A

Brownian Motion

A.1 Historical Background

The botanist Robert Brown, in 1828, observed the irregular movements of particles of pollen suspended in water. In 1877, Delsaux explained the ceaseless changes of direction in the particles' paths by the collisions between the particles of pollen and the water molecules. A motion of this type was described as being a "random motion".

In 1900, Bachelier [18], with a view to studying price movements on the Paris exchange, exhibited the "Markovian" nature of Brownian motion: the position of a particle at time $t + s$ depends on its position at time t, and does not depend on its position before time t. It is worth emphasizing that Bachelier was a forerunner in the field, and that the theory of Brownian motion was developed for the financial markets before it was developed for physics.

In 1905, Einstein [134] determined the transition density function of Brownian motion by means of the heat equation, and so linked Brownian motion to partial differential equations of the parabolic type. That same year, Smoluchowski described Brownian motion as a limit of random walks.

The first rigorous mathematical study of Brownian motion was carried out by N. Wiener [366] (1923), who also gave a proof of the existence of Brownian motion. P. Lévy [254], (1948) worked on the finer properties of Brownian paths, without any knowledge of concepts such as filtration or stopping time. Since then, Brownian motion has continued to fascinate probabilists, for the study of its paths just as much as for that of stochastic integration theory. See for example the books Knight [238] and Yor [373, 374, 375].

A.2 Intuition

The easiest Brownian motion to imagine is probably Brownian motion in the plane: at each instant in time, the particle randomly chooses a direction, and then makes a "step" in that direction. However, for an approach that is both intuitive and rigorous, we must study the real Brownian motion: at each instant that is a multiple of Δt, the particle "randomly" chooses to move left or right to a distance Δx from its starting point. To model this "randomness", we turn to a sequence of independent identically distributed random variables $(Y_i\,,\, i \geq 1)$ such that $P(Y_i = \Delta x) = P(Y_i = -\Delta x) = \dfrac{1}{2}$.

At time t, the particle will have made $\left[\dfrac{t}{\Delta t}\right]$ moves (where $[a]$ denotes the integer part of a). The particle's position will be $V_t = Y_1 + Y_2 + \cdots + Y_{[\frac{t}{\Delta t}]}$.

All this takes place on a very small scale: we would like to let both Δt and Δx tend to zero in an appropriate way. Note that $EV_t^2 \simeq (\Delta x)^2 \dfrac{t}{\Delta t}$. In order for this quantity to have a limit, we must impose that $\dfrac{(\Delta x)^2}{\Delta t}$ have a limit. The increment Δt will be "very small" and Δx will be "small", so that $(\Delta x)^2$ will also be "very small". The most straightforward choice is $\Delta x = \sqrt{\Delta t}$ and $\Delta t = \dfrac{1}{n}$.

Let us now give a precise formulation of this approach.

A.3 Random Walk

On a probability space $(\Omega, \mathcal{F}, P)$, let

$$P(X_i = 1) \;=\; P(X_i = -1) \;=\; \frac{1}{2} \quad , \quad i \in \mathbb{N}^* ,$$

be a family of independent identically distributed random variables (the X_i are said to be independent Bernoulli variables). To this family, we associate the sequence $(S_n\,,\, n \geq 0)$ defined by

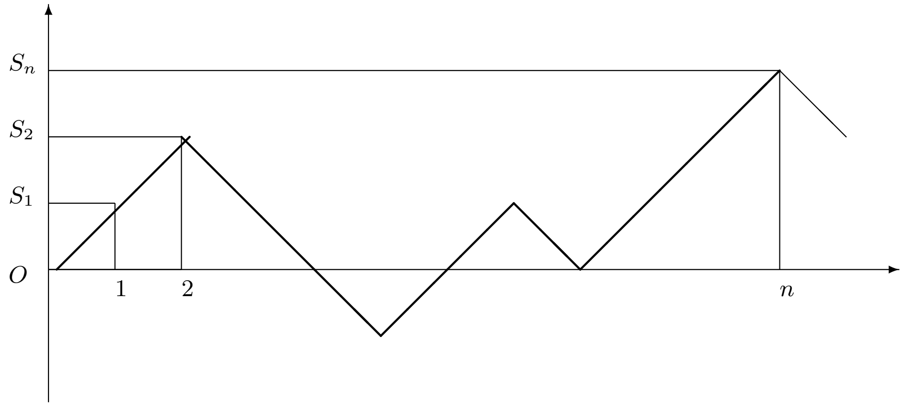
$$\begin{aligned} S_0 &= 0 \\ S_n &= \sum_{i=1}^{n} X_i\,. \end{aligned} \tag{A.1}$$

We have $E(S_n) = 0$, $\mathrm{Var}\,(S_n) = n$. We say that the sequence S_n is a *random walk*. We can interpret it as a game of tossing a coin: a player tosses a coin, he wins one euro if it comes up tails, and loses one euro if it comes up heads. He has no initial wealth ($S_0 = 0$). His wealth at **time** n (after n games) is S_n. We represent the series of the results obtained over N successive games as a graph (see Fig. A.1).

We note that the sequence $(S_m - S_n, m \geq n)$ is independent of $(S_0, S_1, \ldots, S_n)$ and that $S_m - S_n$ has the same probability law as S_{m-n} (the binomial distribution depends only on $m - n$).

We now proceed with a two-fold normalization. Let N be fixed.

Fig. A.1. A random walk



- We transform the time interval $[0, N]$ into the interval $[0, 1]$,
- and we change the scale of values taken by S_n.

More precisely, we define a family of random variables indexed by real numbers of the form $\frac{k}{N}$, $k \in \mathbb{N}$:

$$U_{\frac{k}{N}} = \frac{1}{\sqrt{N}} S_k . \tag{A.2}$$

We move from $U_{\frac{k}{N}}$ to $U_{\frac{k+1}{N}}$ in a "very small" interval of time equal to $\frac{1}{N}$, by making a step of a "small length" $\frac{1}{\sqrt{N}}$ (towards the left or towards the right). We have

$$E\big(U_{\frac{k}{N}}\big) = 0 \quad \text{and} \quad \mathrm{Var}\left(U_{\frac{k}{N}}\right) = \frac{k}{N} .$$

The independence and stationarity properties of the random walk still hold, i.e.,

- if $k \geq k'$, $U_{\frac{k}{N}} - U_{\frac{k'}{N}}$ is independent of $U_{\frac{p}{N}}$ for $p \leq k'$;
- if $k \geq k'$, $U_{\frac{k}{N}} - U_{\frac{k'}{N}}$ has the same probability law as $U_{\frac{k-k'}{N}}$.

We define a continuous-time process, that is, a family of random variables $(U_t\,, t \geq 0)$ starting from $U_{\frac{k}{N}}$, by requiring the function $t \to U_t$ to be affine between times $\dfrac{k}{N}$ and $\dfrac{k+1}{N}$. To do this, for N fixed, we note that for all $t \in \mathbb{R}_+$, there exists a unique $k(t) \in \mathbb{N}$ such that $\dfrac{k(t)}{N} \leq t < \dfrac{k(t)+1}{N}$, and we set

$$U_t^N \;=\; U_{\frac{k}{N}} + N\left(t - \frac{k}{N}\right)\left(U_{\frac{k+1}{N}} - U_{\frac{k}{N}}\right)$$

where $k = k(t)$.

(The process $(U_t,\, t \geq 0)$ does not have independent increments. However, if $t \geq t'$ and $\dfrac{k'+1}{N} > t' \geq \dfrac{k'}{N}$, we have that

$$U_t^N - U_{t'}^N \quad \text{is independent of} \quad U^N_{\frac{p}{N}}\,,\ p \leq k' \quad)\,.$$

Let us return for a brief moment to writing U as a function of the random walk S.

For $t = 1$ we have $U_1^N = \dfrac{1}{\sqrt{N}} S_N$. The central limit theorem then implies that U_1^N converges in distribution to a standard normal random variable.

Exercise A.3.1.

1. Show that U_t^N converges in distribution to a normal random variable with mean 0 and variance t as $N \to \infty$. Notice how $0 \leq t - \dfrac{k(t)}{N} \leq \dfrac{1}{N}$, and how
$$\left|U^N_{\frac{k+1}{N}} - U^N_{\frac{k}{N}}\right| \;\leq\; \frac{1}{\sqrt{N}} \qquad \text{with} \quad k = k(t)\,.$$
2. Show that $(U_{t_1}^N, U_{t_2}^N, \ldots, U_{t_n}^N)$ with $t_1 < t_2 < \cdots < t_n$ converges in distribution to a vector $\left(Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n}\right)$, such that $Z_{t_i} - Z_{t_{i-1}}$ is independent of $\left(Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n}\right)$ and such that $(Z_{t_i} - Z_{t_{i-1}})$ has a normal distribution with mean 0 and variance $(t_i - t_{i-1})$ (use the central limit theorem for vectors).

It can be shown that U^N converges[1] to a process B, which has continuous paths (i.e., for almost all ω, the mapping $t \to B_t(\omega)$ is continuous), and which satisfies

(i) $B_0 = 0$.

[1] In the sense of convergence in distribution. This is stronger than the convergence in distribution of finite families. See Karatzas and Shreve [233]. It is also possible to construct a probability space on which all the random walks S_N are defined, and on which the normalized sums U^N converge a.s. to a Brownian motion (Knight [238]).

(ii) $B_{t+s} - B_t$ has the normal distribution $\mathcal{N}(0, s)$.

(iii) $B_{t+s} - B_t$ is independent of $B_{t_i} - B_{t_{i+1}}$, for $t_0 < \cdots < t_n = t$.

Remark A.3.2. We can show that Brownian motion is the only process satisfying (i), (iii) and

(ii)′ The distribution of $B_{t+s} - B_t$ depends only on s.

We introduce the notation $\Delta B(t) = B(t + \Delta t) - B(t)$ where $B(t) = B_t$ and $\Delta t > 0$. The Brownian motion then satisfies:

- $E[\Delta B(t)] = 0$ $\quad$ $\mathrm{Var}[\Delta B(t)] = \Delta t$ $\quad$ (using (ii))
- $E_t[\Delta B(t)] = 0$ $\quad$ $E_t[(\Delta B(t))^2] = \Delta t$ $\quad$ (using (ii) and (iii))

where E_t is the conditional expectation with respect to $\mathcal{F}_t = \sigma(B_s\, , \, s \leq t)$. The equality $E_t(\Delta B(t)) = 0$ can be interpreted as follows: if the position of the Brownian motion at time t is known, then the average move between times t and $t + \Delta t$ is zero. This property is a result of the independence and of the Gaussian nature of Brownian motion.

A.4 The Stochastic Integral

Brownian motion represents the path of a particle that incessantly changes direction. The graph of such a path has many sharp peaks and troughs, and is not differentiable at these points (the left and right derivatives are not equal). We can prove the following result:

Theorem A.4.1. *For almost all ω, the function $t \mapsto B_t(\omega)$ is a.s. nowhere differentiable (i.e., the set of t for which $B_t(\omega)$ is differentiable has Lebesgue measure zero).*

The lack of differentiability of Brownian paths forbids **the interpretation of the symbol** $\mathrm{d}B_t$ as B'_t, and makes it impossible to define $\int \theta(t)\, \mathrm{d}B_t$ using the usual methods (such as writing $\mathrm{d}B_t = B'(t)\mathrm{d}t$).

As Brownian motion has unbounded variation, the Stieljes integration theory cannot be applied. However, we can draw on the ideas of Riemann integration theory, as long as we carefully check each step along the way. The aim is to define a new integral, in such a way that it is additive with respect to θ, and satisfies

$$\int_{[a,b]} \mathrm{d}B_t \;=\; B(b) - B(a)\ .$$

Hence the idea of defining the integral for a step function[2] θ (i.e., such that $\theta(t) = \theta(t_i)$, $t \in]t_i, t_{i+1}]$, $t_0 = 0 < t_1 \cdots < t_p = T$), as:

$$\int_0^T \theta(s)\,\mathrm{d}B_s \;=\; \sum_{i=0}^{p-1} \theta(t_i)[B(t_{i+1}) - B(t_i)] \;.$$

When θ is a process, we impose conditions of measurability, which are slightly stronger than assuming the process to be adapted to the Brownian motion's filtration. For technical reasons, we also impose integrability conditions on the process θ, in order for $\sum \theta(t_i)(B(t_{i+1}) - B(t_i))$ to converge when the time-step tends to 0 (we approximate the process θ with a step process).

As seen in Chaps. 2 to 4, we are led to study Itô processes, i.e., processes X of the form

$$X_t \;=\; x + \int_0^t \mu(s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s \;. \tag{A.3}$$

It is important to understand that the notation

$$\mathrm{d}X_t \;=\; \mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t \;, \tag{A.4}$$

is only a symbolic notation, with which we can develop a stochastic calculus. The exact meaning of (A.4) is given by writing X in the form of (A.3).

Still working symbolically, and interpreting $\mathrm{d}B_t$ (and $\mathrm{d}X_t$) as small increments ΔB_t of B (or ΔX_t), we obtain

$$E(\mathrm{d}X_t) = \mu(t)\mathrm{d}t, \quad \mathrm{Var}(\mathrm{d}X_t) = \sigma^2(t)\mathrm{d}t$$

and similarly,

$$\begin{aligned} E_t(\mathrm{d}X_t) \;&=\; \mu(t)\,\mathrm{d}t \\ \mathrm{Var}_t\,(\mathrm{d}X_t) \;&=\; E_t[\mathrm{d}X_t - E_t(\mathrm{d}X_t)]^2 \;=\; \sigma^2(t)\,\mathrm{d}t \;. \end{aligned}$$

(Exact calculations would lead to $E_t(\Delta X_t) = \int_t^{t+\Delta t} \mu(s)\,\mathrm{d}s$ and $E_t\left(\Delta X_t - \int_t^{t+\Delta t} \mu(s)\mathrm{d}s\right)^2 = \int_t^{t+\Delta t} \sigma^2(s)\mathrm{d}s$.)

It is worth emphasizing that whilst $\mathrm{d}t$ and $\mathrm{d}B_t$ are both "small", their sizes are of different orders. Indeed, we have "$E(\mathrm{d}B_t) = 0$" and "$E(\mathrm{d}B_t)^2 = \mathrm{d}t$". This symbolic representation of (A.4) has an advantage: it suggests that if we apply Taylor's expansion to a function of X_t, and if we wish to keep the $\mathrm{d}t$ terms, we will need to include terms from the expansion of "$(\mathrm{d}B_t)^2$", which will have a role to play.

[2] Taking functions that are left-continuous. This is a small technical difficulty that we do not dwell upon here.

A.5 Itô's Formula

Using this very intuitive approach to Itô processes, we can persuade ourselves (and persuade the reader) that Itô's lemma is "quite natural". Let

$$\mathrm{d}X_t \;=\; \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t\;, \tag{A.5}$$

be an Itô process, and let f be a function of class C^2. We can apply Taylor's expansion to f :

$$\begin{aligned} f(X_{t+\Delta t}) - f(X_t) \;&=\; (X_{t+\Delta t} - X_t)\,f'(X_t) + \frac{1}{2}(X_{t+\Delta t} - X_t)^2 f^{''}(X_t) \\ &\quad + o(X_{t+\Delta t} - X_t)^2\;. \end{aligned}$$

Setting $\Delta X_t = X_{t+\Delta t} - X_t$ and identifying ΔX_t with $\mathrm{d}X_t$ as we did before, we obtain from the expression for ΔX_t given in (A.5)

$$\begin{aligned} \Delta f(X_t) \;&=\; \mu(t, X_t)f'(X_t)\Delta t + \sigma(t, X_t)f'(X_t)\Delta B_t \\ &\quad + \frac{1}{2}\{\mu^2(t, X_t)(\Delta t)^2 + \sigma^2(t, X_t)(\Delta B_t)^2 \\ &\quad + 2\mu(t, X_t)\sigma(t, X_t)\Delta t\,\Delta B_t\}f''(X_t) + o(\Delta X_t)^2\;. \end{aligned}$$

We saw above that the $(\Delta B_t)^2$ term is "of the same order as" Δt. Therefore, we must keep it in this form. However the $(\Delta t)^2$ and $(\Delta t)\,(\Delta B_t)$ terms are $o(\Delta t)$. It is appropriate to keep only terms of order lesser than or equal to that of Δt. We obtain

$$\begin{aligned} \Delta f(X_t) \;&=\; \mu(t, X_t)f'(X_t)\Delta t + \frac{1}{2}\,\sigma^2(t, X_t)f''(X_t)\,\Delta t \\ &\quad + \sigma(t, X_t)f'(X_t)\,\Delta B_t\;. \end{aligned}$$

Table A.1. Multiplication Table 1

	dt	dB
dt	0	0
dB	0	dt

We remark that a rigorous proof of Itô's lemma rests on the same idea.

Furthermore, we note that this intuitive approach to Itô's formula makes it easy to write down: we take Taylor's expansion of order 2, and use the "multiplication table" in Table A.1.

A similar technique can be used to move up the case of a multi-dimensional Brownian motion. If B^1 and B^2 are two independent Brownian motions, $\Delta B_t^1 \ \Delta B_t^2$ has zero expectation, so we neglect these terms in the Taylor expansion. This leads to the multiplication table in Table A.2.

Table A.2. Multiplication Table 2

	$\mathrm{d}t$	$\mathrm{d}B_t^1$	$\mathrm{d}B_t^2$
$\mathrm{d}t$	0	0	0
$\mathrm{d}B_t^1$	0	$\mathrm{d}t$	0
$\mathrm{d}B_t^2$	0	0	$\mathrm{d}t$

Example A.5.1.

$$\begin{aligned}
\mathrm{d}X_t^1 &= \mu_1 \,\mathrm{d}t + \sigma_1 \,\mathrm{d}B_t^1 \\
\mathrm{d}X_t^2 &= \mu_2 \,\mathrm{d}t + \sigma_2 \,\mathrm{d}B_t^2 \\
\mathrm{d}(X_t^1 X_t^2) &= X_t^1 \,\mathrm{d}X_t^2 + X_t^2 \,\mathrm{d}X_t^1 + \mathrm{d}X_t^1 \,\mathrm{d}X_t^2 \\
&= X_t^1 \,\mathrm{d}X_t^2 + X_t^2 \,\mathrm{d}X_t^1 + \sigma_1\sigma_2 \,\mathrm{d}t \,.
\end{aligned}$$

B

Numerical Methods

We present here several methods for approximating solutions to partial differential equations (PDEs) of the parabolic type that are analogous to those appearing in the Black–Scholes model. We have chosen to give our exposition of these methods in a simple case, assuming the coefficients to be constant, for example. In this case, we know an explicit solution to the Black–Scholes equation, and numerical methods are of little interest. However, we hope to show which are the difficulties that arise, and to make it easier for the reader to access specialist works on the subject such as Cessenat et al. [50], Kloeden and Platen [237], Dupuis and Kushner [131], and Rogers and Talay [314].

To simplify the exposition, we assume that the market includes one riskless bond whose price is given by

$$\mathrm{d}S_t^0 \;=\; S_t^0\, r(t, S_t)\, \mathrm{d}t \;,$$

and a stock whose price satisfies

$$\mathrm{d}S_t \;=\; b(t, S_t)\, \mathrm{d}t + \sigma(t, S_t)\, \mathrm{d}B_t \;,$$

where B is a real-valued Brownian motion.

We showed in Chap. 3 (Sect. 3.4) that to calculate the value of the contingent product $g(S_T)$, we need to solve the following PDE:

$$\begin{cases} \mathcal{L}C(t,x) - r(t,x)C(t,x) \;=\; 0 \\ C(T,x) \;=\; g(x) \end{cases} \tag{B.1}$$

with

$$\mathcal{L}C(t,x) = xr(t,x)\frac{\partial C}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 C}{\partial x^2}(t,x) + \frac{\partial C}{\partial t}(t,x) \;.$$

This solution can be written as

$$C(t,x) \;=\; E\left(\mathrm{e}^{-\phi(T)} g(W_T^{x,t})\right) \tag{B.2}$$

with

$$\phi(s) \;=\; \int_t^s r(u, W_u^{x,t})\,\mathrm{d}u\ .$$

In this formulation, $W_u^{x,t}$ denotes the solution to the following stochastic differential equation

$$\begin{cases} \mathrm{d}W_u^{x,t} = \mu(u, W_u^{x,t})\mathrm{d}u + \sigma(u, W_u^{x,t})\mathrm{d}B_u\ , \\ \;W_t^{x,t} = x\ , \end{cases}$$

where we have set $\mu(t,x) = \; x\, r(t,x)$.

We present two methods for approximating C, the first one using (B.1) and techniques for approximating solutions to parabolic equations, and the second using (B.2) and simulating the process W.

B.1 Finite Difference

We are going to use the fact that C is the unique solution to the partial differential equation (B.1) satisfying conditions of regularity.

Let us give an example of regularity conditions in a particular case. Let us assume that $r(t,x) = r$ and $\sigma(t,x) = x\sigma$. In this case, we solve (B.1) on $[0,T]\times]0,\infty[$. We then have the following result (Karatzas et al. [233]): if Δ is continuous on $[0,T]\times]0,\infty[$, Hölder continuous in x uniformly with respect to (t,x) on a compact set, if g is continuous, and if Δ and g satisfy

$$\begin{cases} \qquad\qquad |g(x)| \;\le\; K(1 + x^{\alpha} + x^{-\alpha}) \\ \max\limits_{0\le t\le T} |\Delta(t,x)| \;\le\; K(1 + x^{\alpha} + x^{-\alpha}) \qquad 0 < x < \infty\ , \end{cases} \tag{B.3}$$

then equation B.1 has a unique solution in the set of $C^{1,2}([0,T]\times]0,\infty[)$ functions satisfying (B.3).

Note that in this particular case, in order to solve

$$\frac{\partial C}{\partial t} + rx\,\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2\,\frac{\partial^2 C}{\partial x^2} - rC \;=\; \Delta$$

we first make a change of variable, setting $H(t,x) = C(t,\mathrm{e}^x)$. We are thus led to solve $\dfrac{\partial H}{\partial t} + \left(r - \dfrac{1}{2}\sigma^2\right)\dfrac{\partial H}{\partial x} + \dfrac{1}{2}\sigma^2\,\dfrac{\partial^2 H}{\partial x^2} - rH \;=\; \Delta$, which has constant coefficients.

Let us suppose therefore that (B.1) **has a unique solution**

The first difficulty is that the domain on which we are studying (B.1) is unbounded. Therefore, let us first solve the problem on $[0, T] \times [-K, +K]$.

To obtain the uniqueness of the solution, boundary conditions are needed. Let us either impose *Dirichlet* conditions: we take $a \in \mathbb{R}$ and impose

$$C(t, K) = C(t, -K) = a \qquad t \in [0, T] , \tag{B.4}$$

or *Neumann* conditions:

$$\frac{\partial C}{\partial x}(t, K) = \frac{\partial C}{\partial x}(t, -K) = b \qquad t \in [0, T] , \tag{B.4bis}$$

with $b \in \mathbb{R}$.

In the case of constant coefficients, $C(t, x)$ can be expressed as a function of the normal distribution. In this case, it is easy to show that

$$C(t, x) \underset{x \to \infty}{\longrightarrow} \infty \qquad t \in [0, T]$$

$$\frac{\partial C}{\partial x}(t, x) \underset{x \to \infty}{\longrightarrow} 1 \qquad t \in [0, T] .$$

In this case, the Neumann conditions with $b = 1$ are best suited to the problem.

B.1.1 Method

We continue our exposition of the method in the cases of Dirichlet and Neumann conditions.

We define a grid on the domain $[0, T] \times [-K, +K]$, with steps of size $h = \dfrac{2}{N+1}$ for the space variable x and of size ε for the time variable t.

We use the notation

$$\begin{aligned} t_n &= n\varepsilon \quad 0 \le n \le M && \text{with} \quad M\varepsilon = T , \\ x_i &= -K + i\frac{2K}{N+1} && \text{with} \quad 0 \le i \le N+1 ; \end{aligned}$$

(both the step sizes h and ε will tend to 0.)

The *finite difference* method is a means of obtaining an approximation to the solution, by using the nodes (t_n, x_i) on the grid. Let $C(t, x)$ be the solution to (B.1). We are looking for a family M of vectors $(C^n(i), 1 \le i \le N)_{n<M}$ such that $C^n(i)$ is close to $C(t_n, x_i)$ for $i = 1, \ldots, N$ and $n = 0, \ldots, M-1$ (from our choice of the boundary condition in time, we know that $C(t_M, x_i) = g(x_i) := C^M(i)$). If we are working with Dirichlet conditions, we impose

$$C(t_n, x_0) = C(t_n, x_{N+1}) = a \qquad n < M$$

i.e.,

$$C^n(0) \;=\; C^n(N+1) \;=\; a \qquad n < M \;.$$

If we are working with Neumann conditions with $b = 0$, we take

$$C(t_n, x_{N+1}) \;=\; C(t_n, x_N) \qquad \text{and} \qquad C(t_n, x_0) \;=\; C(t_n, x_1)$$

i.e.,

$$C^n(N+1) \;=\; C^n(N) \qquad \text{and} \qquad C^n(0) = C^n(1)$$

(if $b \neq 0$ we take for example $C^n(N+1) = C^n(N) + bh$).

Next, we approximate

$$\begin{aligned}
&\frac{\partial C}{\partial t}(t_n, x_i) && \text{by} \quad \frac{C^{n+1}(i) - C^n(i)}{\varepsilon} && \text{(scheme 1)}\\
& && \text{or by} \quad \frac{C^n(i) - C^{n-1}(i)}{\varepsilon} && \text{(scheme 2)}\\
&\frac{\partial C}{\partial x}(t_n, x_i) && \text{by} \quad \frac{C^n(i+1) - C^n(i-1)}{2h} &&\\
&\frac{\partial^2 C}{\partial x^2}(t_n, x_i) && \text{by} \quad \frac{C^n(i+1) - 2C^n(i) + C^n(i-1)}{h^2} \;. &&
\end{aligned}$$

Exercise B.1.1. Why have we chosen these approximations?

B.1.2 The Implicit Scheme Case

By substituting the expressions above into the partial differential equation, we obtain in the case of scheme 1 (called the *implicit scheme*)

$$\begin{aligned}
\frac{C^{n+1}(i) - C^n(i)}{\varepsilon} \;=\; &-\frac{\sigma^2(n,i)}{2}\,\frac{C^n(i+1) - 2C^n(i) + C^n(i-1)}{h^2}\\
&-\mu(n,i)\,\frac{C^n(i+1) - C^n(i-1)}{2h} + r(n,i)C^n(i)
\end{aligned}$$

where $\sigma(n,i) = \sigma(t_n, x_i)$, $\mu(n,i) = \mu(t_n, x_i)$ and $r(n,i) = r(t_n, x_i)$. Hence C^n can be computed as a function of C^{n+1} (remember that it is C^M rather than C^0 that is known at the outset).

Let us carry through our analysis in the case where r, σ and μ depend only on the space variable x. We can write the previous equation in the matrix form:

$$\frac{1}{\varepsilon}(C^{n+1} - C^n) \;=\; AC^n \qquad \text{where} \quad C^n = C^n(i) \;,$$

and where the matrix A is a tridiagonal matrix.

In the case of Dirichlet conditions, $C^n(0)$ and $C^n(N+1)$ are known. It remains to determine $(C^n(i)\;,\; 1 \le i \le N)$, where C^n is a vector of $\mathbb{R}^N$. In the case $a = 0$, the matrix A has the form

$$A = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \\ c_2 & a_2 & b_2 & 0 & \\ 0 & c_3 & a_3 & b_3 & \\ & 0 & & \ddots & 0 \\ & & & c_N & a_N \end{bmatrix}$$

with

$$\begin{aligned} a_i &= \frac{\sigma^2(i)}{h^2} + r(i) \\ b_i &= -\frac{\sigma^2(i)}{2h^2} - \frac{\mu(i)}{2h} \\ c_i &= -\frac{\sigma^2(i)}{2h^2} + \frac{\mu(i)}{2h} \ . \end{aligned}$$

Exercise B.1.2. Modify A in order to study the case $a \neq 0$.

We obtain C^n as a function of C^{n+1} by solving the system $\frac{1}{\varepsilon}(C^{n+1} - C^n) = AC^n$.

In the case of Neumann conditions, we determine $C(n,i)$ for $1 \leq i \leq N$ from the equalities $C(n,0) = C(n,1)$ and $C(n,N+1) = C(n,N)$. Matrix A is written as

$$A = \begin{bmatrix} \alpha_1 & b_1 & 0 & & & \\ c_2 & a_2 & b_2 & & & \\ & c_3 & a_3 & b_3 & & \\ & \cdots & & & \cdots & \cdots \\ & & & & c_N & \alpha_N \end{bmatrix}$$

(only the first and last lines have changed) with

$$\alpha_1 = \frac{\sigma^2(1)}{2h^2} + \frac{\mu(1)}{2h} + r(1) \qquad \alpha_N = \frac{\sigma^2(N)}{2h^2} - \frac{\mu(N)}{2h} + r(N) \ .$$

Scheme 2 seems more straightforward. We can obtain C^{n-1} as a function of C^n using $\frac{1}{\varepsilon}(C^n - C^{n-1}) = AC^n$, so it is no longer necessary to solve a system. This scheme is called *explicit*, but it is not as efficient as scheme 1, for reasons of stability (see for example Ciarlet [60]). Let us return to scheme 1.

B.1.3 Solving the System

Solving the system $\frac{1}{\varepsilon}(C^{n+1} - C^n) = AC^n$ calls on methods for solving the equation

$$(I + \varepsilon A)C^n = C^{n+1}$$

where $I + \varepsilon A$ is a tridiagonal matrix. We can then proceed using the pivot method, which consists in writing $I + \varepsilon A$ as a product of two matrices $L\,U$, where L is upper triangular and U is lower triangular, and in solving $LU(X) = B$ in two steps:

- solve $LY = B$
- solve $UX = Y$.

We can also solve $(I+\varepsilon A)X = Y$ by iterative methods. These are based on the idea that $I+\varepsilon A$ can be written as $C-D$ where C and D are two matrices, with C being invertible (there are a number of possible decompositions).

We then need to solve $CX = Y + DX$. We construct a sequence of vectors $(U^n, n \geq 1)$ defined by recurrence for any fixed U^0, with

$$U^{n+1} \qquad \text{such that} \qquad CU^{n+1} = Y + DU^n .$$

We show that (if the spectral radius of $C^{-1}D$ is smaller than 1) the sequence U^n converges to X, the solution to $(I + \varepsilon A)X = Y$.

B.1.4 Other Schemes

We can use other schemes than schemes 1 and 2. Let us assume that $\mu = r = 0$ and that σ is constant, and describe some of the other possibilities. Scheme 1 is then written

$$\frac{C^{n+1} - C^n}{\varepsilon} = \frac{\sigma^2}{2} A_h C^n$$

where A_h is the operator $[A_h C]_i = -\frac{1}{h^2}\{C(i+1) - 2C(i) + C(i-1)\}$, whose matrix we already know.

We could use the *Crank-Nicholson* scheme:

$$\frac{C^{n+1} - C^n}{\varepsilon} = \frac{\sigma^2}{2} A_h(\theta C^{n+1} + (1-\theta)C^n)$$

(In the case $\theta = 1$, we are back to the implicit case of scheme 1, and the case $\theta = 0$ returns us to the explicit case).

The choice between the different schemes based on error estimation.

B.2 Extrapolation Methods

Let us assume that $\mu(t,x) = \mu x$ and $\sigma(t,x) = \sigma x$. We have seen how a change of variable can bring us back to an equation with constant coefficients. Assume further that $\mu = 1/2$, $\sigma = 1$ and $r = 0$ for our exposition of the method, and that $\Delta = 0$.

B.2.1 The Heat Equation

We want to approximate the solution of

$$\begin{cases} \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 C}{\partial x^2} = 0 \\ C(T,x) = g(x) , \end{cases} \tag{B.5}$$

with boundary conditions of either the Dirichlet or Neumann type. This equation is known as the *heat equation*. Note however that the heat equation is usually written as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(0,x) = g(x) . \end{cases} \qquad \left(\text{or in some cases without the coefficient of } \frac{1}{2} \right)$$

The two forms are equivalent under a change of the time variable.

B.2.2 Approximations

We start with a semi-discrete approximation, i.e., we discretize only the space variable. Thus we replace (B.5) with

$$\begin{cases} \dfrac{\partial C}{\partial t} + A_h C = 0 \\ C(T,x) = g(x) \end{cases} \tag{B.6}$$

where

$$A_h C = \frac{1}{2h^2}\{C(t,x+h) - 2C(t,x) + C(t,x-h)\}$$

and where A_h is a matrix operator. We are led to solve system (B.6) for $x = x_0, x_1, \ldots, x_{N+1}$. We can then apply methods that are specific to differential systems (e.g. Euler, Runge–Kutta).

The exact solution to (B.6) is

$$C(t) = (\exp -A_h t)\, C(0) . \tag{B.7}$$

[When the μ, σ and r coefficients depend on t, this formula is no longer valid, but other analogous methods can be employed.]

To approximate the solution to (B.7), we need to approximate

$$C(t+\Delta t) = (\exp(-A_h\, \Delta t))\; C(t) ,$$

and thus to approximate $\exp(-A_h\, \Delta t)$. We set $A_h = A$ and $\Delta t = \varepsilon$. An approximation of $e^{-\varepsilon A}$ is $(1+\varepsilon A)^{-1}$ where

$$C(t+\varepsilon) \simeq (1+\varepsilon A)^{-1} C(t) .$$

We can check that we thus recover the implicit scheme.

Another approximation of $e^{-\varepsilon A}$ is

$$e^{-\varepsilon A} \simeq \left(1+\frac{\varepsilon}{2}A\right)^{-1} \left(1-\frac{\varepsilon}{2}A\right)$$

(Padé's approximation). This leads to the Crank-Nicholson scheme.

We can also use mixed methods. Expanding the exponential $e^{-2\varepsilon A}$, we get

$$C(t+2\varepsilon) \simeq (1-2\varepsilon A+2\varepsilon^2 A^2)C(t) .$$

The implicit scheme leads to an approximate solution γ where

$$\begin{aligned}\gamma(t+2\varepsilon) &= (1+2\varepsilon A)^{-1} \gamma(t) \\ &\simeq (1-2\varepsilon A+4\varepsilon^2 A^2)\, \gamma(t) .\end{aligned}$$

If we apply the implicit scheme twice (to go from t to $t+\varepsilon$, and then from $t+\varepsilon$ to $t+2\varepsilon$), we obtain

$$\begin{aligned}\Gamma(t+2\varepsilon) &= (1+\varepsilon A)^{-2} \Gamma(t) \\ &\simeq (1-2\varepsilon A+3\varepsilon^2 A^2)\, \Gamma(t) .\end{aligned}$$

Hence the approximation of $(1-2\varepsilon A+2\varepsilon^2 A^2)C(t)$, by

$$2\Gamma(t+2\varepsilon)-\gamma(t+2\varepsilon) ,$$

which leads to the scheme

$$\begin{cases} C^{n+1/3} = (1+2\varepsilon A)^{-1} C^n \\ C^{n+2/3} = (1+2\varepsilon A)^{-2} C^n \\ C^{n+1} = 2C^{n+2/3} - C^{n+1/3} . \end{cases}$$

B.3 Simulation

In this section, we give a brief overview of simulation methods that can be used to approximate solutions to stochastic differential equations as well as the expectations of random variables.

B.3.1 Simulation of the Uniform Distribution on $[0, 1]$

The probability law of a random variable X that is uniformly distributed on $[0, 1]$ is defined by $P(X \in [a, b]) = b - a$ for $0 \leq a < b \leq 1$. A sequence of *"random numbers"* is a series of random variables $X_1, X_2, \ldots, X_n, \ldots$ that are independent, identically distributed, and have the same distribution as X. We would like to simulate this sequence, i.e., we would like to obtain a deterministic sequence of numbers in $[0, 1]$ which has "the same statistical properties" as the sequence $(X_n)_{n\geq 1}$. We do not dwell on methods for simulating these sequences of random numbers here. Most programming languages provide a "random" procedure for generating random numbers. Another approach is to use low discrepancy sequences.

We refer the interested reader to Bouleau [40], Niederreiter [289] and Ripley [310]. These provide various programming methods, as well as a discussion of the meaning of the expression "the same statistical properties".

B.3.2 Simulation of Discrete Variables

To simulate a random variable X, which can take k values $(a_1, a_2, \ldots, a_k)$ with probabilities $P(X = a_i) = p_i$, we can use the random variable

$$Z \ = \ a_1 \mathbf{1}_{U<p_1} + a_2 \mathbf{1}_{p_1 \leq U < p_1 + p_2} + \cdots + a_k \mathbf{1}_{p_1 + \cdots + p_{k-1} \leq U \leq 1}$$

where $\mathbf{1}_{\alpha \leq U < \beta}$ is worth 1 if $\alpha \leq U < \beta$, and 0 otherwise, and where U is a uniformly distributed random variable on $[0, 1]$, which can be simulated as outlined above.

B.3.3 Simulation of a Random Variable

Case of a Random Variable with a Continuous Density Function

Let X be a random variable with probability density function f, which is continuous. We denote by $F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t$ its cumulative distribution function. If f is strictly positive, F has an inverse mapping F^{-1}.

Exercise B.3.1. Show that, whatever the probability density function f, the variable $F(X)$ is uniformly distributed on $[0, 1]$. What is the distribution of $F^{-1}(U)$, if U is uniformly distributed on $[0, 1]$?

Show that if F is the cumulative distribution function of a random variable X (i.e., $F(x) = P(X < x)$) and if $F^{-}(y) = \inf\{x \,|\, y < F(x)\}$, then X has the same distribution as $F^{-}(U)$ where U is uniformly distributed on $[0, 1]$.

We can now simulate X by using $F^{-}(U)$. If we want to simulate $(X_1, X_2, \ldots, X_n)$ where the X_i are independent and identically distributed, we can use $(F^{-}(U_1), F^{-}(U_2), \ldots, F^{-}(U_n))$ where the $U_1, U_2, \ldots, U_n$ are independent random variables that are uniformly distributed on $[0, 1]$.

This method is often long, and requires a subroutine for calculating F^{-}. Therefore the accept/reject method is often used.

The Accept/Reject Method

Suppose that X is a random variable with a bounded continuous density function f with a compact support $[a, b]$.

Consider a pair of random variables (U, V) that are uniformly distributed on the rectangle $[a, b] \times [0, k]$. When the point with coordinates (U, V) is below the curve of f, we accept it, and set $X = U$. Otherwise, it is rejected and a new point is drawn at random. It is easy to check that the variable X thus defined, has probability density function f.

When the support of f is not contained in a compact set, this method is no longer valid, as there is no uniform distribution on an unbounded interval. We then use another density function g, such that

- the variable with probability density g is easy to simulate,
- $kg(x) \geq f(x)$ for a real constant k.

We then simulate a variable Y with density g and a variable U that is uniformly distributed on $[0, 1]$, and we set $Z = kUg(Y)$.

1. If $Z < f(Y)$ we set $X = Y$.
2. Otherwise, we simulate new Y and U, and go back to 1.

The Gaussian Case

Specific methods apply to this case.

Exercise B.3.2. Let U and V be two independent random variables that are uniformly distributed on $[0, 1]$. Show that

$$\begin{aligned} X &= (-2 \log U)^{1/2} \cos 2\pi U \\ Y &= (-2 \log U)^{1/2} \sin 2\pi V \end{aligned}$$

are independent random variables that have the standard normal distribution $N(0, 1)$.

The exercise immediately yields a simulation method. A normally distributed variable with mean m and variance σ^2 can be written $m + \sigma X$, where X follows the distribution $N(0, 1)$.

Further methods are to be found in Bouleau [40] and Ripley [310].

B.3.4 Simulation of an Expectation

Let X be a random variable. We would like to simulate $E(X)$.

Using the Simulation of *X*

If we have at our disposal a program for simulating independent random variables with the same distribution as X, we can simulate $E(X)$ by using the law of large numbers, i.e.,

$$E(X) = \lim_n \frac{1}{n} \sum_1^n X_i \,.$$

The same method can be used to simulate $E(\psi(X))$. The stopping criterion, which determines n in such a way as to get a small enough error, is obtained via the Bienaymé–Chebyshev inequality.

When the Density of *X* is Known

We suppose for the sake of simplicity that the density function f of X has support $[0,1]$. We need to calculate $\int_0^1 g(x)\mathrm{d}x$ with $g(x) = xf(x)$. The law of large numbers shows that if $x_1, x_2, \ldots, x_n$ is a sequence of numbers that are evenly spread on $[0,1]$ (i.e., simulating a sequence of random numbers), $\frac{1}{n}\sum_1^n g(x_i) \to \int_0^1 g(x)\mathrm{d}x$ (the measure $\frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ converges weakly to the Lebesgue measure).

Meanwhile, note that there are sequences $(x_1, \ldots, x_n)$ that converge faster by the method described above than when the x_i are chosen "randomly" and "independently". This is the case with the Van der Corput sequences[1].

B.3.5 Simulation of a Brownian Motion

Random Walks

Brownian motion can be approximated by a random walk[2], i.e., the distribution of B_t can be approximated by the distribution of $\frac{1}{\sqrt{n}}(X_1 + X_2 + \cdots + X_{[nt]}) = S_n$ where the X_i are independent and identically distributed random variables such that $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$, where $[\cdot]$ denotes the integer part of a number. We can then approximate $E(\psi(B_t))$ by $E(\psi(S_n))$ for any continuous bounded function ψ.

[1] See Bouleau [40] p. 228.
[2] See Appendix A.

Using Gaussian Variables

Another method involves using normal distributions: if $(X_i\,,\, i\ \leq n)$ are independent standard Gaussian variables, and if

$$\begin{aligned} S_0 &= 0 \\ S_{n+1} &= S_n + \delta X_n \qquad \text{where} \quad \delta \in \mathbb{R}^+ \end{aligned}$$

then $(S_0, S_1, \ldots, S_n)$ has the same distribution as $(B_0, B_\delta, \ldots, B_{n\delta})$.

B.3.6 Simulation of Solutions to Stochastic Differential Equations

Let X_t be the solution to the stochastic differential equation

$$\mathrm{d}X_t \;=\; \mu(t, X_t)\,\mathrm{d}t \;+ \sigma(t, X_t)\,\mathrm{d}B_t$$

where B_t is a d-dimensional Brownian motion.

When μ and σ are constant, the solution is $X_t = X_0 + \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right]$, and we can then simply simulate the Brownian motion. In the general case, we need to use approximation methods such as the following.

The Euler Scheme

We discretize the stochastic differential equation above, using a scheme of the form

$$\tilde{X}_{t_{k+1}} \;=\; f\big(\tilde{X}_{t_k}, B_{t_{k+1}}, B_{t_k}\big) \qquad k \in \{0, \ldots, N-1\}$$

where the t_k subdivide $[0, T]$ into steps of size $\Delta t = \dfrac{T}{N}$. The simplest scheme is the Euler scheme,

$$\begin{cases} \tilde{X}_{t_{k+1}} \;=\; \tilde{X}_{t_k} + \mu(t_k, \tilde{X}_{t_k})(t_{k+1} - t_k) + \sigma(t_k, \tilde{X}_{t_k})(B_{t_{k+1}} - B_{t_k}) \\ \quad \tilde{X}_0 \;=\; X_0\,. \end{cases}$$

We can show (Maruyama [266]) that this scheme converges on quadratic average to the solution of the stochastic differential equation, in the sense that

$$\exists\, C > 0\,,\ \forall\, k \in \{0, 1, \ldots, N-1\} \qquad E\big|X_{t_k} - \tilde{X}^N_{t_k}\big|^2 \;\leq\; C\Delta t\,.$$

Numerous schemes have been introduced to improve the speed of convergence. Moreover, other criteria of convergence can be used, for example convergence in L^p spaces, or a.s. convergence.

The Milshtein Scheme

Let us consider the one dimensional case where the coefficients μ and σ do not depend on time, and where σ is of class C^1. Using Taylor's expansion to approximate $\sigma(X_t)$, we obtain

$$\begin{aligned} X(t) \; &\simeq \mu(X(0))t + \sigma(X(0))\,(B_t - B_0) \\ &\quad + \sigma(X(0))\,\sigma^T X(0)) \int_0^t \{B(s) - B(0)\}\,\mathrm{d}B(s)\;. \end{aligned}$$

The stochastic integral is easy to evaluate (Exercice 3.1.12). This leads us to the Milshtein scheme

$$\begin{aligned} \overline{X}_{t_{k+1}} \;=\; \overline{X}_{t_k} &+ \mu(\overline{X}_{t_k})\,(t_{k+1} - t_k) \\ &+ \sigma(\overline{X}_{t_k})\,(B_{t_{k+1}} - B_{t_k}) \\ &+ \frac{1}{2}\sigma(\overline{X}_{t_k})\sigma^T(\overline{X}_{t_k})\left[B^2_{t_{k+1}} - B^2_{t_k} - (t_{k+1} - t_k)\right]\;. \end{aligned}$$

We can then show (Milstein [279], Talay [350]) that the scheme converges a.s. and on quadratic average, with greater speed than the Euler scheme.

In higher dimensions than 1, the Milshtein scheme requires restrictions on the matrix σ. The reader can refer to Talay [350], Pardoux and Talay [298] or to the books Kloeden and Platen [237] and Dupuis and Kushner [131].

B.3.7 Calculating $E(f(X_t))$

We would like to give approximations of the term $E(f(X_t))$, when the process X is a solution to a stochastic differential equation

$$\mathrm{d}X_t \;=\; \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t\;.$$

The coefficients μ and σ do not depend on t. Notice that this not a restriction. In the general case, it is enough to consider the process $Y_t = (t, X_t)$ and to write down the SDE satisfied by Y_t.

Calculating the Distribution of X_t

A first method for evaluating $E(f(X_t))$ consists in calculating the distribution of X_t explicitly.

If the coefficients μ and σ are regular, and if $X(0)$ has a density function p_0, then $X(t)$ has a density distribution $p(t,\cdot)$ that solves

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}p \;&=\; L^* p \\ p(0,\cdot) \;&=\; p_0 \end{aligned}$$

where L^* is the adjoint of $L = \sum b_i(x)\dfrac{\partial}{\partial x_i} + \dfrac{1}{2}\sum a_{i,j}(x)\dfrac{\partial^2}{\partial x_i\,\partial x_j}$ with $a = \sigma\sigma^T$, i.e., $L^*p = \sum \dfrac{\partial^2}{\partial x_i \partial x_j}(a_{i,j}\,p) + \sum \dfrac{\partial}{\partial x_i}(b_i\,p)$. We can try to solve this equation numerically, but it is difficult, particularly in spaces of higher dimensions.

The Euler and Milshtein Scheme

A second method consists in using a scheme (Euler's or Milshtein's) to simulate N independent occurrences $\overline{X}_t$ of X_t, which we denote $\overline{X}_t(\omega_i)$, and in calculating

$$\frac{1}{N}\sum_{i=1}^{N} f(\overline{X_t}(\omega_i))\ ,$$

for a t of the form $\dfrac{kT}{n}$. According to the law of large numbers, this provides an approximation of $E(f(\overline{X}_t))$.

We can then show that

$$|E(f(X_T)) - E(f(\overline{X}_T))| \ \leq\ C(T)\,\frac{T}{n}\ .$$

There also exist (Talay [352]) methods that lead to second order schemes.

The General Case

To approximate expressions of the form

$$E\biggl(\int_0^T \Delta(X_s)\,\mathrm{d}s + g(X_T)\biggr)\ ,$$

we can use the process

$$Y_t \ = \ \biggl(\int_0^t \Delta(X_s)\,\mathrm{d}s\ ,\ X_t\biggr)\ ,$$

and then write down the stochastic differential equation that it satisfies, and apply the methods covered in the previous subsection and described by Talay ([351] and [352]).

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