Appendix A

Linear Elliptic Partial Differential Equations

A.1 Sobolev Spaces

We are going to use the integration theory of Lebesgue. Therefore, we shall always identify functions which differ only on a set of measure zero. Thus, when we speak about a *function*, we actually always mean an equivalence class of functions under the above identification. In particular, a statement like "the function f is continuous" is to be interpreted as "f differs from a continuous function at most on a set of measure zero" or equivalently "the equivalence class of f contains a continuous function".

Replacing functions by their equivalence classes is necessary in order to make the L^p- and Sobolev spaces Banach spaces.

Definition A.1.1. $\Omega \subset \mathbb{R}^d$ open, $p \in \mathbb{R}$, $p \geq 1$,

$$L^p(\Omega) := \Big\{ f : \Omega \to \mathbb{R} \cup \{\pm\infty\} \text{ measurable}$$
$$\text{and } \|f\|_{L^p(\Omega)} := \Big(\int_\Omega |f(x)|^p dx \Big)^{\frac{1}{p}} < \infty \Big\},$$
$$L^\infty(\Omega) := \Big\{ f : \Omega \to \mathbb{R} \cup \{\pm\infty\} \text{ measurable}$$
$$\text{and } \|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < \infty \Big\}, \quad \text{with}$$

$$\operatorname*{ess\,sup}_{x\in\Omega} f(x) := \inf\big\{ a \in \mathbb{R} \cup \{\infty\} : f(x) \leq a \text{ for almost all } x \in \Omega \big\}.$$

Theorem A.1.1. *With norm* $\|\cdot\|_{L^p(\Omega)}$, $L^p(\Omega)$ *is a Banach space for* $1 \le p \le \infty$.

Theorem A.1.2 (Hölder's Inequality). *Let* $p, q \ge 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *(*$q = \infty$ *for* $p = 1$ *and vice versa),* $f \in L^p(\Omega), g \in L^q(\Omega)$. *Then* $fg \in L^1(\Omega)$ *and*

$$\int_\Omega |f(x)g(x)|\, dx \le \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q dx\right)^{\frac{1}{q}}.$$

More generally, for $p_1, \dots p_m \ge 1$, $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$, $f_i \in L^{p_i}(\Omega)$, $i = 1, \dots, m$,

$$\int \Big|\prod_{i=1}^m f_i(x)\Big| dx \le \prod_{i=1}^m \left(\int |f_i(x)|^{p_i}\right)^{\frac{1}{p_i}}$$

Theorem A.1.3. *If* $(f_n)_{n\in\mathbb{N}}$ *converges to* f *in* $L^p(\Omega)$, *then a subsequence converges pointwise almost everywhere to* f.

Theorem A.1.4. $C_0^\infty(\Omega)$ *is dense in* $L^p(\Omega)$ *for* $1 \le p < \infty$ *(but not for* $p = \infty$*).*

Theorem A.1.5. *If* $f \in L^2(\Omega)$ *and*

$$\int_\Omega f(x)\varphi(x)\, dx = 0, \quad \textit{for every } \varphi \in C_0^\infty(\Omega),$$

then

$$f = 0.$$

We let

$$L^p_{\mathrm{loc}}(\Omega) := \big\{ f : \Omega \to \mathbb{R} \cup \{\pm\infty\} : f \in L^p(\Omega') \text{ for } \forall\ \Omega' \Subset \Omega \big\}.$$

Definition A.1.2. Let $f \in L^1_{\mathrm{loc}}(\Omega)$. We call $v \in L^1_{\mathrm{loc}}(\Omega)$ the *weak derivative* of f in the direction of x^i, $v = D_i f$, if

$$\int_\Omega v(x)\varphi(x)\, dx = -\int_\Omega f(x)\frac{\partial \varphi(x)}{\partial x^i}\, dx,$$

for all $\varphi \in C_0^1(\Omega)$. Here $x = (x^1, \dots, x^n) \in \mathbb{R}^n$.

Weak derivatives of higher order are similarly defined (notation $D_{\boldsymbol{\alpha}} f$ for a multiindex $\boldsymbol{\alpha}$).

Definition A.1.3. $k \in \mathbb{N}$, $1 \le p \le \infty$. We define the *Sobolev spaces* and *Sobolev norms* as follows:

$$\begin{aligned}
W^{k,p}(\Omega) &:= \{f \in L^p(\Omega) : \forall \boldsymbol{\alpha} \text{ with } |\boldsymbol{\alpha}| \le k : D_{\boldsymbol{\alpha}} f \in L^p(\Omega)\},\\
\|f\|_{W^{k,p}(\Omega)} &:= \left(\sum_{|\boldsymbol{\alpha}| \le k} \int_\Omega |D_{\boldsymbol{\alpha}} f|^p \right)^{\frac{1}{p}} \text{ for } 1 \le p < \infty,\\
\|f\|_{W^{k,\infty}(\Omega)} &:= \sum_{|\boldsymbol{\alpha}| \le k} \operatorname*{ess\,sup}_{\boldsymbol{\alpha} \in \Omega} |D_{\boldsymbol{\alpha}} f(x)|,\\
H_0^{k,p}(\Omega) &:= \text{closure of } C_0^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{W^{k,p}(\Omega)},\\
H^{k,p}(\Omega) &:= \text{closure of } C^\infty(\Omega) \text{ w.r.t. } \|\cdot\|_{W^{k,p}(\Omega)}.
\end{aligned}$$

Theorem A.1.6. *$W^{k,p}(\Omega) = H^{k,p}(\Omega)$ for $1 \le p < \infty, k \in \mathbb{N}$. $W^{k,p}(\Omega)$ is a Banach space for $1 \le p \le \infty$, $k \in \mathbb{N}$.*

Some local properties of Sobolev functions:

Lemma A.1.1. *$\Omega \subset \mathbb{R}^d$ open, $f \in H^{1,1}(\Omega)$, $i \in \{1, \dots, d\}$. Then for almost all $\lambda \in \mathbb{R}, f|_{\{x^i = \lambda\}}$ is absolutely continuous.*

Let $f \in L^1(\Omega)$, Ω open in $\mathbb{R}^d$. Then for almost all $x_0 \in \Omega$,

$$\lim_{r \to 0} \frac{1}{|B(x_0, r)|} \int |f(x) - f(x_0))| \, dx = 0$$

($|B(x_0, r)| = \omega_d r^d$ denotes the Lebesgue measure of the ball $B(x_0, r)$).

An x_0 satisfying this property is called a Lebesgue point. If x_0 is a Lebesgue point, then f is approximately continuous at x_0; this means the following:
For $\varepsilon > 0$, let

$$S_\varepsilon := \{y \in \Omega : |f(y) - f(x_0)| < \varepsilon\}.$$

Then

$$\lim_{r \to 0} \frac{|S_\varepsilon \cap B(x_o, r)|}{|B(x_0, r)|} = 1 \quad \text{for all } \varepsilon > 0.$$

Similarly, $f \in H^{1,1}(\Omega)$ is called approximately differentiable at $x_0 \in \Omega$, with approximate derivative $\nabla f(x_0)$, if for

$$S_\varepsilon^1 := \{y \in \Omega : |f(y) - f(x_0)(y - x_0) - \nabla f(x_0)| \le \varepsilon |y - x_0|\},$$

$$\lim_{r \to 0} \frac{|S_\varepsilon^1 \cap B(x_0, r)|}{|B(x_0, r)|} = 1 \quad \text{for all } \varepsilon > 0.$$

We then have

Lemma A.1.2. *A function $f \in H^{1,1}(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, is approximately differentiable almost everywhere, and the weak derivative coincides with the approximate derivative almost everywhere.*

Lemma A.1.3. *$\Omega \subset \mathbb{R}^d$ open, $\ell : \mathbb{R} \to \mathbb{R}$ Lipschitz, $f \in H^{1,p}(\Omega)$. If $\ell \circ f \in L^p(\Omega)$, then $\ell \circ f \in H^{1,p}(\Omega)$ and for almost all $x \in \Omega$,*

$$D_i(\ell \circ f)(x) = \ell'(f(x)) D_i f(x), \quad i = 1, \ldots, d.$$

Theorem A.1.7 (Sobolev Embedding Theorem). *$\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H_0^{1,p}(\Omega)$. Then*

$$\begin{aligned} f &\in L^{\frac{np}{n-p}} && \text{for } p < n, \\ f &\in C^0(\overline{\Omega}) && \text{for } p > n. \end{aligned}$$

More precisely, $\exists$ constants $c = c(n,p)$:

$$\begin{aligned} \|f\|_{L^{\frac{np}{n-p}}(\Omega)} &\le c \|Df\|_{L^p(\Omega)} && \text{for } p < n, \\ \sup_{x \in \Omega} |f(x)| &\le c \mathrm{Vol}\,(\Omega)^{\frac{1}{n} - \frac{1}{p}} \|Df\|_{L^p(\Omega)} && \text{for } p > n. \end{aligned}$$

For $n = p$, $f \in L^q(\Omega)$ for all $q < \infty$.

Remark. $H^{1,n}(\Omega)$ is not contained in $C^0(\overline{\Omega})$ or $L^\infty(\Omega)$.

Let us consider the following example:
$d \ge 2$, $\Omega = \overset{\circ}{B}(0, \frac{1}{e}) \subset \mathbb{R}^d$, $f(x) := \log \log \frac{1}{|x|}$ is in $H_0^{1,d}(\Omega)$, but has a singularity at $x = 0$ and is unbounded there. Using this example, we may even produce functions in $H^{1,d}$ with a dense set of singular points. For example, take $\Omega = \overset{\circ}{B}(0, \frac{1}{2e}) \subset \mathbb{R}^d$, let $(p_\nu)_{\nu \in \mathbb{N}}$ be a dense sequence of points in Ω and consider

$$g(x) := \sum_\nu 2^{-\nu} f(x - p_\nu).$$

Corollary A.1.1 (Poincaré Inequality). *$\Omega \subset \mathbb{R}^n$ open, bounded,*

$$f \in H_0^{1,2}(\Omega) \Rightarrow \|f\|_{L^2(\Omega)} \le \text{const Vol}\,(\Omega)^{\frac{1}{n}} \|Df\|_{L^2(\Omega)}.$$

Corollary A.1.2. *$\Omega \subset \mathbb{R}^n$ open, bounded, then,*

$$H_0^{k,p}(\Omega) \subset \begin{cases} L^{\frac{np}{n-kp}}(\Omega) & \text{for } kp < n, \\ C^m(\overline{\Omega}) & \text{for } 0 \le m < k - \frac{n}{p}. \end{cases}$$

In particular, if $f \in H_0^{k,p}(\Omega)$ for all $k \in \mathbb{N}$ and some fixed p, then $f \in C^\infty(\overline{\Omega})$.

Theorem A.1.8 (Rellich-Kondrachov Compactness Theorem). *$\Omega \subset \mathbb{R}^n$ open, bounded. Suppose $1 \leq q < \frac{np}{n-p}$ if $p < d$, and $1 \leq q < \infty$ if $p \geq d$. Then $H_0^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$, i.e. if $(f_n)_{n\in\mathbb{N}} \subset H_0^{1,p}(\Omega)$ satisfies*

$$\|f_n\|_{W^{1,p}(\Omega)} \leq \text{const},$$

then a subsequence converges in $L^q(\Omega)$.

Corollary A.1.3. *Ω as before. Then $H_0^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

$H^{k,2}(\Omega)$ is a Hilbert space, the scalar product is

$$(f,g)_{H^{k,2}(\Omega)} := \sum_{|\boldsymbol{\alpha}|\leq k} \int_\Omega D_{\boldsymbol{\alpha}} f(x) D_{\boldsymbol{\alpha}} g(x)\, dx.$$

Finally, we recall the concept of weak convergence:

Let H be a Hilbert space with norm $\|\cdot\|$ and a product $\langle\cdot,\cdot\rangle$. Then $(v_n)_{n\in\mathbb{N}} \subset H$ is called *weakly convergent* to $v \in H$,

$$v_n \rightharpoonup v,$$

iff

$$\langle v_n, w\rangle \to \langle v, w\rangle \quad \text{for all } w \in H.$$

Theorem A.1.9. *Every bounded sequence $(v_n)_{n\in\mathbb{N}}$ in H contains a weakly convergent subsequence, and if the limit is v,*

$$\|v\| \leq \liminf_{n\to\infty} \|v_n\|$$

(where (v_n) now is the weakly convergent subsequence).

Example. Let (e_n) be an orthonormal sequence in an infinite dimensional Hilbert space. Then $e_n \rightharpoonup 0$. In particular, the inequality in Theorem A.1.9 may be strict.

A.2 Existence and Regularity Theory for Solutions of Linear Elliptic Equations

Ω will always be an open subset of $\mathbb{R}^m$.

For technical purposes, one often has to approximate weak derivatives if they are not yet known to exist by difference quotients which are supposed to exist. Thus, let

$$\begin{aligned}&f \in L^2(\Omega,\mathbb{R}),\\ &(e_1,\dots,e_m) \text{ an orthonormal basis of } \mathbb{R}^m,\\ &h\in\mathbb{R},\quad h\neq 0.\end{aligned}$$

We put

$$\Delta_i^h f(x) := \frac{f(x+he_i)-f(x)}{h} \quad (\text{ifdist}\,(x,\partial\Omega)>|h|).$$

If $\varphi \in L^2(\Omega)$, $\operatorname{supp}\varphi \Subset \Omega$, $|h| < \operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega)$, we have

$$\int_\Omega (\Delta_i^h f(x))\varphi(x)\,dx = -\int_\Omega f(x)\Delta_i^{-h}\varphi(x)\,dx. \tag{A.2.1}$$

Lemma A.2.1. *If $f\in H^{1,2}(\Omega)$, $\Omega'\Subset\Omega$, $|h|<\operatorname{dist}(\Omega',\partial\Omega)$, then $\Delta_i^h f\in L^2(\Omega')$ and*

$$\|\Delta_i^h f\|_{L^2(\Omega')} \le \|D_i f\|_{L^2(\Omega)} \quad \text{for } i=1,\dots,m.$$

Conversely,

Lemma A.2.2. *If $f\in L^2(\Omega)$ and if for some $K<\infty$*

$$\|\Delta_i^{h_n} f\|_{L^2(\Omega')} \le K$$

for some sequence $h_n\to 0$ and all $\Omega'\Subset\Omega$ with $h_n<\operatorname{dist}(\Omega',\partial\Omega)$, then the weak derivative $D_i f$ exists and

$$\|D_i f\|_{L^2(\Omega)} \le K.$$

The fundamental elliptic regularity theorems for Sobolev norms may be proved by approximating weak derivatives by difference quotients.

We now formulate the general regularity theorem.
We consider an operator

$$Lf(x) := \frac{\partial}{\partial x^\alpha}\big(a^{\alpha\beta}(x)\frac{\partial}{\partial x^\beta}f(x)\big) \tag{A.2.2}$$

for $x\in\Omega$, $f:\Omega\to\mathbb{R}$, $\Omega\subset\mathbb{R}^m$.

We assume that there exist constants $0<\lambda\le\mu$ with

$$\lambda|\xi|^2 \le a^{\alpha\beta}(x)\xi_\alpha\xi_\beta \le \mu|\xi|^2 \tag{A.2.3}$$

for all $x\in\Omega,\xi\in\mathbb{R}^m$. We say that L *is uniformly elliptic*. Let $k\in L^2(\Omega)$. Then $f\in H^{1,2}(\Omega)$ is called *weak solution* of

$$Lf = k$$

if

$$\int_\Omega a^{\alpha\beta}(x)D_\beta f(x)D_\alpha\varphi(x)\,dx = -\int_\Omega k(x)\varphi(x)\,dx \tag{A.2.4}$$

for all $\varphi\in H_0^{1,2}(\Omega)$.

Theorem A.2.1. *Let* $f \in H^{1,2}(\Omega)$ *be a weak solution of (A.2.4). Suppose* $k \in H^{\nu,2}(\Omega)$, $a^{\alpha\beta} \in C^{\nu+1}(\Omega)$ $(\nu \in \mathbb{N})$.

Then

$$f \in H^{\nu+2,2}(\Omega')$$

for every $\Omega' \Subset \Omega$.

If

$$\|a^{\alpha\beta}\|_{C^{\nu+1}(\Omega)} \leq K_\nu,$$

then

$$\|f\|_{H^{\nu+2,2}(\Omega')} \leq c(\|f\|_{L^2(\Omega)} + \|k\|_{H^{\nu,2}(\Omega)}), \tag{A.2.5}$$

where c *depends on* m, λ, ν, K_ν *and* $\operatorname{dist}(\Omega', \partial\Omega)$.

The Harnack inequalities of Moser are of fundamental importance for the theory of elliptic partial differential equations:

Theorem A.2.2. *Let* L *be a uniformly elliptic operator as in (A.2.2), (A.2.3).*

(i) Let u *be a weak subsolution, i.e.*

$$Lu \geq 0 \quad \text{in a ball } B(x_0, 4R) \subset \mathbb{R}^m$$

$(\int a^{\alpha\beta} D_p u \, D_\beta \varphi \leq 0 \quad$ *for all* $\varphi \in H_0^{1,2}(B(x_0, 4R)))$. *For* $p > 1$ *then*

$$\sup_{B(x_0,R)} u \leq c_1 \Big(\frac{p}{p-1}\Big)^{\frac{2}{p}} \Big(\frac{1}{\omega_m (2R)^m} \int_{B(x_0,2R)} \max(u(x), 0)^p \, dx\Big)^{\frac{1}{p}},$$

where c_1 *depends only on* m *and* $\frac{\mu}{\lambda}$ *in (A.2.3).*

(ii) Let u *be a* **positive** *supersolution, i.e.*

$$Lu \leq 0 \qquad \text{in a ball } B(x_0, 4R) \subset \mathbb{R}^m.$$

For $m \geq 3$ *and* $0 < p < \frac{m}{m-2}$ *then*

$$\Big(\frac{1}{\omega_m (2R)^m} \int_{B(x_0,2R)} u^p\Big)^{\frac{1}{p}} \leq \frac{c_2}{(\frac{m}{m-2} - p)^2} \inf_{B(x_0,R)} u,$$

c_2 *again depending only on* m *and* $\frac{\mu}{\lambda}$. *For* $m = 2$ *and* $0 < p < \infty$, *the same estimate holds when* $\frac{c_2}{(\frac{m}{m-2}-p)^2}$ *is replaced by a constant* c_3 *depending on* p *and* $\frac{\mu}{\lambda}$.

The Harnack inequality also translates into estimates for the fundamental solutions of the Laplace-Beltrami operator, and their generalizations, the Green functions. The Green function $G(x_0, x)$ of a ball $B \subset M$ (or another sufficiently regular domain), for x_0 in the interior of B, is symmetric in x and x_0, smooth for $x \neq x_0$, becomes singular like $\frac{1}{(d-2)\omega_d} d(x,x_0)^{2-d}$ in case $d = \dim M \geq 3$ ($\omega_d = \operatorname{Vol} S^{d-1}$) (and like $\frac{1}{\omega_2} \log d(x_0, x)$ for $d = 2$), vanishes for $x \in \partial B$, and satisfies

$$h(x_0) = \int_B \Delta h(x)\, G(x_0, x)\, d\mathrm{Vol}\,(x) \quad \text{for all}\, h \in C_0^2(B).$$

A geometric approximation of the Green function (that is exact in the Euclidean case) has been investigated in §4.7. An analytic alternative that allows to avoid the singularity is the use of the mollified Green function. For simplicity, and because that typically suffices for applications, we only consider the case of a ball. The mollified Green function $G^R(x_0, x)$ on the ball $B(x_0, R)$ relative to the ball $B(x_0, 2R)$ of double radius, $G^R(x_0, \cdot) \in H^{1,2} \cap C_0^0(B(x_0, 2R))$, satisfies

$$\begin{aligned}\int_{B(x_0,2R)} \Delta\varphi(x)\, G^R(x_0, x)\, d\mathrm{Vol}\,(x) &= \int_{B(x_0,2R)} \langle d\varphi(x), dG^R(x_0, x)\rangle\, d\mathrm{Vol}\,(x) \\ &= \fint_{B(x_0,R)} \varphi(x)\, d\mathrm{Vol}\,(x),\end{aligned}$$

for all $\varphi \in H^{1,2}$ with $\operatorname{supp} \varphi \Subset B(x_0, 2R)$.

For purposes of normalization, it is convenient to consider

$$w^R(x) := \frac{|B(x_0, 2R)|}{R^2} G^R(x_0, x)$$

with $|B| := \operatorname{Vol} B$.

We then have

$$\int_{B(x_0,2R)} \langle d\varphi(x), dw^R(x)\rangle = \frac{1}{R^2} \int_{B(x_0,R)} \varphi(x),$$

for all $\varphi \in H^{1,2}$ with $\operatorname{supp} \varphi \Subset B(x_0, 2R)$.

We then have the estimates

Corollary A.2.1.

$$\begin{aligned} 0 \leq w^R \leq \gamma_1 &\quad \text{in } B(x_0, 2R), \\ w^R \geq \gamma_2 > 0 &\quad \text{in } B(x_0, R), \end{aligned}$$

for constants γ_1, γ_2 that do not depend on R.

The estimates of J. Schauder are also very important:

Theorem A.2.3. *Let L be as in (A.2.2), (A.2.3), and suppose that the coefficients $a^{\alpha\beta}(x)$ are Hölder continuous in Ω, i.e. contained in $C^\sigma(\Omega)$ for some $0 < \sigma < 1$.*

(i) If u is a weak solution of

$$Lu = k$$

and if k is in $L^\infty(\Omega)$, then u is in $C^{1,\sigma}(\Omega)$, and on every $\Omega_0 \Subset \Omega$, its $C^{1,\sigma}$-norm can be estimated in terms of its L^2-norm and the L^∞-norm of k, with a structural constant depending on Ω, Ω_0, m, σ, λ, μ and the C^σ-norm of the $a^{\alpha\beta}(x)$.

(ii) If u is a weak solution of

$$Lu = k$$

for some $k \in C^{\nu,\sigma}(\Omega)$, $\nu = 0, 1, 2, \dots$, $0 < \sigma < 1$, and if the coefficients $a^{\alpha\beta}$ are also in $C^{\nu,\sigma}(\Omega)$, then u is in $C^{\nu+2,\sigma}(\Omega)$, and a similar estimate as in (i) holds, this time involving the $C^{\nu,\sigma}$-norm of k and the $a^{\alpha\beta}$.

Finally, we quote the maximum principle.

Theorem A.2.4. *Let $\Omega \subset \mathbb{R}^m$ (or, more generally, $\Omega \subset M$, M a Riemannian manifold) be open and bounded, $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with*

$$Lf \geq 0 \quad \text{in } \Omega,$$

L as in (A.2.2), (A.2.3). Then f assumes its maximum on the boundary $\partial\Omega$.

All the preceding results naturally apply to the Laplace-Beltrami operator on a ball $B(x_0, r)$ in a Riemannian manifold M, putting

$$L = -\Delta = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\gamma} \gamma^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right),$$

$(\gamma_{\alpha\beta})_{\alpha,\beta=1,\dots,m}$ the metric tensor of M in local coordinates, $(\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}$, $\gamma = \det(\gamma_{\alpha\beta})$.

References for the material in this appendix are: Gilbarg and Trudinger[96], Jost[146] and, with a more elementary presentation, Jost[143]. The results of Corollary A.2.1 about Green functions are systematically derived in [113], and in a more general context in [23]. Some further points about Sobolev spaces can be found in Ziemer[273].

A.3 Existence and Regularity Theory for Solutions of Linear Parabolic Equations

In this section, we consider differential equations on $\Omega \times [0, \infty)$ where Ω is an open subset of $\mathbb{R}^m$ as in A.2, and we continue to use the notations introduced there.

In particular, as before, let the operator L be a *uniformly elliptic* operator of the form

$$Lf(x) := \frac{\partial}{\partial x^\alpha}\left(a^{\alpha\beta}(x)\frac{\partial}{\partial x^\beta} f(x)\right) \tag{A.3.1}$$

with constants $0 < \lambda \le \mu$ satisfying

$$\lambda|\xi|^2 \le a^{\alpha\beta}(x,t)\xi_\alpha\xi_\beta \le \mu|\xi|^2 \tag{A.3.2}$$

for all $x \in \Omega, 0 \le t, \xi \in \mathbb{R}^m$. The equation we wish to study then is

$$\frac{\partial}{\partial t} f(x,t) - Lf(x,t) = k(x,t) \text{ for } x \in \Omega,\ t \ge 0 \tag{A.3.3}$$

$$f(x,0) = \phi(x) \tag{A.3.4}$$

for some continuous function $\phi(x)$ and some bounded function $k(x,t)$ (and suitable boundary conditions, but since in the text, we are interested in compact manifolds M in place of the open domain Ω, these will not play an essential role and consequently are not emphasized here). (A.3.3) is a linear parabolic partial differential equation

We first state the parabolic maximum principle.

Theorem A.3.1. *Let $\Omega \subset \mathbb{R}^m$ (or, more generally, $\Omega \subset M$, M a Riemannian manifold) be open and bounded, $f \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with respect to x and in $C^1((0,T)) \cap C^0([0,T])$ with respect to t, with*

$$\frac{\partial}{\partial t} f - Lf \le 0 \quad \text{in } \Omega \times [0,T]. \tag{A.3.5}$$

Then f assumes its maximum for (x,t) with $x \in \partial\Omega$ or for $t = 0$, that is, either on the spatial boundary or at the initial time. In particular, when M is a compact manifold (without boundary), the supremum of $f(\cdot,t)$ is a decreasing function of t.

We have the following existence and regularity theorem for solutions of (A.3.3), with Schauder type estimates

Theorem A.3.2. *Let L be as in (A.3.1), (A.3.2), and suppose that the coefficients $a^{\alpha\beta}(x,t)$ are Hölder continuous in $\Omega \times [0,\infty)$, i.e. contained in $C^\sigma(\Omega \times [0,\infty))$ for some $0 < \sigma < 1$. If we prescribe some boundary values, say $f(y,t) = g(y)$ for all $y \in \partial\Omega$, for some given, e.g. continuous, function g, the solution of (A.3.3) then exists for all $t \ge 0$.*

Furthermore, we have the following estimates:

(i) If u is a weak solution of

$$Lu = k \tag{A.3.6}$$

and if k is in $L^\infty(\Omega \times [0,\infty))$, then as a function of x, u is in $C^{1,\sigma}(\Omega)$, and for every $\Omega_0 \Subset \Omega$ and $t_0 > 0$, its (spatial) $C^{1,\sigma}(\Omega)$-norm on $\Omega_0 \times [t_0,\infty)$ can be estimated in terms of its L^∞-norm and the L^∞-norm of k, with a structural constant depending on Ω, Ω_0, t_0, m, σ, λ, μ and the C^σ-norm of the $a^{\alpha\beta}(x)$.

(ii) If u is a weak solution of

$$Lu = k$$

for some $k \in C^{\nu,\sigma}(\Omega \times [0,\infty))$, $\nu = 0, 1, 2, \ldots$, $0 < \sigma < 1$, and if the coefficients $a^{\alpha\beta}$ are also in $C^{\nu,\sigma}(\Omega \times [0,\infty))$, then u is in $C^{\nu+2,\sigma}(\Omega)$ with respect to x and of class $C^{\nu+1,\sigma}$ with respect to t, and the corresponding norms can be estimated analogously to (i), this time involving the $C^{\nu,\sigma}$-norm of k and the $a^{\alpha\beta}$.

The restriction to $t \geq t_0 > 0$ can be avoided if the initial values f_0 satisfy appropriate regularity results. The estimates on $[0,\infty)$ will then naturally also involve the corresponding norms of f_0.

Theorem A.3.2 concerns a linear parabolic equation. In the text, we shall encounter nonlinear parabolic equations and systems. For those, the global existence and regularity cannot be deduced from a general result, but rather needs to invoke the detailed structure of the system. What one can deduce from Theorem A.3.2, however, is the short time existence of solutions when the linearization of the differential operator satisfies the assumptions of that theorem. This follows by linearization and the implicit function theorem. That means that for such nonlinear systems, we can obtain the existence of a solution on some interval $[0,T)$ whose length depends on the regularity properties of the initial values. This also implies that the maximal interval of existence for nonlinear parabolic systems is open. For the closedness of the interval of existence, and consequently the existence of a solution for all "time" $t \geq 0$, one then needs to derive specific apriori estimates that prevent solutions from becoming singular in finite time.

A reference for parabolic differential equations and systems is [174]. For a textbook treatment, we refer to [146].

Appendix B

Fundamental Groups and Covering Spaces

In this appendix, we briefly list some topological results. We assume that M is a connected manifold, although the results hold for more general spaces.

A *path* or *curve* in M is a continuous map

$$c : [0, a] \to M \quad (a \geq 0).$$

A *loop* is a path with $c(0) = c(a)$, and that point then is called the *base point* of the loop. The *inverse* of a path c is

$$c^{-1} : [0, a] \to M,$$
$$c^{-1}(t) := c(a - t).$$

If $c_i : [0, a_i] \to M$ are paths $(i = 1, 2)$ with $c_2(0) = c_1(a_1)$, we can define the product $c_1 \cdot c_2$ as the path $c : [0, a_1 + a_2] \to M$,

$$c(t) = \begin{cases} c_1(t) & \text{for } 0 \leq t \leq a_1, \\ c_2(t - a_1) & \text{for } a_1 \leq t \leq a_1 + a_2. \end{cases}$$

Two paths $c_i : [0, a_i]$ with $c_1(0) = c_2(0)$ and $c_1(a_1) = c_2(a_2)$ are called *equivalent* or *homotopic* if there exists a continuous function

$$H : [0, 1] \times [0, 1] \to M$$

with

$$\begin{aligned} H(t,0) &= c_1\Big(\frac{t}{a_1}\Big), \\ H(t,1) &= c_2\Big(\frac{t}{a_2}\Big), \qquad \text{for all } t, \\ H(0,s) &= c_1(0) = c_2(0), \\ H(1,s) &= c_1(a_1) = c_2(a_2), \qquad \text{for all } s. \end{aligned}$$

In particular, $c : [0,a] \to M$ is equivalent to $\tilde{c} : [0,1] \to M$ with $\tilde{c}(t) = c(\frac{t}{a})$, and so we may assume that all paths are parametrized on the unit interval.

We obtain an equivalence relation on the space of all paths. The equivalence class of c is denoted $[c]$, and it is not hard to verify that $[c_1 c_2]$ and $[c^{-1}]$ are independent of the choice of representations. Thus, we may define

$$\begin{aligned} [c_1 \cdot c_2] &=: [c_1] \cdot [c_2], \\ [c^{-1}] &=: [c]^{-1}. \end{aligned}$$

In particular, the equivalence or homotopy classes of loops with fixed base point $p \in M$ form a group $\pi_1(M,p)$, the *fundamental group* of M with *base point* p.

If p and q are in M and $\gamma : [0,1] \to M$ satisfies $\gamma(0) = p$, $\gamma(1) = q$, then for every loop c with base point q, $\gamma^{-1}c\gamma$ is a loop with base point p, and this induces an isomorphism between $\pi_1(M,q)$ and $\pi_1(M,p)$. We may thus speak of the fundamental group $\pi_1(M)$ of M without reference to a base point. M is called *simply connected* if $\pi_1(M) = 0$. A continuous map $f : M \to N$ induces a map $f_\# : \pi_1(M,p) \to \pi_1(N, f(p))$ of fundamental groups.

A continuous map

$$\pi : X \to M$$

is called a *covering map* if each $p \in M$ has a neighborhood U with the property that each connected component of $\pi^{-1}(U)$ is mapped homeomorphically onto U. If $p \in M$ and H is a subgroup of $\pi_1(M,p)$, there exists a covering $\pi : X \to M$ with the property that for any $x \in X$ with $\pi(x) = p$, we have $\pi_*(\pi_1(X,x)) = H$.

If we choose $H = \{1\}$, we obtain a simply connected manifold $\tilde{M}$ and a covering

$$\pi : \tilde{M} \to M.$$

$\tilde{M}$ is called the *universal covering* of M.

If $\pi : X \to M$ is a covering, $c : [0,1] \to M$ a path, $x_0 \in \pi^{-1}(x(0))$, then there exists a unique path

$$\tilde{c} : [0,1] \to X$$

with $\tilde{c}(0) = x_0$ and $c(t) = \pi(\tilde{c}(t))$. $\tilde{c}$ is called the lift of c through x_0.

More generally, if M' is another manifold, $f : M' \to M$ is continuous, $p_0 \in M$, $y_0 \in f^{-1}(p_0)$, $x_0 \in \pi^{-1}(p_0)$, there exists a continuous

$$\tilde{f} : M' \to X$$

with $\tilde{f}(y_0) = x_0$ and $f = \pi \circ \tilde{f}$ if and only if $f_\#(\pi_1(M', y_0)) \subset \pi_\#(\pi_1(X, x_0))$. $\tilde{f}$ is unique if it exists.

Let $\pi : \tilde{M} \to M$ be the universal covering of M. A *deck transformation* is a homeomorphism $\varphi : \tilde{M} \to \tilde{M}$ with

$$\pi = \pi \circ \varphi.$$

Let $\pi(x_0) = p_0$. $\pi_1(M, p_0)$ then bijectively corresponds to $\pi^{-1}(p_0)$. More precisely, $x_1 \in \pi^{-1}(p_0)$ corresponds to the homotopy class of $\pi(\gamma_{x_1})$, where $\gamma_{x_1} : [0, 1] \to \tilde{M}$ is any path with $\gamma_{x_1}(0) = x_0$, $\gamma_{x_1}(1) = x_1$. The deck transformations form a group that acts simply transitively on $\pi^{-1}(p_0)$, and associating to a deck transformation $\varphi(x_0) \in \pi^{-1}(p_0)$ then yields an isomorphism between the group of deck transformations and $\pi_1(M, p_0)$.

If M and N are manifolds with universal coverings $\tilde{M}$ and $\tilde{N}$, resp., and if

$$f : M \to N$$

is a continuous map, we consider the induced homomorphism

$$\rho := f_\sharp : \pi_1(M, p) \to \pi_1(N, f(p))$$

of fundamental groups. If $\pi : \tilde{M} \to M$ is the universal covering, we can lift $f \circ \pi : \tilde{M} \to N$ to a map

$$\tilde{f} : \tilde{M} \to \tilde{N},$$

because the above lifting condition is trivially satisfied as $\pi_1(\tilde{M}) = \{1\}$. $\tilde{f}$ is equivariant w.r.t. the above homomorphism ρ in the sense that for every $\lambda \in \pi_1(M, p)$, acting as a deck transformation on $\tilde{M}$, we have

$$\tilde{f}(\lambda x) = \rho(\lambda)\tilde{f}(x) \quad \text{for every } x \in \tilde{M}, \tag{B.1}$$

where $\rho(\lambda)$ acts as a deck transformation on $\tilde{N}$. We say that $\tilde{f}$ is a ρ-equivariant map between the universal covers $\tilde{M}$ and $\tilde{N}$.

Conversely, given any homomorphism

$$\rho : \pi_1(M, p) \to \pi_1(N, q)$$

and any ρ-equivariant map

$$g : \tilde{M} \to \tilde{N} \quad (\text{with } g(p) = q),$$

not necessarily continuous, then g induces a map

$$g' : M \to N$$

whose lift to universal covers is g. g' is continuous if g is.

Finally, if $\tilde{M}$ is the universal cover of a compact Riemannian manifold M, a so-called fundamental domain $F(M)$ for M in $\tilde{M}$ can be constructed as follows:

For simplicity of notation, we denote the group $\pi_1(M, x_0)$ operating by deck transformations on $\tilde{M}$ by Γ, and its trivial element by e.

Let $d(.,.)$ be the Riemannian distance function on $\tilde{M}$. We select any $z_0 \in \tilde{M}$. We then put

$$F(M) := \{z \in \tilde{M} : d(z, z_0) < d(\gamma z, z_0) \quad \text{for all } \gamma \in \Gamma, \gamma \neq e\}.$$

$F(M)$ is open. Since Γ operates by isometries, i.e.

$$d(\lambda z_1, \lambda z_2) = d(z_1, z_2) \quad \text{for all } \lambda \in \Gamma, z_1, z_2 \in \tilde{M},$$

we may also write

$$F(M) = \{z \in \tilde{M} : d(z, z_0) < d(z, \lambda z_0) \quad \text{for all } \lambda \in \Gamma, \lambda \neq e\}.$$

By its definition, $F(M)$ cannot contain any two points that are equivalent under the operation of Γ. On the other hand, for any $z \in \tilde{M}$, we may find some $\mu \in \Gamma$ such that

$$\mu z \in \overline{F(M)}.$$

Thus, the closure of $F(M)$ contains at least one point from every orbit of Γ in $\tilde{M}$.

If $f : M \to \mathbb{R}$ is an integrable function, and if $\tilde{f} : \tilde{M} \to \mathbb{R}$ is its lift to the universal cover of M, then

$$\int_M f(x)\, d\mathrm{Vol}\,(x) = \int_{F(M)} \tilde{f}(y)\, d\mathrm{Vol}\,(y).$$

Examples of fundamental groups.

1. $\pi_1(\mathbb{R}^n) = \{1\}$ for all n.
2. $\pi_1(S^1) = \mathbb{Z}$.

   A generator is given by

   $$c : [0,1] \to S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$
   $$c(t) = (\cos 2\pi t, \sin 2\pi t).$$

   The universal covering of S^1 is $\mathbb{R}^1$, and the covering map is likewise given by

   $$\pi(t) = (\cos 2\pi t, \sin 2\pi t).$$

3. $\pi_1(S^n) = \{1\}$ for $n \geq 2$.
4. $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for $n \geq 3$.

The preceding results can be found in any reasonable textbook on Algebraic Topology, for example in [99] or [240].

Bibliography

[1] A. Abbondandolo and P. Majer. Morse homology on Hilbert spaces. *Comm. Pure Appl. Math.*, 54:689–760, 2001.

[2] U. Abresch and W. Meyer. Pinching below $\frac{1}{4}$, injectivity radius, and conjugate radius. *J. Diff. Geom.*, 40:643–691, 1994.

[3] U. Abresch and W. Meyer. A sphere theorem with pinching constant below $\frac{1}{4}$. *J. Diff. Geom.*, 44:214–261, 1996.

[4] S.I. Al'ber. On n–dimensional problems in the calculus of variations in the large. *Sov. Math. Dokl.*, 5:700–704, 1964.

[5] S.I. Al'ber. Spaces of mappings into a manifold with negative curvature. *Sov. Math. Dokl.*, 9:6–9, 1967.

[6] F. Almgren. Questions and answers about area–minimizing surfaces and geometric measure theory. *Proc. Symp. Pure Math.*, 54(I):29–53, 1993.

[7] M. Atiyah, R. Bott, and A. Shapiro. Clifford moduls. *Topology*, 3(Suppl. I):3–38, 1964.

[8] M. Atiyah and I. Singer. The index of elliptic operators:III. *Ann. Math.*, 87:546–604, 1963.

[9] T. Aubin. *Nonlinear analysis on manifolds. Monge–Ampère equations.* Springer–Verlag, Berlin, 1982.

[10] W. Ballmann. Der Satz von Ljusternik und Schnirelman. *Bonner Math. Schriften*, 102:1–25, 1978.

[11] W. Ballmann. *Lectures on spaces of nonpositive curvature.* DMV Seminar Vol.25, Birkhäuser, 1995.

[12] W. Ballmann, M. Gromov, and V. Schroeder. *Manifolds of nonpositive curvature.* Birkhäuser, 1985.

[13] V. Bangert. On the existence of closed geodesics on two–spheres. *Internat. J. Math*, 4:1–10, 1993.

[14] H. Bass and J. Morgan (eds.). *The Smith conjecture.* Academic Press, 1984.

[15] V.N. Berestovskij and I.G. Nikolaev. Multidimensional generalized Riemannian spaces. In Y. G. Reshetnyak, editor, *Geometry IV*, pages 165 – 243. Encyclopedia Math. Sciences Vol 70, Springer, 1993.

[16] M. Berger. Les variétés Riemanniennes ($\frac{1}{4}$)–pincées. *Ann. Scuola Norm. Sup. Pisa*, III(14):161–170, 1960.

[17] M. Berger. *A panoramic view of Riemannian geometry.* Springer, 2003.

[18] N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators.* Springer, 1992.

[19] A. Besse. *Einstein manifolds.* Springer, 1987.

[20] G. Besson, G. Courtois, and S. Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *GAFA*, 5:731–799, 1995.

[21] G. Besson, G. Courtois, and S. Gallot. Minimal entropy and Mostow's rigidity theorems. *Ergod. Th.* & *Dynam. Sys*, 16:623–649, 1996.

[22] M. Betz and R.Cohen. Graph moduli spaces and cohomology operations. Preprint, 1995.

[23] M. Biroli and U. Mosco. A Saint-Venant principle for Dirichlet forms on discontinuous media. *Annali Mat. Pura Appl.(IV)*, 169:125–181, 1995.

[24] S. Bochner. Harmonic surfaces in Riemannian metric. *Trans.AMS*, 47:146–154, 1940.

[25] S. Bochner and K. Yano. *Curvature and Betti numbers.* Princeton Univ. Press, 1953.

[26] J. Borchers and W. Garber. Analyticity of solutions of the O(N) nonlinear σ–model. *Comm. Math. Phys*, 71:299–309, 1980.

[27] J.P. Bourguignon. *Eugenio Calabi and Kähler metrics*, pages 61–85. In: P. de Bartolomeis(ed.), Manifolds and geometry, Symp. Math. 36, Cambridge Univ. Press, 1996.

[28] J.P. Bourguignon and H. Karcher. Curvature operators: Pinching estimates and geometric examples. *Ann. scient. Éc. Norm. Sup*, 11:71–92, 1978.

[29] S. Bradlow. Vortices in holomorphic line bundles over closed Kähler manifolds. *Comm. Math. Phys*, 135:1–17, 1990.

[30] S. Bradlow. Special metrics and stability for holomorphic bundles with global sections. *J. Diff. Geom*, 33:169–214, 1991.

[31] S. Brendle and R. Schoen. Manifolds with 1/4-pinched curvature are space forms. *arxiv*, 0705.0766, 2007.

[32] H. Brézis and J. Coron. Large solutions for harmonic maps in two dimensions. *Comm. Math. Phys*, 92:203–215, 1983.

[33] C. Le Brun. Einstein metrics and Mostow rigidity. *Math. Res. Lett.*, 2:1 – 8, 1995.

[34] Yu. Burago, M. Gromov, and G. Perel'man. A. D. Alexandrov's spaces with curvatures bounded from below. *Russ. Math. Surveys*, 42:1–58, 1992.

[35] P. Buser and H. Karcher. *Gromov's almost flat manifolds.* Astérisque, 1981.

[36] L. Caffarelli and Y.S. Yang. Vortex condensation in the Chern-Simons-Higgs Model. An existence theorem. *Comm. Math. Phys*, 168:321–336, 1995.

[37] E. Calabi and E. Vesentini. On compact, locally symmetric Kähler manifolds. *Ann. Math*, 71:472–507, 1960.

[38] H. D. Cao and X. P. Zhu. A complete proof of the Poincaré and geometrization conjectures - application of the Hamilton-Perelman theory of the Ricci flow. *Asian J. Math.*, 10(2):195–492, 2006.

[39] H. D. Cao and X. P. Zhu. Hamilton-Perelman's proof of the Poincaré conjecture and the geometrization conjecture. arXiv:math.DG/0612069, 2006.

[40] J. Cao and F. Xavier. Kähler parabolicity and the Euler number of compact of non–positive sectional curvature. *Math. Ann*, 319:483–491, 2001.

[41] K.C. Chang. Heat flow and boundary value problem for harmonic maps. *Anal. Nonlinéaire*, 6:363–396, 1989.

[42] K.C. Chang. *Infinite dimensional Morse theory and multiple solution problems.* Birkhäuser, 1993.

[43] I. Chavel. *Riemannian geometry - A modern introduction.* Cambridge University Press, 1993.

[44] J. Cheeger. Finiteness theorems for Riemannian manifolds. *Amer. J. Math.*, 92:61–74, 1970.

[45] J. Cheeger and D. Ebin. *Comparison theorems in Riemannian geometry.* North Holland, 1975.

[46] J. Cheeger and D. Gromoll. The splitting theorems for manifolds of nonnegative Ricci curvature. *J. Diff. Geom*, 6:119–129, 1971.

[47] J. Cheeger and M. Gromov. Collapsing Riemannian manifolds while keeping their curvature bounded. *I. J. Diff. Geom.*, 23:309–346, 1983.

[48] J. Cheeger and M. Gromov. Collapsing Riemannian manifolds while keeping their curvature bounded. *II, J. Diff. Geom.*, 32:269–298, 1990.

[49] Q. Chen, J. Jost, J.Y. Li, and G.F. Wang. Regularity theorems and energy identities for Dirac-harmonic maps. *Math. Zeitschr.*, 251:61–84, 2005.

[50] Q. Chen, J. Jost, J.Y. Li, and G.F. Wang. Dirac-harmonic maps. *Math. Zeitschr.*, 254:409–432, 2006.

[51] W.Y. Chen and J. Jost. A Riemannian version of Korn's inequality. *Calc. Var.*, 14:517–530, 2001.

[52] B. Chow and D. Knopf. *The Ricci flow: An introduction.* Amer. Math. Soc., 2004.

[53] B. Chow, P. Lu, and L. Ni. *Hamilton s Ricci flow.* Amer. Math. Soc. and Intern. Press, 2007.

[54] S. N. Chow and J. Hale. *Methods of bifurcation theory.* Springer, 1982.

[55] C. Conley. *Isolated invariant sets and the Morse index.* CBMS Reg. Conf. Ser. Math. 38, AMS, Providence, R.I., 1978.

[56] C. Conley and E. Zehnder. Morse-type index theory for flows and periodic solutions for Hamiltonian equations. *Comm. Pure Appl. Math.*, 37:207–253, 1984.

[57] K. Corlette. Archimedean superrigidity and hyperbolic geometry. *Ann. Math.*, 135:165–182, 1992.

[58] G. dal Maso. *An introduction to* Γ*–convergence.* Birkhäuser, 1993.

[59] D. de Turck and J. Kazdan. Some regularity theorems in Riemannian geometry. *Ann. Sci. École Norm. Sup.*, 4(14):249–260, 1981.

[60] U. Dierkes, S. Hildebrandt, A. Küster, and O. Wohlrab. *Minimal surfaces, 2 volumes.* Springer, 1992.

[61] W.Y. Ding. Lusternik-Schnirelman theory for harmonic maps. *Acta Math. Sinica*, 2:105–122, 1986.

[62] W.Y. Ding, J. Jost, J.Y. Li, X.W. Peng, and G.F. Wang. Self duality equations for Ginzburg-Landau and Seiberg-Witten type functionals with 6^{th} order potentials. *Comm. Math. Phys.*, 217(2):383–407, 2001.

[63] W.Y. Ding, J. Jost, J.Y. Li, and G.F. Wang. An analysis of the two–vortex case in the Chern-Simons-Higgs model. *Calc. Var.*, 7:87–97, 1998.

[64] W.Y. Ding, J. Jost, J.Y. Li, and G.F. Wang. Multiplicity results for the two vortex Chern-Simons-Higgs model on the two-sphere. *Commentarii Math. Helv.*, 74:118–142, 1999.

[65] M. do Carmo. *Riemannian geometry.* Birkhäuser, 1992.

[66] S. Donaldson and P. Kronheimer. *The geometry of four-manifolds*. Oxford Univ. Press, 1990.

[67] H. Donnelly and F. Xavier. On the differential form spectrum of negatively curved Riemannian manifolds. *Amer. J. Math.*, 106:169–185, 1984.

[68] P. Eberlein. *Rigidity problems for manifolds of nonpositive curvature*. Springer LNM 1156, 1984.

[69] P. Eberlein. *Geometry of nonpositively curved manifolds*. Univ. Chicago Press, 1996.

[70] P. Eberlein, U. Hamenstädt, and V. Schroeder. Manifolds of nonpositive curvature. *Proc. Symp. Pure Math*, 54:Part 3, 179–227, 1993.

[71] P. Eberlein and B. O'Neill. Visibility manifolds. *Pacific J. Math.*, 46:45–109, 1973.

[72] J. Eells and L. Lemaire. A report on harmonic maps. *Bull. London Math. Soc.*, 10:1–68, 1978.

[73] J. Eells and L. Lemaire. Selected topics in harmonic maps. *CBMS Reg. Conf. Ser.*, 50, 1983.

[74] J. Eells and L. Lemaire. Another report on harmonic maps. *Bull. London Math. Soc.*, 20:385–524, 1988.

[75] J. Eells and J. Sampson. Harmonic mappings of Riemannian manifolds. *Am. J. Math.*, 85:109–160, 1964.

[76] J. Eschenburg and J. Jost. *Differentialgeometrie und Minimalflächen*. Springer, 2007.

[77] P. Deligne et al. *Quantum fields and strings: a course for mathematicians, Vol. I*. Amer.Math.Soc. and Inst. Adv. Study, Princeton, N. J., 1999.

[78] P. Deligne et al. *Quantum fields and strings: a course for mathematicians, Vol. II*. Amer.Math.Soc. and Inst. Adv. Study, Princeton, N. J., 1999.

[79] F. Fang and X. Rong. Positive pinching, volume and second betti number. *GAFA*, 9:641–674, 1999.

[80] H. Federer. *Geometric measure theory*. Springer, 1979.

[81] A. Floer. Witten's complex and infinite dimensional Morse theory. *J. Diff. Geom.*, 30:207–221, 1989.

[82] A. Floer and H. Hofer. Coherent orientations for periodic orbit problems in symplectic geometry. *Math. Z.*, 212:13–38, 1993.

[83] J. Franks. Morse-Smale flows and homotopy theory. *Topology*, 18:199–215, 1979.

[84] J. Franks. Geodesics on S^2 and periodic points of annulus diffeomorphisms. *Inv. math.*, 108:403–418, 1992.

[85] D. Freed. Special Kähler manifolds. *Comm. Math. Phys.*, 203:31–52, 1999.

[86] D. Freed and K. Uhlenbeck. *Instantons and 4-manifolds*. Springer, 1984.

[87] Th. Friedrich. *Neue Invarianten der 4–dimensionalen Mannigfaltigkeiten*. SFB 288, Berlin, 1995.

[88] Th. Friedrich. *Dirac-Operatoren in der Riemannschen Geometrie*. Vieweg, 1997.

[89] K. Fukaya. A boundary of the set of Riemannian manifolds with bounded curvatures and diameters. *J. Diff. Geom.*, 28:1–21, 1988.

[90] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian geometry*. Springer, 1987.

[91] O. García-Prada. Invariant connections and vortices. *Comm. Math. Phys.*, 156:527–546, 1993.

[92] O. García-Prada. Dimensional reduction of stable bundles, vortices and stable pairs. *Intern. J. Math*, 5:1–52, 1994.

[93] O. García-Prada. Proof for the vortex equations over a compact Riemann surface. *Bull. London Math. Soc.*, 26:88–96, 1994.

[94] M. Giaquinta and E. Giusti. On the regularity of the minima of variational integrals. *Acta Math.*, 148:31–46, 1982.

[95] M. Giaquinta and E. Giusti. The singular set of the minima of certain quadratic functionals. *Ann. Scuola Norm. Sup. Pisa, Cl. Sci.*, (4) 11:45–55, 1984.

[96] D. Gilbarg and N.Trudinger. *Elliptic partial differential equations of second order*. Springer, 1977.

[97] M. Goresky and R. MacPherson. *Stratified Morse theory*. Ergebnisse 14, Springer, 1988.

[98] M. Grayson. Shortening embedded curves. *Ann. Math.*, 120:71–112, 1989.

[99] M. Greenberg. *Lectures on algebraic topology*. Benjamin, Reading, Mass., 1967.

[100] R. Greene and H. Wu. Lipschitz convergence of Riemannian manifolds. *Pacific J. Math.*, 131:119–141, 1988.

[101] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Wiley–Interscience, 1978.

[102] D. Gromoll. Spaces of nonnegative curvature. *Proc. Sym. Pure Math.*, 54(Part 3):337–356, 1993.

[103] D. Gromoll, W. Klingenberg, and W. Meyer. *Riemannsche Geometrie im Großen.* Springer LNM 55, ^{2}, 1975.

[104] D. Gromoll and J. Wolf. Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of nonpositive curvature. *Bull. AMS*, 77:545–552, 1971.

[105] M. Gromov. *Structures métriques pour les variétés riemanniennes.* Rédigé par J. Lafontaine and P. Pansu. Cedic-Nathan, Paris, 1980.

[106] M. Gromov. Pseudoholomorphic curves in symplectic geometry. *Inv. math.*, 82:307–347, 1985.

[107] M. Gromov. Kähler hyperbolicity and L^2-Hodge theory. *J. Diff. Geom.*, 33:263–292, 1991.

[108] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces.* Birkhäuser, 1999.

[109] M. Gromov and B. Lawson. The classification of simply connected manifolds of positive scalar curvature. *Ann. Math.*, 111:423–434, 1980.

[110] M. Gromov and B. Lawson. Spin and scalar curvature in the presence of a fundamental group. *Ann. Math.*, 111:209–230, 1980.

[111] M. Gromov and R. Schoen. Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. *Publ. Math. IHES*, 76:165–246, 1992.

[112] M. Grüter. Regularity of weak H-surfaces. *J. reine angew. Math.*, 329:1–15, 1981.

[113] M. Grüter and K.-O. Widman. The Green function for uniformly elliptic equations. *man. math.*, 37:303–342, 1982.

[114] M. Günther. Zum Einbettungssatz von J. Nash. *Math. Nachr.*, 144:165–187, 1989.

[115] R. Hamilton. *Harmonic maps of manifolds with boundary.* Springer LNM 471, 1975.

[116] R. Hamilton. Three–manifolds with positive Ricci curvature. *J. Diff. Geom.*, 17:255–306, 1982.

[117] R. Hamilton. The Harnack estimate for the Ricci flow. *J. Diff. Geom.*, 37:225–243, 1993.

[118] R. Hardt and L. Simon. Boundary regularity and embedded solutions for the oriented plateau problem. *Ann. Math.*, 110:439–486, 1979.

[119] P. Hartman. On homotopic harmonic maps. *Can. J. Math.*, 19:673–687, 1967.

[120] S. Hawking and Ellis. *The large scale structure of space–time.* Cambridge University Press, 1973.

[121] F. Hélein. Régularité des applications faiblement harmoniques entre une surface et une varieté riemannienne. *C.R. Acad. Sci. Paris*, 312:591–596, 1991.

[122] F. Hélein. *Harmonic maps, conservation laws and moving frames.* Cambridge Univ. Press, 2002.

[123] S. Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces.* Academic Press, 1978.

[124] S. Hildebrandt, H. Kaul, and K.O. Widman. Harmonic mappings into Riemannian manifolds with non–positive sectional curvature. *Math. Scand.*, 37:257–263, 1975.

[125] S. Hildebrandt, H. Kaul, and K.O. Widman. An existence theorem for harmonic mappings of Riemannian manifolds. *Acta Math.*, 138:1–16, 1977.

[126] N. Hingston. On the growth of the number of closed geodesics on the two–sphere. *Intern. Math. Res.*, 9:253–262, 1993.

[127] N. Hitchin. The self–duality equations on a Riemann surface. *Proc. London Math. Soc.*, 55:19–126, 1987.

[128] M.C. Hong, J. Jost, and M. Struwe. Asymptotic limits of a Ginzburg–Landau type functional. In J. Jost, editor, *Geometric Analysis and the Calculus of Variations for Stefan Hildebrandt*, pages 99–123. International Press, Boston, 1996.

[129] D. Husemoller. *Fibre bundles.* Springer, GTM 20, 1975.

[130] T. Ishihara. A mapping of Riemannian manifolds which preserves harmonic functions. *J. Math. Kyoto Univ.*, 19:215–229, 1979.

[131] W. Jäger and H. Kaul. Uniqueness and stability of harmonic maps and their Jacobi fields. *Man. math.*, 28:269–291, 1979.

[132] J. Jost. *Harmonic mappings between Riemannian manifolds.* ANU–Press, Canberra, 1984.

[133] J. Jost. The Dirichlet problem for harmonic maps from a surface with boundary onto a 2–sphere with non–constant boundary values. *J. Diff. Geom.*, 19:393–401, 1984.

[134] J. Jost. The geometric calculus of variations: A short survey and a list of open problems. *Expo. Math.*, 6:111–143, 1988.

[135] J. Jost. A nonparametric proof of the theorem of Ljusternik and Schnirelman. *Arch. Math.*, 53:497–509, 1989. Correction in *Arch. Math.* 56 (1991), 624.

[136] J. Jost. *Two-dimensional geometric variational problems.* Wiley–Interscience, 1991.

[137] J. Jost. Unstable solutions of two–dimensional geometric variational problems. *Proc. Symp. Pure Math*, 54(I):205–244, 1993.

[138] J. Jost. Equilibrium maps between metric spaces. *Calc. Var.*, 2:173–204, 1994.

[139] J. Jost. Convex functionals and generalized harmonic maps into spaces of nonpositive curvature. *Comment. Math. Helv.*, 70:659–673, 1995.

[140] J. Jost. Generalized harmonic maps between metric spaces. In J. Jost, editor, *Geometric analysis and the calculus of variations for Stefan Hildebrandt*, pages 143–174. Intern. Press, Boston, 1996.

[141] J. Jost. Generalized Dirichlet forms and harmonic maps. *Calc. Var.*, 5:1–19, 1997.

[142] J. Jost. *Nonpositive curvature: Geometric and analytic aspects.* Birkhäuser, 1997.

[143] J. Jost. *Postmodern analysis.* Springer, 1998, ^{3}2005.

[144] J. Jost. *Geometry and physics.* Springer, 2008.

[145] J. Jost. *Nonlinear methods in Riemannian and Kählerian geometry.* Birkhäuser, ^{2}1991.

[146] J. Jost. *Partial differential equations.* Springer, ^{2}2007.

[147] J. Jost. *Compact Riemann surfaces.* Springer, ^{3}2006.

[148] J. Jost and H. Karcher. Geometrische Methoden zur Gewinnung von a-priori-Schranken für harmonische Abbildungen. *Man. math.*, 40:27–77, 1982.

[149] J. Jost and X. Li-Jost. *Calculus of variations.* Cambridge Univ. Press, 1998.

[150] J. Jost, X.W. Peng, and G.F. Wang. Variational aspects of the Seiberg-Witten functional. *Calc. Var.*, 4:205–218, 1996.

[151] J. Jost and M. Struwe. Morse-Conley theory for minimal surfaces of varying topological type. *Inv. math.*, 102:465–499, 1990.

[152] J. Jost and L. Todjihounde. Harmonic nets in metric spaces. *Pacific J. Math.*, 2007.

[153] J. Jost and Y. L. Xin. Vanishing theorems for L^2-cohomology groups. *J. reine angew. Math.*, 525, 2000.

[154] J. Jost and S.T. Yau. Harmonic mappings and Kähler manifolds. *Math. Ann.*, 262:145–166, 1983.

[155] J. Jost and S.T. Yau. A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry. *Acta Math.*, 170:221–254, 1993. Corr. in *Acta Math.* 173 (1994), 307.

[156] J. Jost and S.T. Yau. Harmonic maps and superrigidity. *Proc. Symp. Pure Math.*, 54(I):245–280, 1993.

[157] J. Jost and K. Zuo. Harmonic maps of infinite energy and rigidity results for Archimedean and non-Archimedean representations of fundamental groups of quasiprojective varieties. *J. Diff. Geom.*, 47:469–503, 1997.

[158] J. Jost and K. Zuo. Vanishing theorems for L^2-cohomology on infinite coverings of compact Kähler manifolds and applications in algebraic geometry. *Comm. Anal. Geom.*, 8:1–30, 2000.

[159] E. Kähler. Über eine bemerkenswerte Hermitesche Metrik. *Abh. Math. Sem. Univ. Hamburg*, 9:173–186, 1933.

[160] E. Kähler. Der innere Differentialkalkül. *Rend. Mat. Appl.(5)*, 21:425–523, 1962.

[161] E. Kähler. *Mathematische Werke – Mathematical Works.* edited by R. Berndt and O. Riemenschneider: ; de Gruyter, 2003.

[162] H. Karcher. Riemannian center of mass and mollifier smoothing. *Comm. Pure Appl. Math.*, 30:509–541, 1977.

[163] W. Kendall. Convexity and the hemisphere. *J. London Math. Soc.*, 43(2):567–576, 1991.

[164] Kervaire. A manifold which does not admit any differentiable structure. *Comment. Math. Helv.*, 34:257–270, 1960.

[165] B. Kleiner and J. Lott. Notes on Perelman s papers. *arXiv*, math.DG/0605667. p:47–54, 2006.

[166] W. Klingenberg. Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung. *Comm. Math. Helv.*, 35:47–54, 1961.

[167] W. Klingenberg. *Lectures on closed geodesics.* Springer, 1978.

[168] W. Klingenberg. *Riemannian geometry.* de Gruyter, 1982.

[169] S. Kobayashi and K. Nomizu. *Foundations of differential geometry, I.* Wiley–Interscience, 1963.

[170] S. Kobayashi and K. Nomizu. *Foundations of differential geometry, II.* Wiley–Interscience, 1969.

[171] N. Korevaar and R. Schoen. Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.*, 1:561–659, 1993.

[172] N. Korevaar and R. Schoen. Global existence theorems for harmonic maps to non–locally compact spaces. *Comm. Anal. Geom.*, 5:213–266, 1997.

[173] P. Kronheimer and T. Mrowka. The genus of embedded surfaces in the projective plane. *Math. Res. Letters*, 1:797–808, 1994.

[174] O.A. Ladyzhenskaya, V. A. Solonnikov, and N.N. Ural'ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian, Amer. Math. Soc., Providence, R. I., 1968.

[175] O.A. Ladyzhenskaya and N.N. Ural'ceva. *Linear and quasilinear elliptic equations*. Translated from the Russian, Academic Press, 1968.

[176] H.B. Lawson and M.L. Michelsohn. *Spin geometry*. Princeton University Press, Princeton, 1989.

[177] H.B. Lawson and S.T. Yau. Compact manifolds of nonpositive curvature. *J. Diff. Geom.*, 7:211–228, 1972.

[178] L. Lemaire. Applications harmoniques de surfaces riemanniennes. *J. Diff. Geom.*, 13:51–78, 1978.

[179] F.H. Lin. Analysis on singular spaces. In Ta-Tsien Li, editor, *Geometry, Analysis and Mathematical Physics, in honor of Prof. Chaohao Gu*, pages 114–126. World Scientific, 1997.

[180] J. Lohkamp. Negatively Ricci curved manifolds. *Bull. AMS*, 27:288–292, 1992.

[181] J. Lohkamp. Metrics of negative Ricci curvature. *Ann. Math.*, 140:655–683, 1994.

[182] U. Ludwig. Stratified Morse theory with tangential conditions.

[183] G.A. Margulis. Discrete groups of motion of manifolds of nonpositive curvature. *AMS Transl.*, 190:33–45, 1977.

[184] G.A. Margulis. *Discrete subgroups of semisimple Lie groups*. Springer, 1991.

[185] Y. Matsushima. On the first Betti number of compact quotient spaces of higher-dimensional symmetric spaces. *Ann. Math.*, 75:312–330, 1962.

[186] W. Meeks and S.T. Yau. The classical Plateau problem and the topology of three-dimensional manifolds. *Top.*, 21:409–440, 1982.

[187] M. Micallef and J.D. Moore. Minimal two–spheres and the topology of manifolds with positive curvature on totally isotropic two–planes. *Ann. Math.*, 127:199–227, 1988.

[188] A. Milgram and P. Rosenbloom. Harmonic forms and heat conduction, I: Closed Riemannian manifolds. *Proc. Nat. Acad. Sci.*, 37:180–184, 1951.

[189] J. Milnor. On manifolds homeomorphic to the 7-sphere. *Ann. Math.*, 64:399–405, 1956.

[190] J. Milnor. Differentiable structures on spheres. *Am. J. Math.*, 81:962–972, 1959.

[191] J. Milnor. *Morse theory, Ann. Math. Studies 51.* Princeton Univ. Press, 1963.

[192] J. Milnor. *Lectures on the h-cobordism theorem.* Princeton Univ. Press, 1965.

[193] N. Mok. Geometric Archimedean superrigidity in the Hermitian case.

[194] N. Mok. Aspects of Kähler geometry on arithmetic varieties. *Proc.Symp. Pure Math.*, 52:335–396, 1991.

[195] N. Mok, Y.T. Siu, and S.K. Yeung. Geometric superrigidity. *Inv. math.*, 113:57–84, 1993.

[196] J. Morgan. *The Seiberg-Witten equations and applications to the topology of smooth four manifolds.* Princeton University Press, 1996.

[197] J. Morgan, Z. Szabó, and C. Taubes. A product formula for the Seiberg-Witten invariants and the generalized Thom conjecture. *J. Diff. Geom.*, 44:706–788, 1996.

[198] J. Morgan and G. Tian. *Ricci Flow and the Poincare Conjecture.* arXiv:math/0607607v2, 2006.

[199] C. Morrey. The problem of Plateau on a Riemannian manifold. *Ann. Math.*, 49:807–851, 1948.

[200] G. Mostow. *Strong rigidity of locally symmetric spaces.* Ann. Math. Studies 78, Princeton Univ. Press, 1973.

[201] A. Nadel. Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature. *Ann. Math.*, 132:549–596, 1990.

[202] I.G. Nikolaev. Smoothness of the metric of spaces with curvature that is bilaterally bounded in the sense of A.D. Aleksandrov. *Sib. Math. J.*, 24:247–263, 1983.

[203] I.G. Nikolaev. Bounded curvature closure of the set of compact Riemannian manifolds. *Bull. AMS*, 24:171–177, 1991.

[204] I.G. Nikolaev. *Synthetic methods in Riemannian geometry.* Lecture Notes. Univ. Illinois at Urbana–Champaign, 1992.

[205] J. Nitsche. *Lectures on minimal surfaces, Vol. I.* Cambridge Univ. Press, 1989.

[206] G. Perel'man. The entropy formula for the Ricci flow and its geometric application. arxiv:math.DG/0211159, 2002.

[207] G. Perel'man. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. arxiv:math.DG/0307245, 2003.

[208] G. Perel'man. Ricci flow with surgery on three-manifolds. arxiv:math.DG/0303109, 2003.

[209] S. Peters. Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds. *J. reine angew. Math.*, 394:77–82, 1984.

[210] S. Peters. Convergence of Riemannian manifolds. *Compos. Math.*, 62:3–16, 1987.

[211] P. Petersen. *Riemannian geometry.* Springer, 1998.

[212] A. Petrunin, X. Rong, and W. Tuschmann. Collapsing vs. positive pinching. *GAFA*, 9:699–735, 1999.

[213] A. Petrunin and W. Tuschmann. Diffeomorphism finiteness, positive pinching, and second homotopy. *GAFA*, 9:736–774, 1999.

[214] D. Quillen. Determinants of Cauchy-Riemann operators over a Riemann surface. *Funct. Anal. Appl.*, 19:31–34, 1985.

[215] Y.G. Reshetnyak. Inextensible mappings in a space of curvature no greater than K. *Sib. Math. J.*, 9:683–689, 1968.

[216] E. Ruh. Almost flat manifolds. *J. Diff. Geom.*, 17:1–14, 1982.

[217] I.K. Sabitov and S.Z. Shefel'. Connections between the order of smoothness of a surface and that of its metric. *Sib. Math. J.*, 17:687–694, 1976.

[218] R. Sachs and H. Wu. *General relativity for mathematicians.* Springer GTM, 1977.

[219] J. Sacks and K. Uhlenbeck. The existence of minimal immersions of 2-spheres. *Ann. Math.*, 113:1–24, 1981.

[220] T. Sakai. *Riemannian geometry.* Amer. Math. Soc., 1995.

[221] D. Salamon. *Spin geometry and Seiberg-Witten invariants.* 1995.

[222] J. Sampson. Applications of harmonic maps to Kähler geometry. *Contemp. Math.*, 49:125–134, 1986.

[223] M. Schlicht. Another proof of Bianchi's identity in arbitrary bundles. *Ann. Global Anal. Geom.*, 13:19–22, 1995.

[224] R. Schoen and K. Uhlenbeck. A regularity theory for harmonic maps. *J. Diff. Geom.*, 17:307–335, 1982.

[225] R. Schoen and K. Uhlenbeck. Boundary regularity and miscellaneous results on harmonic maps. *J. Diff. Geom.*, 18:253–268, 1983. Correction in J. Diff. Geom. 18 (1983), 329.

[226] R. Schoen and S.T. Yau. Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with nonnegative scalar curvature. *Ann. Math.*, 110:127–142, 1979.

[227] R. Schoen and S.T. Yau. The structure of manifolds with positive scalar curvature. *Man. math.*, 28:159–183, 1979.

[228] J.A. Schouten. *Der Riccikalkül*. Springer, 1924.

[229] J.A. Schouten. *Ricci-calculus*. Springer, 1954.

[230] M. Schwarz. *Morse homology*. Birkhäuser, 1993.

[231] M. Schwarz. Equivalences for Morse homology. *Contemp. Math.*, 246:197–216, 1999.

[232] N. Seiberg and E. Witten. Electromagnetic duality, monopol condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory. *Nucl. Phys.*, 431(B):581–640, 1994.

[233] N. Seiberg and E. Witten. Monopoles, duality and chiral symmetry breaking in $n = 2$ supersymmetric QCD. *Nucl. Phys. B*, 431:581–640, 1994.

[234] Y.T. Siu. The complex analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds. *Ann. Math.*, 112:73–111, 1980.

[235] Y.T. Siu. The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and suitable finite symmetry group. *Ann.Math.*, 127:585–627, 1988.

[236] Y.T. Siu and S.T. Yau. Compact Kähler manifolds of positive bisectional curvature. *Inv. math.*, 59:189–204, 1980.

[237] S. Smale. On gradient dynamical systems. *Ann. Math.*, 74:199–206, 1961.

[238] E. Spanier. *Algebraic topology*. McGraw Hill, 1966.

[239] K. Steffen. *An introduction to harmonic mappings*. SFB 256, Vorlesungsreihe, Vol. 18, Bonn, 1991.

[240] R. Stoecker and H. Zieschang. *Algebraische Topologie*. Teubner, ^{2}1994.

[241] S. Stolz. Simply connected manifolds of positive scalar curvature. *Ann. Math.*, 136:511–540, 1992.

[242] M. Struwe. On the evolution of harmonic mappings. *Comm. Math. Helv.*, 60:558–581, 1985.

[243] M. Struwe. *Variational methods*. Springer, ^{3}1999.

[244] G. Tarantello. Multiple condensate solutions for the Chern-Simons-Higgs theory. *J. Math. Phys.*, 37:3769–3796, 1996.

[245] C. Taubes. Arbitrary n–vortex solutions to the first order Ginzburg-Landau equations. *Comm. Math. Phys.*, 72:227–292, 1980.

[246] C. Taubes. The Seiberg-Witten invariant and symplectic forms. *Math. Res. Letters*, 1:809–822, 1994.

[247] C. Taubes. More constraints on symplectic manifolds from Seiberg-Witten invariants. *Math. Res. Letters*, 2:9–14, 1995.

[248] C. Taubes. The Seiberg-Witten and the Gromov invariants. *Math. Res. Letters*, 9:809–822, 1995.

[249] C. Taubes. Counting pseudoholomorphic submanifolds in dimension 4. *J. Diff. Geom.*, 44:818 – 893, 1996.

[250] R. Thom. Sur une partition cellules associés à une fonction sur une variété. *C. R. Acad. Sci. Paris*, 228:973–975, 1949.

[251] W. Thurston. Hyperbolic structures on 3-manifolds. Preprint, 1980.

[252] W. Thurston. *Three-dimensional geometry and topology, Vol. 1.* Princeton University Press, 1997.

[253] G. Tian. On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$. *Inv. math.*, 89:225–246, 1987.

[254] G. Tian. Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.*, 130:1–39, 1997.

[255] G. Tian and S.T. Yau. Kähler-Einstein metrics on complex surfaces with $c_1(M) > 0$. *Comm. Math. Phys.*, 112:175–203, 1987.

[256] W. Tuschmann. Collapsing, solvmanifolds, and infrahomogenous spaces. *Diff. Geom. Appl.*, 7:251–264, 1997.

[257] W. Tuschmann. Endlichkeitssätze und positive Krümmung. Habilitation thesis, Leipzig, 2000.

[258] J. Weber. The Morse-Witten complex via dynamical systems. arXiv:math.GT/0411465, 2004.

[259] R. Wells. *Differential analysis on complex manifolds*. Springer, ^{2}1980.

[260] H. Weyl. *Die Idee der Riemannschen Fläche*. Teubner, Leipzig, Berlin, 1913.

[261] H. Weyl. *Space, time, matter*. Dover, 1952. Translated from the German.

[262] H. Weyl. *Das Kontinuum und andere Monographien*. Chelsea, New York, 1973. Reprint.

[263] B. Wilking. Manifolds with positive sectional curvature almost everywhere. *Invent.Math.*, 148:117–141, 2002.

[264] E. Witten. Supersymmetry and Morse theory. *J. Diff. Geom*, 17:661–692, 1982.

[265] E. Witten. Monopoles and 4-manifolds. *Math. Res. Letters*, 1:764–796, 1994.

[266] J. Wolf. *Spaces of constant curvature*. Publish or Perish, Boston, 1974.

[267] H. Wu. The Bochner technique in differential geometry. *Math. Rep.*, 3(2):289–538, 1988.

[268] Y.L. Xin. *Geometry of harmonic maps*. Birkhäuser, 1996.

[269] Y.L. Xin. *Minimal submanifolds and related topics*. World Scientific, 2004.

[270] S.T. Yau. On the fundamental group of compact manifolds of non-positive curvature. *Ann. Math.*, 93:579–585, 1971.

[271] S.T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I. *Comm. Pure Appl. Math.*, 31:339–411, 1978.

[272] S.T. Yau. Open problems in geometry. *Proc. Symp. Pure Math.*, 54(I):1–28, 1993.

[273] W. Ziemer. *Weakly differentiable functions*. Springer, 1989.

[274] R. Zimmer. *Ergodic theory and semisimple groups*. Birkhäuser, 1984.

Index

Universitext

Aguilar, M.; Gitler, S.; Prieto, C.: Algebraic Topology from a Homotopical Viewpoint
Aksoy, A.; Khamsi, M. A.: Methods in Fixed Point Theory
Alevras, D.; Padberg M. W.: Linear Optimization and Extensions
Andersson, M.: Topics in Complex Analysis
Aoki, M.: State Space Modeling of Time Series
Arnold, V. I.: Lectures on Partial Differential Equations
Arnold, V. I.; Cooke, R.: Ordinary Differential Equations
Audin, M.: Geometry
Aupetit, B.: A Primer on Spectral Theory
Bachem, A.; Kern, W.: Linear Programming Duality
Bachmann, G.; Narici, L.; Beckenstein, E.: Fourier and Wavelet Analysis
Badescu, L.: Algebraic Surfaces
Balakrishnan, R.; Ranganathan, K.: A Textbook of Graph Theory
Balser, W.: Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations
Bapat, R.B.: Linear Algebra and Linear Models
Benedetti, R.; Petronio, C.: Lectures on Hyperbolic Geometry
Benth, F. E.: Option Theory with Stochastic Analysis
Berberian, S. K.: Fundamentals of Real Analysis
Berger, M.: Geometry I, and II
Bhattacharya, R.; Waymire, E. C.: A Basic Course in Probability Theory
Bliedtner, J.; Hansen, W.: Potential Theory
Blowey, J. F.; Coleman, J. P.; Craig, A. W. (Eds.): Theory and Numerics of Differential Equations
Blowey, J. F.; Craig, A.; Shardlow, T. (Eds.): Frontiers in Numerical Analysis, Durham 2002, and Durham 2004
Blyth, T. S.: Lattices and Ordered Algebraic Structures
Börger, E.; Grädel, E.; Gurevich, Y.: The Classical Decision Problem
Böttcher, A; Silbermann, B.: Introduction to Large Truncated Toeplitz Matrices
Boltyanski, V.; Martini, H.; Soltan, P. S.: Excursions into Combinatorial Geometry
Boltyanskii, V. G.; Efremovich, V. A.: Intuitive Combinatorial Topology
Bonnans, J. F.; Gilbert, J. C.; Lemaréchal, C.; Sagastizábal, C. A.: Numerical Optimization
Booss, B.; Bleecker, D. D.: Topology and Analysis
Borkar, V. S.: Probability Theory
Bridges/Vita: Techniques of Constructive Analysis
Brunt B. van: The Calculus of Variations
Bühlmann, H.; Gisler, A.: A Course in Credibility Theory and its Applications
Carleson, L.; Gamelin, T. W.: Complex Dynamics
Cecil, T. E.: Lie Sphere Geometry: With Applications of Submanifolds
Chae, S. B.: Lebesgue Integration
Chandrasekharan, K.: Classical Fourier Transform
Charlap, L. S.: Bieberbach Groups and Flat Manifolds
Chern, S.: Complex Manifolds without Potential Theory
Chorin, A. J.; Marsden, J. E.: Mathematical Introduction to Fluid Mechanics
Cohn, H.: A Classical Invitation to Algebraic Numbers and Class Fields
Curtis, M. L.: Abstract Linear Algebra
Curtis, M. L.: Matrix Groups
Cyganowski, S.; Kloeden, P.; Ombach, J.: From Elementary Probability to Stochastic Differential Equations with MAPLE
Da Prato, G.: An Introduction to Infinite Dimensional Analysis
Dalen, D. van: Logic and Structure
Das, A.: The Special Theory of Relativity: A Mathematical Exposition

Debarre, O.: Higher-Dimensional Algebraic Geometry

Deitmar, A.: A First Course in Harmonic Analysis. 2nd edition

Demazure, M.: Bifurcations and Catastrophes

Devlin, K. J.: Fundamentals of Contemporary Set Theory

DiBenedetto, E.: Degenerate Parabolic Equations

Diener, F.; Diener, M.(Eds.): Nonstandard Analysis in Practice

Dimca, A.: Sheaves in Topology

Dimca, A.: Singularities and Topology of Hypersurfaces

DoCarmo, M. P.: Differential Forms and Applications

Duistermaat, J. J.; Kolk, J. A. C.: Lie Groups

Dumortier.: Qualitative Theory of Planar Differential Systems

Dundas, B. I.; Levine, M.; Østvaer, P. A.; Röndip, O.; Voevodsky, V.: Motivic Homotopy Theory

Edwards, R. E.: A Formal Background to Higher Mathematics Ia, and Ib

Edwards, R. E.: A Formal Background to Higher Mathematics IIa, and IIb

Emery, M.: Stochastic Calculus in Manifolds

Emmanouil, I.: Idempotent Matrices over Complex Group Algebras

Endler, O.: Valuation Theory

Engel, K.-J.; Nagel, R.: A Short Course on Operator Semigroups

Erez, B.: Galois Modules in Arithmetic

Everest, G.; Ward, T.: Heights of Polynomials and Entropy in Algebraic Dynamics

Farenick, D. R.: Algebras of Linear Transformations

Foulds, L. R.: Graph Theory Applications

Franke, J.; Härdle, W.; Hafner, C. M.: Statistics of Financial Markets: An Introduction

Frauenthal, J. C.: Mathematical Modeling in Epidemiology

Freitag, E.; Busam, R.: Complex Analysis

Friedman, R.: Algebraic Surfaces and Holomorphic Vector Bundles

Fuks, D. B.; Rokhlin, V. A.: Beginner's Course in Topology

Fuhrmann, P. A.: A Polynomial Approach to Linear Algebra

Gallot, S.; Hulin, D.; Lafontaine, J.: Riemannian Geometry

Gardiner, C. F.: A First Course in Group Theory

Gårding, L.; Tambour, T.: Algebra for Computer Science

Godbillon, C.: Dynamical Systems on Surfaces

Godement, R.: Analysis I, and II

Goldblatt, R.: Orthogonality and Spacetime Geometry

Gouvêa, F. Q.: p-Adic Numbers

Gross, M. et al.: Calabi-Yau Manifolds and Related Geometries

Grossman, C.; Roos, H.-G.; Stynes, M.: Numerical Treatment of Partial Differential Equations

Gustafson, K. E.; Rao, D. K. M.: Numerical Range. The Field of Values of Linear Operators and Matrices

Gustafson, S. J.; Sigal, I. M.: Mathematical Concepts of Quantum Mechanics

Hahn, A. J.: Quadratic Algebras, Clifford Algebras, and Arithmetic Witt Groups

Hájek, P.; Havránek, T.: Mechanizing Hypothesis Formation

Heinonen, J.: Lectures on Analysis on Metric Spaces

Hlawka, E.; Schoißengeier, J.; Taschner, R.: Geometric and Analytic Number Theory

Holmgren, R. A.: A First Course in Discrete Dynamical Systems

Howe, R., Tan, E. Ch.: Non-Abelian Harmonic Analysis

Howes, N. R.: Modern Analysis and Topology

Hsieh, P.-F.; Sibuya, Y. (Eds.): Basic Theory of Ordinary Differential Equations

Humi, M., Miller, W.: Second Course in Ordinary Differential Equations for Scientists and Engineers

Hurwitz, A.; Kritikos, N.: Lectures on Number Theory

Huybrechts, D.: Complex Geometry: An Introduction

Isaev, A.: Introduction to Mathematical Methods in Bioinformatics

Istas, J.: Mathematical Modeling for the Life Sciences

Iversen, B.: Cohomology of Sheaves

Jacod, J.; Protter, P.: Probability Essentials

Jennings, G. A.: Modern Geometry with Applications

Jones, A.; Morris, S. A.; Pearson, K. R.: Abstract Algebra and Famous Inpossibilities

Jost, J.: Compact Riemann Surfaces

Jost, J.: Dynamical Systems. Examples of Complex Behaviour

Jost, J.: Postmodern Analysis

Jost, J.: Riemannian Geometry and Geometric Analysis

Kac, V.; Cheung, P.: Quantum Calculus

Kannan, R.; Krueger, C. K.: Advanced Analysis on the Real Line

Kelly, P.; Matthews, G.: The Non-Euclidean Hyperbolic Plane

Kempf, G.: Complex Abelian Varieties and Theta Functions

Kitchens, B. P.: Symbolic Dynamics

Kloeden, P.; Ombach, J.; Cyganowski, S.: From Elementary Probability to Stochastic Differential Equations with MAPLE

Kloeden, P. E.; Platen; E.; Schurz, H.: Numerical Solution of SDE Through Computer Experiments

Koralov, L. B.; Sinai, Ya. G.: Theory of Probability and Random Processes. 2nd edition

Kostrikin, A. I.: Introduction to Algebra

Krasnoselskii, M. A.; Pokrovskii, A. V.: Systems with Hysteresis

Kuo, H.-H.: Introduction to Stochastic Integration

Kurzweil, H.; Stellmacher, B.: The Theory of Finite Groups. An Introduction

Kyprianou, A.E.: Introductory Lectures on Fluctuations of Lévy Processes with Applications

Lang, S.: Introduction to Differentiable Manifolds

Lefebvre, M.: Applied Stochastic Processes

Lorenz, F.: Algebra I: Fields and Galois Theory

Luecking, D. H., Rubel, L. A.: Complex Analysis. A Functional Analysis Approach

Ma, Zhi-Ming; Roeckner, M.: Introduction to the Theory of (non-symmetric) Dirichlet Forms

Mac Lane, S.; Moerdijk, I.: Sheaves in Geometry and Logic

Marcus, D. A.: Number Fields

Martinez, A.: An Introduction to Semiclassical and Microlocal Analysis

Matoušek, J.: Using the Borsuk-Ulam Theorem

Matsuki, K.: Introduction to the Mori Program

Mazzola, G.; Milmeister G.; Weissman J.: Comprehensive Mathematics for Computer Scientists 1

Mazzola, G.; Milmeister G.; Weissman J.: Comprehensive Mathematics for Computer Scientists 2

Mc Carthy, P. J.: Introduction to Arithmetical Functions

McCrimmon, K.: A Taste of Jordan Algebras

Meyer, R. M.: Essential Mathematics for Applied Field

Meyer-Nieberg, P.: Banach Lattices

Mikosch, T.: Non-Life Insurance Mathematics

Mines, R.; Richman, F.; Ruitenburg, W.: A Course in Constructive Algebra

Moise, E. E.: Introductory Problem Courses in Analysis and Topology

Montesinos-Amilibia, J. M.: Classical Tessellations and Three Manifolds

Morris, P.: Introduction to Game Theory

Mortveit, H.; Reidys, C.: An Introduction to Sequential Dynamical Systems

Nicolaescu, L.: An Invitation to Morse Theory

Nikulin, V. V.; Shafarevich, I. R.: Geometries and Groups
Oden, J. J.; Reddy, J. N.: Variational Methods in Theoretical Mechanics
Øksendal, B.: Stochastic Differential Equations
Øksendal, B.; Sulem, A.: Applied Stochastic Control of Jump Diffusions. 2nd edition
Orlik, P.; Welker, V.: Algebraic Combinatorics
Poizat, B.: A Course in Model Theory
Polster, B.: A Geometrical Picture Book
Porter, J. R.; Woods, R. G.: Extensions and Absolutes of Hausdorff Spaces
Procesi, C.: Lie Groups
Radjavi, H.; Rosenthal, P.: Simultaneous Triangularization
Ramsay, A.; Richtmeyer, R. D.: Introduction to Hyperbolic Geometry
Rautenberg, W.: A concise Introduction to Mathematical Logic
Rees, E. G.: Notes on Geometry
Reisel, R. B.: Elementary Theory of Metric Spaces
Rey, W. J. J.: Introduction to Robust and Quasi-Robust Statistical Methods
Ribenboim, P.: Classical Theory of Algebraic Numbers
Rickart, C. E.: Natural Function Algebras
Rotman, J. J.: Galois Theory
Rubel, L. A.: Entire and Meromorphic Functions
Ruiz-Tolosa, J. R.; Castillo E.: From Vectors to Tensors
Runde, V.: A Taste of Topology
Rybakowski, K. P.: The Homotopy Index and Partial Differential Equations
Sagan, H.: Space-Filling Curves
Samelson, H.: Notes on Lie Algebras
Sauvigny, F.: Partial Differential Equations I
Sauvigny, F.: Partial Differential Equations II
Schiff, J. L.: Normal Families
Schirotzek, W.: Nonsmooth Analysis
Sengupta, J. K.: Optimal Decisions under Uncertainty
Séroul, R.: Programming for Mathematicians
Seydel, R.: Tools for Computational Finance
Shafarevich, I. R.: Discourses on Algebra
Shapiro, J. H.: Composition Operators and Classical Function Theory
Simonnet, M.: Measures and Probabilities
Smith, K. E.; Kahanpää, L.; Kekäläinen, P.; Traves, W.: An Invitation to Algebraic Geometry
Smith, K. T.: Power Series from a Computational Point of View
Smorynski: Self-Reference and Modal Logic
Smoryński, C.: Logical Number Theory I. An Introduction
Srivastava: A Course on Mathematical Logic
Stichtenoth, H.: Algebraic Function Fields and Codes
Stillwell, J.: Geometry of Surfaces
Stroock, D. W.: An Introduction to the Theory of Large Deviations
Sunder, V. S.: An Invitation to von Neumann Algebras
Tamme, G.: Introduction to Étale Cohomology
Tondeur, P.: Foliations on Riemannian Manifolds
Toth, G.: Finite Möbius Groups, Minimal Immersions of Spheres, and Moduli
Tu, L. W.: An Introduction to Manifolds
Verhulst, F.: Nonlinear Differential Equations and Dynamical Systems
Weintraub, S. H.: Galois Theory
Wong, M. W.: Weyl Transforms
Xambó-Descamps, S.: Block Error-Correcting Codes
Zaanen, A.C.: Continuity, Integration and Fourier Theory
Zhang, F.: Matrix Theory
Zong, C.: Sphere Packings
Zong, C.: Strange Phenomena in Convex and Discrete Geometry
Zorich, V. A.: Mathematical Analysis I
Zorich, V. A.: Mathematical Analysis II