

Concluding Remarks

Let us conclude these lecture notes with some remarks. Although we have covered many aspects of conformal field theory, we could only scratch the surface and provide an introduction to a collection of CFT issues. For further reading and study, we have provided a list of essential references at the end of these notes.

However, coming back to our introduction, this course was meant to accompany a string theory lecture as part of the 2007 newly established “Theoretical and Mathematical Physics” master programme at the LMU Munich. As a consequence, we put special emphasis on computational techniques in CFT which are important for string theory and had to neglect directions in CFT which are also important but have their roots in Statistical Physics or pure Mathematical Physics. For the interested reader, let us give a (incomplete) list of developments not covered in these notes:

- We have focused on unitary CFTs, as they are important for string theory, though, it is well known that non-unitary CFTs with negative central charge play a very important role for statistical integrable models in two dimensions. These issues are discussed for instance in the book by di Francesco, Mathieu, Sénéchal.
- We have only mentioned the basics about symmetry algebras in CFT. In particular, the field of Kač–Moody algebras would have deserved a much more detailed discussion, as they also play a very important role in mathematics. Their generalisation to \hat{e}_{10} and \hat{e}_{11} might turn out to be essential for a non-perturbative formulation of String and M-Theory, respectively. Similarly, the vast field of \mathcal{W} algebras could only be touched.
- We have discussed some aspects of free field CFT, however, interacting CFTs can be constructed from free fields by allowing for a non-vanishing background charge. This is the celebrated Feigin–Fuks construction which we also did not cover.
- Again related to non-unitary CFTs, we did not touch the very much discussed Logarithmic conformal field theories.
- There exist a number of interesting attempts to develop an axiomatic approach to CFT which we did not mention, since our emphasis was on applications of CFT techniques to string theory.

General Books on CFT and String Theory

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6. J. Fuchs, "Lectures on conformal field theory and Kac-Moody algebras," hep-th/9702194.
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8. M. R. Gaberdiel and P. Goddard, "Axiomatic conformal field theory," *Commun. Math. Phys.* **209** (2000) 549–594, hep-th/9810019.
9. M. R. Gaberdiel, "An introduction to conformal field theory," *Rept. Prog. Phys.* **63** (2000) 607–667, hep-th/9910156.
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16. J. Polchinski, "String theory. vol. 2: Superstring theory and beyond,". Cambridge, UK: University Press (1998) 531p.
17. B. Zwiebach, "A first course in string theory,". Cambridge, UK: University Press (2004) 558p.
18. K. Becker, M. Becker, and J. H. Schwarz, "String theory and M-theory: A modern introduction,". Cambridge, UK: Cambridge University Press (2007) 739p.
19. E. Kiritsis, "String theory in a nutshell,". Princeton, USA: University Press (2007) 588p.
20. B. R. Greene, "String theory on Calabi-Yau manifolds," hep-th/9702155.

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