Appendix A
Rank Theory

Definitions and Basic Properties

Definable ranks are fundamental tools in Descriptive Set Theory. In our presentation we will concentrate only on the first level of the projective hierarchy.

Definition A.1. Let X be a Polish space and B be a $\mathbf{\Pi}^1_1$ subset of X. A map $\phi : B \to \omega_1$ is said to be a $\mathbf{\Pi}^1_1$-*rank on* B if there are relations $\leq_\Sigma, \leq_\Pi \subseteq X \times X$ in $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$, respectively, such that for every $y \in B$ we have

$$\begin{aligned} \phi(x) \leq \phi(y) &\Leftrightarrow (x \in B) \text{ and } \phi(x) \leq \phi(y) \\ &\Leftrightarrow x \leq_\Sigma y \Leftrightarrow x \leq_\Pi y. \end{aligned}$$

The basic properties of $\mathbf{\Pi}^1_1$-ranks are summarized below.

Theorem A.2. *Let X be a Polish space, B a $\mathbf{\Pi}^1_1$ subset of X, and $\phi : B \to \omega_1$ a $\mathbf{\Pi}^1_1$-rank on B. Then the following hold:*

(i) For every countable ordinal ξ the set $B_\xi = \{x \in B : \phi(x) \leq \xi\}$ is Borel.
(ii) For every analytic subset A of B we have $\sup\{\phi(x) : x \in A\} < \omega_1$.
(iii) B is Borel if and only if $\sup\{\phi(x) : x \in B\} < \omega_1$.

Property (i) in Theorem A.2 follows from the fact that $\mathbf{\Delta}^1_1 = B(X)$. For every $\xi < \omega_1$ the set B_ξ is called the ξ-*resolvent* of B. Property (ii) is known as *boundedness*. It is a consequence of a more general result concerning the length of definable well-founded relations due to K. Kunen and D.A. Martin (see [Ke, Theorems 35.23 and 31.1]). Part (iii) follows easily by part (ii). The following fact will be useful in the discussion below.

Fact A.3. *Let X be a Polish space, B a $\mathbf{\Pi}^1_1$ subset of X, and $\phi : B \to \omega_1$. Then ϕ is a $\mathbf{\Pi}^1_1$-rank on B if and only if there are relations $\leq'_\Sigma, <'_\Sigma \subseteq X \times X$ both in $\mathbf{\Sigma}^1_1$ such that for every $y \in B$ we have*

$$(x \in B) \text{ and } \phi(x) \leq \phi(y) \Leftrightarrow x \leq'_\Sigma y$$

and

$$(x \in B) \text{ and } \phi(x) < \phi(y) \Leftrightarrow x <'_\Sigma y.$$

Proof. First, assume that ϕ is a $\mathbf{\Pi}^1_1$-rank on B and let $\leq_\Sigma, \leq_\Pi$ be the associated relations described in Definition A.1. Define $<'_\Sigma$ by

$$x <'_\Sigma y \Leftrightarrow (x \leq_\Sigma y) \text{ and } \neg(y \leq_\Pi x)$$

and set $\leq'_\Sigma = \leq_\Sigma$. It is easy to see that $\leq'_\Sigma$ and $<'_\Sigma$ are as desired.

Conversely, set $\leq_\Sigma = \leq'_\Sigma$ and define

$$x \leq_\Pi y \Leftrightarrow (x \in B) \text{ and } \neg(y <'_\Sigma x).$$

Again it is easily verified that the relations $\leq_\Sigma$ and $\leq_\Pi$ witness that ϕ is a $\mathbf{\Pi}^1_1$-rank on B. The proof is completed. □

Well-Founded Trees

The following theorem provides the archetypical example of a $\mathbf{\Pi}^1_1$-rank.

Theorem A.4. *Let Λ be a countable set. Then the set $\mathrm{WF}(\Lambda)$ is $\mathbf{\Pi}^1_1$ and the map $T \mapsto o(T)$ is a $\mathbf{\Pi}^1_1$-rank on $\mathrm{WF}(\Lambda)$.*

Proof. To see that $\mathrm{WF}(\Lambda)$ is $\mathbf{\Pi}^1_1$ notice that

$$T \in \mathrm{WF}(\Lambda) \Leftrightarrow \forall \sigma \in \Lambda^{\mathbb{N}}\ \exists k \in \mathbb{N} \text{ with } \sigma|k \notin T.$$

We proceed to show that $T \mapsto o(T)$ is a $\mathbf{\Pi}^1_1$-rank on $\mathrm{WF}(\Lambda)$. We will use Fact A.3. Specifically, consider the relations $\leq_\Sigma$ and $<_\Sigma$ in $\mathrm{Tr}(\Lambda) \times \mathrm{Tr}(\Lambda)$ defined by

$$S \leq_\Sigma T \Leftrightarrow T \notin \mathrm{WF}(\Lambda) \text{ or } [S, T \in \mathrm{WF}(\Lambda) \text{ and } o(S) \leq o(T)]$$

and

$$S <_\Sigma T \Leftrightarrow T \notin \mathrm{WF}(\Lambda) \text{ or } [S, T \in \mathrm{WF}(\Lambda) \text{ and } o(S) < o(T)].$$

By Proposition 1.5, we have

$$S \leq_\Sigma T \Leftrightarrow \exists f : S \to T \text{ monotone}.$$

and so the relation $\leq_\Sigma$ is $\mathbf{\Sigma}^1_1$. For every $T \in \mathrm{Tr}(\Lambda)$ and every $\lambda \in \Lambda$ we set $T_\lambda = \{t : \lambda^\frown t \in T\}$. Observe that if $T \in \mathrm{WF}(\Lambda)$, then $o(T) = \sup\{o(T_\lambda) : \lambda \in \Lambda\} + 1$, while if $T \in \mathrm{IF}(\Lambda)$, then there exists $\lambda \in \Lambda$ such that $T_\lambda \in \mathrm{IF}(\Lambda)$. Using these remarks and invoking again Proposition 1.5 we see that

$$S <_\Sigma T \Leftrightarrow \exists \lambda \in \Lambda \text{ and } \exists f : S \to T_\lambda \text{ monotone}.$$

The proof is completed. □

Let A be an analytic subset of WF. By Theorems A.2(ii) and A.4, there exists a well-founded tree S on $\mathbb{N}$ such that $o(T) \leq o(S)$ for every $T \in A$. The following parameterized version of this fact is useful in applications.

Theorem A.5. *Let X be a standard Borel space and $A \subseteq X \times \mathrm{Tr}$ be analytic. Then there exists a Borel map $f : X \to \mathrm{Tr}$ such that for every $x \in X$, if the section $A_x = \{T : (x,T) \in A\}$ of A at x is a subset of* WF*, then $f(x) \in \mathrm{WF}$ and $o\big(f(x)\big) \geq \sup\{o(T) : T \in A_x\}$, while if $A_x \cap \mathrm{IF} \neq \varnothing$, then $f(x) \in \mathrm{IF}$.*

Proof. All uncountable standard Borel spaces are Borel isomorphic (see [Ke, Theorem 15.6]). Hence, we may assume that $X = \mathbb{N}^{\mathbb{N}}$. In this case we will show that the map f can be chosen to be continuous. So let $A \subseteq \mathbb{N}^{\mathbb{N}} \times \mathrm{Tr}$ be analytic. There exists $F \subseteq \mathbb{N}^{\mathbb{N}} \times \mathrm{Tr} \times \mathbb{N}^{\mathbb{N}}$ closed with $A = \mathrm{proj}_{\mathbb{N}^{\mathbb{N}} \times \mathrm{Tr}} F$. For every $x \in \mathbb{N}^{\mathbb{N}}$ we define $T_x \in \mathrm{Tr}(\mathbb{N} \times \mathbb{N})$ by

$$T_x = \big\{(t,s) : |t| = |s| = n \text{ and } \exists (y,T,z) \in F \text{ with } x|n = y|n,\ t \in T \text{ and } s = z|n\big\}.$$

The map $h : \mathbb{N}^{\mathbb{N}} \to \mathrm{Tr}(\mathbb{N} \times \mathbb{N})$ defined by $h(x) = T_x$ is easily seen to be continuous.

Claim A.6. *For every $x \in \mathbb{N}^{\mathbb{N}}$ we have $T_x \in \mathrm{WF}(\mathbb{N} \times \mathbb{N})$ if and only if $A_x \subseteq \mathrm{WF}$.*

Proof of Claim A.6. Fix $x \in \mathbb{N}^{\mathbb{N}}$. First assume that T_x is well-founded. For every $T \in A_x$ we select $z \in \mathbb{N}^{\mathbb{N}}$ such that $(x,T,z) \in F$. Define $\phi : T \to T_x$ by $\phi(t) = (t, z|n)$ where $n = |t|$. Then ϕ is a well-defined monotone map. As $T_x \in \mathrm{WF}(\mathbb{N} \times \mathbb{N})$, by Proposition 1.5, we see that $T \in \mathrm{WF}$ and that $o(T) \leq o(T_x)$.

Conversely, assume that $T_x \in \mathrm{IF}(\mathbb{N} \times \mathbb{N})$. Let $\big((t_n, s_n)\big)$ be an infinite branch of T_x. For every $n \in \mathbb{N}$ there exist $y_n \in \mathbb{N}^{\mathbb{N}}$, $T_n \in \mathrm{Tr}$, and $z_n \in \mathbb{N}^{\mathbb{N}}$ such that $(y_n, T_n, z_n) \in F$ and $y_n|n = x|n$, $t_n \in T_n$ and $z_n|n = s_n$. It follows that $y_n \to x$ and $z_n \to z$ where $z = \bigcup_n s_n \in \mathbb{N}^{\mathbb{N}}$. Moreover, by passing to subsequences if necessary, we may assume that there exists $T \in \mathrm{Tr}$ such that $T_n \to T$ (the space of trees is compact). The set F is closed. Hence $(x,T,z) \in F$ and so $T \in A_x$. As every $T \in \mathrm{Tr}$ is downwards closed, we see that $t_n \in T_k$ for every $k \geq n$. This implies that $t_n \in T$ for every $n \in \mathbb{N}$; that is, the tree T is ill-founded. The claim is proved. □

Notice that by the proof of the above claim, we have that if $A_x \subseteq \mathrm{WF}$, then $\sup\{o(T) : T \in A_x\} \leq o(T_x)$. Now let $g : \mathrm{Tr}(\mathbb{N} \times \mathbb{N}) \to \mathrm{Tr}$ be any continuous map satisfying the following:

(a) We have $T \in \mathrm{WF}(\mathbb{N} \times \mathbb{N})$ if and only if $g(T) \in \mathrm{WF}$.
(b) For every $T \in \mathrm{Tr}(\mathbb{N} \times \mathbb{N})$ we have $o(T) \leq o\big(g(T)\big)$.

We define $f : \mathbb{N}^{\mathbb{N}} \times \mathrm{Tr}$ by $f(x) = g(T_x)$. Clearly f is as desired. The proof of Theorem A.5 is completed. □

Reductions

Let us recall the following notion.

Definition A.7. Let X and Y be Polish spaces, $A \subseteq X$ and $B \subseteq Y$. We say that A is *Wadge* (respectively, *Borel*) *reducible to* B if there exists a continuous (respectively, Borel) map $f : X \to Y$ such that $f^{-1}(B) = A$.

The link between the concept of Borel reducibility and $\mathbf{\Pi}^1_1$-ranks is given in the following fact. Its proof is straightforward.

Fact A.8. *Let X and Y be Polish spaces, $A \subseteq X$ and $B \subseteq Y$. Assume that A is Borel reducible to B via a Borel map $f : X \to Y$. Assume, moreover, that B is $\mathbf{\Pi}^1_1$ and that $\phi : B \to \omega_1$ is a $\mathbf{\Pi}^1_1$-rank on B. Then A is $\mathbf{\Pi}^1_1$ and the map $\psi : A \to \omega_1$ defined by $\psi(x) = \phi\big(f(x)\big)$ is a $\mathbf{\Pi}^1_1$-rank on A.*

Theorem A.4 combined with Fact A.8 gives us a powerful method for constructing $\mathbf{\Pi}^1_1$-ranks on $\mathbf{\Pi}^1_1$ sets. Simply find a reduction of the set in question to WF and then assign to every point the order of the well-founded tree to which the point is reduced. We will illustrate this method by showing the following fundamental result.

Theorem A.9. *Let X be a Polish space and $B \subseteq X$ be a $\mathbf{\Pi}^1_1$ set. Then there exists a $\mathbf{\Pi}^1_1$-rank on B.*

Proof. We have already indicated that, by Theorem A.4 and Fact A.8, it is enough to find a Borel reduction of B to WF. Invoking the fact that all uncountable Polish spaces are Borel isomorphic, we may assume that X is the Baire space $\mathbb{N}^{\mathbb{N}}$. In this case we will show that B is Wadge reducible to WF. In particular, by Theorem 1.6 and the fact that B is $\mathbf{\Pi}^1_1$, there exists a pruned tree T on $\mathbb{N} \times \mathbb{N}$ such that $B^{\mathrm{c}} = p[T]$. For every $\sigma \in \mathbb{N}^{\mathbb{N}}$ we let

$$T(\sigma) = \{t \in \mathbb{N}^{<\mathbb{N}} : |t| = k \text{ and } (\sigma|k, t) \in T\} \in \mathrm{Tr}.$$

The tree $T(\sigma)$ is usually called the section tree of T at σ. It is easy to see that the map $f : \mathbb{N}^{\mathbb{N}} \to \mathrm{Tr}$ defined by $f(\sigma) = T(\sigma)$ is continuous. Observe that

$$\sigma \notin B \Leftrightarrow \exists \tau \in \mathbb{N}^{\mathbb{N}} \text{ with } (\sigma, \tau) \in [T] \Leftrightarrow T(\sigma) \in \mathrm{IF}$$

and so $f^{-1}(\mathrm{WF}) = B$. The proof is completed. □

Derivatives

The most frequently met construction of an ordinal ranking in Analysis involves some kind of derivation procedure. The main result in this section asserts that if the derivative is sufficiently definable (in particular, if it is Borel), then the associated rank is actually a $\mathbf{\Pi}^1_1$-rank. A typical example is the Cantor–Bendixson derivative of a compact subset K of a Polish space X and the corresponding Cantor–Bendixson rank on the set of all countable compact subsets of X.

Precisely, let X be a Polish space. A map $D : K(X) \to K(X)$ is said to be a *derivative* on $K(X)$ if $D(K) \subseteq K$ for all $K \in K(X)$ and $D(K_1) \subseteq D(K_2)$ if $K_1 \subseteq K_2$. For every $K \in K(X)$ by transfinite recursion we define the *iterated derivatives* $\big(D^\xi(K)\big)$ $(\xi < \omega_1)$ of K by

$$D^0(K) = K,\ D^{\xi+1}(K) = D\big(D^\xi(K)\big) \text{ and } D^\lambda(K) = \bigcap_{\xi<\lambda} D^\xi(K) \text{ if } \lambda \text{ is limit.}$$

Clearly $\big(D^\xi(K)\big)$ $(\xi < \omega_1)$ is a transfinite decreasing sequence of compact subsets of X, and so, it is eventually constant. The *D-rank* of K, denoted by $|K|_D$, is defined to be the least ordinal ξ such that $D^\xi(K) = D^{\xi+1}(K)$. Also we set $D^\infty(K) = D^{|K|_D}(K)$.

In applications we often need to deal with parameterized derivatives. In particular, let X and Y be Polish spaces. A map $\mathbb{D} : Y \times K(X) \to K(X)$ is said to be a *parameterized derivative* if for every $y \in Y$ the map $\mathbb{D}_y : K(X) \to K(X)$ defined by $\mathbb{D}_y(K) = \mathbb{D}(y, K)$, is a derivative on $K(X)$. We have the following.

Theorem A.10. **[KW1]** *Let X and Y be Polish spaces and $\mathbb{D} : Y \times K(X) \to K(X)$ be a parameterized derivative. Assume that $\mathbb{D}$ is Borel. Then the set*

$$\Omega_{\mathbb{D}} = \big\{(y, K) \in Y \times K(X) : \mathbb{D}^\infty_y(K) = \varnothing\big\}$$

is $\mathbf{\Pi}^1_1$ and the map $(y, K) \mapsto |K|_{\mathbb{D}_y}$ is a $\mathbf{\Pi}^1_1$-rank on $\Omega_{\mathbb{D}}$.

Theorem A.10 will be frequently used in the following form.

Theorem A.11. *Let X be a Polish space and $D_n : K(X) \to K(X)$ $(n \in \mathbb{N})$ be a sequence of Borel derivatives on $K(X)$. Then the set*

$$\Omega = \{K \in K(X) : D^\infty_n(K) = \varnothing\ \forall n \in \mathbb{N}\}$$

is $\mathbf{\Pi}^1_1$ and the map $K \mapsto \sup\{|K|_{D_n} : n \in \mathbb{N}\}$ is a $\mathbf{\Pi}^1_1$-rank on Ω.

Proof. Let $n \in \mathbb{N}$ be arbitrary. Applying Theorem A.10 for $Y = \{n\}$ and $\mathbb{D} = D_n$, we see that the set $\Omega_{D_n} = \{K \in K(X) : D^\infty_n(K) = \varnothing\}$ in $\mathbf{\Pi}^1_1$. As $\Omega = \bigcap_n \Omega_{D_n}$, we get that Ω is $\mathbf{\Pi}^1_1$.

For notational convenience we set $\phi(K) = \sup\{|K|_{D_n} : n \in \mathbb{N}\}$ for every $K \in \Omega$. Let $Y = \mathbb{N}$ equipped with the discrete topology and consider the map $\mathbb{D} : Y \times K(X) \to K(X)$ defined by $\mathbb{D}(n, K) = D_n(K)$. Then $\mathbb{D}$ is a parameterized Borel derivative. Invoking Theorem A.10 again, we see that the map $(n, K) \mapsto |K|_{\mathbb{D}_n} = |K|_{D_n}$ is a $\mathbf{\Pi}^1_1$-rank on $\Omega_{\mathbb{D}}$. Let $\leq_\Sigma$ and $\leq_\Pi$ be the associated relations. Observe that for every $K \in \Omega$ we have

$$\begin{aligned}(H \in \Omega) \text{ and } \phi(H) \leq \phi(K) &\Leftrightarrow \forall n \in \mathbb{N}\ \exists m \in \mathbb{N} \text{ with } (n, H) \leq_\Sigma (m, K)\\ &\Leftrightarrow \forall n \in \mathbb{N}\ \exists m \in \mathbb{N} \text{ with } (n, H) \leq_\Pi (m, K).\end{aligned}$$

Hence ϕ is a $\mathbf{\Pi}^1_1$-rank on Ω. The proof is completed. □

For the proof of Theorem A.10 we need some preliminary results. We start with the following.

Lemma A.12. *Let X be a Polish space. Then the map $\bigcap : K(X)^{\mathbb{N}} \to K(X)$ defined by $\bigcap\big((K_n)\big) = \bigcap_n K_n$, is Borel.*

Proof. By Proposition 1.4, it is enough to show that the set $A_U = \big\{(K_n) \in K(X)^{\mathbb{N}} : U \cap \big(\bigcap_n K_n\big) \neq \varnothing\big\}$ is Borel for every open subset U of X. So let U be one. Write U as $\bigcup_m F_m$ where each F_m is closed. Notice that

$$(K_n) \in A_U \Leftrightarrow \exists m \in \mathbb{N}\ \forall n \in \mathbb{N} \text{ we have } F_m \cap K_0 \cap \ldots \cap K_n \neq \varnothing.$$

Therefore, A_U is Borel. The proof is completed. □

Let $\alpha \in 2^{\mathbb{N}\times\mathbb{N}}$, that is, α is the characteristic function of a binary relation on $\mathbb{N}$. The *field* $F(\alpha)$ of α is the set $\{n \in \mathbb{N} : \alpha(n, n) = 1\}$. For every $\alpha \in 2^{\mathbb{N}\times\mathbb{N}}$ we define $\leq_\alpha \subseteq \mathbb{N} \times \mathbb{N}$ by

$$n \leq_\alpha m \Leftrightarrow n, m \in F(\alpha) \text{ and } \alpha(n, m) = 1.$$

Let LO^* be the subset of $2^{\mathbb{N}\times\mathbb{N}}$ consisting of all $\alpha \in 2^{\mathbb{N}\times\mathbb{N}}$ with $0 \in F(\alpha)$ and such that $\leq_\alpha$ is a linear ordering on $F(\alpha)$ with 0 as the least element. Notice that the set LO^* is closed in $2^{\mathbb{N}\times\mathbb{N}}$ as

$$\begin{aligned}\alpha \in \mathrm{LO}^* \Leftrightarrow\ & 0 \in F(\alpha) \text{ and } (\forall n \in F(\alpha)\ 0 \leq_\alpha n) \text{ and}\\ & \forall n, m \in F(\alpha)\ [n \leq_\alpha m \text{ or } m \leq_\alpha n] \text{ and}\\ & \forall n, m \in F(\alpha)\ [n \leq_\alpha m \text{ and } m \leq_\alpha n \Rightarrow n = m] \text{ and}\\ & \forall n, m, k \in F(\alpha)\ [n \leq_\alpha m \text{ and } m \leq_\alpha k \Rightarrow n \leq_\alpha k].\end{aligned}$$

Also let WO^* be the subset of LO^* consisting of all $\alpha \in \mathrm{LO}^*$ for which $\leq_\alpha$ is a well-ordering on $F(\alpha)$. For every $\alpha \in \mathrm{WO}^*$ by $|\alpha|$ we denote the unique countable ordinal which is isomorphic to $(F(\alpha), <_\alpha)$, where $<_\alpha$ is the strict part of $\leq_\alpha$ on $F(\alpha)$, that is,

$$n <_\alpha m \Leftrightarrow n \neq m \text{ and } n \leq_\alpha m.$$

Notice that $\{|\alpha| : \alpha \in \mathrm{WO}^*\} = \omega_1 \setminus \{0\}$. It can be shown (but it is of no use in the argument below) that the set WO^* is $\mathbf{\Pi}^1_1$ and that the map $\alpha \mapsto |\alpha|$ is a $\mathbf{\Pi}^1_1$-rank on WO^* (see [Ke, Theorem 34.4]).

We fix a Borel map $h : \mathrm{LO}^* \to \mathrm{LO}^*$ such that:

(a) $\alpha \in \mathrm{WO}^*$ if and only if $h(\alpha) \in \mathrm{WO}^*$.
(b) For every $\alpha \in \mathrm{WO}^*$ we have $|h(\alpha)| = |\alpha| + 1$.

We are ready to proceed to the proof of Theorem A.10.

Proof of Theorem A.10. First we notice that the set $\Omega_{\mathbb{D}}$ is $\mathbf{\Pi}^1_1$ as

$$(y, K) \notin \Omega_{\mathbb{D}} \Leftrightarrow \exists H \in K(X)\ [\mathbb{D}(y, H) = H \text{ and } H \subseteq K \text{ and } H \neq \varnothing].$$

It remains to show that the map $(y, K) \mapsto |K|_{\mathbb{D}_y}$ is a $\mathbf{\Pi}^1_1$-rank on $\Omega_{\mathbb{D}}$. To this end it is enough to find $\mathbf{\Sigma}^1_1$ relations R and S in $\mathrm{LO}^* \times Y \times K(X)$ such that the following are satisfied:

(P1) If $(y, K) \in \Omega_{\mathbb{D}}$ with $K \neq \varnothing$, then for every $\alpha \in \mathrm{LO}^*$ we have

$$(\alpha, y, K) \in R \Leftrightarrow \alpha \in \mathrm{WO}^* \text{ and } |\alpha| \leq |K|_{\mathbb{D}_y}. \tag{A.1}$$

(P2) If $\alpha \in \mathrm{WO}^*$, then for every $(y, K) \in Y \times K(X)$ we have

$$(\alpha, y, K) \in S \Leftrightarrow (y, K) \in \Omega_{\mathbb{D}} \text{ and } |K|_{\mathbb{D}_y} = |\alpha|. \tag{A.2}$$

Indeed, assuming that the relations R and S have been defined, we complete the proof as follows. We define $\leq_\Sigma$ and $<_\Sigma$ by

$$(z, H) \leq_\Sigma (y, K) \Leftrightarrow (K = \varnothing \Rightarrow H = \varnothing) \text{ and } \big[(H = \varnothing) \text{ or } (\exists \alpha \in \mathrm{LO}^* \text{ with } [(\alpha, y, K) \in R \text{ and } (\alpha, z, H) \in S])\big]$$

and

$$(z, H) <_\Sigma (y, K) \Leftrightarrow (K \neq \varnothing) \text{ and } \big[(H = \varnothing) \text{ or } (\exists \alpha \in \mathrm{LO}^* \text{ with } [(h(\alpha), y, K) \in R \text{ and } (\alpha, z, H) \in S])\big].$$

Then $\leq_\Sigma$ and $<_\Sigma$ are both $\mathbf{\Sigma}^1_1$ as the relations R and S are $\mathbf{\Sigma}^1_1$ and the map h is Borel. Moreover, invoking properties (P1) and (P2) above, we see that for every $(y, K) \in \Omega_{\mathbb{D}}$ we have

$$(z, H) \in \Omega_{\mathbb{D}} \text{ and } |H|_{\mathbb{D}_z} \leq |K|_{\mathbb{D}_y} \Leftrightarrow (z, H) \leq_\Sigma (y, K)$$

and

$$(z, H) \in \Omega_{\mathbb{D}} \text{ and } |H|_{\mathbb{D}_z} < |K|_{\mathbb{D}_y} \Leftrightarrow (z, H) <_\Sigma (y, K).$$

By Fact A.3 we conclude that the map $(y, K) \mapsto |K|_{\mathbb{D}_y}$ is a $\mathbf{\Pi}^1_1$-rank on $\Omega_{\mathbb{D}}$.

We proceed to define the relations R and S. For the first one we set

$$(\alpha, y, K) \in R \Leftrightarrow \exists p \in K(X)^{\mathbb{N}} \text{ with } \Big(p(0) = K \text{ and } \Big[\forall m \in F(\alpha) \\ \big(p(m) \neq \varnothing \text{ and } [m \neq 0 \Rightarrow p(m) \subseteq \bigcap_{n <_\alpha m} \mathbb{D}(y, p(n))]\big)\Big]\Big).$$

By Lemma A.12 and our assumptions on the map $\mathbb{D}$, we see that R is $\mathbf{\Sigma}^1_1$. We will check that it satisfies property (P1). So let $(y, K) \in \Omega_{\mathbb{D}}$ with $K \neq \varnothing$ and $\alpha \in \mathrm{LO}^*$. For notational convenience let us set $\zeta = |K|_{\mathbb{D}_y}$. First notice that if $\alpha \in \mathrm{WO}^*$ with $|\alpha| \leq \zeta$, then clearly $(\alpha, y, K) \in R$. Conversely, assume that $(\alpha, y, K) \in R$. Let $p \in K(X)^{\mathbb{N}}$ witnessing this fact. We observe that for every $m \in F(\alpha)$ with $m \neq 0$ there exists $\xi < \zeta$ such that $\bigcap_{n <_\alpha m} \mathbb{D}(y, p(n)) \nsubseteq \mathbb{D}^{\xi+1}_y(K)$. For if not, there would existed $m \in F(\alpha)$ with $m \neq 0$ and such that for every $\xi < \zeta$

$$\varnothing \neq p(m) \subseteq \bigcap_{n <_\alpha m} \mathbb{D}(y, p(n)) \subseteq \mathbb{D}^{\xi+1}_y(K).$$

This implies that

$$\mathbb{D}^{\infty}_y(K) = \bigcap_{\xi < \zeta} \mathbb{D}^{\xi+1}_y(K) \supseteq p(m) \neq \varnothing,$$

contradicting the fact that $(y, K) \in \Omega_{\mathbb{D}}$. We define $f : F(\alpha) \to \{\xi : \xi < \zeta\}$ as follows. We set $f(0) = 0$. If $m \in F(\alpha)$ with $m \neq 0$, then we let

$$f(m) = \text{least } \xi < \zeta \text{ such that } \bigcap_{n <_\alpha m} \mathbb{D}(y, p(n)) \nsubseteq \mathbb{D}^{\xi+1}_y(K).$$

Claim A.13. *For every $m, k \in F(\alpha)$ with $m <_\alpha k$ we have $f(m) < f(k)$.*

Proof of Claim A.13. First we notice that for every $k \in F(\alpha)$ with $k \neq 0$ we have

$$\bigcap_{n <_\alpha k} \mathbb{D}(y, p(n)) \subseteq \mathbb{D}(y, p(0)) = \mathbb{D}^1_y(K).$$

Hence $f(k) > 0 = f(0)$. Thus, in what follows we may assume that $m \in F(\alpha)$ and $m \neq 0$. By the definition of the relation R and the definition of the map f, we have

$$p(m) \subseteq \bigcap_{n <_\alpha m} \mathbb{D}(y, p(n)) \subseteq \bigcap_{\xi < f(m)} \mathbb{D}^{\xi+1}_y(K) = \mathbb{D}^{f(m)}_y(K).$$

Hence $\mathbb{D}(y,p(m)) = \mathbb{D}_y(p(m)) \subseteq \mathbb{D}_y^{f(m)+1}(K)$ as $\mathbb{D}_y$ is a derivative on $K(X)$. It follows that for every $k \in F(\alpha)$ with $m <_\alpha k$ we have $\bigcap_{n<_\alpha k} \mathbb{D}(y,p(n)) \subseteq \mathbb{D}(y,p(m)) \subseteq \mathbb{D}_y^{f(m)+1}(K)$. By the definition of the map f, we see that $f(k) \geq f(m)+1 > f(m)$. The claim is proved. □

By the above claim, we conclude that the map f is order preserving from $(F(\alpha),<_\alpha)$ to ζ. Hence $\alpha \in \mathrm{WO}^*$ and $|\alpha| \leq \zeta = |K|_{\mathbb{D}_y}$. This completes the proof that the relation R satisfies property (P1).

It remains to define the relation S. We set

$$(\alpha,y,K) \in S \Leftrightarrow \exists p \in K(X)^{\mathbb{N}} \text{ with } \Big(p(0)=K \text{ and } \Big[\forall m \in F(\alpha) \\ \big(p(m) \neq \varnothing \text{ and } [m \neq 0 \Rightarrow p(m) = \bigcap_{n<_\alpha m} \mathbb{D}(y,p(n))]\big)\Big] \\ \text{and } \bigcap_{m \in F(\alpha)} \mathbb{D}(y,p(m)) = \varnothing\Big).$$

Invoking Lemma A.12 and the Borelness of the map $\mathbb{D}$, we see that S is $\boldsymbol{\Sigma}_1^1$. Moreover, it is easily verified that S satisfies property (P2). The proof of Theorem A.10 is completed. □

We close this appendix by mentioning the following result concerning sets in product spaces with compact sections. Although it not related to the notion of a $\boldsymbol{\Pi}_1^1$-rank, it is a very useful tool for checking that various derivatives are Borel. Its proof can be found in [Ke, Theorem 28.8].

Theorem A.14. *Let X and Y be Polish spaces and $A \subseteq Y \times X$ be such that for every $y \in Y$ the section $A_y = \{x \in X : (y,x) \in A\}$ of A at y is compact. Consider the map $\Phi_A : Y \to K(X)$ defined by $\Phi_A(y) = A_y$. Then the set A is Borel if and only if Φ_A is a Borel map.*

Appendix B
Banach Space Theory

B.1 Schauder Bases

Definition B.1. A sequence (x_n) in a Banach space X is said to be a *Schauder basis* of X if for every $x \in X$ there exists a unique sequence (a_n) of scalars such that $x = \sum_{n\in\mathbb{N}} a_n x_n$. A sequence (x_n) which is a Schauder basis of its closed linear span is called a *basic sequence*.

Let (x_n) be a Schauder basis of a Banach space X. By (x_n^*) we shall denote the sequence of *bi-orthogonal functionals* associated to (x_n). For every subset F of $\mathbb{N}$ by P_F we shall denote the natural projection onto $\overline{\mathrm{span}}\{x_n : n \in F\}$. The *basis constant* of (x_n) is defined to be the number $\sup\{\|P_{\{0,\dots,n\}}\| : n \in \mathbb{N}\}$. If $x = \sum_{n\in\mathbb{N}} a_n x_n$ is a vector in X, then the support $\mathrm{supp}(x)$ of x is defined to be the set $\{n \in \mathbb{N} : a_n \neq 0\}$.

Definition B.2. Let (x_n) be a Schauder basis of a Banach space X and $C \geq 1$:

(1) The basis (x_n) is said to be *monotone* if its basis constant is 1. It is said to be *bi-monotone* if $\|P_I\| = 1$ for every interval I of $\mathbb{N}$.
(2) The basis (x_n) is said to be C-*unconditional* if $\|P_F\| \leq C$ for every subset F of $\mathbb{N}$. The basis (x_n) is said to be *unconditional* if it is K-unconditional for some $K \geq 1$.
(3) The basis (x_n) is said to be *shrinking* if the sequence (x_n^*) of bi-orthogonal functionals associated to (x_n) is a Schauder basis of X^*.
(4) The basis (x_n) is said to be *boundedly complete* if for every sequence (a_n) of scalars such that $\sup\big\{\|\sum_{n=0}^k a_n x_n\| : k \in \mathbb{N}\big\} < +\infty$ we have that the series $\sum_{n\in\mathbb{N}} a_n x_n$ converges.
(5) A sequence (v_k) in X is said to be *block* if $\max\{n : n \in \mathrm{supp}(v_k)\} < \min\{n : n \in \mathrm{supp}(v_{k+1})\}$ for every $k \in \mathbb{N}$.

Two sequences (x_n) and (y_n), in two Banach spaces X and Y, respectively, are said to be C-*equivalent*, where $C \geq 1$, if for every $k \in \mathbb{N}$ and every $a_0, \dots, a_k \in \mathbb{R}$ we have

$$\frac{1}{C} \cdot \Big\| \sum_{n=0}^{k} a_n y_n \Big\|_Y \leq \Big\| \sum_{n=0}^{k} a_n x_n \Big\|_X \leq C \cdot \Big\| \sum_{n=0}^{k} a_n y_n \Big\|_Y .$$

The following stability result is classical and asserts that basic sequences are invariant under small perturbations.

Proposition B.3. *Let X be a Banach space and (x_n) be a normalized basic sequence in X with basis constant $K \geq 1$. If $(y_n)_{n=0}^{l}$ is a finite sequence in X such that*

$$\|x_n - y_n\| \leq \frac{1}{2K} \cdot \frac{1}{2^{n+2}}$$

for every $n \in \{0, \dots, l\}$, then $(y_n)_{n=0}^{l}$ is 2-equivalent to $(x_n)_{n=0}^{l}$.

B.2 Operators on Banach Spaces

Let X and Y be Banach spaces. By $\mathcal{L}(X,Y)$ we denote the Banach space of all bounded linear operators from X to Y. For every $T \in \mathcal{L}(X,Y)$ by $T^* \in \mathcal{L}(Y^*, X^*)$ we shall denote the *dual* operator of T defined by

$$T^*(y^*)(x) = y^*\big(T(x)\big) \text{ for all } x \in X.$$

We recall the following classes of operators.

Definition B.4. Let X and Y be Banach spaces and $T \in \mathcal{L}(X,Y)$:

(1) T is said to be a *finite rank operator* if $\dim\big(T(X)\big) < \infty$.
(2) T is said to be a *compact operator* if $T(B_X)$ is a relatively compact subset of Y.
(3) T is said to be a *weakly compact operator* if $T(B_X)$ is a relatively weakly compact subset of Y.
(4) T is said to be a *strictly singular operator* if for every infinite-dimensional subspace Z of X the operator $T : Z \to Y$ is not an isomorphic embedding.

Clearly every finite rank operator is compact and every compact operator is weakly compact. Strictly singular operators possess some of the strong stability properties of compact operators (they form, for instance, an operator ideal) though they are not well-behaved under duality, as the dual operator of a strictly singular one is not necessarily strictly singular. The following result of T. Kato shows that strictly singular operators are very "near" to the compact ones (see [LT, Proposition 2.c.4]).

Proposition B.5. *Let X and Y be Banach spaces. Let $T \in \mathcal{L}(X,Y)$ be an operator such that the restriction of T to any finite co-dimensional subspace Z of X is not an isomorphic embedding. Then for every $\varepsilon > 0$ there exists an* *infinite-dimensional subspace Z of X such that $T|_Z$ is compact and $\|T|_Z\| < \varepsilon$. Moreover, if X has a Schauder basis (x_n), then Z can be chosen to be a block subspace.*

Related to Proposition B.5 is the following useful fact which is essentially a Ramsey-type statement.

Lemma B.6. *Let X be a Banach space and Y be a closed subspace of X. Then for every subspace Z of X there exists a further subspace Z' of Z such that Z' is isomorphic either to a subspace of Y or to a subspace of X/Y.*

In particular, if $Y, X_0, \dots, X_k$ are Banach spaces and $T : Y \to \sum_{n=0}^{k} \oplus X_n$ is a continuous linear operator which is not strictly singular, then there exist a subspace Y' of Y and $i \in \{0, \dots, k\}$ such that the operator $P_i \circ T : Y' \to X_i$ is an isomorphic embedding, where $P_i : \sum_{n=0}^{k} \oplus X_n \to X_i$ stands for the natural projection.

B.3 Interpolation Method

Definitions and Basic Properties

Let $(X, \|\cdot\|)$ be a Banach space and W be a closed, convex, bounded, and symmetric subset of X. For every $n \in \mathbb{N}$ with $n \geq 1$ let $\|\cdot\|_n$ be the Minkowski gauge of the set $2^n W + 2^{-n} B_X$. That is

$$\|x\|_n = \inf \big\{ \lambda > 0 : \frac{x}{\lambda} \in 2^n W + 2^{-n} B_X \big\}.$$

Clearly $\|\cdot\|_n$ is an equivalent norm on X. Let $1 < p < +\infty$. For every $x \in X$ we define

$$|x|_p = \Big(\sum_{n=1}^{\infty} \|x\|_n^p \Big)^{1/p}. \tag{B.1}$$

We notice that the map $|\cdot|_p$ is not necessarily a norm on X (and in fact it is not for most interesting cases). It is, however, a norm on the vector subspace of X consisting of all $x \in X$ for which $|x|_p < +\infty$. This is essentially the content of the following definition due to W.J. Davis, T. Fiegel, W.B. Johnson, and A. Pełczyński.

Definition B.7. **[DFJP]** Let X, W, and $1 < p < +\infty$ be as above. The *p-interpolation space of the pair* (X, W), denoted by $\Delta_p(X, W)$, is defined to be the vector space

$$\{x \in X : |x|_p < +\infty\}$$

equipped with the $|\cdot|_p$ norm.

By $J : (\Delta_p(X, W), |\cdot|_p) \to (X, \|\cdot\|)$ we denote the inclusion map. Respectively, for every $n \in \mathbb{N}$ by $J_n : (\Delta_p(X, W), |\cdot|_p) \to (X, \|\cdot\|_n)$ we denote the inclusion map.

For every pair (X, W) as above and every $1 < p < +\infty$ consider the space

$$Z = \Big(\sum_{n=1}^{\infty} \oplus (X, \|\cdot\|_n) \Big)_{\ell_p}.$$

The space $(\Delta_p(X, W), |\cdot|_p)$ is naturally identified with the "diagonal" subspace $\Delta = \{(x, x, \dots) \in Z : x \in X\}$ of Z via the isometry

$$(\Delta_p(X, W), |\cdot|_p) \ni x \mapsto (x, x, \dots) \in \Delta.$$

It follows that the p-interpolation space of the pair (X, W) is a Banach space. This fact is isolated in the following proposition in which we also gather some of the basic properties of the interpolation space.

Proposition B.8. *Let $(X, \|\cdot\|)$ be a Banach space, W be a closed, convex, bounded, and symmetric subset of X, and $1 < p < +\infty$. Let $Y = (\Delta_p(X, W), |\cdot|_p)$. Then the following hold:*

(i) $W \subseteq B_Y$.
(ii) The space Y is a Banach space and J is continuous.
(iii) The operator $J^{**} : Y^{**} \to X^{**}$ is one-to-one and $(J^{**})^{-1}(X) = Y$.
(iv) The space Y is reflexive if and only if W is weakly compact.
(v) Let τ_X and τ_Y be the relative topologies on B_Y of (X, w) and (Y, w), respectively. Then $\tau_X = \tau_Y$.
(vi) The operator $J^* : X^* \to Y^*$ has norm dense range.

Parts (i)–(v) in the above proposition are essentially the content of [DFJP, Lemma 1]. Part (vi) is also well-known. It is a consequence of the fact that the operator J^{**} is one-to-one.

Spaces with a Schauder Basis

In what follows X will be a Banach space with a Schauder basis (x_n). For every $n \in \mathbb{N}$ let P_n be the natural projection onto $\mathrm{span}\{x_k : k \leq n\}$. Also let W be a closed, convex, bounded, and symmetric subset of X. The following proposition provides sufficient conditions on W so that the p-interpolation space $\Delta_p(X, W)$ will have a basis.

Proposition B.9. **[DFJP]** *Let X and W be as above and $1 < p < +\infty$. Assume that $P_n(W) \subseteq W$ and $\lambda_n x_n \in \mathrm{span}\{W\}$ for some $\lambda_n \in \mathbb{R}$ and every $n \in \mathbb{N}$. We set $z_n = J^{-1}(x_n)$ for every $n \in \mathbb{N}$. Then the sequence (z_n) is a monotone Schauder basis (not normalized) of $\Delta_p(X, W)$. Moreover, if (x_n) is shrinking, then so is (z_n).*

Now assume that X is a Banach space with a *shrinking* Schauder basis (x_n). Let W be a closed, convex, bounded, and symmetric subset of X. If W is weakly compact, then by Proposition B.8(iv) for every $1 < p < +\infty$ the space $\Delta_p(X, W)$ is reflexive. However, the space $\Delta_p(X, W)$ does not necessarily have a basis unless W satisfies $P_n(W) \subseteq W$ for every $n \in \mathbb{N}$. The following lemma shows that we can assume that W has this property without harming the basic topological assumption on W.

Lemma B.10. **[DFJP]** *Let X be a Banach space with a shrinking Schauder basis (x_n) and W be a weakly compact subset of X. Then the set*

$$W' = W \cup \bigcup_{n \in \mathbb{N}} P_n(W)$$

is also weakly compact.

B.4 Local Theory of Infinite-Dimensional Banach Spaces

Let X and Y be two isomorphic Banach spaces (not necessarily infinite-dimensional). The *Banach–Mazur distance* between X and Y is defined by

$$d(X, Y) = \inf \big\{ \|T\| \cdot \|T^{-1}\| : T : X \to Y \text{ is an isomorphism} \big\}. \tag{B.2}$$

Now let $\lambda \geq 1$. An infinite-dimensional Banach space X is said to be a $\mathcal{L}_{\infty,\lambda}$*-space* if for every finite-dimensional subspace F of X there exists a finite-dimensional subspace G of X with $F \subseteq G$ and such that $d(G, \ell_\infty^n) \leq \lambda$, where $n = \dim(G)$. The space X is said to be a $\mathcal{L}_{\infty,\lambda+}$*-space* if X is a $\mathcal{L}_{\infty,\theta}$-space for any $\theta > \lambda$. Finally, the space X is said to be a $\mathcal{L}_\infty$*-space* if X is a $\mathcal{L}_{\infty,\lambda}$-space for some $\lambda \geq 1$. The class of $\mathcal{L}_\infty$-spaces was introduced by Lindenstrauss and Pełczyński [LP1].

It follows readily by the above definition that if X is a separable $\mathcal{L}_{\infty,\lambda}$-space, then there exists an increasing (with respect to inclusion) sequence (G_n) of finite-dimensional subspaces of X with $\bigcup_n G_n$ dense in X and such that $d(G_n, \ell_\infty^{m_n}) \leq \lambda$, where $m_n = \dim(G_n)$ for every $n \in \mathbb{N}$. It is relatively easy to see that this property actually characterizes separable $\mathcal{L}_\infty$-spaces. Precisely, we have the following.

Fact B.11. *Let X be a separable Banach space and $\lambda \geq 1$. Assume that there exists an increasing sequence (F_n) of finite-dimensional subspaces of X with $\bigcup_n F_n$ dense in X and such that $d(F_n, \ell_\infty^{m_n}) \leq \lambda$, where $m_n = \dim(F_n)$. Then X is a $\mathcal{L}_{\infty,\lambda+}$-space.*

Recall that if F is a finite-dimensional subspace of a Banach space X such that $d(F, \ell_\infty^n) \leq \lambda$, where $n = \dim(F)$, then there exists a projection

$P : X \to F$ with $\|P\| \le \lambda^2$. Therefore, every separable $\mathcal{L}_\infty$-space has a finite-dimensional decomposition. In fact, the following stronger structural property is valid due to W.B. Johnson, H.P. Rosenthal, and M. Zippin.

Theorem B.12. **[JRZ]** *Every separable $\mathcal{L}_\infty$-space has a Schauder basis.*

The book of Bourgain [Bou3] contains a presentation of the theory of $\mathcal{L}_\infty$-spaces and a discussion of many remarkable examples. Further structural properties of $\mathcal{L}_\infty$-spaces, and in particular refinements of Theorem B.12, can be found in [Ro4].

B.5 Theorem 6.13: The Radon-Nikodym Property

Our aim in this section is to complete the proof of Theorem 6.13. In particular, we will show the following:

> Let η be such that $0 < \eta < 1$. Let (F_n, j_n) be a system of isometric embeddings where the sequence (F_n) consists of finite-dimensional Banach spaces and for every $n \in \mathbb{N}$ the isometric embedding $j_n : F_n \to F_{n+1}$ is η-admissible. Then the inductive limit of the system (F_n, j_n) has the Radon-Nikodym property.

The Radon-Nikodym property can be defined in many equivalent ways, either geometric or probabilistic. We refer to the monograph of Diestel and Uhl [DU] for more details. We will use, below, the following probabilistic characterization: *a Banach space X has the Radon-Nikodym property if and only if for every probability space (Ω, Σ, μ) and every martingale (M_k) in $L_1(\mu, X)$ satisfying* $\sup_k \int \|M_k\| d\mu < +\infty$*, the martingale (M_k) converges in X μ-almost everywhere.*

The following definition, due to J. Bourgain and G. Pisier, will be our basic conceptual tool.

Definition B.13. **[BP]** Let X be a Banach space, Y be a subspace of X and denote by $q : X \to X/Y$ the natural quotient map. Also let $\delta > 0$. We say that Y is *δ-well-placed* inside X if for every probability space (Ω, Σ, μ) and every $f \in L_1(\mu, X)$ such that $\int f d\mu \in Y$ we have

$$\int \|f\| d\mu \ge \|\int f d\mu\| + \delta \int \|q \circ f\| d\mu. \tag{B.3}$$

We will isolate some basic properties of δ-well-placed subspaces. To this end we recall the following standard notation. Given a probability space (Ω, Σ, μ), a sub-σ-algebra Σ' of Σ, a Banach space Z and a function $g \in L_1(\mu, Z)$, by $\mathbb{E}(g, \Sigma')$ we shall denote the conditional expectation of g relative to Σ'.

Lemma B.14. **[BP]** *Let X be a Banach space, Y be a subspace of X and denote by $q : X \to X/Y$ the natural quotient map. Let $\delta > 0$ and assume* *that Y is δ-well-placed inside X. Let (Ω, Σ, μ) be a probability space. Then the following are satisfied:*

(i) *For every $g \in L_1(\mu, X)$ we have*

$$\int \|g\| d\mu \geq \| \int g d\mu \| + \delta \int \|q \circ g\| d\mu - (2+\delta) \cdot \|q\Big(\int g d\mu\Big)\|.$$

(ii) *If Σ' is a sub-σ-algebra of Σ, then for every $g \in L_1(\mu, X)$ we have*

$$\mathbb{E}(\|g\|, \Sigma') \geq \|\mathbb{E}(g, \Sigma')\| + \delta \cdot \mathbb{E}(\|q \circ g\|, \Sigma') - (2+\delta) \cdot \|q \circ \mathbb{E}(g, \Sigma')\|$$

μ-almost everywhere.

(iii) *If Σ' is a sub-σ-algebra of Σ, then for every $g \in L_1(\mu, X)$ we have*

$$\int \|g\| d\mu \geq \int \|\mathbb{E}(g, \Sigma')\| d\mu + \delta \int \|q \circ g\| d\mu - (2+\delta) \int \|q \circ \mathbb{E}(g, \Sigma')\| d\mu.$$

Proof. (i) Let $g \in L_1(\mu, X)$ and set $x = \int g d\mu$. Also let $\varepsilon > 0$ be arbitrary. We select $y \in Y$ such that

$$\|x - y\| \leq \|q(x)\| + \varepsilon. \tag{B.4}$$

Define $f \in L_1(\mu, X)$ by $f = g - (x-y)\chi_\Omega$ and notice that $\int f d\mu = y \in Y$. Applying inequality (B.3) to f we get

$$\int \|f\| d\mu \geq \|y\| + \delta \int \|q \circ g - q(x)\| d\mu. \tag{B.5}$$

Since $g = f - (y-x)\chi_\Omega$ and $y = x - (x-y)$, we see that

$$\int \|g\| d\mu \geq \int \|f\| d\mu - \|x - y\|, \tag{B.6}$$

$$\|y\| \geq \|x\| - \|x - y\|. \tag{B.7}$$

Finally, notice that

$$\int \|q \circ g - q(x)\| d\mu \geq \int \|q \circ g\| d\mu - \|q(x)\|. \tag{B.8}$$

Combining, successively, inequalities (B.4)–(B.8), we conclude that

$$\int \|g\| d\mu \geq \| \int g d\mu \| + \delta \int \|q \circ g\| d\mu - (2+\delta) \cdot \|q\Big(\int g d\mu\Big)\| - 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the proof of part (i) is completed.
(ii) Notice that if g is a simple function, then the desired estimate follows by the inequality obtained in part (i). The general case follows using this observation and a standard approximation argument.
(iii) Follows immediately by integrating the estimate in part (ii). The proof is completed. □

The link between the notion of a δ-well-placed subspace and the notion of an η-admissible embedding (see Definition 6.7) is given in the following lemma.

Lemma B.15. **[BP]** *Let $0 < \eta \leq 1$ and X, X' be Banach spaces. Also let $J : X \to X'$ be an isometric embedding. Assume that J is η-admissible. Then $J(X)$ is $(1-\eta)$-well-placed inside X'.*

It is easy to see that Lemma B.15 implies Lemma 6.11. The proof given below shows that Lemma B.15 is actually equivalent to Lemma 6.11.

Proof of Lemma B.15. Let (Ω, Σ, μ) be a probability space. Denote by $q : X' \to X'/J(X)$ the natural quotient map. Let $f \in L_1(\mu, X')$ be a simple function such that $\int f d\mu \in J(X)$. Then inequality (6.2) can be reformulated as

$$\int \|f\| \geq \| \int f d\mu\| + (1-\eta) \int \|q \circ f\| d\mu.$$

In other words, inequality (6.2) implies inequality (B.3) for simple functions. The general case follows by a standard approximation argument. Indeed, let $f \in L_1(\mu, X')$ be such that $\int f d\mu \in J(X)$. We may select a sequence (f_n) in $L_1(\mu, X')$ consisting of simple functions, such that $f_n \to f$ μ-almost everywhere and $\int \|f_n - f\| d\mu \to 0$. By passing to a small perturbation of each f_n, we may assume that $\int f_n d\mu \in J(X)$. Hence,

$$\int \|f_n\| \geq \| \int f_n d\mu\| + (1-\eta) \int \|q \circ f_n\| d\mu$$

for every $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ and using the dominated convergence theorem, inequality (B.3) follows. The proof is completed. □

We are ready to proceed to the proof of Theorem 6.13.

Proof of Theorem 6.13: The Radon-Nikodym property. Fix $0 < \eta < 1$ and a system (F_n, j_n) of isometric embeddings such that each F_n is finite-dimensional and for every $n \in \mathbb{N}$ the isometric embedding $j_n : F_n \to F_{n+1}$ is η-admissible. Let X be the inductive limit of the system (F_n, j_n).

As in Sect. 6.3 we start by making some simple observations. In particular, we view the sequence (F_n) as being an increasing (with respect to inclusion) sequence of finite-dimensional subspaces of X such that $\bigcup_n F_n$ is dense in X. For every $n \in \mathbb{N}$ by $q_n : X \to X/F_n$ we shall denote the natural quotient map, while for every pair $n, m \in \mathbb{N}$ with $n < m$ by $I(n,m) : F_n \to F_m$ we shall denote the inclusion operator. As the isometric embedding $I(n, n+1) : F_n \to F_{n+1}$ is η-admissible for every $n \in \mathbb{N}$, by Lemma 6.9, we see that the isometric embedding $I(n, m)$ is also η-admissible for every pair $n, m \in \mathbb{N}$ with $n < m$. Applying Lemma B.15 we see that F_n is $(1-\eta)$-well-placed inside F_m. Using a standard approximation argument we get the following basic fact.

Fact B.16. *For every $n \in \mathbb{N}$ the space F_n is $(1-\eta)$-well-placed inside X.*

We are now in the position to argue that the space X has the Radon-Nikodym property. So, let (Ω, Σ, μ) be a probability space and let (Σ_k) be an increasing sequence of sub-σ-algebras of Σ. Let (M_k) be a martingale in $L_1(\mu, X)$ adapted to (Σ_k) and assume that

$$\sup_k \int \|M_k\| d\mu = C < \infty. \tag{B.9}$$

We have to show that (M_k) converges in X μ-almost everywhere. The main claim is the following.

Claim B.17. **[BP]** *We have*

$$\lim_{n\to\infty} \lim_{k\to\infty} \int \|q_n \circ M_k\| d\mu = 0. \tag{B.10}$$

Proof of Claim B.17. Fix $l \in \mathbb{N}$. Notice that $\lim_n q_n \circ M_l = 0$ μ-almost everywhere. Therefore,

$$\lim_{n\to\infty} \int \|q_n \circ M_l\| d\mu = 0. \tag{B.11}$$

Let $n \in \mathbb{N}$ be arbitrary. Also let $k > l$ be arbitrary. Then $\mathbb{E}(M_k, \Sigma_l) = M_l$. By Fact B.16, the subspace F_n is $(1-\eta)$-well-placed inside X. Hence, applying Lemma B.14(iii) to $Y = F_n$, $g = M_k$ and $\Sigma' = \Sigma_l$ we get

$$\int \|M_k\| d\mu \geq \int \|M_l\| d\mu + (1-\eta) \int \|q_n \circ M_k\| d\mu - (3-\eta) \int \|q_n \circ M_l\| d\mu.$$

Taking the limit above first in k, then in n and finally in l and using (B.11), we conclude that

$$0 \geq (1-\eta) \lim_{n\to\infty} \lim_{k\to\infty} \int \|q_n \circ M_k\| d\mu \geq 0.$$

The claim is proved. □

Let Y be a subspace of X and denote by $Q : X \to X/Y$ the natural quotient map. Notice that the sequence $(Q \circ M_k)$ is a martingale in $L_1(\mu, X/Y)$ adapted to (Σ_k). For every $k \in \mathbb{N}$ we set $g_k = \|Q \circ M_k\| \in L_1(\mu, \mathbb{R})$. The norm $\|\cdot\|$ of X/Y is a convex function, and so, the sequence (g_k) is a submartingale. That is, for every $k \in \mathbb{N}$ the inequality $\mathbb{E}(g_{k+1}, \Sigma_k) \geq g_k$ holds μ-almost everywhere. Moreover, condition (B.9) reduces to the fact that $\sup_k \int |g_k| d\mu \leq C < +\infty$. Hence, by [Bi, Theorem 35.5], we get the following.

Fact B.18. *The following hold:*

(i) The sequence $(\|M_k\|)$ *is convergent μ-almost everywhere.*
(ii) For every $n \in \mathbb{N}$ *the sequence* $(\|q_n \circ M_k\|)$ *is convergent μ-almost everywhere.*

For every $n \in \mathbb{N}$ let $h_n = \sup_k \|q_n \circ M_k\|$. Notice that for μ-almost all $\omega \in \Omega$ the sequence $\big(h_n(\omega)\big)$ is decreasing, and therefore, its limit exists.

Claim B.19. *We have* $\lim_n h_n = 0$ *μ-almost everywhere.*

Proof of Claim B.19. Assume, towards a contradiction, that the claim is false. Hence we may find $A \in \Sigma$ and $\varepsilon, \delta > 0$ such that:

(a) $\mu(A) = \varepsilon$.
(b) $h_n(\omega) > \delta$ for every $n \in \mathbb{N}$ and every $\omega \in A$.

Moreover, by Fact B.18(ii), we may assume that:

(c) The sequence $\big(\|q_n \circ M_k(\omega)\|\big)$ is convergent for every $n \in \mathbb{N}$ and every $\omega \in A$.

Fix $\omega \in A$. For every $n \in \mathbb{N}$ we select $k_n \in \mathbb{N}$ such that $\|q_n \circ M_{k_n}(\omega)\| > \delta$. Notice, first, that there exists an infinite subset L of $\mathbb{N}$ such that $k_n \neq k_m$ for every $n, m \in L$ with $n \neq m$. For if not, by Ramsey's Theorem, there would existed an infinite subset M of $\mathbb{N}$ and $k \in \mathbb{N}$ such that $k_n = k$ for every $n \in M$. This clearly contradicts the fact that $\lim_n \|q_n \circ M_k(\omega)\| = 0$.

Now let $n \in \mathbb{N}$ and $m \in L$ with $n < m$. The sequence (F_n) is increasing with respect to inclusion, and so

$$\|q_n \circ M_{k_m}(\omega)\| \geq \|q_m \circ M_{k_m}(\omega)\| > \delta$$

by the choice of k_m. Therefore for every $\omega \in A$ and every $n \in \mathbb{N}$ the set

$$\big\{k \in \mathbb{N} : \|q_n \circ M_k(\omega)\| > \delta\big\}$$

is infinite. Invoking property (c) above we get that:

(d) For every $\omega \in A$ and every $n \in \mathbb{N}$ there exists $l_n \in \mathbb{N}$ (depending on ω) such that $\|q_n \circ M_k(\omega)\| \geq \delta$ for every $k \in \mathbb{N}$ with $k \geq l_n$.

Combining properties (a) and (d) isolated above we see that

$$\lim_{n\to\infty} \lim_{k\to\infty} \int \|q_n \circ M_k\| d\mu \geq \varepsilon \cdot \delta > 0.$$

This contradicts Claim B.17. The claim is proved. □

Claim B.20. *For μ-almost all $\omega \in \Omega$ the set $\{M_k(\omega) : k \in \mathbb{N}\}$ is a relatively norm compact subset of X.*

Proof of Claim B.20. Let $\omega \in \Omega$ be such that:

(a) $\lim_n h_n(\omega) = 0$.
(b) $\sup\{\|M_k(\omega)\| : k \in \mathbb{N}\} < +\infty$.

We will show that the set $\{M_k(\omega) : k \in \mathbb{N}\}$ is relatively norm compact. By Fact B.18(i) and Claim B.19, this will finish the proof.

To this end we will argue by contradiction. So assume that there exist an infinite subset L of $\mathbb{N}$ and $\varepsilon > 0$ such that

$$\|M_k(\omega) - M_l(\omega)\| > \varepsilon \tag{B.12}$$

for every $k, l \in L$ with $k \neq l$. As $\lim_n h_n(\omega) = 0$ there exists $n_0 \in \mathbb{N}$ with $h_{n_0}(\omega) < \varepsilon/4$. It follows that for every $k \in L$ we may find $x_k \in F_{n_0}$ such that

$$\|x_k - M_k(\omega)\| < \varepsilon/4. \tag{B.13}$$

The sequence $(x_k)_{k \in L}$ is bounded and the space F_{n_0} is finite-dimensional. Therefore, there exists an infinite subset M of L such that

$$\|x_k - x_l\| < \varepsilon/4 \tag{B.14}$$

for every $k, l \in M$ with $k \neq l$. Combining (B.13) and (B.14) we see that $\|M_k(\omega) - M_l(\omega)\| < 3\varepsilon/4$ for every $k, l \in M$ with $k \neq l$. This clearly contradicts (B.12). The claim is proved. □

Now let (x_i^*) be a sequence in B_{X^*} which separates the points in X. Invoking (B.9) we see that for every $i \in \mathbb{N}$ the sequence $(x_i^* \circ M_k)$ is a bounded martingale in $L_1(\mu, \mathbb{R})$. Hence, by [Bi, Theorem 35.5], we get that:

(P) For every $i \in \mathbb{N}$ the sequence $(x_i^* \circ M_k)$ is convergent μ-almost everywhere.

Combining Claim B.10 and property (P) isolated above we conclude that the martingale (M_k) must be convergent in X μ-almost everywhere. This shows that the space X has the Radon-Nikodym property. The proof of Theorem 6.13 is completed. □

Appendix C
The Kuratowski–Tarski Algorithm

We will frequently need to compute the complexity of a given set. To this end we will follow a method, employed by logicians, which is known as the *Kuratowski–Tarski algorithm* (see [Mo, Ke]).

We will comment on the method which relies on the use of logical notations in defining sets and functions. For instance, let X be a Polish space and $P(x)$, $Q(x)$ be expressions defining A and B, respectively, that is, $A = \{x \in X : P(x)\}$ and $B = \{x \in X : Q(x)\}$. Then the expression "$P(x)$ and $Q(x)$" defines the set $A \cap B$, the expression "$P(x)$ or $Q(x)$" defines the set $A \cup B$ while "$\neg P(x)$" defines the set A^{c}. In other words, conjunction corresponds to intersection, disjunction to union and negation to complementation.

Now let X and Y be Polish spaces and let $P(x, y)$ be an expression, where x varies over X and y varies over Y, defining the set $A = \{(x, y) : P(x, y)\}$. In this case, the expression "$\exists y \in Y$ with $P(x, y)$" defines the set $B = \mathrm{proj}_X A$. That is, existential quantification corresponds to projection. On the other hand, universal quantification corresponds to the operation of co-projection as the expression "$\forall y \in Y$ we have $P(x, y)$" defines the set $B = (\mathrm{proj}_X A^{\mathrm{c}})^{\mathrm{c}}$.

We will illustrate by an example the above remarks. So, let X, Y, and Z be Polish spaces and $P(x, y, z)$ and $Q(x, y)$ be two expressions defining two Borel subsets of $X \times Y \times Z$ and $X \times Y$, respectively. Consider the subset A of X defined by

$$x \in A \Leftrightarrow \exists z \in Z \text{ with } [\forall y \in Y \text{ we have } P(x, y, z) \Leftrightarrow Q(x, y)].$$

The expression "$P(x, y, z) \Leftrightarrow Q(x, y)$" is equivalent to

$$\text{“}[P(x, y, z) \text{ and } Q(x, y)] \text{ or } [\neg P(x, y, z) \text{ and } \neg Q(x, y)]\text{”}$$

and so it defines a Borel subset A_1 of $X \times Y \times Z$. The formula

$$\text{“}\forall y \in Y \text{ we have } P(x, y, z) \Leftrightarrow Q(x, y)\text{”}$$

defines the co-projection A_2 of A_1. Hence A_2 is $\mathbf{\Pi}^1_1$. As the final quantifier is existential, we conclude that the set A is the projection of A_2, and so, A is $\mathbf{\Sigma}^1_2$.

We point out that it is the reasoning behind the above mentioned method which justifies the use of the notation $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi, \mathbf{\Delta}^0_\xi$ $(1 \leq \xi < \omega)$ and $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n$ $(n \geq 1)$ for the Borel and projective classes, respectively. We refer to [Mo] and the references therein for a detailed explanation.

Appendix D
Open Problems

1. Let X be a Banach space with property (S) (see Definition 2.28) and with a Schauder basis. Let (e_n) be a normalized Schauder basis of X. By Theorem 2.29, there exists a map $\phi_X : \omega_1 \times \omega_1 \to \omega_1$ such that for every $\xi, \zeta < \omega_1$ and every $Y, Z \in \mathrm{NC}_X$ with $o\big(T_{\mathrm{NC}}(Y, X, (e_n))\big) = \xi$ and $o\big(T_{\mathrm{NC}}(Z, X, (e_n))\big) = \zeta$ we have

$$o\big(T_{\mathrm{NC}}(Y \oplus Z, X, (e_n))\big) \le \phi_X(\xi, \zeta).$$

   Problem 1. Let X be a Banach space with property (S) and with a Schauder basis. Find an explicit upper bound for the map ϕ_X.

   Of particular importance are the cases "$X = \ell_2$" and "$X = C(2^{\mathbb{N}})$."

2. Consider the class

$$\mathrm{SSD} = \{X \in \mathrm{SB} : X^{**} \text{ is separable}\}.$$

   For every $X \in \mathrm{SSD}$ let

$$\varphi_{\mathrm{SSD}}(X) = \max\big\{\mathrm{Sz}(X), \mathrm{Sz}(X^*)\big\}.$$

   The map $\mathrm{SSD} \ni X \mapsto \varphi_{\mathrm{SSD}}(X)$ behaves like a $\mathbf{\Pi}^1_1$-rank for most practical purposes (see [D1]).

   It turned out that the rank ϕ_{SSD} is also well-behaved when restricted on the class REFL of separable reflexive Banach spaces. For instance, E. Odell, Th. Schlumprecht, and A. Zsák [OSZ] have shown that for every countable ordinal ξ the class $\{X \in \mathrm{REFL} : \varphi_{\mathrm{SSD}}(X) \le \xi\}$ is analytic.

   Problem 2. Is the map $\mathrm{REFL} \ni X \mapsto \varphi_{\mathrm{SSD}}(X)$ a $\mathbf{\Pi}^1_1$-rank on REFL?

3. Let (e_n) be a normalized Schauder basis of $C(2^{\mathbb{N}})$. By Corollary 6.28, there exists a map $f : \omega_1 \to \omega_1$ such that for every countable ordinal ξ, every separable Banach space X with $o\big(T_{\mathrm{NC}}(X, C(2^{\mathbb{N}}), (e_n))\big) \le \xi$ embeds into a Banach space Y with a Schauder basis satisfying $o\big(T_{\mathrm{NC}}(Y, C(2^{\mathbb{N}}), (e_n))\big) \le f(\xi)$.

   Problem 3. Find an explicit upper bound for the map f.

4. Let $\mathcal{C}$ be a $\mathbf{\Pi}_1^1$ strongly bounded class of separable Banach spaces. Also let $\phi_{\mathcal{C}} : \mathcal{C} \to \omega_1$ be a canonical $\mathbf{\Pi}_1^1$-rank on $\mathcal{C}$. For every $\xi < \omega_1$ we set

$$\mathcal{C}_\xi = \{Z \in \mathcal{C} : \phi_{\mathcal{C}}(Z) \leq \xi\}$$

and

$$u_{\mathcal{C}}(\xi) = \min\big\{\phi_{\mathcal{C}}(Y) : Y \in \mathcal{C} \text{ and is universal for the class } \mathcal{C}_\xi\big\}.$$

Notice that $u_{\mathcal{C}}(\xi)$ is well-defined.

Problem 4. Find explicit upper bounds for the maps u_{REFL}, u_{SD}, u_{NU}, and u_{NC_X} where X is a minimal Banach space not containing ℓ_1.

No bounds are known for u_{NU} and u_{NC_X}. The problem of estimating the values of these maps is related to Problems 1 and 3.

For the classes REFL and SD there are two results which provide almost optimal upper bounds for the corresponding maps u_{REFL} and u_{SD}. The first one is due to E. Odell, Th. Schlumprecht, and A. Zsák and deals with separable reflexive spaces.

Theorem D.1. [**OSZ**] *Let $\xi < \omega_1$. Then there exists a separable reflexive space Y satisfying* $\max\big\{\mathrm{Sz}(Y), \mathrm{Sz}(Y^*)\big\} \leq \omega^{\xi\cdot\omega+1}$ *and containing an isomorphic copy of every separable reflexive space Z satisfying* $\max\big\{\mathrm{Sz}(Z), \mathrm{Sz}(Z^*)\big\} \leq \omega^{\xi\cdot\omega}$.

The second result is due to D. Freeman, E. Odell, Th. Schlumprecht, and A. Zsák and deals with Banach spaces with separable dual.

Theorem D.2. [**FOSZ**] *Let $\xi < \omega_1$. Then there exists a separable Banach space Y satisfying* $\mathrm{Sz}(Y) \leq \omega^{\xi\cdot\omega+1}$ *and containing an isomorphic copy of every separable space Z satisfying* $\mathrm{Sz}(Z) \leq \omega^{\xi\cdot\omega}$.

5. By Theorem 7.18, the class NC_X is strongly bounded for every minimal Banach space X not containing ℓ_1. The following problems are related with the natural question whether the class NC_{ℓ_1} is also strongly bounded.

Problem 5. Is it true that every separable Banach space X not containing a copy of ℓ_1 embeds into a space Y with a Schauder basis and not containing a copy of ℓ_1?

Problem 6. Let (e_n) be the standard unit vector basis of ℓ_1. Does there exist a map $g : \omega_1 \to \omega_1$ such that for every countable ordinal ξ and every separable Banach space X with $o\big(T_{\mathrm{NC}}(X, \ell_1, (e_n))\big) \leq \xi$ the space X embeds into a Banach space Y with a Schauder basis satisfying $o\big(T_{\mathrm{NC}}(Y, \ell_1, (e_n))\big) \leq g(\xi)$?

Problem 7. Is the class NC_{ℓ_1} strongly bounded?

We notice that an affirmative answer to Problem 6 can be used to provide an affirmative answer to Problem 7 (to see this combine Theorems 2.17, 7.17, and Lemma 2.25).

It seems reasonable to conjecture that Problems 5–7 have an affirmative answer. Our optimism is based on the following facts. Firstly, by Theorem 7.17, Problem 7 is true within the category of Banach spaces with a Schauder basis. Secondly, the "dual" versions of Problems 6 and 7 also have an affirmative answer. Specifically, denoting by (e_n) the standard unit vector basis of ℓ_1, we have the following.

Theorem D.3. **[D4]** *There exists a map* $f:\omega_1\to\omega_1$ *such that for every countable ordinal* ξ *and every separable Banach space* X *with* $o\big(T_{\mathrm{NC}}(X,\ell_1,(e_n))\big)\leq\xi$ *the space* X *is a quotient of a Banach space* Y *with a Schauder basis satisfying* $o\big(T_{\mathrm{NC}}(Y,\ell_1,(e_n))\big)\leq f(\xi)$.

Theorem D.4. **[D4]** *Let* $\mathcal{C}\subseteq\mathrm{SB}$. *Then the following are equivalent:*

(i) *There exists a separable Banach space* Y *not containing a copy of* ℓ_1 *and such that every space* $X\in\mathcal{C}$ *is a quotient of* Y.
(ii) *We have* $\sup\big\{o\big(T_{\mathrm{NC}}(X,\ell_1,(e_n))\big):X\in\mathcal{C}\big\}<\omega_1$.
(iii) *There exists an analytic subset* A *of* NC_{ℓ_1} *such that* $\mathcal{C}\subseteq A$.

6. Let

$$\mathrm{S}=\{X\in\mathrm{SB}:X\text{ has a Schauder basis}\}.$$

By Lemma 2.25, the class S is analytic.

Problem 8. Is the class S analytic non-Borel?

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