Appendix A
Stiffness and Inertia Matrices for a Ramified System Consisting of Rigid Bodies Connected by Beam Elements

Let us consider a system consisting of rigid bodies connected by beam elements. In order to obtain the stiffness and inertia matrices for this system, let us consider first a beam element connecting two rigid bodies [15].

Let us choose a stationary coordinate system ($x_1y_1z_1$) fixed to body 1 (Fig. A.1); the zero of the coordinate system O_1 is located at the center of mass of the body. Let us assume that this body is linked to an elastic beam element; its fixing point O_1' is at a distance from the center of mass equal to radius vector $\boldsymbol{\rho}_1$. Let us now choose a local coordinate system linked by a beam element ($x_1'y_1'z_1'$) whose beginning is at the fixing point O_1' and the coordinate axes run along its main central axes as shown in Fig. A.1. Each node of the system has six degrees of freedom. The ratios between the shifts along the coordinate axes $\mathbf{X}'$ ($x_1'y_1'z_1'$) and the rotations around them $\boldsymbol{\Psi}$, and the corresponding forces $\mathbf{F}_{\mathbf{X}}'$ and the moments $\mathbf{F}_{\boldsymbol{\Psi}}'$, are written with the help of the stiffness matrix $\mathbf{K}_b$ for the beam element in the local coordinate system

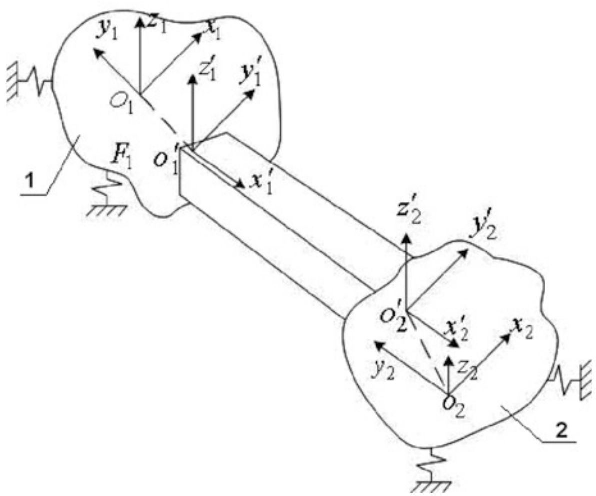

Fig. A.1 Three-dimensional finite beam element and its local coordinates systems

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$$\begin{bmatrix} \mathbf{F}'_{\mathbf{x}} \\ \mathbf{F}'_{\boldsymbol{\Psi}} \end{bmatrix} = \mathbf{K}_b \begin{bmatrix} \mathbf{X}' \\ \boldsymbol{\Psi} \end{bmatrix}, \quad \mathbf{K}_b = \begin{bmatrix} \mathbf{K}_{11b} & \mathbf{K}_{12b} \\ \mathbf{K}_{21b} & \mathbf{K}_{22b} \end{bmatrix}. \tag{A.1}$$

where $\boldsymbol{\Psi} = [\theta_x, \theta_y, \theta_z]$ is the vector of the angular rotations. (Here the first end of the beam reaches body 1 and the second one reaches body 2.)

The relative displacement of the beam element and the body can be determined using the following matrices: $\boldsymbol{\theta}$ is the matrix of the directional cosine angles between axes $(x'y'z')$ and (xyz) for the first and second ends of the beam, respectively

$$\boldsymbol{\theta}^{(i)} = \begin{bmatrix} \alpha_{xx'} & \alpha_{yx'} & \alpha_{zx'} \\ \alpha_{xy'} & \alpha_{yy'} & \alpha_{zy'} \\ \alpha_{xz'} & \alpha_{yz'} & \alpha_{zz'} \end{bmatrix}^{(i)}, \quad (i = 1, 2),$$

and $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ is the matrix of the shifts of the coordinate system origin for the first and second ends of the beam

$$\boldsymbol{\rho}_1 = \begin{bmatrix} 0 & \rho_z & -\rho_y \\ -\rho_z & 0 & \rho_x \\ \rho_y & -\rho_x & 0 \end{bmatrix}_1,$$

$\rho_x = x - x'$, $\rho_y = y - y'$, $\rho_z = z - z'$ (index "$_1$" has been omitted for the sake of simplicity). Analogous ratios can also be written for matrix $\boldsymbol{\rho}_2$ of the second end of the beam.

Matrix $\boldsymbol{\theta}$ is characterized by the rotation of the coordinate axes of the beam element in relation to the body. It can be represented as a product of the three orthogonal matrices $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, $\boldsymbol{\theta}_3$ describing the consecutive angles $\theta_x, \theta_y, \theta_z$ of rotation around the coordinate axes

$$\boldsymbol{\theta}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}, \quad \boldsymbol{\theta}_2 = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix},$$

$$\boldsymbol{\theta}_3 = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{\theta} = \boldsymbol{\theta}_1 \boldsymbol{\theta}_2 \boldsymbol{\theta}_3.$$

If there is no rotation, then $\boldsymbol{\theta} = \mathbf{E}$, where $\mathbf{E}$ is the identity matrix.

In the general case, the shift of the beam in the local coordinate system $(x'y'z')$ can be carried over into a stationary coordinate system $(Oxyz)$ linked to the body using the transformation matrix $\boldsymbol{\Gamma}$, which takes into account both the transfer of the origin of the coordinates and the rotation of the axes. Or in block form

$$\begin{bmatrix} \mathbf{X}' \\ \boldsymbol{\Psi}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{\theta} & \boldsymbol{\theta}\boldsymbol{\rho}_1 \\ \mathbf{0} & \boldsymbol{\theta} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \boldsymbol{\Psi} \end{bmatrix} = \boldsymbol{\Gamma}_1 \begin{bmatrix} \mathbf{X} \\ \boldsymbol{\Psi} \end{bmatrix},$$
$$\boldsymbol{\Gamma}_1 = \begin{bmatrix} \boldsymbol{\theta} & \boldsymbol{\theta}_{\boldsymbol{\rho}_1} \\ \mathbf{0} & \boldsymbol{\theta} \end{bmatrix}. \tag{A.2}$$

for the first end of the beam and, analogously, for the second end. Then the transformation matrix for the whole beam has the following block-diagonal form:

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}_1 & \\ & \boldsymbol{\Gamma}_2 \end{bmatrix}.$$

The transformation of the force factors in the node as written in the stationary coordinate system is

$$\begin{bmatrix} \mathbf{F}'_X \\ \mathbf{F}'_\theta \end{bmatrix} = \begin{bmatrix} \boldsymbol{\theta} & \mathbf{0} \\ \boldsymbol{\theta}_{\boldsymbol{\rho}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_X \\ \mathbf{Q}_\theta \end{bmatrix}.$$

Analogously for the first and second ends of the beam.

Taking into account that

$$\boldsymbol{\theta}^T \boldsymbol{\theta}^{-T}\ ,\ \boldsymbol{\rho}^T = -\boldsymbol{\rho}\,, (\boldsymbol{\rho}\,\boldsymbol{\theta})^T = \boldsymbol{\theta}^T\ \boldsymbol{\rho}^T = -\boldsymbol{\theta}^T\ \boldsymbol{\rho},$$

the reverse transformation has the form

$$\begin{bmatrix} \mathbf{Q}_X \\ \mathbf{Q}_\theta \end{bmatrix} = \begin{bmatrix} \boldsymbol{\theta}^T & \mathbf{0} \\ (\boldsymbol{\theta}\boldsymbol{\rho})^T & \boldsymbol{\theta}^T \end{bmatrix} \begin{bmatrix} \mathbf{F}'_X \\ \mathbf{F}'_\theta \end{bmatrix} = \boldsymbol{\Gamma} \begin{bmatrix} \mathbf{F}'_X \\ \mathbf{F}'_\theta \end{bmatrix}. \tag{A.3}$$

In particular, force $\mathbf{F}_X$, which acts on the beam element, generates a moment (at $\boldsymbol{\theta} = \mathbf{E}$) in the (x,y,z) axes:

$$\mathbf{Q}_\theta = \boldsymbol{\rho} \times \mathbf{F}_x.$$

Or in matrix form

$$\mathbf{Q}_\theta = \boldsymbol{\rho}^T \mathbf{F}_x.$$

From (A.1) and (A.3) we find

$$\begin{bmatrix} \mathbf{Q}_X \\ \mathbf{Q}_\theta \end{bmatrix} = \boldsymbol{\Gamma}^T \mathbf{K}_b \begin{bmatrix} \mathbf{X}' \\ \boldsymbol{\Psi}' \end{bmatrix} = \boldsymbol{\Gamma}^T \mathbf{K}_b \boldsymbol{\Gamma} \begin{bmatrix} \mathbf{X} \\ \boldsymbol{\Psi} \end{bmatrix}. \tag{A.4}$$

From which it follows that the stiffness matrix for the beam element has been carried over to node O:

$$\mathbf{K} = \boldsymbol{\Gamma}^T\, \mathbf{K}_\mathbf{b}\, \boldsymbol{\Gamma}. \tag{A.5}$$

Obviously, this transformation can be written in the following way as a block product:

$$\mathbf{K}_{11} = \mathbf{\Gamma}_1^T \mathbf{K}_{11b} \mathbf{\Gamma}_1 , \quad \mathbf{K}_{12} = \mathbf{\Gamma}_1^T \mathbf{K}_{12b} \mathbf{\Gamma}_2 ,$$
$$\mathbf{K}_{22} = \mathbf{\Gamma}_2^T \mathbf{K}_{22b} \mathbf{\Gamma}_2 , \quad \mathbf{K}_{12} = \mathbf{K}_{21}^T .$$

Using (A.4), a simple algorithm in analytical form for determining the end-element stiffness matrix can be proposed (Appendix B).

The compliance matrix $\mathbf{e} = \mathbf{K}^{-1}$ can be written in the following form in the stationary coordinate system by taking into account (A.5)

$$\mathbf{e} = \mathbf{K}^{-1} = \mathbf{\Gamma}^{-1} \mathbf{e_b} \left(\mathbf{\Gamma}^T\right)^{-1}, \qquad \mathbf{\Gamma}^{-1} = \begin{bmatrix} \boldsymbol{\theta}^T & \\ \boldsymbol{\rho}^T \boldsymbol{\theta}^T & \boldsymbol{\theta}^T \end{bmatrix}. \tag{A.6}$$

Knowing the stiffness (compliance) matrix for each end element, the corresponding matrices for the whole system can be found from the compatibility condition for the forces (shifts) in the decomposition nodes. So the stiffness (compliance) matrix for each ith node is obtained as a sum of the matrices for all beams participating in the given node

$$\hat{\mathbf{K}}_{ii} = \sum_{\mathbf{s}} \mathbf{\Gamma}_{\mathbf{s}}^T \mathbf{K}_{\mathbf{s}} \mathbf{\Gamma}_{\mathbf{s}}, \tag{A.7}$$

where s is the number of end elements included in the ith node.

Blocks $\hat{\mathbf{K}}_{ij}$ reflect the links between nodes i and j

$$\hat{\mathbf{K}}_{ij} = \mathbf{\Gamma}_i^T \mathbf{K}_{ij} \mathbf{\Gamma}_j .$$

In this way, we found all the blocks that form the stiffness matrix of the end-element model.

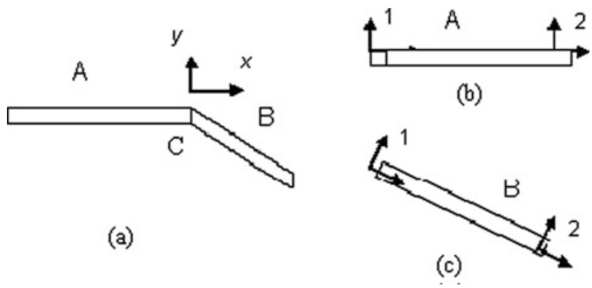
Definition A subsystem is called partial if it was obtained by rigid fixing of all remaining subsystems (nodes).

Therefore, we can consider the i-th node a partial subsystem that is described by matrix $\mathbf{K}_{ii}$.

Example. Let us consider as an example two flat beams linked at point C under a certain angle (e.g., $3\pi/4$) (Fig. A.2)

The general (stationary) coordinate system is chosen at the linkage point as shown in Fig. A.2. Therefore, it coincides with the main inertia axes for beam A. For beam A, the rotation matrix $\boldsymbol{\theta} = \mathbf{E}$ is at end 2. If, in addition to that, the center of inertia of the cross section coincides with point C, then the displacement matrix $\boldsymbol{\rho}_{\mathbf{2}}^{\mathbf{A}} = \mathbf{E}$ and the transformation matrix $\mathbf{\Gamma}_{\mathbf{2}}^{\mathbf{A}} = \mathbf{E}$. End 1 of beam A remains unchanged, so that for it the transformation matrix $\mathbf{\Gamma}_{\mathbf{1}}^{\mathbf{A}} = \mathbf{E}$. As regards beam B, its main central axes at end 1 must be turned around the z axis at an angle of $3\pi/4$ until they coincide with the general coordinate system. Therefore, the rotation matrix $\boldsymbol{\theta}$ will have an angle $\theta_z = \varphi = 3\pi/4$.

Fig. A.2 Flat beams linked under a certain angle



In order to obtain the stiffness matrix of the linked system, it is necessary to perform certain matrix transformations:

– for beam A, its source stiffness matrix (2.29) is:

$$\mathbf{K}^A = \begin{bmatrix} \mathbf{K}_{11}^A & \mathbf{K}_{12}^A \\ \mathbf{K}_{21}^A & \mathbf{K}_{22}^A \end{bmatrix}.$$

For the first end, its transformation matrix is $\boldsymbol{\Gamma}_1^A = \mathbf{E}$, and for the second end – as we have just shown – also $\boldsymbol{\Gamma}_{\mathbf{2}}^{\mathbf{A}} = \mathbf{E}$. Therefore, the stiffness matrix for beam A will not change since $\boldsymbol{\Gamma}^A = \mathbf{E}$.

– For beam *B*, its source stiffness matrix is analogous to

$$\mathbf{K}^B = \begin{bmatrix} \mathbf{K}_{11}^B & \mathbf{K}_{12}^B \\ \mathbf{K}_{21}^B & \mathbf{K}_{22}^B \end{bmatrix}.$$

For the first end of the rotation matrix

$$\boldsymbol{\theta}_1^B = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

As we have presumed above, the shift matrix for it is $\boldsymbol{\rho}_1^B = \mathbf{E}$. Then the transformation matrix for the first end is $\boldsymbol{\Gamma}_1^B = \boldsymbol{\theta}_1^B$. End 2, as above, remains unchanged: for this end, the transformation matrix is $\boldsymbol{\Gamma}_2^B = \mathbf{E}$. Therefore, matrix $\mathbf{K}^{\boldsymbol{B}}$ must be multiplied from the left and the right into the following transformation matrix

$$\boldsymbol{\Gamma}^B = \begin{bmatrix} \boldsymbol{\theta}_1^B & \\ & \mathbf{E} \end{bmatrix},$$

and into its transposed matrix

$$\hat{\mathbf{K}}^{\mathbf{B}} = \begin{bmatrix} \mathbf{A}_1^{B^{\mathbf{tr}}} & \\ & \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{11}^B & \mathbf{K}_{12}^B \\ \mathbf{K}_{21}^B & \mathbf{K}_{22}^B \end{bmatrix} \begin{bmatrix} \mathbf{A}_1^B & \\ & \mathbf{E} \end{bmatrix} = \\ = \begin{bmatrix} \mathbf{A}_1^{B^{\mathbf{tr}}} \mathbf{K}_{11}^B \mathbf{A}_1^B & \mathbf{A}_1^{B^{\mathbf{tr}}} \mathbf{K}_{12}^B \\ \mathbf{A}_1^B \mathbf{K}_{21}^B & \mathbf{K}_{22}^B \end{bmatrix}.$$

And finally, the stiffness matrix for the linked system in Fig. A.2a will take the following form:

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11}^A & \mathbf{K}_{12}^A & \\ \mathbf{K}_{21}^A & \mathbf{K}_{22}^A + \boldsymbol{\theta}_1^{Btr} \mathbf{K}_{11}^B \boldsymbol{\theta}_1^B & \boldsymbol{\theta}_1^{Btr} \mathbf{K}_{12}^B \\ & \mathbf{K}_{21}^B \boldsymbol{\theta}_1^B & \mathbf{K}_{22}^B \end{bmatrix}. \tag{2.27}$$

Appendix B
Stiffness Matrix for Spatial Finite Element

General view for stiffness matrix for spatial beam finite element is (Fig. A.1):

$$
\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{12}^{T} & \mathbf{K}_{22} \end{bmatrix},
$$

$$
\mathbf{K}_{11} = \begin{bmatrix}
\frac{ES}{a} & & & & & \\
& \frac{12EI_z}{a^3} & & & & -\frac{6EI_z}{a^2} \\
& & \frac{12EI_y}{a^3} & & \frac{6EI_y}{a^2} & \\
& & & \frac{GJ_x}{a} & & \\
& & \frac{6EI_y}{a^2} & & \frac{4EI_y}{a} & \\
& -\frac{6EI_z}{a^2} & & & & \frac{4EI_z}{a}
\end{bmatrix}
$$

$$
\mathbf{K}_{12} = \mathbf{K}_{21}^{T} = \begin{bmatrix}
-\frac{ES}{a} & & & & & \\
& -\frac{12EI_z}{a^3} & & & & -\frac{6EI_z}{a^2} \\
& & -\frac{12EI_y}{a^3} & & \frac{6EI_y}{a^2} & \\
& & & -\frac{GJ_x}{a} & & \\
& & -\frac{6EI_y}{a^2} & & \frac{2EI_y}{a} & \\
& \frac{6EI_y}{a^2} & & & & \frac{2EI_z}{a}
\end{bmatrix}
$$

$$
\mathbf{K}_{22} = \begin{bmatrix}
\frac{ES}{a} & & & & & \\
& \frac{12EI_z}{a^3} & & & & \frac{6EI_z}{a^2} \\
& & \frac{12EI_y}{a^3} & & -\frac{6EI_y}{a^2} & \\
& & & \frac{GJ_x}{a} & & \\
& & -\frac{6EI_y}{a^2} & & \frac{4EI_y}{a} & \\
& \frac{6EI_z}{a^2} & & & & \frac{4EI_z}{a}
\end{bmatrix}
$$

Appendix C
Stiffness Matrix Formation Algorithm for a Beam System in Analytical Form

Using the results of Appendix A, an analytical algorithm for the formation of the stiffness matrix for an end-element beam system can be proposed in a general form.

1. In each node i of the system (Figs. A.1 and A.2), it is necessary to define a (general) coordinate system (x_{io},y_{io},z_{io},) linked to this node. The end element t of the beam enters into node i with its first or second end. To be more specific, let us assume that this is end 1.
2. Let us determine the rotation matrix $\boldsymbol{\theta}_{t\,i}$. It depends on the rotation angles θ_x , θ_y , θ_z of the local coordinate system of the t-th beam element and the general coordinate system (x_{io},y_{io},z_{io})

$$\boldsymbol{\theta}_{ti} = \boldsymbol{\theta}_1\boldsymbol{\theta}_2\boldsymbol{\theta}_3\,,$$

$$\boldsymbol{\theta}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}, \ \boldsymbol{\theta}_2 = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix},$$

$$\boldsymbol{\theta}_3 = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Let us calculate the shift matrix $\boldsymbol{\rho}$ for both ends of a beam (if needed):

$$\boldsymbol{\rho}_1 = \begin{bmatrix} 0 & \rho_z & -\rho_y \\ -\rho_z & 0 & \rho_x \\ \rho_y & -\rho_x & 0 \end{bmatrix}_1$$

where ρ_x, ρ_x, ρ_z are the coordinates of the beam ends in the coordinate system (x_{io},y_{io},z_{io}). Let us form the transformation matrix for the first end of the beam included in node i

$$\boldsymbol{\Gamma}_{1i} = \begin{bmatrix} \boldsymbol{\theta} & \boldsymbol{\theta}\mathbf{V}_1 \\ \mathbf{0} & \boldsymbol{\theta} \end{bmatrix}_i,$$

Similarly, let us do the same for the transformation matrix $\mathbf{\Gamma}_{2j}$ for the second end of this beam, which is now included in another node j.

5. Let us find the transformation matrix for beam t

$$\begin{bmatrix} \mathbf{K'}_{11} & \mathbf{K'}_{12} \\ \mathbf{K'}_{21} & \mathbf{K'}_{22} \end{bmatrix}_t = \begin{bmatrix} \mathbf{\Gamma}_{1i} & \\ & \mathbf{\Gamma}_{2j} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}_t \begin{bmatrix} \mathbf{\Gamma}_{1i} & \\ & \mathbf{\Gamma}_{2j} \end{bmatrix}.$$

If node i contains several beams, it is necessary to sum up their transformation matrices $\mathbf{K'}_{11}$. Matrices $\mathbf{K'}_{12}$ and $\mathbf{K'}_{21}$ reflect the link between nodes i and j. Matrix $\mathbf{K'}_{22}$ is now related to node j.

The inertia matrix $\mathbf{M}$ is formed similarly. However, very often it is presumed to be a diagonal one.

Appendix D
Stiffness Matrices for a Planetary Reduction Gear Subsystems

The stiffness matrices for a differential gear and epicycle are

$$\mathbf{K}_S = \begin{bmatrix} a_1 & & \\ & a_1 & \\ & & a_2 \end{bmatrix}, \ \mathbf{K}_{Ep} = \begin{bmatrix} b_1 & & \\ & b_1 & \\ & & b_2 \end{bmatrix}.$$

The linkage matrices for differential gear-satellites are

$$\mathbf{K}_{SC_i} = \begin{bmatrix} -\alpha_i h_1 \cos\gamma & \alpha_i h_1 \sin\gamma & -r_3 \alpha_i h_1 \\ -\beta_i h_1 \cos\gamma & \beta_i h_1 \sin\gamma & -r_3 \beta_i h_1 \\ r_1 h_1 \cos\gamma & -r_1 h_1 \sin\gamma & -r_1 r_3 h_1 \end{bmatrix}.$$

Epicycle-satellite linkage matrix

$$\mathbf{K}_{EpC_i} = \begin{bmatrix} -\alpha_i h_2 \cos\gamma & -\alpha_i h_2 \sin\gamma & r_3 \alpha_i h_2 \\ -\beta_i h_2 \cos\gamma & -\beta_i h_2 \sin\gamma & -r_3 \beta_i h_2 \\ r_2 h_2 \cos\gamma & r_2 h_2 \sin\gamma & -r_3 r_2 h_2 \end{bmatrix},$$

$$\alpha_i = \cos\frac{2\pi(i-1)}{n}, \ \beta_i = \sin\frac{2\pi(i-1)}{n} \ (i = 1\ldots3)$$

$$\alpha_1 = h_1 + h_7\frac{2}{3}, \ \alpha_2 = h_1 r_1^2 + h_{12}\frac{1}{3}$$

$$b_1 = h_2 + h_9\frac{4}{3}, \ b_2 = h_2 r_2^2 + h_9\frac{4}{3}r_2^2.$$

Satellites stiffness matrices

$$\begin{matrix}\mathbf{K}_{C_1} \\ \mathbf{K}_{C_2} \\ \mathbf{K}_{C_3}\end{matrix} = \begin{bmatrix} (h_1 + h_2)Cos^2\gamma + h_6 & (h_1 - h_2)\cos\gamma\sin\gamma & (h_1 - h_2)r_3 Cos\gamma \\ & (h_1 + h_2)\sin^2\gamma + h_6 & (h_1 + h_2)r_3\sin\gamma \\ symm. & & (h_1 + h_2)r_3^2 \end{bmatrix}.$$

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Index

Note: The letters ‘f’ and ‘t’ following locators refer to figures and tables respectively.

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