Appendix A
Numerical Explorations

We complement our analytical study by some *experimental* numerical evaluations of superzeta functions over the *Riemann* zeros, at real arguments. Apart from very few tables for special values (where special formulae apply), we had not seen such functions tabulated or plotted before, and we wanted to view them. (We have not tackled cases with other zeros.)

We found it easier to concentrate on $\mathscr{Z}_0(\sigma) = \mathscr{Z}(\sigma \,|\, 0)$ as the basic case, and to deduce other cases ($t \neq 0$, $\mathscr{L}, \ldots$) by subsequent expansions.

The first step is to sum the defining series $\mathscr{Z}_0(\sigma) = \sum_k \tau_k^{-2\sigma}$ for $\sigma > \frac{1}{2}$. This needs large tables of Riemann zeros: we have used A.M. Odlyzko's list of the 100,000 first zeros he made freely available on the Web [86], for which we express our gratitude to him. Here, the Riemann Hypothesis is de facto implied throughout (there being no numerical counter example).

The other significant step is *numerical analytical continuation*, in order to reach values of $\mathscr{Z}_0(\sigma)$ for some $\sigma < \frac{1}{2}$. We tested two complementary approaches which agreed to several digits whenever they were both usable. However, we did not go below $\sigma = -2$, our numerical precision is rather modest and above all we cannot estimate it reliably. The rounded values we supply are just most likely to be correct; they intend to provide a first glimpse of superzeta functions. Better results are certainly reachable, should there arise a clear need.

A.1 Superzeta Functions of the Second Kind

We now sketch our computations for the function $\mathscr{Z}_0$. We begin by using $\mathscr{Z}_0(\sigma) = \lim_{K\to\infty} \sum_{k \le K} \tau_k^{-2\sigma}$ for $\sigma > \frac{1}{2}$, but the remainders $R_K(\sigma) \stackrel{\text{def}}{=} \sum_{k>K} \tau_k^{-2\sigma}$ ($\propto \tau_K^{1-2\sigma} \log \tau_K$) tend to 0 slowly, more and more so as $\sigma \to \frac{1}{2}^+$, where they diverge. We then approximate the sum defining $R_K(\sigma)$ by an integral $\overline{R}_K(\sigma)$, through an Euler–Maclaurin formula (1.15) for the function $T(k)^{-2\sigma}$ (where $T(k) \stackrel{\text{def}}{=}$ the reciprocal function of $\overline{N}_0(T)$ defined by (4.26)):

$$\mathcal{Z}_0(\sigma) = \lim_{K\to+\infty} S_K(\sigma), \quad S_K(\sigma) \stackrel{\text{def}}{=} \sum_{k=1}^{K-1} \tau_k^{-2\sigma} + \tfrac{1}{2}\tau_K^{-2\sigma} + \overline{R}_K(\sigma), \qquad \text{(A.1)}$$
$$\overline{R}_K(\sigma) \stackrel{\text{def}}{=} \int_{\tau_K}^{+\infty} T^{-2\sigma} \mathrm{d}\overline{N}_0(T) = \frac{1}{2\pi}\frac{\tau_K^{1-2\sigma}}{2\sigma-1}\Big[\log\frac{\tau_K}{2\pi} + \frac{1}{2\sigma-1}\Big]$$

(and likewise for $\mathcal{Z}(\sigma\,|\,t)$, or derivatives, or finite parts at poles, etc.).

The residual error in (A.1), $\widetilde{R}_K(\sigma) \stackrel{\text{def}}{=} \mathcal{Z}_0(\sigma) - S_K(\sigma)$, is $\mathrm{O}(\tau_K^{-2\sigma}S(\tau_K))$ from (4.28) (the error term from (1.15), $\mathrm{O}(\tau_K^{-2\sigma})$, proves smaller). So, $\widetilde{R}_K(\sigma) \ll R_K(\sigma)$: the use of $\overline{R}_K(\sigma)$ in (A.1) does accelerate convergence when $\sigma > \frac{1}{2}$, and extends it to $\{\sigma > 0\}$ [43, p. 116 last line]: therefore, (A.1) gives one method of numerical analytical continuation. The next obstacle to convergence, $\widetilde{R}_K(\sigma)$, differs in type from $R_K(\sigma)$: it *fluctuates* in K, on a root-mean-square scale $\propto \tau_K^{-2\sigma}(\log\log\tau_K)^{1/2}$ according to [87, (2.5.7)], which diverges (as $K \to +\infty$) when $\sigma \le 0$. Higher Euler–Maclaurin corrections are pointless for such fluctuating remainders, only some *damping* can help or restore convergence (as with "chaotic" spectra [3]). Here, *Cesaro averaging* (defined by $\langle S\rangle_K \stackrel{\text{def}}{=} K^{-1}\sum_1^K S_{K'}$) *appears* to work initially – it gives results verifiable at $\sigma = 0$ – but not far: already at $\sigma = -\frac{1}{4}$, even $\langle S\rangle_K(\sigma)$ retains fluctuations $> 10^{-3}$ (in standard deviation) up to $K \approx 10^5$.

So, instead of pursuing ever more severe (and unproven, after all) numerical dampings as σ decreases farther from $\frac{1}{2}$, we switched to another route: to compute the analytical continuation formulae (8.6) and (8.7) themselves. We thus first validated (8.7) against (A.1) for $0 < \sigma < \frac{1}{2}$, then used it for numerical analytical continuation below $\sigma = 0$. On paper this seems to be the obvious thing to do, but in practice it proves arduous: faced with a *Mellin integral*, over an *infinite path* which moreover goes *through a pole*, we achieved decent numerical stability only close to $\sigma = 0$ (and not near $\sigma = \frac{1}{2}$).

Turning to the other interesting case $\mathcal{Z}_*(\sigma) = \mathcal{Z}(s\,|\,t = \frac{1}{2})$, we evaluate it in terms of $\mathcal{Z}_0$ through the expansion (8.19) about $t = 0$. This way, first we capitalize on the preceding calculations, then we benefit from a very rapid numerical convergence: about three terms suffice to match our accuracy for $\mathcal{Z}_0$ itself. Indeed, numerically $\mathcal{Z}_*(\sigma)$ stays very close to $\mathcal{Z}_0(\sigma)$ for $\sigma \ge 0$, thanks to $\tau_k^2 \gg \frac{1}{4}$ $(\forall k)$ (an empirical fact; $\tau_1^2 \approx 199.790455$). In the crudest approximation of (8.19), $\mathcal{Z}_*(\sigma)/\,\mathcal{Z}_0(\sigma) \approx 1 - \sigma/\,4\tau_1^2 \approx 1 - \sigma/800$. We numerically also found $0 < \mathcal{Z}_0(\sigma) - \mathcal{Z}_*(\sigma) < 3\times 10^{-4}$ for all $\sigma > 0$, and

$$A \stackrel{\text{def}}{=} 4[\mathcal{Z}_*'(0) - \mathcal{Z}_0'(0)] \approx -0.0231003495 \;\text{ vs }\; B \equiv -\,\mathcal{Z}_*(1) \approx -0.0230957090; \qquad \text{(A.2)}$$

not only A is small, but given that $A \equiv 4\log \Xi(\frac{1}{2})$ by (8.25) and $B \equiv [\log \Xi]'(0)$ by (4.8), then $|B-A| < 5 \times 10^{-6}$ reflects how *close* the function $\log \Xi(x)$ stays to the parabolic shape $Ax(1-x)$ over the interval $[0,1]$).

A.2 Superzeta Functions of the First Kind

These functions are now most easily reached as follows: first, $\mathscr{Z}_0$ from $\mathcal{Z}_0$ through the confluence identity (5.8) (and the known exact values at $s \in -\mathbb{N}$); then, $\mathscr{Z}(s\,|\,t)$ for $t \neq 0$ by the analog of the expansion (8.19), namely

$$\mathscr{Z}(s\,|\,t) = \sum_{k=1}^{\infty} (\rho - \tfrac{1}{2})^{-s} \left[1 + \frac{t}{\rho - \frac{1}{2}}\right]^{-s} = \sum_{\ell=0}^{\infty} \frac{\Gamma(1-s)}{\Gamma(1-s-\ell)\,\ell!}\,\mathscr{Z}_0(s+\ell)\,t^{\ell}$$

(for $s \in -\mathbb{N}$, one has to cancel the poles of $\mathscr{Z}_0(s+\ell)$ and $\Gamma(1-s-\ell)$ at $\ell = 1-s$ by hand, or else directly use the special values from Table 7.1). Again, for $t = \frac{1}{2}$ about four terms sufficed to match our accuracy for $\mathscr{Z}_0$; for $s > 1$, $\mathscr{Z}_*(s)$ stays very close to $\mathscr{Z}_0(s)$ but no longer for $s \approx 0$, contrary to $\mathcal{Z}_*(s)$ vs $\mathcal{Z}_0(s)$. Here, we could also have used Mellin integral representations like (7.6) for $s < 1$ at any t, but with the same difficulties as above (plus an extra one for $t = \frac{1}{2}$, when the pole merges with the integration endpoint).

We present a rough plot of the two particular functions of first kind $\mathscr{Z}_0(s)$ ($t = 0$) and $\mathscr{Z}_*(s)$ ($t = \frac{1}{2}$) (Fig. A.1). For $s \leq -2$ we show their leading asymptotic form, namely (7.28) cut at its ($n = 2$) term $(-\Gamma(s)^{-1}(\log 2)^s/2^{\frac{1}{2}+t})$.

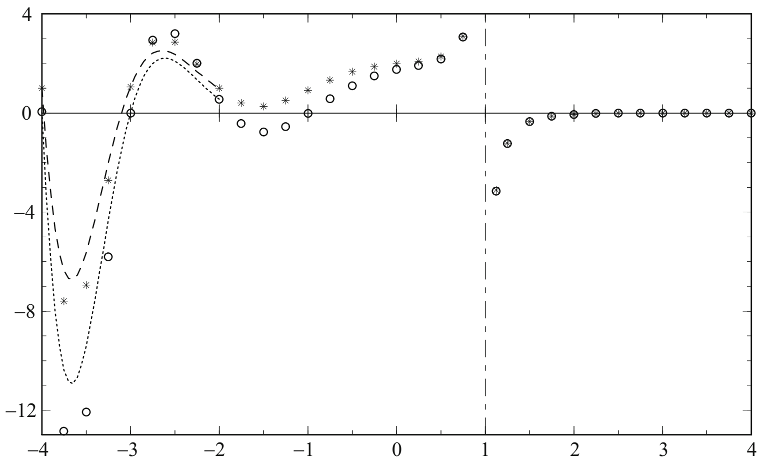

Fig. A.1 The superzeta functions of first kind $\mathscr{Z}_0$ (○) and $\mathscr{Z}_*$ ($*$) over the Riemann zeros. The asymptotic forms (*dashed curves*) are shown for $s \leq -2$; they are exact at integer s

This form is exact at integer $s < 0$ due to the $1/\Gamma(s)$ factor in (7.28), otherwise the next asymptotic term ($n = 3$) is of order unity in the plotted region. (The corresponding graphs for functions of second kind would fall to the σ-axis very fast on the $\sigma > \frac{1}{2}$ side and be chopped up by poles on the whole $\sigma \le \frac{1}{2}$ side, so we skip them.)

A.3 Numerical Tables

Here is a sample of our results, extending [106, Table 2]. We saw earlier such Tables for the *special values* $\sigma_n \stackrel{\text{def}}{=} \mathscr{Z}_*(n)$ $(n = 1, 2, \dots)$ only, albeit with *much* greater accuracy and range ($n_{\max} = 400$) [72, Table 5][64, Appendix].

Table A.1 Numerical values for the superzeta functions of first kind $\mathscr{Z}_0$ and $\mathscr{Z}_*$ over the Riemann zeros. Implied precision is expected to hold, but not guaranteed (*: exact values)

s	$\mathscr{Z}_0(s) = \sum_\rho (\rho - \frac{1}{2})^{-s}$	$\mathscr{Z}_*(s) = \sum_\rho \rho^{-s}$
-2	$\frac{9}{16} = 0.5625^*$	1^*
$-3/2$	-0.76819	0.27288
-1	$-\frac{1}{48}^* \approx -0.0208333$	$\frac{11}{12}^* \approx 0.9166667$
$-1/2$	1.11061	1.66919
0	$\frac{7}{4} = 1.75^*$	2^*
derivative at 0	*0.8118179*	*0.3465736*
$+1/2$	2.19070	2.27780
finite part at 1	*0.9189385*	*0.9420342*
$+1$ (pole!)	0^*	0.0230957
$+3/2$	-0.350386	-0.346357
$+2$	-0.0462100	-0.0461543
$+5/2$	-0.0052348	-0.0055027
$+3$	0^*	-0.0001112
$+7/2$	0.0002259	0.0002022
$+4$	0.0000743	0.0000736

Table A.2 As above but for the functions of second kind $\mathcal{Z}_0$ and $\mathcal{Z}_*$ (here the finite parts at 1/2 have no analytic expressions to our knowledge, and are provided purely numerically)

σ	$\mathcal{Z}_0(\sigma) = \sum_{k=1}^{\infty} \tau_k^{-2\sigma}$	$\mathcal{Z}_*(\sigma) = \sum_{k=1}^{\infty} (\tau_k^2 + \frac{1}{4})^{-\sigma}$
-1	$-\frac{9}{32} = -0.28125^*$	$-\frac{1}{16} = -0.0625^*$
$-3/4$	0.54319	1.69388
$-1/4$	0.785321	0.800805
0	$\frac{7}{8} = 0.875^*$	$\frac{7}{8} = 0.875^*$
Derivative at 0	*0.8118179*	*0.8060429*
$+1/4$	1.549060	1.548829
Finite part at 1/2	*0.251637*	*0.251546*
$+3/4$	0.247760	0.247730
$+1$	0.0231050	0.0230957
$+5/4$	0.0037016	0.0036988
$+3/2$	0.0007295	0.0007287
$+7/4$	0.0001597	0.0001595
$+2$	0.0000372	0.0000371

Appendix B
The Selberg Case

We briefly describe how the our framework of Chap. 10 also successfully adapts to the somewhat different case of superzeta functions built over the zeros of *Selberg zeta functions* for compact hyperbolic surfaces.

The close formal analogy between this "Selberg case" and the general case (Chap. 10) allows us to keep the benefit of the whole procedure used there. However, several differences prevent us from transferring the results and Tables verbatim: mainly, the order of the Selberg zeta functions is $\mu_0 = 2$ vs 1 previously, and some of their nontrivial zeros are *real*, something we forbade before. (A few scattered details prove simpler, though.)

We will only recall selected results, referring to the literature for details, e.g., [12, 50, 91, 97], and to Sect. 6.3.1 for the basic background and notation; we just add that the Gauss–Bonnet formula is implicitly used henceforth: $\mathcal{A}_S/\,4\pi \equiv g-1$, where $g\ (= g_S)$ is the genus of the surface S (an integer ≥ 2).

For a compact hyperbolic surface S (henceforth fixed, with normalized curvature -1), its Selberg zeta function, specified by the Euler product

$$\zeta_S(x) \stackrel{\text{def}}{=} \prod_{\{\varpi\}} \prod_{k=0}^{\infty} \left(1 - \mathrm{e}^{-\ell_\varpi (x+k)}\right) \qquad (\mathrm{Re}\ x > 1) \tag{B.1}$$

continues to an entire function of order $\mu_0 = 2$, which has a functional equation,

$$\zeta_S(\tfrac{1}{2} + t) \equiv \exp\Big[4(g-1) \int_0^t \pi t' \tan \pi t' \,\mathrm{d}t'\Big]\, \zeta_S(\tfrac{1}{2} - t), \tag{B.2}$$

and two classes of zeros:

• Trivial zeros $\{-k\}$ with multiplicities $2(g-1)(2k+1)$, $k \in \mathbb{N}$
• Nontrivial zeros $\rho = \{\frac{1}{2} \pm \mathrm{i}\kappa_k\}_{k=0,1,\ldots}$ where $\frac{1}{4} + \kappa_k^2$ are the eigenvalues of the positive Laplacian on S (both zeros and eigenvalues being counted with multiplicities, as ever). The lowest eigenvalue 0 (the zero-mode), non-degenerate, yields two zeros $\rho = 1$ (simple) and 0 (then of total multiplicity $2(g-1)+1$). To avoid complications, we exclude the exceptional case of an eigenvalue $= \frac{1}{4}$; now, however, finitely many other eigenvalues may exist in $(0, \frac{1}{4})$, adding zeros ρ on the real segment $(0, 1)$ save at $x = \frac{1}{2}$.

So, $\zeta_S(x)$ evokes a general primary function $L(x)$ as in Sect. 10.1, but with order $\mu_0 = 2$ and at least one pair of *real* nontrivial zeros (0 and 1). We can then still define generalized Stieltjes cumulants $g_n^{\mathrm{c}}\langle S\rangle$ here by (10.5), but with $q = -1$:

$$\log \frac{\zeta_S(x)}{x-1} \equiv \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!}\, g_n^{\mathrm{c}}\langle S\rangle\, (x-1)^n \tag{B.3}$$

$$\left(g_0^{\mathrm{c}}\langle S\rangle = -\log \zeta_S'(1),\;\; g_1^{\mathrm{c}}\langle S\rangle = \frac{\zeta_S''(1)}{2\,\zeta_S'(1)},\;\; g_2^{\mathrm{c}}\langle S\rangle = -\frac{\zeta_S'''(1)}{3\,\zeta_S'(1)} + \left[\frac{\zeta_S''(1)}{2\,\zeta_S'(1)}\right]^2, \ldots\right).$$

Note: $\zeta_S'(1) > 0$, because $\zeta_S(x) > 0$ for $x > 1$ by (B.1) and its zero at $x = 1$ is *simple*. For $n \geq 1$, cf. also [47] (where $\gamma^{(n-1)} \equiv (-1)^{n-1} g_n^{\mathrm{c}}\langle S\rangle/(n-1)!$).

B.1 Superzeta Functions of the First Kind

We can define $\mathscr{Z}(s\,|\,t)$ as in (10.6), and then the allowed t-domain Ω_1 also excludes the cut $(-\infty, +\frac{1}{2}]$ due to the real zeros of ζ_S starting with $x = 1$. A real $t < \frac{1}{2}$ now needs to be specified as $t \pm \mathrm{i}0$ (only the special values $\mathscr{Z}(n\,|\,t)$ are continuous on this cut). The confluent case $\mathscr{Z}_0$ accordingly splits into two functions $\mathscr{Z}_{0\pm}$. The limit $t \to \frac{1}{2}$ will require us to remove the zero-mode, as

$$\mathscr{Z}_*(s\,|\,t) = \sum_{\rho \neq (0,1)} (\rho + t - \tfrac{1}{2})^{-s} = \mathscr{Z}(s\,|\,t) - (t - \tfrac{1}{2})^{-s} - (t + \tfrac{1}{2})^{-s}. \tag{B.4}$$

The completion and Hadamard factorization of ζ_S usually involve the Barnes G-function and elaborate fudge factors [37, 95, 103]. Instead, we will decompose ζ_S directly in zeta-regularized factors. Specifically, the sequence

$$x_k = k + \tfrac{1}{2} \quad \text{with multiplicity } (2k+1) \tag{B.5}$$

is *theta-eligible* of order $\mu_0 = 2$, with the Theta function

$$\theta(z) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (2k+1)\,\mathrm{e}^{-(k+1/2)z} = \frac{\mathrm{d}}{\mathrm{d}z}\, \frac{1}{\sinh \frac{1}{2} z} \tag{B.6}$$

$$= \sum_{m=-1}^{\infty} \mathbf{c}_{-2m} z^{2m}, \quad \mathbf{c}_{-2n+2} = \frac{2(2n-1)}{(2n)!}\,(1 - 2^{1-2n}) B_{2n}\,, \tag{B.7}$$

the generalized zeta function

$$\mathbf{z}(s\,|\,t) = \sum_{k=0}^{\infty} (2k+1)(k + \tfrac{1}{2} + t)^{-s} \qquad (\operatorname{Re} s > 2), \tag{B.8}$$

and the zeta-regularized product $\mathbf{d}(\frac{1}{2}+t) = \mathrm{e}^{-\mathbf{z}'(0\,|\,t)}$. The zeta function over the trivial zeros of ζ_S, patterned on (7.3), is then $\mathbf{Z} = 2(g-1)\mathbf{z}$. Thereupon, in analogy with (10.15), we can write [104]

$$\zeta_S(x) \equiv \mathbf{d}(x)^{2(g-1)} \mathscr{D}(x) \equiv \mathbf{d}(x)^{2(g-1)} x(x-1)\, \mathscr{D}_*(x) \qquad (x \equiv \tfrac{1}{2}+t), \quad \text{(B.9)}$$

defining complementary (S-dependent) factors $\mathscr{D}(x)$ (which keeps all the nontrivial zeros of ζ_S), and $\mathscr{D}_*(x)$ (without the zero-mode, as needed in the limit $x \to 1$). The same churning as in the Riemann case (Sect. 7.2.2) then converts (B.9) into integral representations similar to (7.4)–(7.6):

$$\mathscr{Z}(s\,|\,t) = -\,\mathbf{Z}(s\,|\,t) + \frac{\sin \pi s}{\pi} \int_0^\infty \frac{\zeta_S'}{\zeta_S}(\tfrac{1}{2}+t+y)\, y^{-s}\, \mathrm{d}y \qquad (\mathrm{Re}\ s < 1), \quad \text{(B.10)}$$

$$\mathscr{Z}_*(s\,|\,t) = -\,\mathbf{Z}(s\,|\,t) - (t+\tfrac{1}{2})^{-s} + \frac{\sin \pi s}{\pi} \int_0^\infty \left[\frac{\zeta_S'}{\zeta_S}(\tfrac{1}{2}+t+y) - \frac{1}{t-\frac{1}{2}+y} \right] y^{-s} \mathrm{d}y,$$

which are, however, discontinuous across the t-plane cut $(-\infty, +\frac{1}{2}]$.

These entail that $\mathscr{Z}(s\,|\,t)$ and $\mathscr{Z}_*(s\,|\,t)$ have exactly the s-plane singularities of $-\,\mathbf{Z}(s\,|\,t)$, and that $\mathscr{Z}$ has the special values

$$\mathscr{Z}(-n\,|\,t) = -\,\mathbf{Z}(-n\,|\,t) \qquad (n \in \mathbb{N}), \quad \text{(B.11)}$$

$$\mathscr{Z}'(0\,|\,t) = -\,\mathbf{Z}'(0\,|\,t) - \log \zeta_S(\tfrac{1}{2}+t), \quad \text{(B.12)}$$

$$\mathscr{Z}(+n\,|\,t) = -\,\mathbf{Z}(+n\,|\,t) + \frac{(-1)^{n-1}}{(n-1)!} (\log \zeta_S)^{(n)}(\tfrac{1}{2}+t) \quad (n \in \mathbb{N}^*), \quad \text{(B.13)}$$

where *finite parts apply to* $\mathscr{Z}$, $\mathbf{Z}$ for $1 \le n \le \mu_0 = 2$; and similarly for $\mathscr{Z}_*$ using (B.4). Except for the derivative formula (B.12), the special values have (understandably) regained single-valuedness about the real-t axis.

The glitch in the Selberg case is that, unlike before, the functions $\mathbf{z}(s\,|\,t)$ and hence $\mathbf{Z}(s\,|\,t)$ are not known in closed form for general t, making the above formulae of little use. Rescue, however, comes from two directions.

- The *theta-eligibility* trivially transfers to the shifted sequence $\{x_k + t\}$,

$$\sum_{k=0}^{\infty} (2k+1)\, \mathrm{e}^{-(k+1/2+t)z} = \mathrm{e}^{-tz}\, \theta(z) = \sum_{n=-2}^{\infty} \mathbf{c}_{-n}(t)\, z^n \quad \text{(B.14)}$$

with computable polynomial $\mathbf{c}_{-n}(t)$, so all of the algebraic information can now flow from (2.58), yielding that $-\,\mathbf{Z}(s\,|\,t)$ and hence $\mathscr{Z}(s\,|\,t)$ have: *only two poles,* $s=2$ *and* $s=1$, *both simple and of respective residues* $-4(g-1)$ *and* $4(g-1)\,t$, and the rational special values $-2(g-1)(-1)^n n!\,\mathbf{c}_{-n}(t)$ at $s=-n$; the latter result completes Table B.1 for general t, as it is shown.

• The function $\mathbf{Z}(s \,|\, t)$ can be specified for particular values of t, including our favorite locations $t = 0$ and $\frac{1}{2}$, where by simple inspection,

$$\mathbf{z}(s \,|\, 0) = 2(2^{s-1} - 1)\,\zeta(s-1), \qquad \mathbf{z}(s \,|\, \tfrac{1}{2}) = 2\,\zeta(s-1) - \zeta(s), \qquad \text{(B.15)}$$

and this allows us to compute the corresponding special values as displayed in Tables B.2 and B.3. We recall that the confluent-case function now has two principal determinations $\mathscr{Z}_{0\pm}$ corresponding to $\operatorname{Im} t \gtrless 0$ (Table B.2), and that the $t = \frac{1}{2}$ function (Table B.3) is now $\mathscr{Z}_*(s) \stackrel{\text{def}}{=} \mathscr{Z}_*(s \,|\, \frac{1}{2})$, excluding the zero mode which would cause divergence.

As a final remark, such functions of first kind for the Selberg case have never been considered before to our knowledge.

B.2 Superzeta Functions of the Second Kind

We can define $\mathcal{Z}(\sigma \,|\, t) = \sum_{k=0}^{\infty} (\kappa_k^2 + t^2)^{-\sigma}$ as in (10.7), but with the extra cut $(-\infty, +\frac{1}{2}]$ in the allowed t-domain due to the real zeros of $\boldsymbol{\zeta}_S$ starting with $x = 1$. A real $t < \frac{1}{2}$ now needs to be specified as $t \pm \mathrm{i}0$ (except for the special values $\mathcal{Z}(n \,|\, t)$). To accommodate the case $t = \frac{1}{2}$ it is useful to also define the sum without the zero-mode $k = 0$, denoted $\mathcal{Z}_*(\sigma \,|\, t) = \mathcal{Z}(\sigma \,|\, t) - (t^2 - \frac{1}{4})^{-\sigma}$.

In the Selberg case, only this kind of superzeta function has attracted some attention before, and mainly for $t = 0$ [15, 16][105, Sect. 4] and $t = \frac{1}{2}$ [47, 91, 100].

Our successive techniques of Chap. 8 all adapt to the present case.

For $t = 0$ (the confluent case), one must now specify on which side of the cut $(-\infty, \frac{1}{2}]$ the limit $t \to 0^+$ is taken; thus, two principal function determinations $\mathcal{Z}_{0\pm}$ emerge, and each satisfies the same confluence identity (8.2) as before with $\mathscr{Z}_{0\pm}$ (Table B.5; explicit values were given in [16, Sect. 6.4]). For general t, the expansion (8.19) about $t = 0$ works as previously; now it yields that $\mathcal{Z}(\sigma \,|\, t)$ has *a single pole at* $\sigma = 1$, *simple and of residue* $(g-1)$, and rational special values $\mathcal{Z}(-m \,|\, t)$ as given in Table B.4 (upper part).

For the transcendental values, we again use the idea of zeta-regularizing the nontrivial factor $\mathscr{D}$ of $\boldsymbol{\zeta}_S(\frac{1}{2} + t)$ but in the variable $v = t^2$. The canonical large-t expansion (derivable from (B.7), (2.62), and $\log \boldsymbol{\zeta}_S(\frac{1}{2} + t) = \mathrm{O}(t^{-\infty})$),

$$\begin{aligned} &\log \mathscr{D}(\tfrac{1}{2} + t) \sim \tilde{a}_2\, t^2(\log t - \tfrac{3}{2}) + \tilde{a}_0 \log t + \mathrm{O}(t^{-2}), \\ &\tilde{a}_2 = -2(g-1), \quad \tilde{a}_0 = -\tfrac{1}{6}(g-1) \qquad (\text{from } \mathbf{c}_2 = 2,\ \mathbf{c}_0 = \tfrac{1}{12}), \end{aligned} \qquad \text{(B.16)}$$

becomes, in the variable $v = t^2$,

$$\log \mathscr{D}(\tfrac{1}{2} + v^{1/2}) \sim -(g-1)\big[v(\log v - 1) - \underline{2\,v} + \tfrac{1}{12} \log v\big] \ (+\mathrm{O}(v^{-1})), \quad \text{(B.17)}$$

where we have readily underlined a newly born *banned* term in the v variable. This term must be killed to get the zeta-regularized form in v, which therefore has to be $\mathcal{D}(v) = \mathrm{e}^{-2(g-1)v}\, \mathscr{D}(\frac{1}{2}+t)$; this enters a variant of the factorization (B.9), analogous to (8.18) [104],

$$\zeta_S(\tfrac{1}{2}+t) = \left[\mathrm{e}^{t^2}\, \mathbf{d}(\tfrac{1}{2}+t)\right]^{2(g-1)} \mathcal{D}(v), \qquad v \equiv t^2. \tag{B.18}$$

As in the Riemann case (Table 8.2), the transcendental values of second kind are best expressed in terms of those of first kind, already found. We then use

$$\log \mathcal{D}(v) = \log \mathscr{D}(\tfrac{1}{2}+t) - 2(g-1)t^2. \tag{B.19}$$

Immediately, $\log \mathcal{D}(v) \equiv -\,\mathcal{Z}'_{(\sigma)}(0\,|\,t)$ and $\log \mathscr{D}(\frac{1}{2}+t) \equiv -\,\mathscr{Z}'_{(s)}(0\,|\,t)$ entail

$$\mathcal{Z}'_{(\sigma)}(0\,|\,t) = \mathscr{Z}'_{(s)}(0\,|\,t) + 2(g-1)\,t^2. \tag{B.20}$$

The v-derivative of (B.19), using (2.53) and (2.54) with $x = v$ then $x = t$, yields

$$\mathrm{FP}_{\sigma=1}\, \mathcal{Z}(\sigma\,|\,t) = (1/\,2t)\, \mathrm{FP}_{s=1}\, \mathscr{Z}(s\,|\,t) - 2(g-1) \qquad (t \neq 0). \tag{B.21}$$

As for higher v-derivatives, using (2.56) with $\mathcal{D}$ *for which* $\mu_0 = 1$, they give

$$\mathcal{Z}(m\,|\,t) = \frac{(-1)^{m-1}}{(m-1)!}\frac{\mathrm{d}^m}{\mathrm{d}(t^2)^m} \log \mathscr{D}(\tfrac{1}{2}+t) \qquad (m = 2, 3, \ldots), \tag{B.22}$$

which reduces to the same linear combination of the $\log \mathscr{D}^{(n)}(\frac{1}{2}+t)$ as in (8.32). Now it matters that the general formula (2.53) for $\log \mathscr{D}^{(n)}$ outputs $\mathscr{Z}_\infty(n\,|\,t)$ and not $\mathrm{FP}_{s=n}\, \mathscr{Z}(s\,|\,t)$: the two differ when $1 < n \leq \mu_0$, and in the t variable with $\mu_0 = 2$ *there is one such* n, namely $n = 2$. Then by (B.16),

$$\mathscr{Z}_\infty(n\,|\,y) - \mathrm{FP}_{s=n}\, \mathscr{Z}(n\,|\,y) = (-1)^n n H_{n-1}\, \tilde{a}_n(y) = -\delta_{n,2}\, 4(g-1) \tag{B.23}$$

(with $\tilde{a}_2(y) \equiv \tilde{a}_2$ constant here). All that accounts for Table B.4 (lower part).

As an illustration, we fully develop the first few transcendental values of the case $t = \frac{1}{2}$ (Table B.6), referring to our cumulants (B.3) and to Table B.3:

$$\begin{aligned}
\mathcal{Z}'_{*(\sigma)}(0) &= \mathscr{Z}'_{*(s)}(0) + \tfrac{1}{2}(g-1)\\
&= -4(g-1)\,[\zeta'(-1) + \tfrac{1}{4}\log 2\pi] + g_0^{\mathrm{c}}\langle S\rangle + \tfrac{1}{2}(g-1)\\
&= (g-1)\big(-4\zeta'(-1) - \log 2\pi + \tfrac{1}{2}\big) + g_0^{\mathrm{c}}\langle S\rangle,
\end{aligned} \tag{B.24}$$

$$\begin{aligned}
\mathrm{FP}_{\sigma=1}\, \mathcal{Z}_*(\sigma) &= \mathrm{FP}_{s=1}\, \mathscr{Z}_*(s) - 2(g-1)\\
&= 2(g-1)(1+\gamma) + g_1^{\mathrm{c}}\langle S\rangle - 1 - 2(g-1)\\
&= 2(g-1)\,\gamma + g_1^{\mathrm{c}}\langle S\rangle - 1,
\end{aligned} \tag{B.25}$$

$$\begin{aligned}\mathcal{Z}_*(2) &= 2\,\mathscr{Z}_{*,\infty}(1)+\mathscr{Z}_{*,\infty}(2) = 2\,\mathrm{FP}_{\sigma=1}\,\mathscr{Z}_*(s) + (\mathrm{FP}_{\sigma=2}\,\mathscr{Z}_*(s)+2\tilde{a}_2)\\ &= 2\big[2(g-1)(1+\gamma)+g_1^{\mathrm{c}}\langle S\rangle -1\big]\\ &\qquad +\big[-4(g-1)(\gamma-\tfrac{1}{12}\pi^2)+g_2^{\mathrm{c}}\langle S\rangle-1\big]-4(g-1)\\ &= \tfrac{1}{3}(g-1)\pi^2+2g_1^{\mathrm{c}}\langle S\rangle+g_2^{\mathrm{c}}\langle S\rangle-3, \end{aligned} \tag{B.26}$$

using (B.23), and those agree with [100, (20), (21), (25)][47, Theorem A(2)].

Formulae for the general $\mathcal{Z}_*(m)$ are also given in [47, 100] but in recursive form, whereas Table B.6 exhibits a more closed structure.

B.3 Tables of Special-Value Formulae (Selberg Cases)

Table B.1 Special values of the function of first kind $\mathscr{Z}(s\,|\,t)$ over the zeros $\{\rho\}$ of the Selberg zeta function $\zeta_S(x)$ for a compact hyperbolic surface S of genus g. Notation: see (1.4), (1.7), (1.9), (B.1), (B.8); n is an integer

s	$\mathscr{Z}(s\,\vert\,t)=\sum_\rho(\rho+t-\frac{1}{2})^{-s}$
$-n\le 0$	$-\dfrac{4(g-1)}{(n+1)(n+2)}\sum_{0\le m\le \frac{n}{2}+1}\binom{n+2}{2m}(2m-1)(1-2^{1-2m})B_{2m}\,t^{n-2m+2}$
0	$-2(g-1)(t^2+\frac{1}{12})$
0 (*s-derivative*)	$\boxed{\mathscr{Z}'(0\,\vert\,t)=-2(g-1)\,\mathbf{z}'(0\,\vert\,t)-\log\zeta_S(\frac{1}{2}+t)}$
$+n\ge 1$	$-2(g-1)\,\mathbf{z}(n\,\vert\,t)+\dfrac{(-1)^{n-1}}{(n-1)!}(\log\zeta_S)^{(n)}(\frac{1}{2}+t)$ †

† At $n=1,2$: finite parts $\mathrm{FP}_{s=n}$ required for $\mathscr{Z}$ and $\mathbf{z}$

Table B.2 Specialization of Table B.1 to $t=\pm\mathrm{i}0$; all the values except the derivative are actually continuous at $t=0$. Compare with Table 10.5

s	$\mathscr{Z}_{0\pm}(s)\equiv\sum_\rho(\rho\pm\mathrm{i}0-\frac{1}{2})^{-s}$
$-n\le 0$	$-4(g-1)(1-2^{-n-1})\,\dfrac{B_{n+2}}{n+2}$ †
0	$-(g-1)/6$
0 (*s-derivative*)	$\boxed{[\mathscr{Z}_{0\pm}]'(0)=2(g-1)\,[\frac{1}{12}\log 2+\zeta'(-1)]-\log\zeta_S(\frac{1}{2}\pm\mathrm{i}0)}$
1	0 † *
+2 (*finite part*)	$\boxed{\mathrm{FP}_{s=2}\,\mathscr{Z}_{0\pm}(s)=-4(g-1)(\gamma+2\log 2)-(\log\zeta_S)''(\frac{1}{2})}$
$+n\ne 2$	$-4(g-1)(2^{n-1}-1)\,\zeta(n-1)+\dfrac{(-1)^{n-1}}{(n-1)!}(\log\zeta_S)^{(n)}(\frac{1}{2})$ †

† For n odd, $\mathscr{Z}_0(n)\equiv 0$ which makes $(\log\zeta_S)^{(n)}(\frac{1}{2})$ fully explicit, as in (C.4)

* Exceptionally at $t=0$, $s=1$ is a regular point

Table B.3 As Table B.1, but at $t = \frac{1}{2}$ and with the zero-mode removed. For the generalized Stieltjes cumulants $g_n^{\mathrm{c}}\langle S\rangle$, see (B.3). Compare with Table 10.6

s	$\mathscr{Z}_*(s) \equiv \sum_{\rho\neq 0,1} \rho^{-s}$
$-n < 0$	$-2(g-1)(-1)^n \left[2\frac{B_{n+2}}{n+2} + \frac{B_{n+1}}{n+1}\right] - 1$
0	$\frac{2}{3}(g-1) - 2$
0 *(derivative)*	$\boxed{\mathscr{Z}_*'(0) = -4(g-1)\,[\zeta'(-1) + \frac{1}{4}\log 2\pi] + g_0^{\mathrm{c}}\langle S\rangle}$
+1 *(finite part)*	$\boxed{\mathrm{FP}_{s=1}\,\mathscr{Z}_*(s) = 2(g-1)(1+\gamma) + g_1^{\mathrm{c}}\langle S\rangle - 1}$
+2 *(finite part)*	$\boxed{\mathrm{FP}_{s=2}\,\mathscr{Z}_*(s) = -4(g-1)(\gamma - \frac{1}{12}\pi^2) + g_2^{\mathrm{c}}\langle S\rangle - 1}$
$+n > 2$	$-2(g-1)\,[2\,\zeta(n-1) - \zeta(n)] + \frac{g_n^{\mathrm{c}}\langle S\rangle}{(n-1)!} - 1^{\dagger}$

† For k even, $\zeta(k) \equiv (2\pi)^k |B_k|/(2\,k!)$

Table B.4 As Table B.1, but for the function of second kind $\mathcal{Z}(\sigma\,|\,t)$; transcendental special values (lower part) are expressed in terms of those for $\mathscr{Z}(s\,|\,t)$, given by Table B.1; m, n are integers. Compare with Tables 8.1 (upper part) and 8.2 (lower part)

σ	$\mathcal{Z}(\sigma\,|\,t) = \sum_{k=0}^{\infty} (\kappa_k^2 + t^2)^{-\sigma}$	
$-m \leq 0$	$\frac{g-1}{m+1} \sum_{n=0}^{m+1} (-1)^n \binom{m+1}{n} (1 - 2^{1-2n}) B_{2n}\, t^{2(m+1-n)}$	
0	$-(g-1)(t^2 + \frac{1}{12})$	
0 *(σ-/s-derivatives)*	$\boxed{\mathcal{Z}'_{(\sigma)}(0\,|\,t) = \mathscr{Z}'_{(s)}(0\,|\,t) + 2(g-1)\,t^2}$	
+1 *(finite part)*	$\boxed{\mathrm{FP}_{\sigma=1}\,\mathcal{Z}(\sigma\,|\,t) = \frac{1}{2t}\,\mathrm{FP}_{s=1}\,\mathscr{Z}(s\,|\,t) - 2(g-1)}$	$(t \neq 0)^*$
$+m \geq 2$	$\sum_{n=1}^{m} \binom{2m-n-1}{m-1} (2t)^{-2m+n}\, \mathscr{Z}_\infty(n\,|\,t)^{\dagger}$	$(t \neq 0)^*$

* For $t = 0$, see Table B.5 below
† With $\mathscr{Z}_\infty(n\,|\,t) = \mathrm{FP}_{s=n}\,\mathscr{Z}(s\,|\,t) - \delta_{n,2}\,4(g-1)$, see (B.23) (FP is unneeded for $n > 2$)

Table B.5 As Table B.4, but at $t = (1 \pm \mathrm{i})\,0$; all the values except the derivative are actually continuous at $t = 0$. Cf. Table B.2

σ	$\mathcal{Z}_{0\pm}(\sigma) = \sum_{k=0}^{\infty} (\kappa_k^2 \pm \mathrm{i}0)^{-\sigma}$
$-m \leq 0$	$\frac{1}{2}(-1)^m\, \mathscr{Z}_{0\pm}(-2m)$
0 *(σ-/s-derivatives)*	$\boxed{[\mathcal{Z}_{0\pm}]'_{(\sigma)}(0) = [\mathscr{Z}_{0\pm}]'_{(s)}(0)}$
+1 *(finite part)*	$\boxed{\mathrm{FP}_{\sigma=1}\,\mathcal{Z}_{0\pm}(\sigma) = -\frac{1}{2}\mathrm{FP}_{s=2}\,\mathscr{Z}_{0\pm}(s)}$
$+m \geq 2$	$\frac{1}{2}(-1)^m\, \mathscr{Z}_{0\pm}(+2m)$

Table B.6 As Table B.4, but at $t = \frac{1}{2}$ and with the zero-mode removed ($\mathcal{Z}_*(\sigma)$ is the Minakshisundaram–Pleijel zeta function); cf. Table B.3

σ	$\mathcal{Z}_*(\sigma) = \sum_{k=1}^{\infty} (\kappa_k^2 + \frac{1}{4})^{-\sigma} \equiv \sum_{k=1}^{\infty} v_k^{-\sigma}$
$-m < 0$	$\frac{g-1}{m+1} 2^{-2m-1} \sum_{n=0}^{m+1} (-1)^n \binom{m+1}{n} (2^{2n-1} - 1) B_{2n}$
0	$-\frac{1}{3}(g-1) - 1$
0 *(σ-/s-derivatives)*	$\boxed{\mathcal{Z}'_{*(\sigma)}(0) = \mathscr{Z}'_{*(s)}(0) + \frac{1}{2}(g-1)}$
$+1$ *(finite part)*	$\boxed{\mathrm{FP}_{\sigma=1} \mathcal{Z}_*(\sigma) = \mathrm{FP}_{s=1} \mathscr{Z}_*(s) - 2(g-1)}$
$+m \geq 2$	$\sum_{n=1}^{m} \binom{2m-n-1}{m-1} \mathscr{Z}_{*,\infty}(n)$ †

† With $\mathscr{Z}_{*,\infty}(n) = \mathrm{FP}_{s=n} \mathscr{Z}_*(s) - \delta_{n,2} 4(g-1)$, see (B.23) (FP is unneeded for $n > 2$)

Appendix C
On the Logarithmic Derivatives at $\frac{1}{2}$

We briefly revisit the formulae that involve $(\log|L|)^{(n)}(\frac{1}{2})$ from the viewpoint of the primary function L itself when, as in Sect. 10.6, the latter is either the L-function L_χ of a real primitive Dirichlet character χ (= Dirichlet-L cases) or the Dedekind zeta function ζ_K of an algebraic number field K (= Dedekind-ζ cases, which include $\zeta(x)$ for $K=\mathbb{Q}$).

For even $n>0$, those derivatives relate to the corresponding superzeta values $\mathscr{Z}_0(n)$ according to Tables 10.3 and 10.5 (last lines), which is far from an explicit evaluation. At $n=0$, $\log|L|(\frac{1}{2})$ gets related to $\mathscr{Z}_0'(0)$. And even in the basic case: $L=\zeta$, $n=0$, very little seems to be known about $\zeta(\frac{1}{2})$ [81][11, Knuth's sequence pp. 16–17].

In contrast, for odd $n>0$, by virtue of (10.59) and (10.78) (which follow from the functional equation (10.3)), those derivatives *get specified*, as

$$(\log|L|)^{(n)}(\tfrac{1}{2}) \equiv \begin{cases} \frac{1}{2}(n-1)!\,\big[c\,(2^n-1)\,\zeta(n)+c'\,2^n\beta(n)\big], & n=3,5,\dots \\ \frac{1}{2}\,c\,\gamma+\frac{1}{4}\,c'\,\pi+\frac{1}{2}\log\big[(8\pi)^c/d\big], & n=1, \end{cases} \tag{C.1}$$

for suitable (integers) c, c', d: thus in a Dirichlet-L case, by (10.59), $c=1$, $c'=1-2a$ and $d=$ the modulus (or period) of the character χ; while in a Dedekind-ζ case, by (10.78), $c=n_K$, $c'=r_1$, and $d=|d_K|$. These formulae are more explicit than for even n but not yet fully: with n being odd, $\frac{1}{2}(n-1)!\,2^n\beta(n)$ reduces to $\frac{1}{4}\pi^n|E_{n-1}|$ by (3.32), but $\zeta(n)$ (and γ for $n=1$) remain elusive (irreducible); e.g., $(\log|\zeta|)'(\frac{1}{2})=\frac{1}{2}\gamma+\frac{1}{4}\pi+\frac{1}{2}\log 8\pi$.

However, related odd-n identities can manage to eliminate $\zeta(n)$ and γ: if (L_1,L_2) is any *pair* of such functions, each indifferently Dirichlet-L or Dedekind-ζ, then a linear combination (with obvious notation) yields the *fully explicit* relations

$$\begin{aligned} & c_2(\log|L_1|)^{(n)}(\tfrac{1}{2})-c_1(\log|L_2|)^{(n)}(\tfrac{1}{2})\equiv \\ & \qquad (c_2c_1'-c_1c_2')\,\tfrac{1}{4}\pi^n|E_{n-1}|+\delta_{n,1}\,\tfrac{1}{2}\log(d_2^{c_1}/d_1^{c_2}) \qquad \text{for odd } n\ge 1. \end{aligned} \tag{C.2}$$

An immediate example is $L_1 = \zeta,\ L_2 = \beta$, giving [106, (90)]

$$(\log|\zeta|)^{(n)}(\tfrac{1}{2}) - (\log\beta)^{(n)}(\tfrac{1}{2}) \equiv \tfrac{1}{2}\pi^n |E_{n-1}| + \delta_{n,1}\,\log 2 \quad \text{for odd } n \geq 1; \tag{C.3}$$

see also (10.60) for another simple case.

Incidentally, the situation is simpler with Selberg zeta functions $\boldsymbol{\zeta}_S$ as in Appendix B: for any odd $n \geq 1$, either the identity $\mathscr{Z}_0(n) = 0$, or Selberg's functional equation (B.2), entails the *fully* explicit values

$$\left.\begin{aligned}(\log|\boldsymbol{\zeta}_S|)^{(n)}(\tfrac{1}{2}) &\equiv 4(g-1)(n-1)!\,(2^{n-1}-1)\,\zeta(n-1)\\ &= 2(g-1)(2^{n-1}-1)(2\pi)^{n-1}|B_{n-1}|\end{aligned}\right\} \text{ for odd } n \geq 1, \tag{C.4}$$

where in particular $n = 1$ gives $(\log|\boldsymbol{\zeta}_S|)'(\tfrac{1}{2}) \equiv 0$.

Appendix D
On the Zeros of the Zeta Function by Hj. Mellin (1917)

Annotated translation from German, by A. Voros, of the article:

Über die Nullstellen der Zetafunktion
Ann. Acad. Sci. Fennicae **A10** Nr 11 (1917).

We include this English translation of Mellin's seminal paper featuring the first zeta functions over the Riemann zeros to our knowledge. The original is written in German and available in few libraries, hence we believe this Appendix can make that article accessible to a wider audience.

We indicate the errors and misprints we noticed, but we did not carry out a systematic verification. We have strictly preserved the author's notation, which is therefore not consistent with the rest of this book (cf. our discussion of this article in Sect. 5.5). We have not indexed the contents either.

Numbered footnotes are the author's, the others are translator's notes (A.V.).

On the zeros of the zeta function

§ 1.

The series, extended over all the complex zeros ρ of the zeta function,

$$\sum_{\rho} \frac{1}{|\rho|^{\kappa}},$$

is known to converge [1]) for all $\kappa > 1$ and to diverge for $\kappa < 1$.

It follows that the series

$$\sum_{\rho} \frac{x^{\rho}}{\rho^{s}}, \qquad x > 0,$$

where x is a positive parameter, converges absolutely and uniformly in every finite part of the half-plane

$$\Re(s) \geq 1 + \varepsilon \qquad \varepsilon > 0.$$

In an important work [2]) dedicated to the above series, Mr Landau among others proved that the series converges for $0 < s < 1$ if and only if x is neither 1 nor p^m nor p^{-m}, where p is any prime number and m a positive integer.

I will now prove some of the *analytical continuation* properties of the same series in question, that I noticed several years ago.

Let these functions be defined by the following series:

$$Z(s,x) = \sum_{\rho} \frac{x^{\rho}}{\rho^{s}}, \tag{1}$$

$$Z\left(s, \frac{1}{x}\right) = \sum_{\rho} \frac{x^{-\rho}}{\rho^{s}}, \tag{2}$$

$$\bar{Z}(s,x) = \sum_{\rho} \frac{x^{s}}{(-\rho)^{s}},^{\dagger} \tag{3}$$

$$\bar{Z}\left(s, \frac{1}{x}\right) = \sum_{\rho} \frac{x^{-s}}{(-\rho)^{s}},^{\dagger} \tag{4}$$

$$Z(s) = \sum_{\rho} \frac{1}{\rho^{s}}, \tag{5}$$

$$\bar{Z}(s) = \sum_{\rho} \frac{1}{(-\rho)^{s}}. \tag{6}$$

---

[1]) See Landau, Handbuch der Lehre von der Verteilung der Primzahlen [Lessons on the distribution of the prime numbers], I, p. 314.

[2]) See Landau, Über die Nullstellen der Zetafunktion [On the zeros of the Zeta function], Math. Annalen 71, p. 548, 1911. [*Actually*: 1912. (A.V.)]

† *Misprints*: x^s should read x^ρ in (3); x^{-s} should read $x^{-\rho}$ in (4). (A.V.)

I always assume the parameter $x > 1$ in the first four series. The expressions

$$\rho^s,\ (-\rho)^s,\ x^\rho,\ x^{-\rho}$$

are defined through

$$\begin{aligned}\rho^s &= e^{s[\log|\rho| + i\,\mathrm{arc}\,(\rho)]},\\ (-\rho)^s &= e^{s[\log|\rho| + i\,\mathrm{arc}\,(-\rho)]},\\ x^\rho &= \frac{1}{x^{-\rho}} = e^{\rho\log x}\end{aligned}$$

with the arc taken between $-\pi$ and $+\pi$.

It now follows that:

The functions Z and $\bar{Z}$ defined by the first four series (1) to (4) are, for all $x > 1$, entire functions of s, which become rational functions of x for $s = 0$ and for negative integer values $s = -n$, so that

$$(7)\quad \begin{cases} Z(0,x) = x - \sum_{\nu=1}^{\infty} x^{-2\nu} = x - \dfrac{1}{x^2-1}, \\ Z(-n,x) = x - \sum_{\nu=1}^{\infty}(-2\nu)^n x^{-2\nu} = x - \left(x\dfrac{d}{dx}\right)^n \dfrac{1}{x^2-1}, \end{cases}$$

$$(8)\quad \begin{cases} Z\left(0,\dfrac{1}{x}\right) = -1 - \sum_{\nu=1}^{\infty} x^{-2\nu-1} = -1 - \dfrac{1}{x(x^2-1)}, \\ Z\left(-n,\dfrac{1}{x}\right) = -\sum_{\nu=1}^{\infty}(2\nu+1)^n x^{-2\nu-1} = (-1)^{n+1}\left(x\dfrac{d}{dx}\right)^n \dfrac{1}{x(x^2-1)}, \end{cases}$$

$$(9)\quad \begin{cases} \bar{Z}(0,x) = x - \sum_{\nu=1}^{\infty} x^{-2\nu} = x - \dfrac{1}{x^2-1}, \\ \bar{Z}(-n,x) = (-1)^n x - \sum_{\nu=1}^{\infty}(2\nu)^n x^{-2\nu} = (-1)^n\left[x - \left(x\dfrac{d}{dx}\right)^n \dfrac{1}{x^2-1}\right], \end{cases}$$

$$(10)\quad \begin{cases} \bar{Z}\left(0,\dfrac{1}{x}\right) = -1 - \sum_{\nu=1}^{\infty} x^{-2\nu-1} = -1 - \dfrac{1}{x(x^2-1)}, \\ \bar{Z}\left(-n,\dfrac{1}{x}\right) = -(-1)^n\sum_{\nu=1}^{\infty}(2\nu+1)^n x^{-2\nu-1} = -\left(x\dfrac{d}{dx}\right)^n \dfrac{1}{x(x^2-1)}. \end{cases}$$

In these formulas $\left(x\dfrac{d}{dx}\right)^n$ means the n-th iteration of the operator $x\dfrac{d}{dx}$.

The functions Z and $\bar{Z}$ defined by the series (5) and (6) are regular in the whole s-plane, except for the simple pole $s = 1$, with the common residue

$$(11)\qquad \lim_{s=1}\,(s-1)Z(s) = \lim_{s=1}\,(s-1)\bar{Z}(s) = -\frac{1}{2}.$$

For $s = 0$ and for negative integers $s = -n$,

$$(12) \qquad \begin{cases} Z(0) = 1, \; ^{\dagger} \\ Z(-n) = (2^n - 1)\,\zeta(-n), \; ^{\dagger} \end{cases}$$

$$(13) \qquad \begin{cases} \bar{Z}(0) = -\frac{1}{2}, \; ^{\ddagger} \\ \bar{Z}(-n) = (-1)^{n+1} - 2^n\,\zeta(-n). \; ^{\ddagger} \end{cases}$$

§ 2.

The function

$$Z\Big(s, \frac{1}{x}\Big) = \sum_{\rho} \frac{x^{-\rho}}{\rho^s}.$$

I use the well-known formula [1])

$$(14) \qquad F(x,t) = \frac{x^{1-t}}{1-t} + \sum_{\nu=1}^{\infty} \frac{x^{-2\nu-t}}{2\nu+t} - \sum_{\rho} \frac{x^{\rho-t}}{\rho-t} - \frac{\zeta'(t)}{\zeta(t)},$$

which is valid for $x > 1$, and where

$$(15) \qquad F(x,t) = {\sum_{p^m \le x}}' \frac{\log p}{p^{mt}}.$$

The dash means that when x is a prime power, the last term will carry the factor $\frac{1}{2}$.

First, I write (14) as follows:

$$(16) \qquad F(x,t)\,x^t + \frac{\zeta'(t)}{\zeta(t)}\,x^t - \frac{x}{1-t} = \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{2\nu+t} + \sum_{\rho} \frac{x^{\rho}}{-\rho+t}.$$

Since the series considered here converge uniformly [2]) in any finite root-free regions of t, it is permitted to differentiate term by term, which yields:

$$(17) \qquad G(x,t) - \frac{x}{(1-t)^2} = \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu+t)^2} + \sum_{\rho} \frac{x^{\rho}}{(-\rho+t)^2}.$$

† *Error*: (12) should read $Z(0) = 2$, $Z(-n) = (2^n - 1)\zeta(-n) + 1$, upon correcting (26) below. (A.V.)

‡ *Error*: (13) should read $\bar{Z}(0) = \frac{3}{2}$, $\bar{Z}(-n) = (-1)^n - 2^n\zeta(-n)$ upon the corrections in §. 5 below. (A.V.)

[1]) See LANDAU, Handbuch, I. p. 353.

[2]) See LANDAU, Handbuch, I. p. 354.

Here the abbreviation

$$G(x,t) = -\frac{\partial}{\partial t}\Big\{F(x,t)\,x^t + \frac{\zeta'(t)}{\zeta(t)}\,x^t\Big\} \tag{18}$$

is used. Because of (15) and

$$\frac{\zeta'(t)}{\zeta(t)} = -\sum_{p,m} \frac{\log p}{p^{mt}} \qquad \Re(t) > 1,$$

G can be represented in the half-plane

$$\Re(t) \geq 1 + \varepsilon \qquad \varepsilon > 0$$

by the absolutely and uniformly convergent series

$$G(x,t) = \sum_{p^m \geq x} \Big(\frac{x}{p^m}\Big)^t \log p \, \log\Big(\frac{x}{p^m}\Big). \tag{19}$$

In case $x =$ a prime power, the first term in (19) vanishes, so that in the angular sector

$$-\frac{\pi}{2} + \varepsilon \leq \operatorname{arc}(t) \leq +\frac{\pi}{2} - \varepsilon$$

for any constant k and any constant $x > 1$,

$$\lim_{t=\infty} t^k\, G(x,t) = 0 \tag{20}$$

uniformly. The corresponding expression in (16),

$$F(x,t)\,x^t + \frac{\zeta'(t)}{\zeta(t)}\,x^t = -\sum_{p^m \geq x}{}' \Big(\frac{x}{p^m}\Big)^t \log p,$$

does not have this property in the case mentioned, because it contains a t-independent, non-vanishing term.

In (17), I now replace t by $1+t$, multiply by $t^{s-1}dt$ and integrate from $t=0$ to $t=\infty$. By using the formula

$$\int_0^\infty \frac{t^{s-1}dt}{(a+t)^z} = \frac{\Gamma(s)\,\Gamma(z-s)}{\Gamma(z)} \cdot \frac{1}{a^{z-s}} \qquad \begin{array}{c} 0 < \Re(s) < z \\ -\pi < \operatorname{arc}(a) < +\pi \end{array}$$

i.e., in this case

$$\int_0^\infty \frac{t^{s-1}dt}{(a+t)^2} = \frac{\pi(1-s)}{\sin \pi s} \cdot \frac{1}{a^{2-s}},$$

then the result is

(21)

$$\int_0^\infty \big\{G(x,1+t) - xt^{-2}\big\}\, t^{s-1}dt = \frac{\pi(1-s)}{\sin \pi s}\left[\sum_{\nu=1}^\infty \frac{x^{-2\nu}}{(2\nu+1)^{2-s}} + \sum_\rho \frac{x^\rho}{(1-\rho)^{2-s}}\right].$$

The first series on the right-hand side converges when $x > 1$ for all s, the second one is absolutely convergent for $\Re(s) < 1$. The left-hand side integral, which converges for

$$0 < \Re(s) < 2,$$

can be transformed as follows. In the vicinity of $t = 0$,

$$G(x, 1+t) - xt^{-2}$$

can be developed in an ordinary power series:

$$G(x, 1+t) - xt^{-2} = \sum_{\nu=0}^{\infty} C_\nu t^\nu.$$

Let r be positive and smaller than the radius of convergence of this series. Then

$$\int_0^r \left\{G(x, 1+t) - xt^{-2}\right\} t^{s-1} dt = \sum_{\nu=0}^{\infty} C_\nu \frac{r^{s+\nu}}{s+\nu}.$$

The right-hand side is a regular function in the whole s-plane except at the poles $s = 0,\ -1,\ -2, \ldots$. Next,

$$\int_r^\infty \left\{G(x, 1+t) - xt^{-2}\right\} t^{s-1} dt = \int_r^\infty G(x, 1+t)\, t^{s-1} dt + x\frac{r^{s-2}}{s-2}.$$

The right-hand side integral, because of (20), is an entire function $H(s)$ of s. Setting these expressions for

$$\int_0^\infty \{\ \}\, t^{s-1} dt = \int_0^r + \int_r^\infty$$

in (21), one gets the formula

$$\sum_{\nu=0}^{\infty} C_\nu \frac{r^{s+\nu}}{s+\nu} + H(s) + x\frac{r^{s-2}}{s-2} = \frac{\pi(1-s)}{\sin \pi s}\left[\sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu+1)^{2-s}} + \sum_\rho \frac{x^\rho}{(1-\rho)^{2-s}}\right],$$

or, if one substitutes s by $2-s$ and observes that $1-\rho$ simultaneously with ρ run over all nonreal roots,

(22)

$$\sum_{\nu=1}^{\infty} \frac{x^{-2\nu-1}}{(2\nu+1)^s} + \sum_\rho \frac{x^{-\rho}}{\rho^s} = x^{-1}\frac{\sin \pi s}{\pi(1-s)}\left[\sum_{\nu=0}^{\infty} C_\nu \frac{r^{2-s+\nu}}{2-s+\nu} + H(2-s) - x\frac{r^{-s}}{s}\right].$$

The right-hand side is now manifestly an entire function of s, which has the value -1 for $s = 0$ and vanishes for all negative integer values of s. Since the first series on the left $(x > 1)$ is also an entire function of s, which for

$s = 0, -1, -2, -3, \ldots$ yields certain rational functions of x, so the function $Z\left(s, \frac{1}{x}\right)$ has the properties stated in §. 1.

§ 3.

The function

$$\bar{Z}(s, x) = \sum_{\rho} \frac{x^{\rho}}{(-\rho)^s}.$$

In this case I multiply (17) by $t^{s-1}dt$ and integrate along the straight lines

$$0 \;\underline{\qquad\qquad\qquad}\; \infty\, e^{i\theta} \qquad\qquad 0 < \theta < \frac{\pi}{2}.$$

Let θ be small enough so that no root ρ lies in the angle

$$0 < \operatorname{arc}(t) \leq \theta.$$

Then for $0 < \Re(s) < 1$,

$$\begin{aligned}
(23) \qquad \int_0^{\infty e^{i\theta}} &\left\{ G(x, t) - \frac{x}{(1-t)^2} \right\} t^{s-1} dt = \\
&= \int_0^{\infty e^{i\theta}} \left\{ \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu + t)^2} + \sum_{\rho} \frac{x^{\rho}}{(-\rho + t)^2} \right\} t^{s-1} dt \\
&= \int_0^{\infty} \left\{ \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu + t)^2} + \sum_{\rho} \frac{x^{\rho}}{(-\rho + t)^2} \right\} t^{s-1} dt \\
&= \frac{\pi(1-s)}{\sin \pi s} \left[\sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu)^{2-s}} + \sum_{\rho} \frac{x^{\rho}}{(-\rho)^{2-s}} \right].
\end{aligned}$$

The integral (23) can be transformed as follows:

$$\begin{aligned}
\int_0^{\infty e^{i\theta}} \{\ \} t^{s-1} dt &= \int_0^{\infty e^{i\theta}} G(x, t)\, t^{s-1} dt + x \int_0^{\infty e^{i\theta}} \frac{t^{s-1} dt}{(-1+t)^2} \\
&= \int_0^{re^{i\theta}} G\, t^{s-1} dt + \int_{re^{i\theta}}^{\infty e^{i\theta}} G\, t^{s-1} dt + x \int_0^{-\infty} \frac{t^{s-1} dt}{(-1+t)^2} \\
&= \int_0^{re^{i\theta}} G\, t^{s-1} dt + \int_{re^{i\theta}}^{\infty e^{i\theta}} G\, t^{s-1} dt + x\, e^{\pi i s} \int_0^{\infty} \frac{t^{s-1} dt}{(1+t)^2}.
\end{aligned}$$

r is smaller than the radius of convergence of

$$G(x, t) = \sum_{\nu=0}^{\infty} b_{\nu} t^{\nu},$$

so that

$$\int_0^{re^{i\theta}} G\, t^{s-1} dt = \sum_{\nu=0}^{\infty} b_\nu \, \frac{(re^{i\theta})^{s+\nu}}{s+\nu}.$$

The second integral on the right is, because of (20), an entire function: $\bar{H}(s)$ of s:

$$\int_{re^{i\theta}}^{\infty e^{i\theta}} G\, t^{s-1} dt = \bar{H}(s),$$

while

$$\int_0^{\infty} \frac{t^{s-1} dt}{(1+t)^2} = \frac{\pi(1-s)}{\sin \pi s}.$$

This puts (23) in the form

$$\sum_{\nu=0}^{\infty} b_\nu \frac{(re^{i\theta})^{s+\nu}}{s+\nu} + \bar{H}(s) + x e^{\pi i s} \frac{\pi(1-s)}{\sin \pi s} = \frac{\pi(1-s)}{\sin \pi s} \left[\sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu)^{2-s}} + \sum_{\rho} \frac{x^{\rho}}{(-\rho)^{2-s}} \right].$$

Replacing s by $2-s$, one finally gets

(24)

$$\sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu)^s} + \sum_{\rho} \frac{x^{\rho}}{(-\rho)^s} = \frac{\sin \pi s}{\pi(1-s)} \left[\sum_{\nu=0}^{\infty} b_\nu \frac{(re^{i\theta})^{2-s+\nu}}{2-s+\nu} + \bar{H}(2-s) \right] + x\, e^{\pi i s}.$$

The right-hand side is obviously an entire function of s, which is equal to $(-1)^n\, x$ for $s = 0$ and for negative integer values $s = -n$. The first series on the left of (24) is manifestly $(x > 1)$ also an entire function of s, which for $s = 0,\ -1,\ -2, \ldots$ gives some easily determined rational functions of x. It follows directly that $\bar{Z}(s, x)$ has the properties listed in §. 1.

§ 4.

The function

$$Z(s) = \sum_{\rho} \frac{1}{\rho^s}.$$

From the known formula [1])

$$\frac{\zeta'(t)}{\zeta(t)} = C - \frac{1}{t-1} - \frac{1}{2} \frac{\Gamma'(\frac{t}{2}+1)}{\Gamma(\frac{t}{2}+1)} + \sum_{\rho} \Big(\frac{1}{t-\rho} + \frac{1}{\rho} \Big)$$

$$= C_1 - \frac{1}{t-1} + \frac{1}{t+2} + \sum_{\nu=1}^{\infty} \Big(\frac{1}{t+2+2\nu} - \frac{1}{2\nu} \Big) + \sum_{\rho} \Big(\frac{1}{t-\rho} + \frac{1}{\rho} \Big)$$

[1]) See LANDAU, Handbuch, I. p. 316.

it follows by differentiation

$$-\left(\frac{\zeta'(t)}{\zeta(t)}\right)' + \frac{1}{(t-1)^2} = \sum_{\nu=1}^{\infty} \frac{1}{(t+2\nu)^2} + \sum_{\rho} \frac{1}{(t-\rho)^2}. \tag{25}$$

I replace t by $t+1$ and, through integration for $0 < \Re(s) < 1$, obtain

$$-\int_0^\infty \left\{\left(\frac{\zeta'(1+t)}{\zeta(1+t)}\right)' - \frac{1}{t^2}\right\} t^{s-1} dt = \frac{\pi(1-s)}{\sin \pi s}\left[\sum_{\nu=1}^{\infty} \frac{1}{(2\nu+1)^{2-s}} + \sum_{\rho} \frac{1}{(1-\rho)^{2-s}}\right].$$

The integral allows exactly the same transformations as the corresponding integral in §. 2. One thus arrives, after replacing $2-s$ by s, at the end result

$$\begin{aligned} \sum_{\nu=1}^{\infty} \frac{1}{(2\nu+1)^s} + \sum_{\rho} \frac{1}{\rho^s} &= \left(1 - \frac{1}{2^s}\right)\zeta(s) + Z(s)^{\dagger} \\ &= \frac{\sin \pi s}{\pi(1-s)}\left[\sum_{\nu=0}^{\infty} C_\nu \frac{r^{2-s+\nu}}{2-s+\nu} + \frac{r^{-s}}{s} + H(2-s)\right], \end{aligned} \tag{26}$$

where H is an entire function. This results in the properties listed in §. 2 ‡ concerning $Z(s)$.

§ 5.

The function

$$\bar{Z}(s) = \sum_{\rho} \frac{1}{(-\rho)^s}.$$

From (25) one obtains, by integration along the lines

$$0 \;\rule{4cm}{0.4pt}\; \infty\, e^{i\theta} \qquad 0 < \theta < \frac{\pi}{2},$$

$$\begin{aligned} &-\int_0^{\infty e^{i\theta}} \left(\frac{\zeta'(t)}{\zeta(t)}\right)' t^{s-1} dt = \\ &= -\int_0^{\infty e^{i\theta}} \frac{t^{s-1}dt}{(-1+t)^2} + \int_0^{\infty e^{i\theta}} t^{s-1} \sum_{\nu=1}^{\infty} \frac{1}{(2\nu+t)^2}\, dt + \int_0^{\infty e^{i\theta}} t^{s-1} \sum_{\rho} \frac{1}{(-\rho+t)^2}\, dt \\ &= -\int_0^{-\infty} \frac{t^{s-1}dt}{(-1+t)^2} + \int_0^{+\infty} t^{s-1} \sum_{\nu=1}^{\infty} \frac{1}{(2\nu+t)^2}\, dt + \int_0^{+\infty} t^{s-1} \sum_{\rho} \frac{1}{(-\rho+t)^2}\, dt \\ &= \frac{\pi(1-s)}{\sin \pi s}\left[e^{\pi i s} + \sum_{\nu=1}^{\infty} \frac{1}{(2\nu)^{2-s}} + \sum_{\rho} \frac{1}{(-\rho)^{2-s}}\right]^{*}, \end{aligned}$$

† *Error*: the middle member of (26) should read $(1-\frac{1}{2^s})\zeta(s) - 1 + Z(s)$, and this affects the results (12) above. (A.V.)

‡ *Actually*: §. 1. (A.V.)

* *Error*: the term $e^{\pi i s}$ inside the brackets should read $-e^{\pi i s}$, and this error propagates to (27) then to (13). (A.V.)

where θ is so small that it can be presumed there is no ρ in the angle

$$0 < \operatorname{arc}(t) \leq \theta.$$

Just as in §. 3, the first integral can be brought to the form

$$\sum_{\nu=0}^{\infty} C_\nu \frac{(re^{i\theta})^{s+\nu}}{s+\nu} + \bar{H}(s),$$

where $\bar{H}$ is an entire function. The final result is, after replacing $2-s$ by s,

$$(27) \qquad e^{-\pi is} + \sum_{\nu=1}^{\infty} \frac{1}{(2\nu)^s} + \sum_{\rho} \frac{1}{(-\rho)^s} = e^{-\pi is} + \frac{1}{2^s}\zeta(s) + \bar{Z}(s)^{\dagger} = \frac{\sin \pi s}{\pi(1-s)} \left[\sum_{\nu=0}^{\infty} C_\nu \frac{(re^{i\theta})^{2-s+\nu}}{2-s+\nu} + \bar{H}(2-s) \right].$$

There follows the validity of the claims issued in §. 1 concerning $\bar{Z}(s)$.

§ 6.
The function

$$Z(s,x) = \sum_{\rho} \frac{x^\rho}{\rho^s}.$$

In formula (14) I replace t by $-t$ and use the formula

$$\frac{\zeta'(-t)}{\zeta(-t)} = -\log(2\pi) + \frac{\pi}{2}\cot\frac{\pi t}{2} + \frac{\Gamma'(1+t)}{\Gamma(1+t)} + \frac{\zeta'(1+t)}{\zeta(1+t)}.$$

Multiplying the resulting formula by x^{-t} and differentiating, one gets

$$(28) \quad \frac{\partial}{\partial t}\left\{ F(x,-t)\, x^{-t} + x^{-t}\log(2\pi) - x^{-t}\frac{\Gamma'(1+t)}{\Gamma(1+t)} - x^{-t}\left(\frac{\pi}{2}\cot\frac{\pi t}{2} + \frac{\zeta'(1+t)}{\zeta(1+t)}\right)\right\} = -\frac{x}{(1+t)^2} + \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu-t)^2} + \sum_{\rho} \frac{x^\rho}{(\rho+t)^2}.$$

In the vicinity of $t=0$, the left-hand side can be expanded in a standard power series

$$\frac{\partial}{\partial t}\left\{\ \right\} = \sum_{\nu=0}^{\infty} C_\nu t^\nu.$$

† *Error*: both terms $e^{-\pi is}$ in (27) should read $-e^{-\pi is}$, and this affects the results (13) above. (A.V.)

Next, in the angle

$$-\frac{\pi}{2} + \varepsilon \leq \operatorname{arc}(t) \leq +\frac{\pi}{2} - \varepsilon$$

(ε positive, and arbitrarily small), this function behaves everywhere regularly, and upon multiplication with an arbitrary high power of t, it tends to the limit zero uniformly for increasing $|t|$ ($x > 1$). Also, r is positive and smaller than the radius of convergence of the above series, hence along the lines

$$0 \,\rule{4cm}{0.4pt}\, \infty\, e^{i\theta} \qquad\qquad 0 < \theta < \frac{\pi}{2},$$

the integral of the left-hand side of (28) multiplied by $t^{s-1}dt$,

$$\int_0^{\infty e^{i\theta}} t^{s-1} \frac{\partial}{\partial t}\Big\{\quad\Big\}\, dt$$

can be brought to the form

$$\int_0^{\infty e^{i\theta}} t^{s-1} \frac{\partial}{\partial t}\Big\{\quad\Big\}\, dt = \sum_{\nu=0}^{\infty} C_\nu \frac{(re^{i\theta})^{s-\nu}}{s+\nu} + H(s), \;^{\dagger}$$

where H is an entire function. The integrals performed on the right-hand side of (28) give the following results,

$$-x\int_0^{\infty e^{i\theta}} \frac{t^{s-1}}{(1+t)^2} = -x\int_0^{\infty} \frac{t^{s-1}}{(1+t)^2} = -x\,\frac{\pi(1-s)}{\sin \pi s}\,,$$

$$\int_0^{\infty e^{i\theta}} t^{s-1} \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu - t)^2}\, dt = \int_0^{-\infty} t^{s-1} \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu - t)^2}\, dt$$

$$= e^{\pi i s}\int_0^{+\infty} t^{s-1} \sum_{\nu=0}^{\infty} \frac{x^{-2\nu}}{(2\nu + t)^2}\, dt\,^{\ddagger} = e^{\pi i s}\,\frac{\pi(1-s)}{\sin \pi s} \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu)^{2-s}}\,,$$

$$\int_0^{\infty e^{i\theta}} t^{s-1} \sum_{\rho} \frac{x^{\rho}}{(\rho + t)^2}\, dt = \int_0^{+\infty} t^{s-1} \sum_{\rho} \frac{x^{\rho}}{(\rho + t)^2}\, dt = \frac{\pi(1-s)}{\sin \pi s} \sum_{\rho} \frac{x^{\rho}}{\rho^{2-s}}\,,$$

where θ is so small that it can be presumed there is no ρ in the angle $0 < \operatorname{arc}(t) \leq \theta$. One also has, upon replacing $2 - s$ by s:

$$(29)\quad e^{-\pi i s} \sum_{\nu=1}^{\infty} \frac{x^{-2\nu}}{(2\nu)^s} + \sum_{\rho} \frac{x^{\rho}}{\rho^s} = x + \frac{\sin \pi s}{\pi(1-s)}\left[\sum_{\nu=0}^{\infty} C_\nu \frac{(re^{i\theta})^{2-s+\nu}}{2-s+\nu} + H(2-s)\right].$$

† *Misprint*: the exponent in the right-hand side should read $s + \nu$. (A.V.)

‡ *Misprint*: the summation $\sum_{\nu=0}^{\infty}$ should read $\sum_{\nu=1}^{\infty}$. (A.V.)

The right-hand side is obviously an entire function of s, which takes the value x for $s = 0$ and for negative integer values $s = -n$. The first series on the left-hand side is also ($x > 1$) an entire function of s, which for $s = 0$ and for negative integer values $s = -n$ turns into some rational functions of x, so that

$$\sum_{\nu=1}^{\infty} (-2\nu)^n \, x^{-2\nu} = \left(x \frac{d}{dx} \right)^n \frac{1}{x^2 - 1}.$$

Hence without further ado, one obtains the validity of the claims in §. 1 concerning $Z(s, x)$.

§ 7.

The function

$$\bar{Z}\left(s, \frac{1}{x}\right) = \sum_{\rho} \frac{x^{-\rho}}{(-\rho)^s}.$$

Replacing t by $t - 1$ in the formula (28), and bringing the term

$$-\frac{x}{t^2}$$

to the left-hand side, one is led by the above method (bearing in mind that $1 - \rho$ and ρ both run through the complex roots of $\zeta(t)$) to the formula

$$(30) \quad e^{\pi i s} \sum_{\nu=1}^{\infty} \frac{x^{-2\nu-1}}{(2\nu+1)^s} + \sum_{\rho} \frac{x^{-\rho}}{(-\rho)^s}$$
$$= x^{-1} \frac{\sin \pi s}{\pi(1-s)} \left[-x \frac{(re^{i\theta})^{-s}}{s} + \sum_{\nu=0}^{\infty} C_\nu \frac{(re^{i\theta})^{2-s+\nu}}{2-s+\nu} + H(2-s) \right],$$

which yields all that was said in §. 1 regarding $\bar{Z}\left(s, \frac{1}{x}\right)$.

§ 8.

The functions

$$Z_1(s, x) = \sum_{\gamma > 0} \frac{x^\rho}{\rho^s}, \qquad Z_2(s, x) = \sum_{\gamma < 0} \frac{x^\rho}{\rho^s},$$

$$Z_1\left(s, \frac{1}{x}\right) = \sum_{\gamma > 0} \frac{x^{-s}}{\rho^s}, \dagger \qquad Z_2\left(s, \frac{1}{x}\right) = \sum_{\gamma < 0} \frac{x^s}{\rho^s}, \dagger$$

† *Misprints*: both x^{-s} and x^s should read $x^{-\rho}$. (A.V.)

$$Z_1(s) = \sum_{\gamma>0} \frac{1}{\rho^s}, \qquad Z_2(s) = \sum_{\gamma<0} \frac{1}{\rho^s},$$

$$\rho = \beta + i\gamma, \qquad x > 1.$$

The nature of the functions defined by those series is now also easily obtained, if one notes that

$$\left\{\begin{aligned} Z(s,x) &= Z_1(s,x) + Z_2(s,x), \\ \bar{Z}(s,x) &= e^{\pi i s} Z_1(s,x) + e^{-\pi i s} Z_2(s,x); \end{aligned}\right. \tag{31}$$

$$\left\{\begin{aligned} Z\left(s,\frac{1}{x}\right) &= Z_1\left(s,\frac{1}{x}\right) + Z_2\left(s,\frac{1}{x}\right), \\ \bar{Z}\left(s,\frac{1}{x}\right) &= e^{\pi i s} Z_1\left(s,\frac{1}{x}\right) + e^{-\pi i s} Z_2\left(s,\frac{1}{x}\right); \end{aligned}\right. \tag{32}$$

$$\left\{\begin{aligned} Z(s) &= Z_1(s) + Z_2(s), \\ \bar{Z}(s) &= e^{\pi i s} Z_1(s) + e^{-\pi i s} Z_2(s). \end{aligned}\right. \tag{33}$$

From (31), for instance, it follows that

$$2i\, Z_1(s,x) = \frac{\bar{Z}(s,x) - e^{-\pi i s} Z(s,x)}{\sin \pi s}.$$

The numerator is an entire function of s, which vanishes not only at the points $s = 1,\ 2,\ 3, \cdots,\ \infty$ but also at the points $s = 0,\ -1,\ -2, \cdots,\ -\infty$. The former is due to the series [1] for $Z(s,x)$ and $\bar{Z}(s,x)$, the latter to the formulas (7) and (9). The same obviously apply also to the functions $Z_2(s,x)$, $Z_1\left(s,\frac{1}{x}\right)$, $Z_2\left(s,\frac{1}{x}\right)$.

[1]) By a result of VON MANGOLDT, the series

$$\sum_{\rho} \frac{x^\rho}{\rho}$$

converges for all $x > 0$. Concerning the point $s = 1$, one needs to prove that

$$\lim_{s=1}\ Z(s,x) = \sum_{\rho} \frac{x^\rho}{\rho} \quad \text{and} \quad \lim_{s=1}\ \bar{Z}(s,x) = \sum_{\rho} \frac{x^\rho}{-\rho},$$

but I will not elaborate on this further.

The functions $Z_1(s,x)$, $Z_2(s,x)$, $Z_1\left(s,\dfrac{1}{x}\right)$, $Z_2\left(s,\dfrac{1}{x}\right)$ *are also entire functions of* s *for every* $x>1$.

From the formulae (33), in conjunction with (12) and (13), it follows that:

The functions $Z_1(s)$ *and* $Z_2(s)$ *are meromorphic in the whole* s*-plane, with a double pole at the point* $s=1$ *and with simple poles at the points* $s=0,\ -1,\ -2,\cdots,\ -\infty$.

The results developed in this work can be generalized without difficulty. I will only allude to that. Setting

$$Z(s,x,a)=\sum_{\rho}\frac{x^{\rho+a}}{(\rho+a)^s},$$

where x is positive, and initially different from 1, then Z is an entire function of s. If $a=\frac{m}{n}$ is rational, then for $s=0$ and for negative integer s, Z turns into certain rational functions of $\sqrt[n]{x}$.

Helsingfors, January 1917.

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