

Appendix A

Linear Elliptic Partial Differential Equations

A.1 Sobolev Spaces

We are going to use the integration theory of Lebesgue. Therefore, we shall always identify functions which differ only on a set of measure zero. Thus, when we speak about a *function*, we actually always mean an equivalence class of functions under the above identification. In particular, a statement like “the function f is continuous” is to be interpreted as “ f differs from a continuous function at most on a set of measure zero” or equivalently “the equivalence class of f contains a continuous function”.

Replacing functions by their equivalence classes is necessary in order to make the L^p - and Sobolev spaces Banach spaces.

Definition A.1.1. $\Omega \subset \mathbb{R}^d$ open, $p \in \mathbb{R}$, $p \geq 1$,

$$\begin{aligned} L^p(\Omega) &:= \left\{ f : \Omega \rightarrow \mathbb{R} \cup \{\pm\infty\} \text{ measurable} \right. \\ &\quad \left. \text{and } \|f\|_{L^p(\Omega)} := \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}, \\ L^\infty(\Omega) &:= \left\{ f : \Omega \rightarrow \mathbb{R} \cup \{\pm\infty\} \text{ measurable} \right. \\ &\quad \left. \text{and } \|f\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{x \in \Omega} |f(x)| < \infty \right\}, \quad \text{with} \\ \operatorname{ess\,sup}_{x \in \Omega} f(x) &:= \inf \{ a \in \mathbb{R} \cup \{\infty\} : f(x) \leq a \text{ for almost all } x \in \Omega \}. \end{aligned}$$

Theorem A.1.1. With norm $\|\cdot\|_{L^p(\Omega)}$, $L^p(\Omega)$ is a Banach space for $1 \leq p \leq \infty$.

Theorem A.1.2 (Hölder's Inequality). Let $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ ($q = \infty$ for $p = 1$ and vice versa), $f \in L^p(\Omega)$, $g \in L^q(\Omega)$. Then $fg \in L^1(\Omega)$ and

$$\int_{\Omega} |f(x)g(x)| dx \leq \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |g(x)|^q dx \right)^{\frac{1}{q}}.$$

More generally, for $p_1, \dots, p_m \geq 1$, $\frac{1}{p_1} + \dots + \frac{1}{p_m} = 1$, $f_i \in L^{p_i}(\Omega)$, $i = 1, \dots, m$,

$$\int_{\Omega} \left| \prod_{i=1}^m f_i(x) \right| dx \leq \prod_{i=1}^m \left(\int_{\Omega} |f_i(x)|^{p_i} dx \right)^{\frac{1}{p_i}}.$$

Theorem A.1.3. If $(f_n)_{n \in \mathbb{N}}$ converges to f in $L^p(\Omega)$, then a subsequence converges pointwise almost everywhere to f .

Theorem A.1.4. $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$ (but not for $p = \infty$).

Theorem A.1.5. If $f \in L^2(\Omega)$ and

$$\int_{\Omega} f(x)\varphi(x) dx = 0, \quad \text{for every } \varphi \in C_0^\infty(\Omega),$$

then

$$f = 0.$$

We let

$$L_{\text{loc}}^p(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \cup \{\pm\infty\} : f \in L^p(\Omega') \text{ for } \forall \Omega' \Subset \Omega\}.$$

Definition A.1.2. Let $f \in L_{\text{loc}}^1(\Omega)$. We call $v \in L_{\text{loc}}^1(\Omega)$ the *weak derivative* of f in the direction of x^i , $v = D_i f$, if

$$\int_{\Omega} v(x)\varphi(x) dx = - \int_{\Omega} f(x) \frac{\partial \varphi(x)}{\partial x^i} dx,$$

for all $\varphi \in C_0^1(\Omega)$. Here $x = (x^1, \dots, x^n) \in \mathbb{R}^n$.

Weak derivatives of higher order are similarly defined (notation $D_\alpha f$ for a multiindex α).

Definition A.1.3. $k \in \mathbb{N}$, $1 \leq p \leq \infty$. We define the *Sobolev spaces* and *Sobolev norms* as follows:

$$\begin{aligned} W^{k,p}(\Omega) &:= \{f \in L^p(\Omega) : \forall \alpha \text{ with } |\alpha| \leq k : D_\alpha f \in L^p(\Omega)\}, \\ \|f\|_{W^{k,p}(\Omega)} &:= \left(\sum_{|\alpha| \leq k} \int_\Omega |D_\alpha f|^p \right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty, \\ \|f\|_{W^{k,\infty}(\Omega)} &:= \sum_{|\alpha| \leq k} \operatorname{ess\,sup}_{\alpha \in \Omega} |D_\alpha f(x)|, \\ H_0^{k,p}(\Omega) &:= \text{closure of } C_0^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{W^{k,p}(\Omega)}, \\ H^{k,p}(\Omega) &:= \text{closure of } C^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{W^{k,p}(\Omega)}. \end{aligned}$$

Theorem A.1.6. $W^{k,p}(\Omega) = H^{k,p}(\Omega)$ for $1 \leq p < \infty$, $k \in \mathbb{N}$. $W^{k,p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$, $k \in \mathbb{N}$.

Some local properties of Sobolev functions:

Lemma A.1.1. $\Omega \subset \mathbb{R}^d$ open, $f \in H^{1,1}(\Omega)$, $i \in \{1, \dots, d\}$. Then for almost all $\lambda \in \mathbb{R}$, $f|_{\{x^i=\lambda\}}$ is absolutely continuous.

Let $f \in L^1(\Omega)$, Ω open in \mathbb{R}^d . Then for almost all $x_0 \in \Omega$,

$$\lim_{r \rightarrow 0} \frac{1}{|B(x_0, r)|} \int |f(x) - f(x_0)| dx = 0$$

($|B(x_0, r)| = \omega_d r^d$ denotes the Lebesgue measure of the ball $B(x_0, r)$).

An x_0 satisfying this property is called a Lebesgue point. If x_0 is a Lebesgue point, then f is approximately continuous at x_0 ; this means the following:

For $\varepsilon > 0$, let

$$S_\varepsilon := \{y \in \Omega : |f(y) - f(x_0)| < \varepsilon\}.$$

Then

$$\lim_{r \rightarrow 0} \frac{|S_\varepsilon \cap B(x_0, r)|}{|B(x_0, r)|} = 1 \quad \text{for all } \varepsilon > 0.$$

Similarly, $f \in H^{1,1}(\Omega)$ is called approximately differentiable at $x_0 \in \Omega$, with approximate derivative $\nabla f(x_0)$, if for

$$S_\varepsilon^1 := \{y \in \Omega : |f(y) - f(x_0)(y - x_0) - \nabla f(x_0)| \leq \varepsilon|y - x_0|\},$$

$$\lim_{r \rightarrow 0} \frac{|S_\varepsilon^1 \cap B(x_0, r)|}{|B(x_0, r)|} = 1 \quad \text{for all } \varepsilon > 0.$$

We then have

Lemma A.1.2. A function $f \in H^{1,1}(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, is approximately differentiable almost everywhere, and the weak derivative coincides with the approximate derivative almost everywhere.

Lemma A.1.3. $\Omega \subset \mathbb{R}^d$ open, $\ell : \mathbb{R} \rightarrow \mathbb{R}$ Lipschitz, $f \in H^{1,p}(\Omega)$. If $\ell \circ f \in L^p(\Omega)$, then $\ell \circ f \in H^{1,p}(\Omega)$ and for almost all $x \in \Omega$,

$$D_i(\ell \circ f)(x) = \ell'(f(x))D_i f(x), \quad i = 1, \dots, d.$$

Theorem A.1.7 (Sobolev Embedding Theorem). $\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H_0^{1,p}(\Omega)$. Then

$$\begin{aligned} f &\in L^{\frac{np}{n-p}} && \text{for } p < n, \\ f &\in C^0(\overline{\Omega}) && \text{for } p > n. \end{aligned}$$

More precisely, \exists constants $c = c(n, p)$:

$$\begin{aligned} \|f\|_{L^{\frac{np}{n-p}}(\Omega)} &\leq c \|Df\|_{L^p(\Omega)} && \text{for } p < n, \\ \sup_{x \in \Omega} |f(x)| &\leq c \text{Vol}(\Omega)^{\frac{1}{n} - \frac{1}{p}} \|Df\|_{L^p(\Omega)} && \text{for } p > n. \end{aligned}$$

For $n = p$, $f \in L^q(\Omega)$ for all $q < \infty$.

Remark. $H^{1,n}(\Omega)$ is not contained in $C^0(\overline{\Omega})$ or $L^\infty(\Omega)$.

Let us consider the following example:

$d \geq 2$, $\Omega = \overset{\circ}{B}(0, \frac{1}{e}) \subset \mathbb{R}^d$, $f(x) := \log \log \frac{1}{|x|}$ is in $H_0^{1,d}(\Omega)$, but has a singularity at $x = 0$ and is unbounded there. Using this example, we may even produce functions in $H^{1,d}$ with a dense set of singular points. For example, take $\Omega = \overset{\circ}{B}(0, \frac{1}{2e}) \subset \mathbb{R}^d$, let $(p_\nu)_{\nu \in \mathbb{N}}$ be a dense sequence of points in Ω and consider

$$g(x) := \sum_{\nu} 2^{-\nu} f(x - p_\nu).$$

Corollary A.1.1 (Poincaré Inequality). $\Omega \subset \mathbb{R}^n$ open, bounded,

$$f \in H_0^{1,2}(\Omega) \Rightarrow \|f\|_{L^2(\Omega)} \leq \text{const Vol}(\Omega)^{\frac{1}{n}} \|Df\|_{L^2(\Omega)}.$$

Corollary A.1.2. $\Omega \subset \mathbb{R}^n$ open, bounded, then,

$$H_0^{k,p}(\Omega) \subset \begin{cases} L^{\frac{np}{n-kp}}(\Omega) & \text{for } kp < n, \\ C^m(\overline{\Omega}) & \text{for } 0 \leq m < k - \frac{n}{p}. \end{cases}$$

In particular, if $f \in H_0^{k,p}(\Omega)$ for all $k \in \mathbb{N}$ and some fixed p , then $f \in C^\infty(\overline{\Omega})$.

Theorem A.1.8 (Rellich-Kondrachov Compactness Theorem). $\Omega \subset \mathbb{R}^n$ open, bounded. Suppose $1 \leq q < \frac{np}{n-p}$ if $p < d$, and $1 \leq q < \infty$ if $p \geq d$. Then $H_0^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$, i.e. if $(f_n)_{n \in \mathbb{N}} \subset H_0^{1,p}(\Omega)$ satisfies

$$\|f_n\|_{W^{1,p}(\Omega)} \leq \text{const},$$

then a subsequence converges in $L^q(\Omega)$.

Corollary A.1.3. Ω as before. Then $H_0^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$.

$H^{k,2}(\Omega)$ is a Hilbert space, the scalar product is

$$(f, g)_{H^{k,2}(\Omega)} := \sum_{|\alpha| \leq k} \int_{\Omega} D_{\alpha} f(x) D_{\alpha} g(x) dx.$$

Finally, we recall the concept of weak convergence:

Let H be a Hilbert space with norm $\|\cdot\|$ and a product $\langle \cdot, \cdot \rangle$. Then $(v_n)_{n \in \mathbb{N}} \subset H$ is called *weakly convergent* to $v \in H$,

$$v_n \rightharpoonup v,$$

iff

$$\langle v_n, w \rangle \rightarrow \langle v, w \rangle \quad \text{for all } w \in H.$$

Theorem A.1.9. Every bounded sequence $(v_n)_{n \in \mathbb{N}}$ in H contains a weakly convergent subsequence, and if the limit is v ,

$$\|v\| \leq \liminf_{n \rightarrow \infty} \|v_n\|$$

(where (v_n) now is the weakly convergent subsequence).

Example. Let (e_n) be an orthonormal sequence in an infinite dimensional Hilbert space. Then $e_n \rightharpoonup 0$. In particular, the inequality in Theorem A.1.9 may be strict.

A.2 Existence and Regularity Theory for Solutions of Linear Elliptic Equations

Ω will always be an open subset of \mathbb{R}^m .

For technical purposes, one often has to approximate weak derivatives if they are not yet known to exist by difference quotients which are supposed to exist. Thus, let

$$\begin{aligned} f &\in L^2(\Omega, \mathbb{R}), \\ (e_1, \dots, e_m) &\text{ an orthonormal basis of } \mathbb{R}^m, \\ h &\in \mathbb{R}, \quad h \neq 0. \end{aligned}$$

We put

$$\Delta_i^h f(x) := \frac{f(x + he_i) - f(x)}{h} \quad (\text{if } \text{dist}(x, \partial\Omega) > |h|).$$

If $\varphi \in L^2(\Omega)$, $\text{supp } \varphi \Subset \Omega$, $|h| < \text{dist}(\text{supp } \varphi, \partial\Omega)$, we have

$$\int_{\Omega} (\Delta_i^h f(x)) \varphi(x) dx = - \int_{\Omega} f(x) \Delta_i^{-h} \varphi(x) dx. \quad (\text{A.2.1})$$

Lemma A.2.1. *If $f \in H^{1,2}(\Omega)$, $\Omega' \Subset \Omega$, $|h| < \text{dist}(\Omega', \partial\Omega)$, then $\Delta_i^h f \in L^2(\Omega')$ and*

$$\|\Delta_i^h f\|_{L^2(\Omega')} \leq \|D_i f\|_{L^2(\Omega)} \quad \text{for } i = 1, \dots, m.$$

Conversely,

Lemma A.2.2. *If $f \in L^2(\Omega)$ and if for some $K < \infty$*

$$\|\Delta_i^{h_n} f\|_{L^2(\Omega')} \leq K$$

for some sequence $h_n \rightarrow 0$ and all $\Omega' \Subset \Omega$ with $h_n < \text{dist}(\Omega', \partial\Omega)$, then the weak derivative $D_i f$ exists and

$$\|D_i f\|_{L^2(\Omega)} \leq K.$$

The fundamental elliptic regularity theorems for Sobolev norms may be proved by approximating weak derivatives by difference quotients.

We now formulate the general regularity theorem.

We consider an operator

$$Lf(x) := \frac{\partial}{\partial x^\alpha} (a^{\alpha\beta}(x) \frac{\partial}{\partial x^\beta} f(x)) \quad (\text{A.2.2})$$

for $x \in \Omega$, $f : \Omega \rightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^m$.

We assume that there exist constants $0 < \lambda \leq \mu$ with

$$\lambda |\xi|^2 \leq a^{\alpha\beta}(x) \xi_\alpha \xi_\beta \leq \mu |\xi|^2 \quad (\text{A.2.3})$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^m$. We say that L is *uniformly elliptic*. Let $k \in L^2(\Omega)$. Then $f \in H^{1,2}(\Omega)$ is called *weak solution* of

$$Lf = k$$

if

$$\int_{\Omega} a^{\alpha\beta}(x) D_\beta f(x) D_\alpha \varphi(x) dx = - \int_{\Omega} k(x) \varphi(x) dx \quad (\text{A.2.4})$$

for all $\varphi \in H_0^{1,2}(\Omega)$.

Theorem A.2.1. *Let $f \in H^{1,2}(\Omega)$ be a weak solution of (A.2.4). Suppose $k \in H^{\nu,2}(\Omega)$, $a^{\alpha\beta} \in C^{\nu+1}(\Omega)$ ($\nu \in \mathbb{N}$).*

Then

$$f \in H^{\nu+2,2}(\Omega')$$

for every $\Omega' \Subset \Omega$.

If

$$\|a^{\alpha\beta}\|_{C^{\nu+1}(\Omega)} \leq K_\nu,$$

then

$$\|f\|_{H^{\nu+2,2}(\Omega')} \leq c(\|f\|_{L^2(\Omega)} + \|k\|_{H^{\nu,2}(\Omega)}), \tag{A.2.5}$$

where c depends on m, λ, ν, K_ν and $\text{dist}(\Omega', \partial\Omega)$.

The Harnack inequalities of Moser are of fundamental importance for the theory of elliptic partial differential equations:

Theorem A.2.2. *Let L be a uniformly elliptic operator as in (A.2.2), (A.2.3).*

(i) Let u be a weak subsolution, i.e.

$$Lu \geq 0 \quad \text{in a ball } B(x_0, 4R) \subset \mathbb{R}^m$$

($\int a^{\alpha\beta} D_\alpha u D_\beta \varphi \leq 0$ for all $\varphi \in H_0^{1,2}(B(x_0, 4R))$). For $p > 1$ then

$$\sup_{B(x_0, R)} u \leq c_1 \left(\frac{p}{p-1} \right)^{\frac{2}{p}} \left(\frac{1}{\omega_m (2R)^m} \int_{B(x_0, 2R)} \max(u(x), 0)^p dx \right)^{\frac{1}{p}},$$

where c_1 depends only on m and $\frac{\mu}{\lambda}$ in (A.2.3).

*(ii) Let u be a **positive** supersolution, i.e.*

$$Lu \leq 0 \quad \text{in a ball } B(x_0, 4R) \subset \mathbb{R}^m.$$

For $m \geq 3$ and $0 < p < \frac{m}{m-2}$ then

$$\left(\frac{1}{\omega_m (2R)^m} \int_{B(x_0, 2R)} u^p \right)^{\frac{1}{p}} \leq \frac{c_2}{\left(\frac{m}{m-2} - p\right)^2} \inf_{B(x_0, R)} u,$$

c_2 again depending only on m and $\frac{\mu}{\lambda}$. For $m = 2$ and $0 < p < \infty$, the same estimate holds when $\frac{c_2}{\left(\frac{m}{m-2} - p\right)^2}$ is replaced by a constant c_3 depending on p and $\frac{\mu}{\lambda}$.

The Harnack inequality also translates into estimates for the fundamental solutions of the Laplace-Beltrami operator, and their generalizations, the Green functions. The Green function $G(x_0, x)$ of a ball $B \subset M$ (or another sufficiently regular domain), for x_0 in the interior of B , is symmetric in x and x_0 , smooth for $x \neq x_0$, becomes

singular like $\frac{1}{(d-2)\omega_d}d(x, x_0)^{2-d}$ in case $d = \dim M \geq 3$ ($\omega_d = \text{Vol } S^{d-1}$) (and like $\frac{1}{\omega_2} \log d(x_0, x)$ for $d = 2$), vanishes for $x \in \partial B$, and satisfies

$$h(x_0) = \int_B \Delta h(x) G(x_0, x) d\text{Vol}(x) \quad \text{for all } h \in C_0^2(B).$$

A geometric approximation of the Green function (that is exact in the Euclidean case) has been investigated in §4.7. An analytic alternative that allows to avoid the singularity is the use of the mollified Green function. For simplicity, and because that typically suffices for applications, we only consider the case of a ball. The mollified Green function $G^R(x_0, x)$ on the ball $B(x_0, R)$ relative to the ball $B(x_0, 2R)$ of double radius, $G^R(x_0, \cdot) \in H^{1,2} \cap C_0^0(B(x_0, 2R))$, satisfies

$$\begin{aligned} \int_{B(x_0, 2R)} \Delta \varphi(x) G^R(x_0, x) d\text{Vol}(x) &= \int_{B(x_0, 2R)} \langle d\varphi(x), dG^R(x_0, x) \rangle d\text{Vol}(x) \\ &= \int_{B(x_0, R)} \varphi(x) d\text{Vol}(x), \end{aligned}$$

for all $\varphi \in H^{1,2}$ with $\text{supp } \varphi \Subset B(x_0, 2R)$.

For purposes of normalization, it is convenient to consider

$$w^R(x) := \frac{|B(x_0, 2R)|}{R^2} G^R(x_0, x)$$

with $|B| := \text{Vol } B$.

We then have

$$\int_{B(x_0, 2R)} \langle d\varphi(x), dw^R(x) \rangle = \frac{1}{R^2} \int_{B(x_0, R)} \varphi(x),$$

for all $\varphi \in H^{1,2}$ with $\text{supp } \varphi \Subset B(x_0, 2R)$.

We then have the estimates

Corollary A.2.1.

$$\begin{aligned} 0 \leq w^R &\leq \gamma_1 \quad \text{in } B(x_0, 2R), \\ w^R &\geq \gamma_2 > 0 \quad \text{in } B(x_0, R), \end{aligned}$$

for constants γ_1, γ_2 that do not depend on R .

The estimates of J. Schauder are also very important:

Theorem A.2.3. *Let L be as in (A.2.2), (A.2.3), and suppose that the coefficients $a^{\alpha\beta}(x)$ are Hölder continuous in Ω , i.e. contained in $C^\sigma(\Omega)$ for some $0 < \sigma < 1$.*

(i) *If u is a weak solution of*

$$Lu = k$$

and if k is in $L^\infty(\Omega)$, then u is in $C^{1,\sigma}(\Omega)$, and on every $\Omega_0 \Subset \Omega$, its $C^{1,\sigma}$ -norm can be estimated in terms of its L^2 -norm and the L^∞ -norm of k , with a structural constant depending on Ω , Ω_0 , m , σ , λ , μ and the C^σ -norm of the $a^{\alpha\beta}(x)$.

(ii) If u is a weak solution of

$$Lu = k$$

for some $k \in C^{\nu,\sigma}(\Omega)$, $\nu = 0, 1, 2, \dots$, $0 < \sigma < 1$, and if the coefficients $a^{\alpha\beta}$ are also in $C^{\nu,\sigma}(\Omega)$, then u is in $C^{\nu+2,\sigma}(\Omega)$, and a similar estimate as in (i) holds, this time involving the $C^{\nu,\sigma}$ -norm of k and the $a^{\alpha\beta}$.

Finally, we quote the maximum principle.

Theorem A.2.4. *Let $\Omega \subset \mathbb{R}^m$ (or, more generally, $\Omega \subset M$, M a Riemannian manifold) be open and bounded, $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with*

$$Lf \geq 0 \quad \text{in } \Omega,$$

L as in (A.2.2), (A.2.3). Then f assumes its maximum on the boundary $\partial\Omega$.

All the preceding results naturally apply to the Laplace-Beltrami operator on a ball $B(x_0, r)$ in a Riemannian manifold M , putting

$$L = -\Delta = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\gamma} \gamma^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right),$$

$(\gamma_{\alpha\beta})_{\alpha,\beta=1,\dots,m}$ the metric tensor of M in local coordinates, $(\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}$, $\gamma = \det(\gamma_{\alpha\beta})$.

References for the material in this appendix are: Gilbarg and Trudinger[96], Jost[146] and, with a more elementary presentation, Jost[143]. The results of Corollary A.2.1 about Green functions are systematically derived in [113], and in a more general context in [23]. Some further points about Sobolev spaces can be found in Ziemer[273].

A.3 Existence and Regularity Theory for Solutions of Linear Parabolic Equations

In this section, we consider differential equations on $\Omega \times [0, \infty)$ where Ω is an open subset of \mathbb{R}^m as in A.2, and we continue to use the notations introduced there.

In particular, as before, let the operator L be a *uniformly elliptic* operator of the form

$$Lf(x) := \frac{\partial}{\partial x^\alpha} \left(a^{\alpha\beta}(x) \frac{\partial}{\partial x^\beta} f(x) \right) \quad (\text{A.3.1})$$

with constants $0 < \lambda \leq \mu$ satisfying

$$\lambda |\xi|^2 \leq a^{\alpha\beta}(x, t) \xi_\alpha \xi_\beta \leq \mu |\xi|^2 \quad (\text{A.3.2})$$

for all $x \in \Omega, 0 \leq t, \xi \in \mathbb{R}^m$. The equation we wish to study then is

$$\frac{\partial}{\partial t} f(x, t) - Lf(x, t) = k(x, t) \text{ for } x \in \Omega, t \geq 0 \quad (\text{A.3.3})$$

$$f(x, 0) = \phi(x) \quad (\text{A.3.4})$$

for some continuous function $\phi(x)$ and some bounded function $k(x, t)$ (and suitable boundary conditions, but since in the text, we are interested in compact manifolds M in place of the open domain Ω , these will not play an essential role and consequently are not emphasized here). (A.3.3) is a linear parabolic partial differential equation

We first state the parabolic maximum principle.

Theorem A.3.1. *Let $\Omega \subset \mathbb{R}^m$ (or, more generally, $\Omega \subset M$, M a Riemannian manifold) be open and bounded, $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with respect to x and in $C^1((0, T)) \cap C^0([0, T])$ with respect to t , with*

$$\frac{\partial}{\partial t} f - Lf \leq 0 \quad \text{in } \Omega \times [0, T]. \quad (\text{A.3.5})$$

Then f assumes its maximum for (x, t) with $x \in \partial\Omega$ or for $t = 0$, that is, either on the spatial boundary or at the initial time. In particular, when M is a compact manifold (without boundary), the supremum of $f(\cdot, t)$ is a decreasing function of t .

We have the following existence and regularity theorem for solutions of (A.3.3), with Schauder type estimates

Theorem A.3.2. *Let L be as in (A.3.1), (A.3.2), and suppose that the coefficients $a^{\alpha\beta}(x, t)$ are Hölder continuous in $\Omega \times [0, \infty)$, i.e. contained in $C^\sigma(\Omega \times [0, \infty))$ for some $0 < \sigma < 1$. If we prescribe some boundary values, say $f(y, t) = g(y)$ for all $y \in \partial\Omega$, for some given, e.g. continuous, function g , the solution of (A.3.3) then exists for all $t \geq 0$.*

Furthermore, we have the following estimates:

(i) *If u is a weak solution of*

$$Lu = k \quad (\text{A.3.6})$$

and if k is in $L^\infty(\Omega \times [0, \infty))$, then as a function of x , u is in $C^{1,\sigma}(\Omega)$, and for every $\Omega_0 \Subset \Omega$ and $t_0 > 0$, its (spatial) $C^{1,\sigma}(\Omega)$ -norm on $\Omega_0 \times [t_0, \infty)$ can be estimated in terms of its L^∞ -norm and the L^∞ -norm of k , with a structural constant depending on $\Omega, \Omega_0, t_0, m, \sigma, \lambda, \mu$ and the C^σ -norm of the $a^{\alpha\beta}(x)$.

(ii) If u is a weak solution of

$$Lu = k$$

for some $k \in C^{\nu,\sigma}(\Omega \times [0, \infty))$, $\nu = 0, 1, 2, \dots$, $0 < \sigma < 1$, and if the coefficients $a^{\alpha\beta}$ are also in $C^{\nu,\sigma}(\Omega \times [0, \infty))$, then u is in $C^{\nu+2,\sigma}(\Omega)$ with respect to x and of class $C^{\nu+1,\sigma}$ with respect to t , and the corresponding norms can be estimated analogously to (i), this time involving the $C^{\nu,\sigma}$ -norm of k and the $a^{\alpha\beta}$.

The restriction to $t \geq t_0 > 0$ can be avoided if the initial values f_0 satisfy appropriate regularity results. The estimates on $[0, \infty)$ will then naturally also involve the corresponding norms of f_0 .

Theorem A.3.2 concerns a linear parabolic equation. In the text, we shall encounter nonlinear parabolic equations and systems. For those, the global existence and regularity cannot be deduced from a general result, but rather needs to invoke the detailed structure of the system. What one can deduce from Theorem A.3.2, however, is the short time existence of solutions when the linearization of the differential operator satisfies the assumptions of that theorem. This follows by linearization and the implicit function theorem. That means that for such nonlinear systems, we can obtain the existence of a solution on some interval $[0, T)$ whose length depends on the regularity properties of the initial values. This also implies that the maximal interval of existence for nonlinear parabolic systems is open. For the closedness of the interval of existence, and consequently the existence of a solution for all “time” $t \geq 0$, one then needs to derive specific a priori estimates that prevent solutions from becoming singular in finite time.

A reference for parabolic differential equations and systems is [174]. For a textbook treatment, we refer to [146].

Appendix B

Fundamental Groups and Covering Spaces

In this appendix, we briefly list some topological results. We assume that M is a connected manifold, although the results hold for more general spaces.

A *path* or *curve* in M is a continuous map

$$c : [0, a] \rightarrow M \quad (a \geq 0).$$

A *loop* is a path with $c(0) = c(a)$, and that point then is called the *base point* of the loop. The *inverse* of a path c is

$$\begin{aligned} c^{-1} &: [0, a] \rightarrow M, \\ c^{-1}(t) &:= c(a - t). \end{aligned}$$

If $c_i : [0, a_i] \rightarrow M$ are paths ($i = 1, 2$) with $c_2(0) = c_1(a_1)$, we can define the product $c_1 \cdot c_2$ as the path $c : [0, a_1 + a_2] \rightarrow M$,

$$c(t) = \begin{cases} c_1(t) & \text{for } 0 \leq t \leq a_1, \\ c_2(t - a_1) & \text{for } a_1 \leq t \leq a_1 + a_2. \end{cases}$$

Two paths $c_i : [0, a_i]$ with $c_1(0) = c_2(0)$ and $c_1(a_1) = c_2(a_2)$ are called *equivalent* or *homotopic* if there exists a continuous function

$$H : [0, 1] \times [0, 1] \rightarrow M$$

with

$$\begin{aligned} H(t, 0) &= c_1\left(\frac{t}{a_1}\right), \\ H(t, 1) &= c_2\left(\frac{t}{a_2}\right), \quad \text{for all } t, \\ H(0, s) &= c_1(0) = c_2(0), \\ H(1, s) &= c_1(a_1) = c_2(a_2), \quad \text{for all } s. \end{aligned}$$

In particular, $c : [0, a] \rightarrow M$ is equivalent to $\tilde{c} : [0, 1] \rightarrow M$ with $\tilde{c}(t) = c(\frac{t}{a})$, and so we may assume that all paths are parametrized on the unit interval.

We obtain an equivalence relation on the space of all paths. The equivalence class of c is denoted $[c]$, and it is not hard to verify that $[c_1 c_2]$ and $[c^{-1}]$ are independent of the choice of representations. Thus, we may define

$$\begin{aligned} [c_1 \cdot c_2] &=: [c_1] \cdot [c_2], \\ [c^{-1}] &=: [c]^{-1}. \end{aligned}$$

In particular, the equivalence or homotopy classes of loops with fixed base point $p \in M$ form a group $\pi_1(M, p)$, the *fundamental group* of M with *base point* p .

If p and q are in M and $\gamma : [0, 1] \rightarrow M$ satisfies $\gamma(0) = p$, $\gamma(1) = q$, then for every loop c with base point q , $\gamma^{-1}c\gamma$ is a loop with base point p , and this induces an isomorphism between $\pi_1(M, q)$ and $\pi_1(M, p)$. We may thus speak of the fundamental group $\pi_1(M)$ of M without reference to a base point. M is called *simply connected* if $\pi_1(M) = 0$. A continuous map $f : M \rightarrow N$ induces a map $f_{\#} : \pi_1(M, p) \rightarrow \pi_1(N, f(p))$ of fundamental groups.

A continuous map

$$\pi : X \rightarrow M$$

is called a *covering map* if each $p \in M$ has a neighborhood U with the property that each connected component of $\pi^{-1}(U)$ is mapped homeomorphically onto U . If $p \in M$ and H is a subgroup of $\pi_1(M, p)$, there exists a covering $\pi : X \rightarrow M$ with the property that for any $x \in X$ with $\pi(x) = p$, we have $\pi_*(\pi_1(X, x)) = H$.

If we choose $H = \{1\}$, we obtain a simply connected manifold \tilde{M} and a covering

$$\pi : \tilde{M} \rightarrow M.$$

\tilde{M} is called the *universal covering* of M .

If $\pi : X \rightarrow M$ is a covering, $c : [0, 1] \rightarrow M$ a path, $x_0 \in \pi^{-1}(c(0))$, then there exists a unique path

$$\tilde{c} : [0, 1] \rightarrow X$$

with $\tilde{c}(0) = x_0$ and $c(t) = \pi(\tilde{c}(t))$. \tilde{c} is called the *lift* of c through x_0 .

More generally, if M' is another manifold, $f : M' \rightarrow M$ is continuous, $p_0 \in M$, $y_0 \in f^{-1}(p_0)$, $x_0 \in \pi^{-1}(p_0)$, there exists a continuous

$$\tilde{f} : M' \rightarrow X$$

with $\tilde{f}(y_0) = x_0$ and $f = \pi \circ \tilde{f}$ if and only if $f_{\#}(\pi_1(M', y_0)) \subset \pi_{\#}(\pi_1(X, x_0))$. \tilde{f} is unique if it exists.

Let $\pi : \tilde{M} \rightarrow M$ be the universal covering of M . A *deck transformation* is a homeomorphism $\varphi : \tilde{M} \rightarrow \tilde{M}$ with

$$\pi = \pi \circ \varphi.$$

Let $\pi(x_0) = p_0$. $\pi_1(M, p_0)$ then bijectively corresponds to $\pi^{-1}(p_0)$. More precisely, $x_1 \in \pi^{-1}(p_0)$ corresponds to the homotopy class of $\pi(\gamma_{x_1})$, where $\gamma_{x_1} : [0, 1] \rightarrow \tilde{M}$ is any path with $\gamma_{x_1}(0) = x_0$, $\gamma_{x_1}(1) = x_1$. The deck transformations form a group that acts simply transitively on $\pi^{-1}(p_0)$, and associating to a deck transformation $\varphi(x_0) \in \pi^{-1}(p_0)$ then yields an isomorphism between the group of deck transformations and $\pi_1(M, p_0)$.

If M and N are manifolds with universal coverings \tilde{M} and \tilde{N} , resp., and if

$$f : M \rightarrow N$$

is a continuous map, we consider the induced homomorphism

$$\rho := f_{\#} : \pi_1(M, p) \rightarrow \pi_1(N, f(p))$$

of fundamental groups. If $\pi : \tilde{M} \rightarrow M$ is the universal covering, we can lift $f \circ \pi : \tilde{M} \rightarrow N$ to a map

$$\tilde{f} : \tilde{M} \rightarrow \tilde{N},$$

because the above lifting condition is trivially satisfied as $\pi_1(\tilde{M}) = \{1\}$. \tilde{f} is equivariant w.r.t. the above homomorphism ρ in the sense that for every $\lambda \in \pi_1(M, p)$, acting as a deck transformation on \tilde{M} , we have

$$\tilde{f}(\lambda x) = \rho(\lambda)\tilde{f}(x) \quad \text{for every } x \in \tilde{M}, \tag{B.1}$$

where $\rho(\lambda)$ acts as a deck transformation on \tilde{N} . We say that \tilde{f} is a ρ -equivariant map between the universal covers \tilde{M} and \tilde{N} .

Conversely, given any homomorphism

$$\rho : \pi_1(M, p) \rightarrow \pi_1(N, q)$$

and any ρ -equivariant map

$$g : \tilde{M} \rightarrow \tilde{N} \quad (\text{with } g(p) = q),$$

not necessarily continuous, then g induces a map

$$g' : M \rightarrow N$$

whose lift to universal covers is g . g' is continuous if g is.

Finally, if \tilde{M} is the universal cover of a compact Riemannian manifold M , a so-called fundamental domain $F(M)$ for M in \tilde{M} can be constructed as follows:

For simplicity of notation, we denote the group $\pi_1(M, x_0)$ operating by deck transformations on \tilde{M} by Γ , and its trivial element by e .

Let $d(., .)$ be the Riemannian distance function on \tilde{M} . We select any $z_0 \in \tilde{M}$. We then put

$$F(M) := \{z \in \tilde{M} : d(z, z_0) < d(\gamma z, z_0) \text{ for all } \gamma \in \Gamma, \gamma \neq e\}.$$

$F(M)$ is open. Since Γ operates by isometries, i.e.

$$d(\lambda z_1, \lambda z_2) = d(z_1, z_2) \text{ for all } \lambda \in \Gamma, z_1, z_2 \in \tilde{M},$$

we may also write

$$F(M) = \{z \in \tilde{M} : d(z, z_0) < d(z, \lambda z_0) \text{ for all } \lambda \in \Gamma, \lambda \neq e\}.$$

By its definition, $F(M)$ cannot contain any two points that are equivalent under the operation of Γ . On the other hand, for any $z \in \tilde{M}$, we may find some $\mu \in \Gamma$ such that

$$\mu z \in \overline{F(M)}.$$

Thus, the closure of $F(M)$ contains at least one point from every orbit of Γ in \tilde{M} .

If $f : M \rightarrow \mathbb{R}$ is an integrable function, and if $\tilde{f} : \tilde{M} \rightarrow \mathbb{R}$ is its lift to the universal cover of M , then

$$\int_M f(x) d\text{Vol}(x) = \int_{F(M)} \tilde{f}(y) d\text{Vol}(y).$$

Examples of fundamental groups.

1. $\pi_1(\mathbb{R}^n) = \{1\}$ for all n .
2. $\pi_1(S^1) = \mathbb{Z}$.

A generator is given by

$$\begin{aligned} c : [0, 1] &\rightarrow S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}, \\ c(t) &= (\cos 2\pi t, \sin 2\pi t). \end{aligned}$$

The universal covering of S^1 is \mathbb{R}^1 , and the covering map is likewise given by

$$\pi(t) = (\cos 2\pi t, \sin 2\pi t).$$

3. $\pi_1(S^n) = \{1\}$ for $n \geq 2$.
4. $\pi_1(\text{SO}(n)) = \mathbb{Z}_2$ for $n \geq 3$.

The preceding results can be found in any reasonable textbook on Algebraic Topology, for example in [99] or [240].

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Index

- L^2 -distance, 496
- ρ -equivariant, 496
- Γ -convergence, 456
- Γ -limit, 456
- ρ -equivariant, 547

- a priori estimate, 475
- a-priori estimates, 519
- abelian, 284
- abelian subspace, 284, 285, 289, 291
- action, 15
- adjoint, 88
- adjoint representation, 269, 270
- antiselfdual, 129
- arc length, 19
- area, 463
- asymptotic, 227, 293
- asymptotic geometry, 227
- atlas, 2
- autonomous ordinary differential equation, 322
- autoparallel, 119

- base, 37, 60
- Betti number, 101, 256, 303, 374
- Bianchi identity, 117, 128, 130, 141
- Bieberbach Theorem, 237
- Bochner method, 162, 489
- Bochner Theorem, 161
- Bonnet-Myers Theorem, 162, 198
- boundary operator, 302, 335, 336, 338, 339, 359, 365, 368
- bounded geometry, 469
- broken trajectory, 333, 338
- bundle chart, 37
- bundle homomorphism, 39
- bundle metric, 41, 122

- calculus of variations, 304
- canonical orientation, 359
- Cartan decomposition, 288, 293
- Cartan involution, 288
- Cartesian product, 39
- catenoid, 172
- Cauchy polar decomposition, 278
- Cauchy-Riemann equation, 413
- Cauchy-Riemann operator, 156
- center of mass, 221, 222, 226, 449, 517
- chain complex, 335, 337, 365
- chain rule, 490
- Chern class, 133, 252
- Chern-Simons functional, 135
- chirality operator, 70, 154
- Christoffel symbols, 18, 112, 118, 120, 139, 251, 407
- Clifford algebra, 62, 65, 68
- Clifford bundle, 82, 154
- Clifford multiplication, 62, 70, 77, 155, 157, 160
- closed forms, 94
- closed geodesic, 29, 31, 181, 378, 380, 384, 386, 489, 524
- coboundary operator, 335, 340
- cogeodesic flow, 51
- coherent, 358
- coherent orientation, 358, 360
- cohomologous, 94
- cohomology class, 94
- cohomology group, 94, 99, 250
- cohomology of $\mathbb{C}P^n$, 243
- cohomology theory, 340
- commutative diagram, 370

- compact (noncompact) type, 274
- complete, 35, 215, 431, 434, 520
- complex Clifford algebra, 70
- complex manifold, 4, 10
- complex projective space, 241
- complex spin group, 70, 71
- complex tangent space, 10
- complex vector bundle, 45
- conformal, 415, 417, 419, 438–440, 444, 445, 462, 468
- conformal coordinates, 173
- conformal map, 174
- conformal metric, 392, 414, 415, 440
- conformal structure, 173, 414
- conformally invariant, 417
- conjugate point, 191, 193, 194, 198
- connected by the flow, 331, 332
- connecting trajectory, 330
- connection, 112, 114
- constant sectional curvature, 143
- continuous map, 1
- contravariant, 40
- convergence theorem, 239
- convex, 151, 216, 491
- coordinate change, 42
- coordinate chart, 2, 413
- coordinate representation, 26
- cotangent bundle, 39, 42, 50
- cotangent space, 39
- cotangent vector, 39
- Courant-Lebesgue Lemma, 430, 433, 462, 468, 472
- covariant, 40
- covariant derivative, 112, 113
- covariant tensor, 145
- critical point, 304, 366, 409, 420, 459, 487
- critical point of the volume function, 170
- critical set, 299
- cup product, 348
- curvature, 117, 124
- curvature operator, 118
- curvature tensor, 118, 140, 143, 260, 267
- curves of steepest descent, 300
- de Rham cohomology group, 94, 161
- deck transformation, 496
- deformation retract, 243
- degree of line bundle, 395
- density, 464, 467, 472
- derivative, 6
- determinant, 352
- determinant line, 353, 355, 357
- determinant line bundle, 81, 399
- diameter, 198
- diffeomorphism, 4
- difference quotient, 454
- differentiable, 2, 137
- differentiable manifold, 2, 56
- differentiable map, 4, 8
- differential equation, 382
- differential operators, 102
- dimension, 1
- Dirac operator, 156, 157, 159, 160, 399
- Dirichlet integral, 409, 460, 462
- Dirichlet problem, 440
- Dirichlet's principle, 95
- distance, 15
- distance function, 16, 198, 430
- divergence, 91
- dual basis, 40
- dual bundle, 115
- dual space, 39
- dualization, 335
- Einstein manifold, 143
- Einstein summation convention, 6
- elliptic, 103
- ellipticity condition, 103
- embedding, 10
- energy, 18, 409, 417, 420, 432, 451, 459, 461, 487, 523
- energy density, 407
- energy functional, 377, 448, 450
- energy minimizing, 428, 429, 431, 433, 435, 436, 443, 444
- Enneper's surface, 172
- equicontinuity, 443

- equicontinuous, 435, 436, 438, 444
- equivariant, 547
- estimates of J. Schauder, 540
- Euclidean type, 274
- Euler characteristic, 301, 303, 374
- Euler class, 348
- Euler-Lagrange equation, 18, 95, 184, 391, 400, 409, 411
- exact form, 94
- exact sequence, 369
- exponential map, 20, 30, 57, 120, 188, 189, 201, 276, 283, 291
- extended index, 195
- exterior p -form, 43
- exterior derivative, 43, 46, 116, 117
- exterior product, 39, 43

- finite energy, 486
- finiteness theorem, 238
 - π_2 -, 239
- first Betti number theorem, 236
- first Chern class, 252
- first fundamental form, 165
- first order differential equation, 47
- flat, 143, 285, 289
- flat connection, 121
- flat Riemannian manifold, 143
- flow, 49
- flow line, 300, 308, 323
- formally selfadjoint, 157
- frame field, 43
- Fredholm operator, 349, 350, 353, 355, 358
- Friedrichs mollification, 225
- Fubini-Study metric, 246
- fundamental class, 347
- fundamental domain, 548

- gauge group, 127, 128
- gauge transformation, 127
- Gauss curvature, 143, 166
- Gauss equations, 166
- Gauss lemma, 189
- Gauss-Bonnet Theorem, 233
- Gauss-Kronecker curvature, 165, 167

- generalized Morse-Smale-Floer condition, 345
- generic homotopy, 343
- geodesic, 19, 50, 119, 139, 172, 179, 183, 188, 194, 196, 201, 214, 258, 259, 262, 289, 487, 490
- geodesic of shortest length, 24, 30, 35
- geodesic ray, 227, 228
- geodesically complete, 34, 35, 258
- Ginzburg-Landau functional, 392, 402
- gradient, 91, 300, 308
- gradient flow, 307, 386
- graph flow, 345
- Green function, 539
- group of diffeomorphisms, 49

- Hadamard manifold, 230
- Hadamard-Cartan Theorem, 215
- half spin bundle, 399
- half spinor bundles, 81
- half spinor representation, 77
- Hamiltonian flow, 51
- harmonic, 415–417, 419, 421, 425, 428, 431, 435, 436, 438, 439, 443–445, 486–488, 490–492, 523
- harmonic form, 89, 94, 161, 162
- harmonic function, 90, 409, 412
- harmonic map, 172, 409, 412
- harmonic spinor field, 159, 163
- Harnack inequality, 539
- Harnack inequality, 507, 510
- Hartman-Wintner-Lemma, 422, 465
- Hartmann-Grobman-Theorem, 312
- Hausdorff property, 1
- heat flow, 31
- helicoid, 172
- Hermitian line bundle, 392
- Hermitian metric, 245, 246, 415
- Hessian, 150, 490
- Hilbert space, 95
- Hodge $*$ operator, 78
- Hodge decomposition theorem, 256
- holomorphic, 242, 413–417, 439, 445
- holomorphic quadratic differential, 417, 418, 421, 422, 439, 445

- holomorphic tangent space, 10
- holomorphic vector bundle, 46
- holomorphic vector field, 419
- homeomorphism, 1
- homoclinic orbit, 323
- homogeneous, 259
- homogeneous coordinates, 242
- homology group, 101, 302, 335, 339, 365, 366
- homology theory, 369
- homotopic, 28, 427, 428, 436, 440, 523
- homotopy, 29, 342, 345, 357
- Hopf map, 244
- Hopf-Rinow Theorem, 34, 198, 259
- hyperbolic, 143
- hyperbolic space, 200, 201, 258
- hyperplane, 242

- immersed minimal submanifold, 171
- immersion, 10
- index, 195
- index form, 183
- induced connection, 116
- infinite dimensional Riemannian manifold, 28, 376
- infinitesimal isometry, 55
- injectivity radius, 27, 429, 431, 440, 469
- instanton, 130
- integral curve, 48
- invariant k -form, 131
- invariant polynomial, 131
- involution, 247, 258
- isometric immersion, 171
- isometry, 26
- isometry group, 259
- isotropy group, 295
- Iwasawa Decomposition, 291
- Iwasawa decomposition, 291

- Jacobi equation, 184, 188, 201
- Jacobi field, 183–186, 188, 201, 206, 208, 261, 262, 470
- Jacobi identity, 52, 55, 270, 288

- Kähler form, 245, 246, 249
- Kähler identities, 253
- Kähler metric, 246, 250
- Karcher's constructions, 221, 229
- Killing field, 55, 56, 153, 188, 262, 263, 274
- Killing form, 125, 271, 277
- Korn's inequality, 154

- Lagrangian, 391
- Laplace operator, 101
- Laplace-Beltrami operator, 89, 90, 149, 251, 412, 415, 491, 507, 511
- left invariant Riemannian metric, 60
- left translation, 59
- length, 15
- length minimizing, 193
- lens space, 260
- level hypersurface, 323
- Levi-Civita connection, 138, 149, 164, 258, 408, 485
- Lichnerowicz Theorem, 163
- Lie algebra, 52, 53, 56, 60, 63, 269, 275
- Lie bracket, 52, 60, 116, 275
- Lie derivative, 54, 55, 145, 152
- Lie group, 56, 269, 275
- linear elliptic equation, 537
- linear parabolic equation, 541
- linear subspace, 242
- local 1-parameter group, 55
- local 1-parameter group of diffeomorphisms, 49
- local conformal parameter, 414
- local coordinates, 2, 41, 120, 378, 407, 417, 425, 459
- local flow, 48
- local information, 367
- local isometry, 26
- local minimum, 384
- local product structure, 332
- local stable manifold, 313
- local triviality, 37
- local unstable manifold, 313
- local variation, 168
- locally symmetric, 260
- locally symmetric space, 262

- lower semicontinuity of the energy, 455
- manifold, 1
- maximum principle, 542
- maximum principle, 398, 419, 440
- Mayer-Vietoris sequence, 243
- mean curvature, 165, 170
- metric bundle chart, 42
- metric connection, 122, 124, 125
- metric tensor, 145
- minimal 2-sphere, 438
- minimal 2-sphere, 445
- minimal submanifold, 170, 413
- minimal submanifolds of Euclidean space, 172
- minimal surface, 173, 465
- minimal surfaces in \mathbb{R}^3 , 172
- minimizers of convex functionals, 499
- minimizing, 181
- minimizing sequence, 432
- minimum, 304
- model space, 201
- modulus of continuity, 431, 436, 475
- mollification, 225–227
- monotonicity formula, 464, 472
- Moreau-Yosida approximation, 499
- Morse function, 300, 307, 343
- Morse index, 307, 366
- Morse index theorem, 196
- Morse inequalities, 374
- Morse-Floer cohomology, 335
- Morse-Floer theory, 330
- Morse-Palais-Lemma, 309
- Morse-Smale-Floer condition, 330–333, 337, 367
- Morse-Smale-Floer flow, 339
- Morse-Smale-Floer function, 359, 374
- Moser's Harnack inequality, 539
- Myers and Steenrod Theorem, 269
- negative basis, 85
- negative gradient flow, 300, 307, 311, 317, 322, 325, 382
- negative sectional curvature, 181, 199, 489, 524
- noncompact type, 274
- nondegenerate, 100, 307, 313, 325
- nonnegative Ricci curvature, 161
- nonpositive curvature, 523
- nonpositive sectional curvature, 214, 215, 487, 488, 520, 524
- normal bundle, 45
- normal coordinates, 21
- nullity, 195
- one-form, 39
- one-parameter subgroup, 259, 263, 283
- open set, 1
- orbit, 308, 323
- orientable, 2
- orientable flow, 339
- orientation, 85, 337, 338, 355
- orthonormal basis, 42
- Palais-Smale condition, 304, 325, 330, 380, 382, 384, 420
- Palais-Smale sequence, 385
- parabolic differential equation, 31, 542
- parabolic estimates, 32, 542
- parabolic maximum principle, 33, 542
- paracompact, 1
- parallel form, 161
- parallel sections, 113
- parallel transport, 113, 122, 205
- parametric minimal surface, 174, 415, 419
- partition of unity, 5, 381
- perturbed functional, 402
- Poincaré duality, 347
- Poincaré inequality, 475, 503, 506, 507, 536
- polar coordinates, 23
- positive basis, 85
- positive gradient flow, 336
- positive Ricci curvature, 162
- positive root, 291
- positive sectional curvature, 182
- potential, 391
- Preissmann's theorem, 524
- principal G -bundle, 60

- principal bundle, 61
- principal curvatures, 165
- probability measure, 221
- projection, 37, 60
- proper, 304
- pulled back bundle, 38
- Pythagoras inequality, 221

- quadrilateral comparison theorem, 218
- quaternion algebra, 68

- rank, 37
- rank of a symmetric space, 285
- Rauch comparison theorem, 201, 430, 432
- real on the boundary (holomorphic quadratic differential), 421, 422
- real projective space, 261
- regular, 289
- regular geodesic, 289
- regular homotopy, 343
- regularity, 439, 475, 478, 502, 537, 538, 541
- relative homology group, 365
- relative index, 331
- relative Morse index, 307
- Rellich compactness theorem, 96, 195, 197
- Rellich-Kondrachov compactness theorem, 537
- removable singularity, 445, 460
- representation formula, 212
- Reshetnyak's quadrilateral comparison theorem, 218
- Riccati equation, 210
- Ricci curvature, 142, 143, 198, 488, 520, 524
- Ricci form, 251
- Ricci tensor, 142, 251
- Riemann surface, 174, 392, 413, 415, 417, 420, 425, 428, 434, 436, 438, 439, 463, 469
- Riemannian metric, 13, 40, 42, 300, 308, 414
- Riemannian normal coordinates, 21
- Riemannian polar coordinates, 23, 24
- right translation, 59
- root, 287, 289

- saddle point, 304
- scalar curvature, 142, 252
- scalar product, 125
- Schauder estimates, 542
- Schur, 143
- second covariant derivative, 149
- second fundamental form, 165, 166, 208
- second fundamental tensor, 164, 165
- second variation, 179
- second variation of energy, 484, 486, 487
- section, 38
- sectional curvature, 142, 143, 268, 429, 431, 469, 520
- Seiberg-Witten equations, 402, 404
- Seiberg-Witten functional, 399, 402
- selfdual, 129, 130
- selfdual form, 403
- selfduality, 397, 402
- selfduality equations, 397
- semisimple, 273, 274, 277
- short time existence, 543
- shortest curve, 29
- shortest geodesic, 497
- singular, 289
- singular geodesic, 289
- singular hyperplanes, 290
- smoothing, 225
- smoothness of critical points, 410
- Sobolev curve, 376
- Sobolev embedding theorem, 377, 379, 483, 536
- Sobolev norm, 95
- Sobolev space, 88, 95, 96, 102, 195, 376, 428, 451, 535
- space form, 143
- sphere, 12, 25–27, 144, 164, 167, 186, 188, 189, 194, 198, 201, 202, 243, 257, 418
- sphere at infinity, 227
- sphere theorem, 234

- spherical, 143
- spin group, 65
- spin manifold, 80, 158, 163
- spin structure, 80
- spin^c manifold, 81, 160, 399
- spin^c structure, 81
- spinor bundle, 81, 155
- spinor field, 81, 158
- spinor representation, 77, 79
- spinor space, 75, 76
- splitting off of minimal 2-sphere, 437
- splitting theorem, 236
- stable foliation, 318, 332
- stable manifold, 308, 317, 325, 331
- star operator, 85, 87
- stratification, 330
- strictly convex, 151
- strictly convex function, 492
- structural conditions, 473, 475, 481, 483
- structure group, 38, 41, 42, 61
- subbundle, 39
- subharmonic, 491, 492
- submanifold, 11, 45, 166
- symmetric, 247
- symmetric space, 247, 258, 259, 261, 274, 282
- Synge Theorem, 182
- system of differential equations, 47
- system of first order ODE, 113

- tangent bundle, 10, 38, 41
- tangent space, 7, 9
- tangent vector, 7
- tension field, 411, 490
- tensor, 40
- tensor field, 40
- tensor product, 39
- theorem of Lyusternik and Fet, 386
- theorem of Picard-Lindelöf, 308, 310
- theorem of Reeb, 389
- theorema egregium, 166
- Tits building, 291
- topological invariant, 403
- topology of Riemannian manifolds, 162
- torsion, 120
- torsion free, 120
- torus, 3, 26, 27, 101
- total space, 37
- totally geodesic, 166, 167, 285, 489, 490, 492, 524
- transformation behavior, 40, 42, 115, 118
- transformation formula for p -forms, 44
- transition map, 37
- translation, 259
- transversal intersection, 330, 331
- transversality, 341

- unitary group, 244
- universal covering, 547
- unstable foliation, 319
- unstable manifold, 300, 308, 317, 325, 331, 366–368

- variation of volume, 169
- vector bundle, 37
- vector field, 38, 47
- vector representation, 65
- volume form, 87, 249

- weak convergence, 537
- weak derivative, 534, 538
- weak minimal surface, 462–465, 468, 469
- weak solution, 538, 539
- weakly harmonic, 459, 461, 462, 468, 474, 483
- Weitzenböck formula, 149, 159
- Weyl chamber, 290, 291, 293

- Yang-Mills connection, 126, 128
- Yang-Mills equation, 130
- Yang-Mills functional, 125, 126, 128, 134

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