. Let $\boldsymbol{x}^* \in \mathcal{X}$ be a point satisfying (A.31). Then $\boldsymbol{x}^*$ is a *stationary point* in $\mathcal{X}$.

A.8 Differentiable Manifolds

The local method developed in Sect. 3.5 makes use of a number of concepts from differential geometry, namely differentiable manifolds and parameterizations. Within the scope of this monograph, it is sufficient to consider manifolds that are topological subspaces of $\mathbb{R}^M$. Accordingly, in the following, $\mathcal{M}$ always denotes a topological subspace of $\mathbb{R}^M$.

Definition A.8.1 (Local Parameterization). Let $\mathcal{X}$ be a nonempty open subset of $\mathbb{R}^N$ and $\phi : \mathcal{X} \to \mathcal{M}$ a homeomorphism. Then $(\mathcal{X}, \phi)$ is a local parameterization of $\mathcal{M}$.

Definition A.8.2 (Global Parameterization). Let $(\mathcal{X}, \phi)$ be a local parameterization of $\mathcal{M}$. If $\phi(\mathcal{X}) = \mathcal{M}$, then $(\mathcal{X}, \phi)$ is a global parameterization of $\mathcal{M}$.

Definition A.8.3 (Manifold). Let $\{(\mathcal{X}_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{I}}$ be a collection of local parameterizations of $\mathcal{M}$ such that

$$\mathcal{M} = \bigcup_{\alpha \in \mathcal{I}} \phi_\alpha(\mathcal{X}_\alpha),$$

where $\mathcal{X}_\alpha \subseteq \mathbb{R}^N$. Then $\mathcal{M}$ is a topological manifold of dimension N.

Note that the general definition of a manifold requires that $\mathcal{M}$ be Hausdorff and second-countable [66]. These two properties are automatically fulfilled by any topological subspace of $\mathbb{R}^M$, and are therefore omitted in Definition A.8.3.

Definition A.8.4. A function $\boldsymbol{f} : \mathcal{X} \to \mathbb{R}^M$, with $\mathcal{X} \subseteq \mathbb{R}^N$ open, is of class C^1 if the partial derivatives $\frac{\partial f_m}{\partial x_n} : \mathcal{X} \to \mathbb{R}$ are continuous. The function $\boldsymbol{f}$ is of class C^r, $r \geq 2$, if the partial derivatives $\frac{\partial f_m}{\partial x_n} : \mathcal{X} \to \mathbb{R}$ are of class C^{r-1}.

Definition A.8.5. Two local parameterizations $(\mathcal{X}_\alpha, \phi_\alpha)$ and $(\mathcal{X}_\beta, \phi_\beta)$ are C^r-compatible if $\phi_\alpha(\mathcal{X}_\alpha) \cap \phi_\beta(\mathcal{X}_\beta) \neq \emptyset$ implies that $\phi_\alpha^{-1} \circ \phi_\beta$ is of class C^r.

Definition A.8.6. Let $\mathcal{M}$ be a topological manifold, and let $\mathcal{P} = \{(\mathcal{X}_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{I}}$ be a collection of local parameterizations that covers $\mathcal{M}$. Moreover, assume that all elements of $\mathcal{P}$ are pairwise C^r compatible. A C^r structure on $\mathcal{M}$ is a collection $\mathcal{D}$ of local parameterizations of $\mathcal{M}$ such that:

1. $\mathcal{D} \supseteq \mathcal{P}$,
2. If $(\mathcal{X}, \phi)$ is a local parameterization of $\mathcal{M}$ that is C^r compatible with all elements of $\mathcal{D}$, then $(\mathcal{X}, \phi)$ is in $\mathcal{D}$.

The C^r structure $\mathcal{D}$ is obtained by adding all C^r compatible parameterizations to $\mathcal{P}$. Accordingly, $\mathcal{D}$ is said to be *generated* by $\mathcal{P}$.

Definition A.8.7 (Differentiable Manifold). Let $\mathcal{M}$ be a topological manifold, and let $\mathcal{D}$ be a C^r structure on $\mathcal{M}$. Then $(\mathcal{M}, \mathcal{D})$ is a differentiable manifold of class C^r.

One of the simplest examples of a differentiable manifold of class C^r is $\left(\mathbb{R}^M, \mathcal{D}\right)$, where $\mathcal{D}$ is a C^r structure generated by $\mathcal{P} = \left\{(\mathbb{R}^M, \mathrm{id})\right\}$.

Lemma A.8.1. *Suppose that $\mathcal{D}$ is a C^r structure on $\mathbb{R}^M$ satisfying $(\mathbb{R}^M, \mathrm{id}) \in \mathcal{D}$. Let $(\mathcal{X}, \phi) \in \mathcal{D}$. Then ϕ is of class C^r.*

Proof. The parameterization $(\mathcal{X}, \phi)$ is C^r compatible with $(\mathbb{R}^M, \mathrm{id})$. By definition, $\mathcal{X}$ is nonempty, hence $\mathcal{X} \cap \mathbb{R}^M \neq \emptyset$, implying that $\mathrm{id}^{-1} \circ \phi = \phi$ is of class C^r. □

In the context of this monograph, parameterizations $(\mathcal{X}, \phi)$ with ϕ of class C^1 are of particular importance, cf. Sect. 3.5. However, the assumption that $(\mathcal{M}, \mathcal{D})$ is a manifold of class C^1 is not sufficient for a parameterization of $\mathcal{M}$ to be of class C^1. As an example, let $(\mathcal{X}, \phi)$ be a global parameterization of $\mathcal{M}$. Then $\{(\mathcal{X}, \phi)\}$ is a collection that covers $\mathcal{M}$ and $\phi^{-1} \circ \phi$ is of class C^∞, hence $(\mathcal{M}, \mathcal{D})$, with $\mathcal{D} \supseteq \{(\mathcal{X}, \phi)\}$, is a differentiable manifold, regardless of the differentiability class of ϕ. A sufficient condition for the existence of a local parameterization such that ϕ is of class C^1 is provided by Proposition A.8.2. Proposition A.8.2 relies on the concept of a regular submanifold, which is developed next.

Definition A.8.8. A subset $\mathcal{N}$ of a C^r manifold $(\mathcal{M}, \mathcal{D})$ has the *N-submanifold property* if for each $\boldsymbol{y} \in \mathcal{N}$ there exists a parameterization $(\mathcal{X}, \boldsymbol{\phi}) \in \mathcal{D}$ such that $\boldsymbol{y} \in \boldsymbol{\phi}(\mathcal{X})$ and

$$\boldsymbol{\phi}^{-1}(\mathcal{N} \cap \boldsymbol{\phi}(\mathcal{X})) \subseteq \mathbb{R}^N \times \{\mathbf{0}\}. \tag{A.32}$$

Parameterizations that fulfill these requirements are called *preferred* parameterizations (relative to $\mathcal{N}$).

If a set $\mathcal{N}$ has the N-submanifold property, it can be parameterized by using the first N coordinates of the preferred parameterizations—according to (A.32), the remaining $M - N$ coordinates are identically to zero, hence do not carry any information. Intuitively, a local parameterization of $\mathcal{N}$ can be obtained by removing the coordinates that are identical to zero. In order to do so, let $M > N$ and define maps $\boldsymbol{\pi}$ and $\boldsymbol{\theta}$ as follows:

$$\boldsymbol{\pi} : \mathbb{R}^M \to \mathbb{R}^N, (x_1, \ldots, x_M) \mapsto (x_1, \ldots, x_N),$$
$$\boldsymbol{\theta} : \mathbb{R}^N \to \mathbb{R}^M, \boldsymbol{y} \mapsto (\boldsymbol{y}, \mathbf{0}).$$

While $\boldsymbol{\pi}$ projects on the first N coordinates of $\boldsymbol{x} \in \mathbb{R}^M$, $\boldsymbol{\theta}$ appends $M - N$ zeros to $\boldsymbol{y} \in \mathbb{R}^N$. Suppose that $\mathcal{N}$ is a topological space that has the N-submanifold property. Let $(\mathcal{X}, \boldsymbol{\phi})$ be a preferred parameterization relative to $\mathcal{N}$. Then $\boldsymbol{\psi} : \mathcal{Y} \to \mathcal{N}$, with

$$\mathcal{Y} = \boldsymbol{\pi}(\boldsymbol{\phi}^{-1}(\mathcal{N} \cap \boldsymbol{\phi}(\mathcal{X}))), \tag{A.33}$$

$$\boldsymbol{\psi}(\boldsymbol{y}) = \boldsymbol{\phi}(\boldsymbol{\theta}(\boldsymbol{y})), \tag{A.34}$$

is a homeomorphism, provided that $\mathcal{N} \cap \boldsymbol{\phi}(\mathcal{X})$ is an open set in the topology of $\mathcal{N}$. The set $\boldsymbol{\phi}(\mathcal{X})$ is open in $\mathcal{M}$, hence $\mathcal{N} \cap \boldsymbol{\phi}(\mathcal{X})$ is open in $\mathcal{N}$ if $\mathcal{N}$ is equipped with the subspace topology. As a result, if $\mathcal{N}$ has the subspace topology then $(\mathcal{Y}, \boldsymbol{\psi})$ is a local parameterization of $\mathcal{N}$. Let $\mathcal{P}$ be the collection of local parameterizations obtained from all the preferred parameterizations. By Definition A.8.8, $\mathcal{P}$ covers $\mathcal{N}$, hence $\mathcal{N}$ equipped with the subspace topology is a topological manifold of dimension N. Moreover, the parameterizations in $\mathcal{P}$ are pairwise C^r compatible. Let $(\mathcal{Y}_\alpha, \boldsymbol{\psi}_\alpha), (\mathcal{Y}_\beta, \boldsymbol{\psi}_\beta) \in \mathcal{P}$ such that $\boldsymbol{\psi}_\alpha(\mathcal{Y}_\alpha) \cap \boldsymbol{\psi}_\beta(\mathcal{Y}_\beta) \neq \emptyset$. Then

$$\boldsymbol{\psi}_\alpha^{-1} \circ \boldsymbol{\psi}_\beta = \boldsymbol{\pi} \circ \boldsymbol{\phi}_\alpha^{-1} \circ \boldsymbol{\phi}_\beta \circ \boldsymbol{\theta},$$

where $(\mathcal{X}_\alpha, \boldsymbol{\phi}_\alpha), (\mathcal{X}_\beta, \boldsymbol{\phi}_\beta) \in \mathcal{D}$ are the corresponding preferred parameterizations. But $\boldsymbol{\phi}_\alpha^{-1} \circ \boldsymbol{\phi}_\beta$ is C^r, and $\boldsymbol{\pi}$ and $\boldsymbol{\theta}$ are C^∞, hence $\boldsymbol{\psi}_\alpha^{-1} \circ \boldsymbol{\psi}_\beta$ is of class C^r. By Definition A.8.6, $\mathcal{P}$ generates a C^r structure on $\mathcal{N}$. In the following, let $\mathcal{F}$ denote the C^r structure generated by $\mathcal{P}$.

Definition A.8.9 (Regular Submanifold). Let $(\mathcal{M}, \mathcal{D})$ be a differentiable manifold of class C^r. Suppose $\mathcal{N} \subset M$ is equipped with the subspace topology and has the N-submanifold property. Then $(\mathcal{N}, \mathcal{F})$ is a *regular submanifold* of $(\mathcal{M}, \mathcal{D})$.

Proposition A.8.2. *Let $(\mathcal{N}, \mathcal{F})$ be a regular submanifold of $(\mathbb{R}^M, \mathcal{D})$, where $\mathcal{D}$ is a C^r structure satisfying $(\mathbb{R}^M, \mathrm{id}) \in \mathcal{D}$. For each $\boldsymbol{s} \in \mathcal{N}$ there exists a local parameterization $(\mathcal{Y}, \psi) \in \mathcal{F}$ such that $\boldsymbol{s} \in \psi(\mathcal{Y})$ and ψ is of class C^r.*

Proof. By Definition A.8.9, for each $\boldsymbol{s} \in \mathcal{N}$ there exists a preferred parameterization $(\mathcal{X}, \phi) \in \mathcal{D}$ such that $\boldsymbol{s} \in \phi(\mathcal{X})$. By (A.33) and (A.34), $(\mathcal{X}, \phi)$ induces a local parameterization $(\mathcal{Y}, \psi)$ of $\mathcal{N}$, with $\psi(\boldsymbol{y}) = \phi(\boldsymbol{\theta}(\boldsymbol{y}))$. By Lemma A.8.1, ϕ is of class C^r, while $\boldsymbol{\theta}$ is of class C^∞, hence the composition of ϕ and $\boldsymbol{\theta}$ is of class C^r. Finally, $\psi(\mathcal{Y}) = \mathcal{N} \cap \phi(\mathcal{X})$, thus $\boldsymbol{s} \in \psi(\mathcal{Y})$. □

By Proposition A.8.2, regular submanifolds of $(\mathbb{R}^M, \mathcal{D})$, where $\mathcal{D}$ is a C^r structure satisfying $(\mathbb{R}^M, \mathrm{id}) \in \mathcal{D}$, have the attractive property that they have parameterizations of class C^r. Due to this property, such manifolds play an important role in the development of a local method in Sect. 3.5.

Definition A.8.10. Let $(\mathbb{R}^M, \mathcal{D})$ be a differentiable manifold of class C^r, with $(\mathbb{R}^M, \mathrm{id}) \in \mathcal{D}$. Suppose $(\mathcal{N}, \mathcal{F})$ is a regular submanifold of $(\mathbb{R}^M, \mathcal{D})$. Then $\mathcal{N}$ is denoted as C^r submanifold of $\mathbb{R}^M$.

Note that when saying that $\mathcal{N}$ is a C^r submanifold of $\mathbb{R}^M$, it is implied that the differentiable structures $\mathcal{D}$ and $\mathcal{F}$ are used, where $\mathcal{D}$ is a C^r structure on $\mathbb{R}^M$ satisfying $(\mathbb{R}^M, \mathrm{id}) \in \mathcal{D}$, and $\mathcal{F}$ is the C^r structure on $\mathcal{N}$ obtained from the preferred parameterizations relative to $\mathcal{N}$.

Proposition A.8.3. *Let $\mathcal{N}$ be an N-dimensional manifold. Furthermore, suppose $\mathcal{N}$ is a C^r submanifold of $\mathbb{R}^M$. The tangent cone of $\mathcal{N}$ at $\boldsymbol{s} \in \mathcal{N}$ is an N-dimensional linear subspace of $\mathbb{R}^M$ and is given by*

$$\mathcal{T}_{\mathcal{N}}(\boldsymbol{s}) = \mathrm{range}\left(\nabla\phi(\boldsymbol{\mu})^{\mathrm{T}}\right),$$

where $(\mathcal{X}, \phi)$ is a local parameterization of $\mathcal{N}$ such that $\boldsymbol{s} = \phi(\boldsymbol{\mu})$ and ϕ is of class C^r.

Proof. Existence of a parameterization $(\mathcal{X}, \phi)$ such that $\boldsymbol{s} \in \phi(\mathcal{X})$ and ϕ is of class C^r follows from Proposition A.8.2. Let $\boldsymbol{\mu} = \phi^{-1}(\boldsymbol{s})$. Due to the fact that ϕ is of class C^r, $\phi(\boldsymbol{\mu}')$ can be written as

$$\phi(\boldsymbol{\mu}') = \phi(\boldsymbol{\mu}) + \nabla\phi(\boldsymbol{\mu})^{\mathrm{T}}(\boldsymbol{\mu}' - \boldsymbol{\mu}) + \boldsymbol{z}(\boldsymbol{\mu}'), \tag{A.35}$$

where $\boldsymbol{z}$ is of class C^r with

$$\nabla\boldsymbol{z}(\boldsymbol{\mu}) = \boldsymbol{0}. \tag{A.36}$$

Let $\left\{t^{(n)}\right\}$ be a sequence such that $t^{(n)} \in \mathbb{R}_{++}$ and

$$\lim_{n\to\infty} t^{(n)} = 0.$$

Let

$$\boldsymbol{\mu}^{(n)} = \boldsymbol{\mu} + t^{(n)}\Delta\boldsymbol{\mu},$$

with $\Delta\boldsymbol{\mu} \in \mathbb{R}^N$. For n large enough $\boldsymbol{\mu}^{(n)} \in \mathcal{X}$. Moreover, by continuity of $\boldsymbol{\phi}$,

$$\lim_{n\to\infty} \boldsymbol{\phi}(\boldsymbol{\mu}^{(n)}) = \boldsymbol{\phi}(\boldsymbol{\mu}).$$

Plugging (A.35) into (A.30) yields

$$\nabla\boldsymbol{\phi}(\boldsymbol{\mu})^{\mathrm{T}}\Delta\boldsymbol{\mu} + \lim_{n\to\infty} \frac{\boldsymbol{z}(\boldsymbol{\mu}^{(n)})}{t^{(n)}} = \boldsymbol{v}. \tag{A.37}$$

By (A.36) and l'Hôpital's rule,

$$\lim_{n\to\infty} \frac{\boldsymbol{z}(\boldsymbol{\mu}^{(n)})}{t^{(n)}} = \mathbf{0}.$$

Moreover, (A.37) holds for any $\Delta\boldsymbol{\mu} \in \mathbb{R}^N$, hence

$$\mathcal{T}_{\mathcal{N}}(\boldsymbol{s}) = \operatorname{range}\left(\nabla\boldsymbol{\phi}(\boldsymbol{\mu})^{\mathrm{T}}\right).$$

The map $\boldsymbol{\phi}$ is a homeomorphism, hence $\operatorname{rank}\left(\nabla\boldsymbol{\phi}(\boldsymbol{\mu})^{\mathrm{T}}\right) = N$. □

Proposition A.8.4. *Let $\mathcal{N}$ be an $M-1$-dimensional C^r submanifold of $\mathbb{R}^M$. There exists a continuous map $\boldsymbol{n} : \mathcal{N} \to \mathbb{R}^M$ such that*

$$\|\boldsymbol{n}(\boldsymbol{s})\|_2 = 1,$$
$$\mathcal{T}_{\mathcal{N}}(\boldsymbol{s}) = \operatorname{null}\left(\boldsymbol{n}(\boldsymbol{s})^{\mathrm{T}}\right).$$

Proof. By Proposition A.8.3, $\mathcal{T}_{\mathcal{N}}(\boldsymbol{s})$ is an $M-1$ dimensional subspace of $\mathbb{R}^M$. Hence, at each $\boldsymbol{s} \in \mathcal{N}$, there exists $\boldsymbol{n} \in \mathbb{R}^M$ such that

$$\mathcal{T}_{\mathcal{N}}(\boldsymbol{s}) = \operatorname{null}\left(\boldsymbol{n}^{\mathrm{T}}\right).$$

Moreover, at each $\boldsymbol{s} \in \mathcal{N}$ there exists a local parameterization $(\mathcal{X}, \boldsymbol{\phi})$ such that

$$\operatorname{range}\left(\nabla\boldsymbol{\phi}(\boldsymbol{\phi}^{-1}(\boldsymbol{s}))^{\mathrm{T}}\right) = \operatorname{null}\left(\boldsymbol{n}^{\mathrm{T}}\right).$$

Equivalently, $\boldsymbol{n}$ is a nontrivial solution of the following system of equations:

$$\nabla\boldsymbol{\phi}(\boldsymbol{\phi}^{-1}(\boldsymbol{s}))\boldsymbol{n} = \mathbf{0}. \tag{A.38}$$

The parameterization $\boldsymbol{\phi}$ in Proposition A.8.3 is of class C^r, thus $\nabla\boldsymbol{\phi} \circ \boldsymbol{\phi}^{-1}$ is continuous. As a result, it is possible to find a continuous map $\boldsymbol{n} : \mathcal{N} \to \mathbb{R}^M$ such that $\boldsymbol{n}(\boldsymbol{s})$ is a nontrivial solution of (A.38). Moreover, if $\boldsymbol{n}$ is a solution of (A.38), then $\boldsymbol{n}/\|\boldsymbol{n}\|_2$ is also a solution. The map $\omega : \boldsymbol{x} \mapsto \boldsymbol{x}/\|\boldsymbol{x}\|_2$ is continuous on $\mathbb{R}^M \setminus \{\mathbf{0}\}$, hence $\boldsymbol{\omega} \circ \boldsymbol{n}$ is continuous. □

A.9 Yates' Framework for Power Control

Results from Yates' work [2] are repeatedly used throughout this monograph. For completeness, this appendix reproduces the relevant definitions and results from [2].

Definition A.9.1 (Standard Interference Function). A function $\boldsymbol{f} : \mathbb{R}_+^K \to \mathbb{R}_+^K$ is a *standard interference function* if it satisfies the following properties:

$$\boldsymbol{f}(\boldsymbol{p}) \gg \boldsymbol{0},$$
$$\boldsymbol{p} \geq \boldsymbol{p}' \implies \boldsymbol{f}(\boldsymbol{p}) \geq \boldsymbol{f}(\boldsymbol{p}'),$$
$$\alpha \boldsymbol{f}(\boldsymbol{p}) \gg \boldsymbol{f}(\alpha \boldsymbol{p}'), \forall \alpha > 1.$$

Theorem A.9.1. *Let $\boldsymbol{f}$ be a standard interference function. If there exists $\boldsymbol{p}^* \in \mathbb{R}_+^K$ such that*

$$\boldsymbol{p}^* = \boldsymbol{f}(\boldsymbol{p}^*),$$

then $\boldsymbol{p}^*$ is unique.

Lemma A.9.2. *Let $\boldsymbol{f}$ be a standard interference function and define a sequence $\{\boldsymbol{p}^{(n)}\}$ as follows:*

$$\boldsymbol{p}^{(n+1)} = \boldsymbol{f}(\boldsymbol{p}^{(n)}).$$

Assume that $\boldsymbol{p}^{(0)} \geq \boldsymbol{f}(\boldsymbol{p}^{(0)})$. Then

$$\boldsymbol{p}^{(n+1)} \leq \boldsymbol{p}^{(n)},$$
$$\boldsymbol{p}^{(n)} \geq \boldsymbol{f}(\boldsymbol{p}^{(n)}),$$
$$\lim_{n\to\infty} \boldsymbol{p}^{(n)} = \boldsymbol{p}^*,$$

where $\boldsymbol{p}^*$ is a unique fixed point such that $\boldsymbol{p}^* = \boldsymbol{f}(\boldsymbol{p}^*)$.

Corollary A.9.3. *Assume there exists $\boldsymbol{p}^*$ such that $\boldsymbol{p}^* = \boldsymbol{f}(\boldsymbol{p}^*)$. Then*

$$\boldsymbol{p} > \boldsymbol{f}(\boldsymbol{p}) \implies \boldsymbol{p} > \boldsymbol{p}^*.$$

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