

## A

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### Appendix: Further Topics

In this final chapter we briefly indicate some concepts and developments in nonsmooth analysis that have not been treated in the preceding text. For a comprehensive discussion we refer to the monograph [189] by Rockafellar and Wets (finite-dimensional theory) and the monograph [136, 137] by Mordukhovich (infinite-dimensional theory).

In the sequel, unless otherwise specified, let  $E$  be a normed vector space,  $f : E \rightarrow \overline{\mathbb{R}}$  a proper functional, and  $\bar{x} \in \text{dom } f$ .

(I) Clarke [34] defines the subdifferential

$$\partial_{\uparrow} f(\bar{x}) := \{x^* \in E^* \mid (x^*, -1) \in N_C(\text{epi } f, (\bar{x}, f(\bar{x})))\}.$$

This set is always  $\sigma(E^*, E)$ -closed but may be empty. However, if  $\bar{x}$  is a local minimizer of  $f$ , then  $0 \in \partial_{\uparrow} f(\bar{x})$ . Moreover, Corollary 11.3.2 shows that  $\partial_{\uparrow} f(\bar{x}) = \partial_{\circ} f(\bar{x})$  for any  $\bar{x} \in E$  whenever  $f$  is locally L-continuous on  $E$ .

(II) Rockafellar [184] (see also [186, 189]) defines the (*regular*) *subderivative*  $f^{\uparrow}(\bar{x}, \cdot) : E \rightarrow \overline{\mathbb{R}}$  of  $f$  at  $\bar{x}$  as

$$f^{\uparrow}(\bar{x}, y) := \lim_{\epsilon \downarrow 0} \limsup_{(x, \alpha) \rightarrow_f \bar{x}} \inf_{z \in B(y, \epsilon)} \frac{f(x + \tau z) - \alpha}{\tau}.$$

In this context,  $(x, \alpha) \rightarrow_f \bar{x}$  means  $(x, \alpha) \in \text{epi } f$ ,  $x \rightarrow \bar{x}$ , and  $\alpha \rightarrow f(\bar{x})$ . Rockafellar shows that  $f^{\uparrow}(\bar{x}, \cdot)$  is l.s.c. and that  $f^{\uparrow}(\bar{x}, 0) = -\infty$  if and only if  $\partial_{\uparrow} f(\bar{x}) = \emptyset$ . If  $f^{\uparrow}(\bar{x}, 0) > -\infty$ , then  $f^{\uparrow}(\bar{x}, 0) = 0$ ,  $f^{\uparrow}(\bar{x}, \cdot)$  is sublinear, and

$$\text{epi } f^{\uparrow}(\bar{x}, \cdot) = T_C(\text{epi } f, (\bar{x}, f(\bar{x}))).$$

It then follows that  $f^{\uparrow}(\bar{x}, \cdot)$  is the support functional of the Clarke subdifferential, i.e.,

$$\partial_{\uparrow} f(\bar{x}) = \{x^* \in E^* \mid \langle x^*, y \rangle \leq f^{\uparrow}(\bar{x}, y) \quad \forall y \in E\}.$$

The functional  $f$  is said to be *directionally Lipschitz* at  $\bar{x}$  if  $\text{epi } f$  is epi-Lipschitz at  $(\bar{x}, f(\bar{x}))$ . Moreover,  $f$  is said to be *subdifferentially regular* if

$$f^\uparrow(\bar{x}, y) = \underline{f}_H(\bar{x}, y) \quad \forall y \in E;$$

this is equivalent to  $\text{epi } f$  being a tangentially regular set. If  $f$  is locally L-continuous around  $\bar{x}$ , then  $f$  is subdifferentially regular if and only if  $f$  is regular in the sense of Clarke. Rockafellar shows that the sum rule

$$\partial_\uparrow(f + g)(\bar{x}) \subseteq \partial_\uparrow f(\bar{x}) + \partial_\uparrow g(\bar{x}) \quad (\diamond)$$

holds if  $g$  is directionally Lipschitz at  $\bar{x}$  and  $\text{dom } f^\uparrow(\bar{x}, \cdot) \cap \text{int dom } g^\uparrow(\bar{x}, \cdot)$  is nonempty. The sum rule holds with equality if, in addition,  $f$  and  $g$  are subdifferentially regular. Certain chain rules are also established.

**(III)** Michel and Penot [129,130] study several types of directional derivatives and associated subdifferentials. Their aim is to generalize the G-derivative rather than the strict H-derivative as does the Clarke subdifferential (cf. Remark 7.3.10).

Using the epi-limit convergence concept, Michel and Penot define, among others, the *pseudo-strict derivative* of  $f$  at  $\bar{x}$  as

$$f^\wedge(\bar{x}, y) := \sup_{z \in E} \text{eLim sup}_{\substack{\tau \downarrow 0, z' \rightarrow z \\ (\bar{x} + \tau z', \alpha) \rightarrow_f \bar{x}}} \frac{1}{\tau} (f(\bar{x} + \tau y + \tau z') - \alpha).$$

The functional  $f^\wedge(\bar{x}, \cdot)$  is shown to be l.s.c. and sublinear. If  $f$  is locally L-continuous around  $\bar{x}$ , then  $f^\wedge(\bar{x}, \cdot)$  coincides with  $f^\diamond(\bar{x}, \cdot)$ . If the directional G-derivative  $f_G(\bar{x}, \cdot)$  exists, is finite and sublinear, then  $f^\wedge(\bar{x}, \cdot) = f_G(\bar{x}, \cdot)$ . The associated subdifferential, defined as

$$\partial^\wedge f(\bar{x}) := \{x^* \in E^* \mid \langle x^*, y \rangle \leq f^\wedge(\bar{x}, y) \quad \forall y \in E\},$$

thus satisfies  $\partial^\wedge f(\bar{x}) = \{f'(\bar{x})\}$  whenever  $f$  is G-differentiable at  $\bar{x}$ . If  $\bar{x}$  is a local minimizer of  $f$ , then  $0 \in \partial^\wedge f(\bar{x})$ . Under certain regularity assumptions, the sum rule  $\partial^\wedge(f + g)(\bar{x}) \subseteq \partial^\wedge f(\bar{x}) + \partial^\wedge g(\bar{x})$  and a chain rule are established. Finally, multiplier rules are derived in [130].

**(IV)** Various other generalized derivative concepts were introduced and studied. Some of them are Halkin's *screen* [83], Treiman's *B-derivatives* [206, 207], and Warga's *derivate containers* [213, 214]. Much was done by Hiriart-Urruty [84, 85, 87] in clarifying and refining these derivative concepts.

**(V)** Concerning sensitivity analysis, we refer in the finite-dimensional case to Klatte and Kummer [111], Luderer et al. [126], and Rockafellar and Wets [189], and in the infinite-dimensional case to Bonnans and Shapiro [16], Clarke et al. [39], and Mordukhovich [136] as well as to the literature cited in these books.

(VI) Following Borwein and Zhu [23], we started the differential analysis of lower semicontinuous functionals with Zhu’s nonlocal fuzzy sum rule which, in turn, is based on the Borwein–Preiss smooth variational principle. The basis of Mordukhovich’s work constitutes his extremal principle. Another fundamental result in nonsmooth analysis is a multidirectional mean value theorem (such as Theorem 9.6.2) that originally goes back to Clarke and Ledyaev [37, 38]. In [39] this result is applied to establish, among others, non-smooth implicit and inverse function theorems. Mordukhovich and Shao [140] showed the equivalence of the extremal principle and the local fuzzy sum rule, and Zhu [226] proved the equivalence of the latter to the nonlocal fuzzy sum rule and to the multidirectional mean value theorem.

(VII) Ioffe systematically investigated *approximate subdifferentials*; see, among others, [94–98, 100].

Let  $\mathcal{F}$  denote the collection of all finite-dimensional vector subspaces of  $E$ . If  $S$  is a nonempty subset of  $E$ , we write

$$f_{(S)}(x) := \begin{cases} f(x) & \text{if } x \in S, \\ +\infty & \text{if } x \in E \setminus S. \end{cases}$$

Ioffe starts with the lower directional Hadamard derivative, which he calls *lower directional Dini derivative*,

$$\underline{f}_H(\bar{x}, y) := \liminf_{\tau \downarrow 0, z \rightarrow y} \frac{1}{\tau} (f(\bar{x} + \tau z) - f(\bar{x})) \quad \forall y \in E.$$

He then defines the *Hadamard subdifferential* (or *Dini subdifferential*)

$$\partial^- f(\bar{x}) := \{x^* \in E^* \mid \langle x^*, y \rangle \leq \underline{f}_H(\bar{x}, y) \quad \forall y \in E\}$$

and the *A-subdifferential*

$$\partial_A f(\bar{x}) := \bigcap_{L \in \mathcal{F}} \text{Lim sup}_{x \rightarrow_f \bar{x}} \partial^- f_{(x+L)}(x).$$

Here, the Painlevé–Kuratowski upper limit is taken with respect to the norm topology in  $E$  and the weak star topology in  $E^*$ . Observe that Ioffe utilizes the topological form of the Painlevé–Kuratowski upper limit whereas Mordukhovich utilizes the sequential form. The interrelation between the concepts of Ioffe and Mordukhovich is discussed by Mordukhovich and Shao [142], see also Mordukhovich [136].

Ioffe further defines the *G-normal cone* to the set  $M \subseteq E$  at  $\bar{x} \in M$  as

$$N_G(M, \bar{x}) := \text{cl}^* \bigcup_{\lambda > 0} \lambda \partial_A \text{d}_M(\bar{x})$$

and, respectively, the  $G$ -subdifferential and the *singular*  $G$ -subdifferential of  $f$  at  $\bar{x}$  as

$$\begin{aligned}\partial_G f(\bar{x}) &:= \{x^* \in E^* \mid (x^*, -1) \in N_G(\text{epi } f, (\bar{x}, f(\bar{x})))\}, \\ \partial_G^\infty f(\bar{x}) &:= \{x^* \in E^* \mid (x^*, 0) \in N_G(\text{epi } f, (\bar{x}, f(\bar{x})))\}.\end{aligned}$$

The  $G$ -subdifferential and the  $A$ -subdifferential coincide for any function on finite-dimensional normed vector spaces and for directionally Lipschitz functions on arbitrary normed vector spaces. For convex functionals on any normed vector space, the  $G$ -subdifferential (in contrast to the  $A$ -subdifferential) coincides with the subdifferential of convex analysis. In general, the sets  $\partial_A f(\bar{x})$  and  $\partial_G f(\bar{x})$  are not convex. However, they are minimal in a certain sense among all “reasonable” subdifferential constructions. In particular, for any proper functional  $f$  one has  $\partial_C f(\bar{x}) = \overline{\text{co}}^*(\partial_G f(\bar{x}) + \partial_G^\infty f(\bar{x}))$ .

Ioffe develops an extensive calculus for these objects. For instance, if  $M_1$  and  $M_2$  are closed sets one of them being epi-Lipschitz at  $\bar{x}$ , then an appropriate regularity assumption ensures that

$$N_G(M_1 \cap M_2, \bar{x}) \subseteq N_G(M_1, \bar{x}) + N_G(M_2, \bar{x}),$$

with equality if  $N_G(M_i, \bar{x}) = T(M_i, \bar{x})^\circ$  for  $i = 1, 2$ . This result is verified with the aid of Ekeland’s variational principle. It is then applied to derive the following sum rule. If the functionals  $f$  and  $g$  are l.s.c. and one of them is directionally Lipschitz at  $\bar{x}$ , then under the regularity assumption  $\partial_G^\infty f(\bar{x}) \cap (-\partial_G^\infty g(\bar{x})) = \{o\}$ , the sum rule ( $\diamond$ ) holds with  $\partial_\uparrow$  replaced by  $\partial_G$ . An analogous result is obtained for the  $A$ -subdifferential. For an extended chain rule in terms of  $\partial_A$  see Jourani and Thibault [106]. We also refer to the survey paper by Hiriart-Urruty et al. [90].

Multiplier rules in terms of  $G$ -subdifferentials were established by Glover et al. [74]. For an alternative proof of these multiplier rules and an interesting discussion of the absence of constraint qualifications we refer to Pühl [172]. Modified multiplier rules were obtained by Glover and Craven [73].

**(VIII)** In these lectures, we confined ourselves to *first-order* necessary optimality conditions (which are also sufficient in the convex case). For *second-order* necessary and/or sufficient conditions we refer in the smooth case to Bonnans and Shapiro [16] and the literature cited therein. Second-order optimality conditions in terms of generalized derivatives are also treated in many papers. As a small selection we refer to Ben-Tal and Zowe [14], Casas and Tröltzsch [29], Chaney [31], Cominetti [40], Mordukhovich [135], Mordukhovich [136], Mordukhovich and Outrata [138], and Rockafellar [188].

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## References

1. R. A. Adams. *Sobolev Spaces*. Academic, Boston, 1978
2. C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis*. Springer, Berlin Heidelberg New York, 1994
3. E. Asplund. Fréchet differentiability of convex functions. *Acta Math.*, 121: 31–47, 1968
4. K. Atkinson and W. Han. *Theoretical Numerical Analysis*. Springer, Berlin Heidelberg New York, 2001
5. J. -P. Aubin. Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions. In L. Nachbin, editor, *Advances in Mathematics, Supplementary Studies*, pages 160–232. Academic, New York, 1981
6. J. -P. Aubin. *Optima and Equilibria*. Springer, Berlin Heidelberg New York, 1993
7. J. -P. Aubin and I. Ekeland. *Applied Nonlinear Analysis*. Wiley, New York, 1984
8. J. -P. Aubin and H. Frankowska. *Set-Valued Analysis*. Birkhäuser, Boston, 1990
9. V. Barbu and Th. Precupanu. *Convexity and Optimization in Banach Spaces*. Sijthoff and Noordhoff, Alphen aan de Rijn, 1978
10. M. S. Bazaraa, J. J. Goode, and M. Z. Nashed. On the cones of tangents with applications to mathematical programming. *J. Optim. Theory Appl.*, 13: 389–426, 1974
11. M. S. Bazaraa, H. D. Sherali, and C. M. Shetty. *Nonlinear Programming: Theory and Algorithms*. Wiley, New York, 1993
12. M. S. Bazaraa and C. M. Shetty. *Foundations of Optimization*, volume 122 of *Lecture Notes in Economics and Mathematical Systems*. Springer, Berlin Heidelberg New York, 1976
13. B. Beauzamy. *Introduction to Banach Spaces and Their Geometry*. North-Holland, Amsterdam, 1985
14. A. Ben-Tal and J. Zowe. A unified theory of first and second order conditions for extremum problems in topological vector spaces. *Math. Progr. Stud.*, 19: 39–76, 1982
15. E. Bishop and R. R. Phelps. A proof that every Banach space is subreflexive. *Bull. Am. Math. Soc.*, 67:97–98, 1961

16. J. F. Bonnans and A. Shapiro. *Perturbation Analysis of Optimization Problems*. Springer, Berlin Heidelberg New York, 2000
17. J. M. Borwein and A. D. Ioffe. Proximal analysis in smooth spaces. *Set-Valued Anal.*, 4:1–24, 1996
18. J. M. Borwein and A. S. Lewis. *Convex Analysis and Nonlinear Optimization*. Springer, Berlin Heidelberg New York, 2000
19. J. M. Borwein and D. Preiss. A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions. *Trans. Am. Math. Soc.*, 303:517–527, 1987
20. J. M. Borwein and H. M. Strójas. Proximal analysis and boundaries of closed sets in Banach space. I: Theory. *Can. J. Math.*, 38:431–452, 1986
21. J. M. Borwein, J. S. Treiman, and Q. J. Zhu. Necessary conditions for constrained optimization problems with semicontinuous and continuous data. *Trans. Am. Math. Soc.*, 350:2409–2429, 1998
22. J. M. Borwein and Q. J. Zhu. Viscosity solutions and viscosity subderivatives in smooth Banach spaces with applications to metric regularity. *SIAM J. Control Optim.*, 34:1568–1591, 1996
23. J. M. Borwein and Q. J. Zhu. A survey of subdifferential calculus with applications. *Nonlinear Anal.*, 38:687–773, 1999
24. J. M. Borwein and Q. J. Zhu. *Techniques of Variational Analysis*. Springer, Berlin Heidelberg New York, 2005
25. J. M. Borwein and D. M. Zhuang. Verifiable necessary and sufficient conditions for openness and regularity of set-valued and single-valued maps. *J. Math. Anal. Appl.*, 134:441–459, 1988
26. G. Bouligand. Sur les surfaces dépourvues de points hyperlimites. *Ann. Soc. Polon. Math.*, 9:32–41, 1930
27. D. Braess. *Nonlinear Approximation Theory*. Springer, Berlin Heidelberg New York, 1986
28. A. Brøndsted. Conjugate convex functions in topological vector spaces. *Fys. Medd. Dans. Vid. Selsk.*, 34:1–26, 1964
29. E. Casas and F. Tröltzsch. Second-order necessary and sufficient optimality conditions for optimization problems and applications to control theory. *SIAM J. Optim.*, 13:406–431, 2002
30. L. Cesari. *Optimization – Theory and Applications*. Springer, Berlin Heidelberg New York, 1983
31. R. W. Chaney. A general sufficiency theorem for nonsmooth nonlinear programming. *Trans. Am. Math. Soc.*, 276:235–245, 1983
32. I. Cioranescu. *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*. Kluwer, Dordrecht, 1990
33. F. H. Clarke. *Necessary Conditions for Nonsmooth Problems in Optimal Control and the Calculus of Variations*. Ph.D. Thesis, University of Washington, 1973
34. F. H. Clarke. Generalized gradients and applications. *Trans. Am. Math. Soc.*, 205:247–262, 1975
35. F. H. Clarke. A new approach to Lagrange multipliers. *Math. Oper. Res.*, 1: 165–174, 1976
36. F. H. Clarke. *Optimization and Nonsmooth Analysis*. SIAM, Philadelphia, 1990
37. F. H. Clarke and Yu. S. Ledyev. Mean value inequalities. *Proc. Am. Math. Soc.*, 122:1075–1083, 1994

38. F. H. Clarke and Yu. S. Ledyev. Mean value inequalities in Hilbert space. *Trans. Am. Math. Soc.*, 344:307–324, 1994
39. F. H. Clarke, Yu. S. Ledyev, R. J. Stern, and P. R. Wolenski. *Nonsmooth Analysis and Control Theory*. Springer, Berlin Heidelberg New York, 1998
40. R. Cominetti. Metric regularity, tangent sets and second-order optimality conditions. *Appl. Math. Optim.*, 21:265–287, 1990
41. M. G. Crandall, L. C. Evans, and P. -L. Lions. Some properties of viscosity solutions of Hamilton–Jacobi equations. *Trans. Am. Math. Soc.*, 282:487–502, 1984
42. M. G. Crandall and P. -L. Lions. Viscosity solutions of Hamilton–Jacobi equations. *Trans. Am. Math. Soc.*, 277:1–41, 1983
43. B. D. Craven. *Mathematical Programming and Control Theory*. Chapman and Hall, London, 1978
44. B. D. Craven. *Control and Optimization*. Chapman and Hall, London, 1995
45. B. D. Craven, J. Gwinner, and V. Jeyakumar. Nonconvex theorems of the alternative and minimization. *Optimization*, 18:151–163, 1987
46. M. Degiovanni and F. Schuricht. Buckling of nonlinearly elastic rods in the presence of obstacles treated by nonsmooth critical point theory. *Math. Ann.* 311, 675–728, 1998
47. V. F. Demyanov and A. M. Rubinov. *Quasidifferentiable Calculus*. Optimization Software, Publications Division, New York, 1986
48. V. F. Demyanov and A. M. Rubinov. *Foundations of Nonsmooth Analysis and Quasidifferentiable Calculus*. Nauka, Moscow, 1990 (in Russian)
49. R. Deville, G. Godefroy, and V. Zizler. A smooth variational principle with applications to Hamilton–Jacobi equations in infinite dimensions. *J. Funct. Anal.*, 111:197–212, 1993
50. R. Deville, G. Godefroy, and V. Zizler. *Smoothness and Renormings in Banach Spaces*. Pitman Monographs and Surveys in Pure and Applied Mathematics, No. 64. Wiley, New York, 1993
51. J. Diestel. *Geometry of Banach Spaces. Lecture Notes in Mathematics*. Springer, Berlin Heidelberg New York, 1974
52. J. Diestel. *Sequences and Series in Banach Spaces*. Springer, Berlin Heidelberg New York, 1983
53. J. Dieudonné. *Foundations of Modern Analysis*. Academic, New York, 1960
54. A. V. Dmitruk, A. A. Milyutin, and N. P. Osmolovskii. Lyusternik’s theorem and the theory of extrema. *Russ. Math. Surv.*, 35:11–51, 1980
55. S. Dolecki. A general theory of necessary optimality conditions. *J. Math. Anal. Appl.*, 78:267–308, 1980
56. I. Ekeland. On the variational principle. *J. Math. Anal. Appl.*, 47:324–353, 1974
57. I. Ekeland. Nonconvex minimization problems. *Bull. Am. Math. Soc. (N.S.)*, 1:443–474, 1979
58. I. Ekeland and R. Temam. *Convex Analysis and Variational Problems*. North-Holland, Amsterdam, 1976
59. K. -H. Elster, R. Reinhardt, M. Schäuble, and G. Donath. *Einführung in die nichtlineare Optimierung*. Teubner, Leipzig, 1977
60. J. Elstrodt. *Maß- und Integrationstheorie*. Springer, Berlin Heidelberg New York, 1999
61. E. Ernst, M. Théra, and C. Zalinescu. Slice-continuous sets in reflexive Banach spaces: convex constrained optimization and strict convex separation. *J. Funct. Anal.*, 223:179–203, 2005

62. H. Eschrig. *The Fundamentals of Density Functional Theory*. Edition am Gutenbergplatz, Leipzig, 2003
63. L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*. CRC, Boca Raton, 1992
64. K. Fan. Asymptotic cones and duality of linear relations. In O. Shisha, editor, *Inequalities*, volume 2, pages 179–186. Academic, London, 1970
65. W. Fenchel. On conjugate convex functions. *Can. J. Math.*, 1:73–77, 1949
66. W. Fenchel. *Convex Cones, Sets and Functions. Lecture Notes*. Princeton University, Princeton, 1951
67. L. A. Fernandez. On the limits of the Lagrange multiplier rule. *SIAM Rev.*, 39:292–297, 1997
68. O. Ferrero. Theorems of the alternative for set-valued functions in infinite-dimensional spaces. *Optimization*, 20:167–175, 1989
69. D. G. Figueiredo. *Lectures on the Ekeland Variational Principle with Applications and Detours*. Springer (Published for the Tata Institute of Fundamental Research, Bombay), Berlin Heidelberg New York, 1989
70. F. Giannessi. Theorems of the alternative for multifunctions with applications to optimization: general results. *J. Optim. Theory Appl.*, 55:233–256, 1987
71. M. Giaquinta and S. Hildebrandt. *Calculus of Variations I*. Springer, Berlin Heidelberg New York, 1996
72. M. Giaquinta and S. Hildebrandt. *Calculus of Variations II*. Springer, Berlin Heidelberg New York, 1996
73. B. M. Glover and B. D. Craven. A Fritz John optimality condition using the approximate subdifferential. *J. Optim. Theory Appl.*, 82:253–265, 1994
74. B. M. Glover, B. D. Craven, and S. D. Flåm. A generalized Karush–Kuhn–Tucker optimality condition without constraint qualification using the approximate subdifferential. *Numer. Funct. Anal. Optim.*, 14:333–353, 1993
75. B. M. Glover, V. Jeyakumar, and W. Oettli. A Farkas lemma for difference sublinear systems and quasidifferentiable programming. *Math. Oper. Res.*, 63:109–125, 1994
76. A. Göpfert. *Mathematische Optimierung in allgemeinen Vektorräumen*. Teubner, Leipzig, 1973
77. A. Göpfert, Chr. Tammer, H. Riahi, and C. Zalinescu. *Variational Methods in Partially Ordered Spaces*. Springer, Berlin Heidelberg New York, 2003
78. A. Göpfert, Chr. Tammer, and C. Zalinescu. On the vectorial Ekeland’s variational principle and minimal points in product spaces. *Nonlinear Anal.*, 39:909–922, 2000
79. L. M. Graves. Some mapping theorems. *Duke Math. J.*, 17:111–114, 1950
80. K. Groh. On monotone operators and forms. *J. Convex Anal.*, 12:417–429, 2005
81. Ch. Großmann and H. -G. Roos. *Numerical Treatment of Partial Differential Equations*. Springer, Berlin Heidelberg, New York, to appear
82. H. Halkin. Implicit functions and optimization problems without continuous differentiability of the data. *SIAM J. Control*, 12:229–236, 1974
83. H. Halkin. Mathematical programming without differentiability. In D. L. Russell, editor, *Calculus of Variations and Control Theory*, pages 279–288. Academic, New York, 1976
84. J. -B. Hiriart-Urruty. Refinements of necessary optimality conditions in non-differentiable programming I. *Appl. Math. Optim.*, 5:63–82, 1979
85. J. -B. Hiriart-Urruty. Tangent cones, generalized gradients and mathematical programming in Banach spaces. *Math. Oper. Res.*, 4:79–97, 1979



86. J. -B. Hiriart-Urruty. Mean value theorems in nonsmooth analysis. *Numer. Funct. Anal. Optim.*, 2:1–30, 1980
87. J. -B. Hiriart-Urruty. Refinements of necessary optimality conditions in non-differentiable programming II. *Math. Progr. Stud.*, 19:120–139, 1982
88. J. -B. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms. I: Fundamentals*. Springer, Berlin Heidelberg New York, 1993
89. J. -B. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms. II: Advanced Theory and Bundle Methods*. Springer, Berlin Heidelberg New York, 1993
90. J. -B. Hiriart-Urruty, M. Moussaoui, A. Seeger, and M. Volle. Subdifferential calculus without constraint qualification, using approximate subdifferentials: a survey. *Nonlinear Anal.*, 24:1727–1754, 1995
91. R. B. Holmes. *A Course on Optimization and Best Approximation*. Springer, Berlin Heidelberg New York, 1972
92. R. B. Holmes. *Geometric Functional Analysis and Its Applications*. Springer, Berlin Heidelberg New York, 1975
93. A. D. Ioffe. Regular points of Lipschitz functions. *Trans. Am. Math. Soc.*, 251:61–69, 1979
94. A. D. Ioffe. Nonsmooth analysis: differential calculus of nondifferentiable mappings. *Trans. Am. Math. Soc.*, 266:1–56, 1981
95. A. D. Ioffe. Approximate subdifferentials and applications. I. The finite dimensional theory. *Trans. Am. Math. Soc.*, 281(1):389–416, 1984
96. A. D. Ioffe. Approximate subdifferentials and applications. II. *Mathematika*, 33:11–128, 1986
97. A. D. Ioffe. Approximate subdifferentials and applications. III. The metric theory. *Mathematika*, 36:1–38, 1989
98. A. D. Ioffe. Proximal analysis and approximate subdifferentials. *J. Lond. Math. Soc.*, 41:175–192, 1990
99. A. D. Ioffe. A Lagrange multiplier rule with small convex-valued subdifferentials for nonsmooth problems of mathematical programming involving equality and nonfunctional constraints. *Math. Progr. Stud.*, 58:137–145, 1993
100. A. D. Ioffe. Nonsmooth subdifferentials: their calculus and applications. In V. Lakshmikantham, editor, *Proceedings of the First World Congress of Nonlinear Analysts*, pages 2299–2310. De Gruyter, Berlin, 1996
101. A. D. Ioffe and V. Tikhomirov. *Theory of Extremal Problems*. North-Holland, New York, 1978 (German ed.: Deutscher Verlag der Wissenschaften, Berlin, 1979; original Russian ed.: Nauka, Moskow, 1974)
102. J. Jahn. *Vector Optimization: Theory, Applications and Extensions*. Springer, Berlin Heidelberg New York, 2004
103. G. J. O. Jameson. *Topology and Normed Spaces*. Chapman and Hall, London, 1974
104. V. Jeyakumar. Convexlike alternative theorems and mathematical programming. *Optimization*, 16:643–652, 1985
105. F. John. Extremum problems with inequalities as side conditions. In K. O. Friedrichs, O. E. Neugebauer, and J. J. Stoker, editors, *Studies and Essays: Courant Anniversary Volume*, pages 187–204. Wiley-Interscience, New York, 1948
106. A. Jourani and L. Thibault. The approximate subdifferential of composite functions. *Bull. Aust. Math. Soc.*, 47:443–455, 1993

107. A. Jourani and L. Thibault. Metric regularity for strongly compactly Lipschitzian mappings. *Nonlinear Anal.*, 24:229–240, 1995
108. M. Kadec. Spaces isomorphic to a locally uniformly convex space. *Izv. Vysš. Učebn. Zaved. Mat.*, 6:51–57, 1959 (in Russian)
109. W. Karush. *Minima of Functions of Several Variables with Inequalities as Side Conditions*. Master's Thesis, University of Chicago, Chicago, 1939
110. D. Klatte and B. Kummer. Contingent derivatives of implicit (multi-)functions and stationary points. *Ann. Oper. Res.*, 101:313–331, 2001
111. D. Klatte and B. Kummer. *Nonsmooth Equations in Optimization. Regularity, Calculus, Methods and Applications*. Kluwer, Dordrecht, 2002
112. W. Krabs. *Optimierung und Approximation*. Teubner, Stuttgart, 1975
113. A. Y. Kruger and B. S. Mordukhovich. Extremal points and Euler equations in nonsmooth optimization. *Dokl. Akad. Nauk BSSR*, 24:684–687, 1980 (in Russian)
114. H. W. Kuhn and A. W. Tucker. Nonlinear programming. In J. Neyman, editor, *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, pages 481–492. University of California Press, Berkeley, 1951
115. G. Köthe. *Topologische lineare Räume I*. Springer, Berlin Heidelberg New York, 1960
116. M. Landsberg and W. Schirotzek. Mazur–Orlicz type theorems with some applications. *Math. Nachr.*, 79:331–341, 1977
117. J. B. Lasserre. A Farkas lemma without a standard closure condition. *SIAM J. Control Optim.*, 35:265–272, 1997
118. P. -J. Laurent. *Approximation et Optimisation*. Hermann, Paris, 1972
119. G. Lebourg. Valeur moyenne pour gradient généralisé. *C. R. Acad. Sci. Paris*, 281:795–797, 1975
120. Yu. S. Ledyaev and Q. J. Zhu. Implicit multifunction theorems. *Set-Valued Anal.*, 7:209–238, 1999
121. Yu. S. Ledyaev and Q. J. Zhu. Implicit multifunction theorems. Preprint (revised version), <http://homepages.wmich.edu/~zhu/papers/implicit.html>, 2006
122. U. Ledzewicz and S. Walczak. On the Lyusternik theorem for nonsmooth operators. *Nonlinear Anal.*, 22:121–128, 1994
123. P. D. Loewen. *Optimal Control via Nonsmooth Analysis*. American Mathematical Society, Providence, 1993
124. P. D. Loewen. A mean value theorem for Fréchet subgradients. *Nonlinear Anal.*, 23:1365–1381, 1994
125. P. D. Loewen and X. Wang. A generalized variational principle. *Can. J. Math.*, 53(6):1174–1193, 2001
126. B. Luderer, L. Minchenko, and T. Satsura. *Multivalued Analysis and Nonlinear Programming Problems with Perturbations*. Kluwer, Dordrecht, 2002
127. D. Luenberger. *Optimization by Vector Space Methods*. Wiley, New York, 1969
128. L. A. Lyusternik. On constrained extrema of functionals. *Mat. Sb.*, 41:390–401, 1934 (in Russian)
129. P. Michel and J. -P. Penot. Calcul sous-différentiel pour les fonctions lipschitziennes et non lipschitziennes. *C. R. Acad. Sci. Paris*, 298:269–272, 1984
130. P. Michel and J. -P. Penot. A generalized derivative for calm and stable functions. *Diff. Int. Eqs.*, 5:433–454, 1992
131. H. Minkowski. *Theorie der konvexen Körper, insbesondere Begründung ihres Oberflächenbegriffs*. Teubner, Leipzig, 1911

132. B. S. Mordukhovich. Maximum principle in the problem of time optimal control with nonsmooth constraints. *J. Appl. Math. Mech.*, 40:960–969, 1976
133. B. S. Mordukhovich. Metric approximations and necessary optimality conditions for general classes of nonsmooth extremal problems. *Sov. Math. Dokl.*, 22:526–530, 1980
134. B. S. Mordukhovich. *Approximation Methods in Problems of Optimization and Control*. Nauka, Moscow, 1988 (in Russian)
135. B. S. Mordukhovich. Sensitivity analysis in nonsmooth optimization. In D. A. Field and V. Komkov, editors, *Theoretical Aspects of Industrial Design*, volume 58 of *Proceedings in Applied Mathematics*, pages 32–46. SIAM, Philadelphia, 1992
136. B. S. Mordukhovich. *Variational Analysis and Generalized Differentiation I: Basic Theory*. Springer, Berlin Heidelberg New York, 2006
137. B. S. Mordukhovich. *Variational Analysis and Generalized Differentiation II: Applications*. Springer, Berlin Heidelberg New York, 2006
138. B. S. Mordukhovich and J. V. Outrata. On second-order subdifferentials and their applications. *SIAM J. Optim.*, 12:139–169, 2001
139. B. S. Mordukhovich and Y. Shao. Differential characterizations of covering, metric regularity, and Lipschitzian properties of multifunctions between Banach spaces. *Nonlinear Anal.*, 25:1401–1424, 1995
140. B. S. Mordukhovich and Y. Shao. Extremal characterizations of Asplund spaces. *Proc. Am. Math. Soc.*, 124:197–205, 1996
141. B. S. Mordukhovich and Y. Shao. Nonconvex coderivative calculus for infinite-dimensional multifunctions. *Set-Valued Anal.*, 4:205–236, 1996
142. B. S. Mordukhovich and Y. Shao. Nonsmooth sequential analysis in Asplund spaces. *Trans. Am. Math. Soc.*, 348:1235–1280, 1996
143. B. S. Mordukhovich and Y. Shao. Mixed coderivatives of set-valued mappings in variational analysis. *J. Appl. Anal.*, 4:269–294, 1998
144. B. S. Mordukhovich, J. S. Treiman, and Q. J. Zhu. An extended extremal principle with applications to multiobjective optimization. *SIAM J. Optim.*, 14:359–379, 2003
145. B. S. Mordukhovich and B. Wang. Extensions of generalized differential calculus in Asplund spaces. *J. Math. Anal. Appl.*, 272:164–186, 2002
146. B. S. Mordukhovich and B. Wang. Necessary suboptimality and optimality conditions via variational principles. *SIAM J. Control Optim.*, 41:623–640, 2002
147. J. -J. Moreau. Fonctions convexes duales et points proximaux dans un espace hilbertien. *C. R. Acad. Sci. Paris*, 255:2897–2899, 1962
148. J. -J. Moreau. Propriété des applications ‘prox’. *C. R. Acad. Sci. Paris*, 256:1069–1071, 1963
149. M. M. Mäkelä and P. Neittaanmäki. *Nonsmooth Optimization*. World Scientific, Singapore, 1992
150. L. W. Neustadt. *Optimization: A Theory of Necessary Conditions*. Princeton University Press, Princeton, 1976
151. H. V. Ngai, D. T. Luc, and M. Théra. Extensions of Fréchet  $\epsilon$ -subdifferential calculus and applications. *J. Math. Anal. Appl.*, 268:266–290, 2002
152. N. V. Ngai and M. Théra. Metric inequality, subdifferential calculus and applications. *Set-Valued Anal.*, 9:187–216, 2001
153. N. V. Ngai and M. Théra. A fuzzy necessary optimality condition for non-Lipschitz optimization in Asplund spaces. *SIAM J. Optim.*, 12:656–668, 2002

154. N. V. Ngai and M. Théra. Error bounds and implicit multifunction theorem in smooth Banach spaces and applications to optimization. *Set-Valued Anal.*, 12:195–223, 2004
155. D. Pallaschke. Ekeland’s variational principle, convex functions and Asplund spaces. In W. Krabs and J. Zowe, editors, *Modern Methods of Optimization*, pages 274–312. Springer, Berlin Heidelberg New York, 1992
156. D. Pallaschke and S. Rolewicz. *Foundations of Mathematical Optimization: Convex Analysis Without Linearity*. Kluwer, Dordrecht, 1997
157. P. D. Panagiotopoulos. *Hemivariational Inequalities*. Springer, Berlin Heidelberg New York, 1993
158. N. S. Papageorgiou and L. Gasinski. *Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems*. Chapman and Hall, Boca Raton, 2004
159. J. -P. Penot. Calcul sous-différentiel et optimisation. *J. Funct. Anal.*, 27: 248–276, 1978
160. J. -P. Penot. On regularity conditions in mathematical programming. *Math. Progr. Stud.*, 19:167–199, 1982
161. J. -P. Penot. Open mapping theorems and linearization stability. *Numer. Funct. Anal. Optim.*, 8:21–35, 1985
162. J. -P. Penot. The drop theorem, the petal theorem and Ekeland’s variational principle. *Nonlinear Anal.*, 10:813–822, 1986
163. J. -P. Penot. On the mean value theorem. *Optimization*, 19:147–156, 1988
164. J. -P. Penot. Metric regularity, openness and Lipschitzian behavior of multifunctions. *Nonlinear Anal.*, 13:629–643, 1989
165. R. R. Phelps. *Convex Functions, Monotone Operators and Differentiability*, volume 1364 of *Lecture Notes in Mathematics*. Springer, Berlin Heidelberg New York, 1993
166. D. Preiss. Fréchet derivatives of Lipschitzian functions. *J. Funct. Anal.*, 91:312–345, 1990
167. D. Preiss and L. Zajíček. Fréchet differentiation of convex functions in a Banach space with a separable dual. *Proc. Am. Math. Soc.*, 91(2):202–204, 1984
168. B. N. Pšeničnyi. *Convex Analysis and Extremal Problems*. Nauka, Moscow, 1969 (in Russian)
169. B. N. Pšeničnyj. *Notwendige Optimalitätsbedingungen*. Teubner, Leipzig, 1972
170. B. N. Pshenichnyi. *Necessary Conditions for an Extremum*. Dekker, New York, 1971 (German ed.: Teubner, Leipzig, 1972; original Russian ed.: Nauka, Moscow, 1969)
171. H. Pühl. Convexity and openness with linear rate. *J. Math. Anal. Appl.*, 227:382–395, 1998
172. H. Pühl. *Nichtdifferenzierbare Extremalprobleme in Banachräumen: Regularitätsbedingungen sowie die Approximation von Niveaumengen*. Doctoral Thesis, Technische Universität Dresden, Dresden, 1999
173. H. Pühl and W. Schirotzek. Linear semi-openness and the Lyusternik theorem. *Eur. J. Oper. Res.*, 157:16–27, 2004
174. A. Roberts and D. Varberg. *Convex Functions*. Academic, New York, 1973
175. S. M. Robinson. Regularity and stability for convex multivalued functions. *Math. Oper. Res.*, 1:130–143, 1976
176. S. M. Robinson. Stability theory for systems of inequalities. Part II. Differentiable nonlinear systems. *SIAM J. Numer. Anal.*, 13:497–513, 1976
177. R. T. Rockafellar. *Convex Functions and Dual Extremum Problems*. Ph.D. Thesis, Harvard University, Cambridge, 1963

178. R. T. Rockafellar. Level sets and continuity of conjugate convex functions. *Trans. Am. Math. Soc.*, 123:46–63, 1966
179. R. T. Rockafellar. Duality and stability in extremum problems involving convex functions. *Pac. J. Math.*, 21:167–187, 1967
180. R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, 1970
181. R. T. Rockafellar. On the maximal monotonicity of subdifferential mappings. *Pac. J. Math.*, 33:209–216, 1970
182. R. T. Rockafellar. On the maximality of sums of nonlinear monotone operators. *Trans. Am. Math. Soc.*, 149:75–88, 1970
183. R. T. Rockafellar. Directionally Lipschitzian functions and subdifferential calculus. *Proc. Lond. Math. Soc.*, 39:331–355, 1979
184. R. T. Rockafellar. Generalized directional derivatives and subgradients of nonconvex functions. *Can. J. Math.*, 32:157–180, 1980
185. R. T. Rockafellar. Proximal subgradients, marginal values and augmented Lagrangians in nonconvex optimization. *Math. Oper. Res.*, 6:424–436, 1981
186. R. T. Rockafellar. *The Theory of Subgradients and Its Applications to Problems of Optimization: Convex and Nonconvex Functions*. Heldermann, Berlin, 1981
187. R. T. Rockafellar. Extensions of subgradient calculus with applications to optimization. *Nonlinear Anal.*, 9:665–698, 1985
188. R. T. Rockafellar. Second-order optimality conditions in nonlinear programming obtained by way of epi-derivatives. *Math. Oper. Res.*, 14:462–484, 1989
189. R. T. Rockafellar and R. J. -B. Wets. *Variational Analysis*. Springer, Berlin Heidelberg New York, 1998
190. H. -P. Scheffler. *Beiträge zur Analysis von nichtglatten Optimierungsproblemen*. Doctoral Thesis, Technische Universität Dresden, Dresden, 1987
191. H. -P. Scheffler. Mean value properties of nondifferentiable functions and their application in nonsmooth analysis. *Optimization*, 20:743–759, 1989
192. H. -P. Scheffler and W. Schirotzek. Necessary optimality conditions for nonsmooth problems with operator constraints. *Z. Anal. Anwend.*, 7:419–430, 1988
193. W. Schirotzek. On Farkas type theorems. *Commentat. Math. Univ. Carol.*, 22:1–14, 1981
194. W. Schirotzek. On a theorem of Ky Fan and its application to nondifferentiable optimization. *Optimization*, 16:353–366, 1985
195. W. Schirotzek. Nonasymptotic necessary conditions for nonsmooth infinite optimization problems. *J. Math. Anal. Appl.*, 118:535–546, 1986
196. W. Schirotzek. *Differenzierbare Extremalprobleme*. Teubner, Leipzig, 1989
197. L. Schwartz. *Analyse II. Calcul Différentiel et Équations Différentielles*. Hermann, Paris, 1997
198. S. Simons. The least slope of a convex function and the maximal monotonicity of its subdifferential. *J. Optim. Theory Appl.*, 71:127–136, 1991
199. I. Singer. Generalizations of methods of best approximation to convex optimization in locally convex spaces I: extension of continuous linear functionals and characterizations of solutions of continuous convex programs. *Rev. Roum. Math. Pures Appl.*, 19:65–77, 1974
200. C. Stegall. The Radon–Nikodým property in conjugate Banach spaces. *Trans. Am. Math. Soc.*, 264:507–519, 1984
201. J. Stoer and C. Witzgall. *Convexity and Optimization in Finite Dimensions*. Springer, Berlin Heidelberg New York, 1970

202. M. Studniarski. Mean value theorems and sufficient optimality conditions for nonsmooth functions. *J. Math. Anal. Appl.*, 111:313–326, 1985
203. M. Studniarski. Mean value theorems for functions possessing first order convex approximations. Applications in optimization theory. *Z. Anal. Anwend.*, 4:125–132, 1985
204. A. I. Subbotin. *Generalized Solutions of First-Order PDEs. The Dynamical Optimization Perspective*. Birkhäuser, Basel, 1994
205. K. Sundaresan and S. Swaminathan. *Geometry and Nonlinear Analysis in Banach Spaces. Lecture Notes in Mathematics*. Springer, Berlin Heidelberg New York, 1985
206. J. S. Treiman. Finite dimensional optimality conditions: B-gradients. *J. Optim. Theory Appl.*, 62:139–150, 1989
207. J. S. Treiman. Optimal control with small generalized gradients. *SIAM J. Control Optim.*, 28:720–732, 1990
208. S. Troyanski. On locally uniformly convex and differentiable norms in certain non-separable spaces. *Stud. Math.*, 37:173–180, 1971
209. F. Tröltzsch. *Optimality Conditions for Parabolic Control Problems and Applications*. Teubner, Leipzig, 1984
210. H. Tuy. *Convex Inequalities and the Hahn–Banach Theorem*. Number 97 in *Dissertationes Mathematicae (Rozprawy Matematyczne)*, Warszawa, 1972
211. C. Ursescu. Multifunctions with convex closed graph. *Czech. Math. J.*, 25:438–441, 1975
212. W. Walter. *Analysis 1*. Springer, Berlin Heidelberg New York, 2004
213. J. Warga. Derivate containers, inverse functions, and controllability. In D. L. Russell, editor, *Calculus of Variations and Control Theory*, pages 13–46. Academic, New York, 1976
214. J. Warga. Fat homeomorphisms and unbounded derivate containers. *J. Math. Anal. Appl.*, 81:545–560, 1981
215. D. Werner. *Funktionalanalysis*. Springer, Berlin Heidelberg New York, 1997
216. S. Yamamuro. *Differential Calculus in Topological Linear Spaces*, volume 374 of *Lecture Notes in Mathematics*. Springer, Berlin Heidelberg New York, 1974
217. J. J. Ye. Multiplier rules under mixed assumptions of differentiability and Lipschitz continuity. *SIAM J. Control Optim.*, 39:1441–1460, 1999
218. K. Yosida. *Functional Analysis*. Springer, Berlin Heidelberg New York, 1965
219. D. Yost. Asplund spaces for beginners. *Acta Univ. Carol. Math. Phys.*, 34:159–177, 1993
220. D. Zagrodny. Approximate mean value theorem for upper subderivatives. *Nonlinear Anal.*, 12:1413–1428, 1988
221. E. Zeidler. *Nonlinear Functional Analysis and Its Applications III: Variational Methods and Optimization*. Springer, Berlin Heidelberg New York, 1984
222. E. Zeidler. *Nonlinear Functional Analysis and Its Applications I. Fixed-Point Theory*. Springer, Berlin Heidelberg New York, 1986
223. E. Zeidler. *Nonlinear Functional Analysis and Its Applications II A: Linear Monotone Operators*. Springer, Berlin Heidelberg New York, 1990
224. E. Zeidler. *Nonlinear Functional Analysis and Its Applications II B: Nonlinear Monotone Operators*. Springer, Berlin Heidelberg New York, 1990
225. Q. J. Zhu. Clarke–Ledyev mean value inequality in smooth Banach spaces. *Nonlinear Anal.*, 32:315–324, 1998
226. Q. J. Zhu. The equivalence of several basic theorems for subdifferentials. *Set-Valued Anal.*, 6:171–185, 1998

227. Q. J. Zhu. Hamiltonian necessary conditions for a multiobjective optimal control problem with endpoint constraints. *SIAM J. Control Optim.*, 39:97–112, 2000
228. W. P. Ziemer. *Weakly Differentiable Functions*. Springer, Berlin Heidelberg New York, 1989
229. J. Zowe and S. Kurcyusz. Regularity and stability for the mathematical programming problem in Banach spaces. *Appl. Math. Optim.*, 5:49–62, 1979

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