APPENDIX A

The Algebraic Riccati Equation

Consider the state-space model of the form

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \qquad (A.1)$$

where $x \in \Re^n$, $y \in \Re^m$, $u \in \Re^r$ and A, B, C, D, are time-invariant matrices of compatible sizes. This system is denoted (A,B,C,D) and in all further results, it is assumed to be a minimal realization of its transfer matrix:

$$G(s) \triangleq C(sI - A)^{-1}B + D \qquad (A.2)$$

The two particular Algebraic Riccati Equations (AREs) of interest in this work are the Generalized Control Algebraic Riccati Equation (GCARE):

$$(A - BS^{-1}D^*C)^*X + X(A - BS^{-1}D^*C) - XBS^{-1}B^*X + C^*R^{-1}C = 0 \qquad (A.3)$$

and the Generalized Filtering Algebraic Riccati Equation (GFARE):

$$(A - BD^*R^{-1}C)Z + Z(A - BD^*R^{-1}C)^* - ZC^*R^{-1}CZ + BS^{-1}B^* = 0 \qquad (A.4)$$

where

$$R \triangleq (I + DD^*) \qquad (A.5)$$

$$S \triangleq (I + D^*D) \qquad (A.6)$$

and by inspection $R^{-1} = I - DS^{-1}D^*, S^{-1} = I - D^*R^{-1}D$, $DS^{-1} = R^{-1}D$ and $DS = RD$.

Associated with these Riccati equations are the closed-loop control and filtering matrices, defined respectively as:

$$A^c \triangleq A + BF \qquad (A.7)$$

$$A^o \triangleq A + HC \tag{A.8}$$

Where F, the control gain, and H, the filter gain are defined:

$$F \triangleq -S^{-1}(D^*C + B^*X) \tag{A.9}$$

$$H \triangleq -(BD^* + ZC^*)R^{-1} \tag{A.10}$$

Noting that the controllability of $(A - BS^{-1}D^*C, BS^{-1/2})$ is uniquely implied by the controllability of (A, B) and that the observability of $(R^{-1/2}C, A - BD^*R^{-1}C)$ is uniquely implied by the observability of (C, A) (both can be shown by simple PBH tests), then the following theorems give sufficient (but not necessary) conditions for the existence and uniqueness of particular solutions to GCARE.

Theorem A.1 (Kalman, 1960)

If (A,B) is completely controllable, and (C,A) is completely observable, then there exists a unique solution, $X = X^* > 0$ to GCARE and the eigenvalues of A^c have strictly negative real parts.

Remark A.2

It should be noted that considerably weaker conditions would be sufficient to yield the solutions stated in Theorem A.1. The condition of minimality is assumed as this is compatible with assumptions made in the rest of the paper.

Remark A.3

Theorem A.1 can be applied directly to GFARE and equivalent results obtained, if the system (A, B, C, D) is replaced by (A^*, C^*, B^*, D^*), X replaced by Z, and hence A^c replaced by A^o.

It can also be shown that X and Z defined in (A.3), (A.4) respectively, solve:

$$(A-BS^{-1}D^*C)^*Z^{-1}+Z^{-1}(A-BS^{-1}D^*C)+Z^{-1}BS^{-1}B^*Z^{-1}-C^*R^{-1}C = 0 \tag{A.11}$$

$$(A - BD^*R^{-1}C)X^{-1} + X^{-1}(A - BD^*R^{-1}C)^* + X^{-1}C^*R^{-1}CX^{-1} - BS^{-1}B^* = 0 \tag{A.12}$$

It is also possible to relate the stabilizing solutions of GCARE and GFARE.

Theorem A.4 (Bucy,[2])

$$A^o = (I + ZX)A^c(I + ZX)^{-1} \tag{A.13}$$

$$(A^o)^* = (I + XZ)^{-1}(A^c)^*(I + XZ). \tag{A.14}$$

■

(These were proven by Bucy, (1972) for the case $D = 0$, but can be readily extended to the $D \neq 0$ case as well.)

Finally for completeness, the stabilizing solutions of GCARE and GFARE can be shown to satisfy the following related Lyapunov equations:

$$\begin{aligned} XA^c + (A^c)^*X &= -(C + DF)^*(C + DF) - F^*F \\ &= -C^*R^{-1}C - XBS^{-1}B^*X \end{aligned} \tag{A.15}$$

$$\begin{aligned} A^oZ + Z(A^o)^* &= -(B + HD)(B + HD)^* - HH^* \\ &= -BS^{-1}B^* - ZC^*R^{-1}CZ \end{aligned} \tag{A.16}$$

These are a direct result of (A.3) and (A.4). A further two Lyapunov equations can be obtained by combining (A.3) with (A.14) and (A.4) with (A.14):

$$(Z^{-1} + X)^{-1}(A^c)^* + A^c(Z^{-1} + X)^{-1} = -BS^{-1}B^* \tag{A.17}$$

$$(X^{-1} + Z)^{-1}A^o + (A^o)^*(X^{-1} + Z)^{-1} = -C^*R^{-1}C \tag{A.18}$$

APPENDIX B

Suboptimal Nehari Extensions

A state-space characterization will be derived here for all sub-optimal extensions of an unstable function that is constrained to satisfy an 'inner' requirement. We firstly state a more general result characterizing all sub-optimal extensions of any unstable function. This is derived from Glover, (1987).

Lemma B.1

All sub-optimal extensions of a function R, $R^* \in RH_\infty^{m\times p}$, of degree n, with state-space form $R = (A, B, C, D)$, given by

$$\|R + Q\|_\infty \leq \alpha$$

can be written $Q \in RH_\infty^{p\times m}$, where

$$Q = \mathcal{T}_V[\phi], \tag{B.1}$$

$\phi \in RH_\infty^{p\times m}$, $\|\phi\|_\infty \leq 1$, *and*

$$V = \left[\begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{21} & V_{22} \end{array}\right] = \left[\begin{array}{c|c|c} -A^* & W^{*-1}C^* & \alpha^{-1}W^{*-1}QB \\ \hline -(CP + DB^*) & I & -\alpha^{-1}D \\ \hline B^* & 0 & \alpha^{-1}I \end{array}\right] \tag{B.2}$$

where -Q (resp -P) is the controllability (resp) observability Gramian of R^* , and $W \triangleq (PQ - \alpha^2 I)$. ■

We now characterize all sub-optimal extensions of an unstable function R satisfying $R^*R = I$. (That is R^* is co-inner.)

Lemma B.2

Given a co-inner function $R^* \in RH_\infty^{m\times p}$, $m \leq p$, of degree n, with R having state-space realization $R = (A, B, C, D)$, then all transfer functions $Q \in RH_\infty^{p\times m}$, achieving

$$\|R + Q\|_\infty \leq \alpha \tag{B.3}$$

can be written

$$Q = \mathcal{T}_U[\Phi] \tag{B.4}$$

where Φ is an arbitrary transfer function constrained to satisfy $\Phi \in RH_\infty^{p\times m}$, $\|\Phi\|_\infty \le 1$, and

$$U = \left[\begin{array}{cc|c} U_{11} & -\alpha U_{12} & U_{12} \\ \hline U_{21} & I - \alpha U_{22} & U_{22} \end{array}\right] \tag{B.5}$$

with state-space form

$$\left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right] = \left[\begin{array}{c|c|c} -A^* & W^{*-1}C^*D_\perp & \alpha^{-1}W^{*-1}QB \\ \hline -(CP+DB^*) & D_\perp & -\alpha^{-1}D \\ \hline B^* & 0 & \alpha^{-1}I \end{array}\right] \tag{B.6}$$

and -Q (resp -P) is the controllability (resp observability) Gramian of R^*, $W \triangleq (PQ - \alpha^2 I)$, and $D_\perp$ is the unitary completion of D i.e. $[D_\perp, D]$ is square unitary.

Proof

Noting that in Lemma B.1 that ϕ is an *arbitrary* contraction, V_{11}, V_{21} can be postmultiplied by a unitary matrix without changing the parametrization in (B.1). Next note that $R^*R = I \Rightarrow D^*D = I$. Hence, if the unitary completion of D, $D_\perp$, is chosen, so that the matrix

$$S = [D_\perp, D] \tag{B.7}$$

is unitary, i.e. $SS^* = S^*S = I$, then a matrix U, where

$$U = \left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right] = \left[\begin{array}{c|c} V_{11}S & V_{12} \\ \hline V_{21}S & V_{22} \end{array}\right]$$

will also parametrize all Q in (B.1) and $\Phi = S^{-1}\phi$ again satisfies $\|\Phi\|_\infty \le 1$. Note from (B.2) that

$$\begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} S = \left[\begin{array}{c|cc} -A^* & W^{*-1}C^*D_\perp & W^{*-1}C^*D \\ \hline -(CP+DB^*) & D_\perp & D \\ B^* & 0 & 0 \end{array}\right]. \tag{B.8}$$

Further, R^* has a state-space realization given by $(-A^*, C^*, -B^*, D^*)$, and noting from Proposition 2.8 that for a co-inner function $C^*D = -QB$, then the result in (B.6) is immediate. ∎

APPENDIX C

Proof of Miscellaneous Results

1. Proof of Lemma 4.4

The proof is straightforward by noting the following Lemma due to Redheffer:

Lemma C.1 (Redheffer, 1960)

For $J, K \in RL_\infty$ with $\|J\|_\infty \leq \sigma$, $\|J_{22}K\|_\infty < 1$ then $\|\mathcal{F}_L(J,K)\|_\infty \leq \sigma$ if $\|K\|_\infty \leq \sigma^{-1}$. ■

Proof of Lemma 4.4: First, to prove (a), let F be defined as

$$F = \mathcal{F}_L(J, \begin{bmatrix} E_1 \\ E_2 \end{bmatrix})$$

where

$$J = \left[\begin{array}{c|cc} 0 & \alpha^{-1}I & 0 \\ \hline \sqrt{\alpha^{-2}-1}I & 0 & I \end{array}\right]$$

Noting that as $JJ^* = \alpha^{-2}I$ then

$$\|F\| \leq \alpha^{-1} \quad \text{(by Lemma C.1)}.$$

Now, by Definition 2.13,

$$F = \alpha^{-1}\sqrt{\alpha^{-2}-1}\; E_1(I-E_2)^{-1}$$

$$\Rightarrow \|E_1(I-E_2)^{-1}\|_\infty \leq \alpha(1-\alpha^2)^{-1/2}.$$

To prove (b), note that the selection $E_1 = (1-\alpha^2)F$, $E_2 = \alpha I$ demonstrates that for any $F \in RH_\infty$ satisfying $\|F\|_\infty \leq \alpha(1-\alpha^2)^{-1/2}$ there exists E_1, $E_2 \in RH_\infty$ such that $E_1(I-E_2)^{-1} = F$ satisfying

$$\left\| \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \right\|_\infty \leq \alpha.$$

■

2. Proof of Lemma 4.5

The proof of this Lemma uses a well known result from Hankel operator theory:

Lemma C.2 (Francis,1987,p70)

Let

$$\|R - X\|_\infty = \|R^*\|_H \triangleq \sigma_1(R^*) \tag{C.1}$$

where R^*, $X \in RH_\infty$. Then, there exists vectors $g(s)$ and $f(s) \in RH_2$, independent of X, such that

$$(R - X)g(s) = \sigma_1(R^*)f(-s) \tag{C.2}$$

■

Proof of Lemma 4.5:

(i) It is well known that $\|[\tilde{N}, \tilde{M}]\|_H \leq \|[\tilde{N}, \tilde{M}]\|_\infty = 1$.

(ii) Suppose $\|[\tilde{N}, \tilde{M}]\|_H = 1$ then from Lemma C.2, there exists $g(s)$,$f(s) \in RH_2$ such that

$$\begin{bmatrix} \tilde{N}^* \\ \tilde{M}^* \end{bmatrix} g(s) = f(-s) \tag{C.3}$$

(i.e. $R = \begin{bmatrix} \tilde{N}^* \\ \tilde{M}^* \end{bmatrix}$, and $X = 0$ in Lemma C.2)

But, one of the requirements for a coprime factorization (see Def. 2.15) is that there exists U, $V \in RH_\infty$:

$$U^*\tilde{N}^* + V^*\tilde{M}^* = I \tag{C.4}$$

premultiplying (C.3) by $[U^*, V^*]$ yields

$$g(s) = [U^*, V^*]f(-s)$$

which is a contradiction as the right hand side $\notin RH_2$ ■

3. Proof of Theorem 4.26

The proof here follows that of Opdenacker and Jonckheere, (1988, Theorem 2), where we extend the result of that theorem to the case where $D \neq 0$.

To begin, let $G \doteq (A, B, C, D)$ represent the passive system in question. Next we define the so called 'scattering system' associated with G by $S \doteq (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ where

$$\tilde{A} = A - B(I + D)^{-1}C \tag{C.5}$$

$$\tilde{B} = \sqrt{2}B(I + D)^{-1} \tag{C.6}$$

$$\tilde{C} = \sqrt{2}(I + D)^{-1}C \tag{C.7}$$

$$\tilde{D} = (D - I)(I + D)^{-1}. \tag{C.8}$$

Then by Opdenacker and Jonckheere, (1988), we have the following result.

Lemma C.3 (Opdenacker and Jonckheere, 1988)

The following statements are equivalent

(i) G is positive real (hence stable).

(ii) S is bounded real. i.e. S is asymptotically stable and $\|S\|_\infty \leq 1$.

We next state an elementary result for bounded real systems.

Lemma C.4

Let $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be a state-space realization of a bounded real system S, and let P (resp. Q) be the associated controllability (resp. observability) Gramian. Then P and Q solve

$$P\tilde{A}^* + \tilde{A}P + \tilde{B}\tilde{B}^* = 0 \tag{C.9}$$

$$Q\tilde{A} + \tilde{A}^*Q + \tilde{C}^*\tilde{C} = 0 \tag{C.10}$$

and

$$\lambda_{max}(PQ) \leq 1. \tag{C.11}$$

Proof:

This is immediate by noting that

$$\lambda_{max}(PQ) = \|S\|_H \leq \|S\|_\infty = 1.$$

∎

We now return to the proof of Theorem 4.31. Firstly, note that as S is bounded real (and hence asymptotically stable) we have from (C.5) that

$$Re[\lambda_i(\tilde{A})] = Re[\lambda_i(A - B(I + D)^{-1}C)] < 0$$

for all $i = 1, \ldots, n$. This implies that a state feedback of the form

$$F_s = -(I + D)^{-1}C \qquad (C.12)$$

applied to G would be stabilizing.

Next, following the proof in Opdenacker and Jonckheere, (1988), we consider the Linear Quadratic Regulator (LQR) problem of minimizing the cost criterion

$$\mathcal{J} = \int_0^\infty (y(t)^* y(t) + u(t)^* u(t))dt$$

subject to the conditions $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_o$, $y(t) = Cx(t) + Du(t)$. The optimal cost for this problem, denoted $\mathcal{J}_o$, is given by

$$\mathcal{J}_o = x_o^* X x_o, \qquad X > 0, \qquad (C.13)$$

where X solves GCARE in (A.3). By (A.15), this equation can be rewritten as

$$X(A + BF) + (A + BF)^* X + (C + DF)^*(C + DF) + F^* F = 0 \qquad (C.14)$$

and

$$F = -S^{-1}(D^* C + B^* X). \qquad (C.15)$$

We also note that, as F_s in (C.12) is a stabilizing feedback, by Kwakernnak and Sivan, (1972, p322), this can be shown to have an associated LQR cost,

$$\mathcal{J}_s = x_o^* X_s x_o \qquad (C.16)$$

where X_s is the positive definite solution to

$$X_s(A+BF_s)+(A+BF_s)^*X_s+(C+DF_s)^*(C+DF_s)+F_s^*F_s=0. \qquad (C.17)$$

By (C.12) we can rewrite (C.17) as

$$\begin{aligned}
&X_s(A-B(I+D)^{-1}C)+(A-B(I+D)^{-1}C)^*X_s \\
&+(C-D(I+D)^{-1}C)^*(C-D(I+D)^{-1}C)+C^*(I+D)^{*-1}(I+D)^{-1}C=0 \\
\Rightarrow &X_s\tilde{A}+\tilde{A}^*X_s+\tilde{C}^*\tilde{C}=0.
\end{aligned}$$

Hence comparing with (C.10), we can see that $X_s=Q$, the observability Gramian of the bounded real system S. Further, noting that as $\mathcal{J}_o \le \mathcal{J}_s$, we have that

$$X \le X_s. \qquad (C.18)$$

Similarly, if we were to apply an identical analysis to the dual LQR cost criterion

$$\mathcal{J}_1=\int_0^\infty (y_1(t)^*y_1(t)+u_1(t)^*u_1(t))dt$$

subject to the conditions $\dot{x}_1(t)=A^*x_1(t)+C^*u_1(t)$, $x_1(0)=x_{1_o}$, $y_1(t)=B^*x_1(t)+D^*u_1(t)$, it can be shown that the optimal cost is achieved using Z, the solution to GFARE, and that the inequality

$$Z \le Z_s \qquad (C.19)$$

holds, where $Z_s=P$ is the controllability Gramian of the bounded real system S.

Then, by Lemma C.4, (C.18), and (C.19) we have that

$$\mu_i \triangleq \lambda_i(XZ) \le \lambda_i(X_sZ_s) \le 1$$

for all $i=1,\ldots,n$. This completes the proof of Theorem 4.31. ■

APPENDIX D

State-Space Systems for Chapter 7

D.1 State-Space Matrices for Design Example 1

1. Controller for Design (1) - K_1

```
A=

-23.7320    1.0000  -23.7320         0
 -4.4097   -0.2308   -4.3847   -0.0212
 -8.4894         0   -8.4894    1.0000
-19.3781   -5.0461  -21.1999   -0.4724

B=

1.0e+04 *

4.7464
0.8750
1.6979
3.7242

C=

1.0000    6.6643    0.2780    0.6117

D=

0
```

2. Controller for Design (2) - K_2

```
A=

0    10.6591    -4.3091    -3.3354    -7.8802
0  -100.5513    72.1135   -73.5563  -299.0902
0     0.6352    -0.3807     1.0774     2.0607
0     0.3630    -0.6036    -0.1976    -0.7720
0     0.7674    -0.6002    -0.4013    -1.6053

B=

       0
384.8650
 -1.3706
  0.5944
  0.7300

C=

1.0e+03 *

0.0200    5.3296    -2.1545    -1.6677    -3.9401

D=

0
```

D.2 State-Space Matrices for Design Example 2

1. Nominal Space Platform Plant

A=

Columns 1 through 7

0	1.0000	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	1.0000	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	1.0000	0
0	0	0	0	-0.1187	-0.0069	0
0	0	0	0	0	0	0
0	0	0	0	0	0	-0.2819
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Columns 8 through 10

0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
1.0000	0	0
-0.0106	0	0
0	0	1.0000
0	-0.5805	-0.0152

B=

0	0	0	0
0.0017	-0.0010	-0.0131	0.0002
0	0	0	0
0.0018	0.0011	0.0064	0.0067
0	0	0	0
0.0013	0.0005	-0.0114	0.0088
0	0	0	0
0.0028	0.0003	0.0058	0.0038
0	0	0	0
0.0038	-0.0015	0.0025	-0.0063

C=

Columns 1 through 7

0.0017	0	0.0018	0	0.0013	0	0.0028
-0.0010	0	0.0011	0	0.0005	0	0.0003
0	0.0017	0	0.0018	0	0.0013	0
0	-0.0010	0	0.0011	0	0.0005	0
0	0.0017	0	0.0018	0	0.0064	0

Columns 8 through 10

0	0.0038	0
0	-0.0015	0
0.0028	0	0.0038
0.0003	0	-0.0015
-0.0006	0	-0.0055

D=

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

2. Perturbed Space Platform Plant

A=

Columns 1 through 7

0	1.0000	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	1.0000	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	1.0000	0
0	0	0	0	-0.1300	-0.0036	0
0	0	0	0	0	0	0
0	0	0	0	0	0	-0.2876
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Columns 8 through 10

0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
1.0000	0	0
-0.0054	0	0
0	0	1.0000
0	-0.6387	-0.0080

B=

0	0	0	0
0.0008	-0.0012	-0.0135	-0.0021
0	0	0	0
0.0021	0.0006	0.0011	0.0059
0	0	0	0
0.0016	0.0004	-0.0109	0.0086
0	0	0	0
0.0025	0.0004	0.0062	0.0037
0	0	0	0
0.0038	-0.0016	0.0033	-0.0064

C=

Columns 1 through 7

0.0008	0	0.0021	0	0.0016	0	0.0025
-0.0012	0	0.0006	0	0.0004	0	0.0004
0	0.0008	0	0.0021	0	0.0016	0
0	-0.0012	0	0.0006	0	0.0004	0
0	0.0008	0	0.0021	0	0.0060	0

Columns 8 through 10

0	0.0038	0
0	-0.0016	0
0.0025	0	0.0038
0.0004	0	-0.0016
-0.0005	0	-0.0058

D=

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

3. Controller (4 Plant Inputs) - K_3

```
A=

Columns 1 through 7
 -9.4152    -0.0402   -24.4362   -11.8265    -5.5490    17.7206    -4.3232
  0.0123    -1.2286    -0.2333    -0.8251     0.7558     0.7295    -1.2174
  0.0920     0.1171    -0.6752    -0.5095    -0.1901    -0.3531    -0.4163
 -0.0479     0.0953     0.4523    -0.1250    -0.3075    -0.0484    -0.1107
  0.0489    -0.0101     0.0403     0.2887    -0.0520     0.1410     0.0666
  0.1149     0.0338    -0.3611     0.1867    -0.2583    -0.3844    -0.0593
  0.0534     0.0096    -0.2121     0.1220    -0.0755    -0.3125    -0.1647
 -0.0143    -0.0931     0.3833     0.0731    -0.0177     0.0854     0.5616
  0.0126    -0.1144     0.0355     0.0983    -0.0121    -0.0149    -0.1109
  0.0599     0.0105    -0.0902     0.0560    -0.0531    -0.1986    -0.1346
       0          0          0          0          0          0          0
       0          0          0          0          0          0          0

Columns 8 through 12
-36.3849   -11.5446   -38.4824  -123.5177   150.2042
 -1.1755     3.2577     0.3765    -8.6696    -8.1468
  0.0363     0.0633    -0.3457    -0.7840     0.4844
 -0.3636     0.1915    -0.2169    -0.4882     0.0376
 -0.0206    -0.0550    -0.1206    -0.0121     0.2407
  0.4905     0.0650     0.0627     0.0933    -0.0898
 -0.3039     0.2274     0.0370    -0.1706    -0.0911
 -0.3865     0.3303    -0.3291    -0.2695     0.1559
 -0.3149    -0.2715     0.0743     0.1605     0.1634
  0.1532    -0.0528    -0.1880    -0.0843     0.0907
       0          0          0          0          0
       0          0          0          0          0

B=

 1.0e+05 *

 -5.1877     6.3086    -2.5018     1.9381    -0.7728
 -0.3641    -0.3422    -0.1462    -0.0769    -0.0784
 -0.0329     0.0204    -0.0094     0.0069     0.0573
 -0.0205     0.0016     0.0113     0.0036    -0.0068
 -0.0005     0.0101     0.0074    -0.0038     0.0048
  0.0039    -0.0038     0.0069    -0.0019     0.0303
 -0.0072    -0.0038    -0.0037     0.0004     0.0125
 -0.0113     0.0065     0.0031    -0.0045    -0.0155
  0.0067     0.0069    -0.0041    -0.0017    -0.0002
 -0.0035     0.0038     0.0023    -0.0013     0.0008
  0.0042          0          0          0          0
       0     0.0042          0          0          0

C=

Columns 1 through 7
 -0.0739    -0.1040    -0.0700     0.0521     0.0543     0.0518    -0.0236
  0.0219    -0.0113     0.0390     0.0027    -0.0165    -0.0061     0.0102
  0.1423    -0.0613    -0.1911     0.1201    -0.0701    -0.2236    -0.1392
 -0.0041    -0.1781     0.2375     0.0508    -0.0180     0.1030     0.0758

Columns 8 through 12
  0.0573    -0.0867     0.0503          0          0
 -0.0406    -0.0060    -0.0122          0          0
  0.0643    -0.0766    -0.0910          0          0
 -0.2206    -0.1179     0.0187          0          0

D=

     0     0     0     0     0
     0     0     0     0     0
     0     0     0     0     0
     0     0     0     0     0
```

4. Controller (2 Plant Inputs) - Design (1) - K_4

A=

Columns 1 through 7

-0.1979	-0.0092	-0.6512	-0.2228	-0.7871	-0.2439	-0.5546
-0.0471	-1.2110	4.4231	0.0570	-5.3421	-0.7464	-2.5997
-0.0218	0.3036	-1.4531	0.9250	1.1961	0.1351	0.5212
-0.0056	0.0305	-0.1853	-0.2451	0.0901	-0.0286	0.0343
-0.0478	-0.2973	0.9353	-0.0512	-1.6978	0.7367	-0.8882
-0.0087	-0.0324	0.0733	-0.0479	-0.2282	-0.2843	-0.1153
-0.0030	-0.0088	0.0165	-0.0040	-0.0627	-0.0110	-0.0352
-0.0025	-0.0021	-0.0131	-0.0629	-0.0388	-0.1666	-0.1363
-0.0060	-0.0037	-0.0267	-0.0108	-0.0627	-0.0149	-0.0408
-0.0053	-0.0017	-0.0441	-0.2014	-0.0664	-0.2672	-0.0318
-0.0050	0.0046	-0.0554	-0.0104	-0.0175	-0.0087	-0.0177
-0.0052	0.0054	-0.0903	-0.4532	-0.0285	-0.1493	-0.0154
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Columns 8 through 14

-0.4061	-1.1635	-0.2208	-1.5145	-0.0853	19.7945	0.9215
-1.4243	-1.9116	0.0094	6.4576	1.6477	4.7068	121.1035
0.2904	0.1149	-0.0728	-2.3217	-0.4959	2.1783	-30.3639
-0.0155	-0.0213	-0.0475	-0.2745	-0.1152	0.3939	-2.9432
-0.4780	-0.9259	-0.0674	1.1620	0.3844	4.7819	29.7327
-0.1207	-0.1321	-0.0559	0.0752	-0.0006	0.6884	3.1305
0.9734	-0.0464	0.0003	0.0088	0.0244	0.3023	0.8841
-0.0804	-0.0251	-0.0329	-0.0220	-0.0114	0.1193	0.1572
-0.0335	-0.0756	0.9945	-0.0729	0.0168	0.6039	0.3717
-0.0855	-0.3327	-0.0941	-0.0607	-0.1069	0.2530	0.1466
-0.0146	-0.0536	-0.0097	-0.1111	0.9919	0.5035	-0.4647
-0.0240	-0.0369	-0.1121	-0.6719	-0.2533	0.1380	-0.3850
0	0	0	0	0	0	0
0	0	0	0	0	0	0

B=

1.0e+05 *

0.8314	0.0387	0.0880	0.0015	0.0457
0.1977	5.0863	-0.0025	0.3726	0.1947
0.0915	-1.2753	0.0276	-0.0996	-0.0434
0.0165	-0.1236	0.0050	-0.0100	-0.0030
0.2008	1.2488	0.0267	0.0959	0.0620
0.0289	0.1315	0.0053	0.0104	0.0080
0.0127	0.0371	0.0004	0.0031	0.0039
0.0050	0.0066	0.0020	0.0009	0.0050
0.0254	0.0156	0.0027	0.0015	0.0046
0.0106	0.0062	0.0074	0.0009	0.0001
0.0212	-0.0195	0.0039	-0.0012	0.0016
0.0058	-0.0162	0.0095	-0.0044	-0.0108
0.0042	0	0	0	0
0	0.0042	0	0	0

C=

Columns 1 through 7

0.0100	0.0000	0.1265	0.7800	0.1340	0.8012	0.0509
0.0000	0.0100	-0.1011	-0.9858	0.1039	0.9723	0.0228

Columns 8 through 14

0.2574	0.0683	0.2206	0.1075	0.3007	0	0
0.1026	0.0048	0.0244	-0.0197	-0.1358	0	0

D=

0	0	0	0	0
0	0	0	0	0

5. Controller (2 Plant Inputs) - Design (2) - K_5

A=

Columns 1 through 7

-0.2152	0.0397	-0.9822	-0.1850	-0.6992	-0.1799	-0.5458
-0.0625	-4.8685	19.9674	0.4000	-21.8654	-1.7798	-10.3954
-0.0177	0.9271	-4.0923	0.8873	4.0239	0.3052	1.8581
-0.0032	0.0681	-0.3246	-0.1926	0.2787	-0.0094	0.1272
-0.0422	-0.8844	3.4874	0.0351	-4.2883	0.6319	-2.0952
-0.0053	-0.0691	0.2558	-0.0277	-0.3615	-0.2255	-0.1727
-0.0016	-0.0122	0.0413	-0.0005	-0.0666	-0.0063	-0.0342
-0.0011	-0.0030	0.0022	-0.0447	-0.0291	-0.1256	-0.1299
-0.0027	-0.0047	0.0009	-0.0028	-0.0417	-0.0054	-0.0247
-0.0024	-0.0023	-0.0170	-0.1528	-0.0412	-0.2039	-0.0159
-0.0022	0.0069	-0.0445	-0.0033	0.0143	-0.0006	0.0027
-0.0025	0.0072	-0.0694	-0.3551	0.0040	-0.1198	0.0040
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Columns 8 through 14

-0.3209	-1.3067	-0.1644	-2.1094	-0.0678	21.5200	-3.9688
-3.0814	-6.5633	0.0150	29.9580	3.9161	6.2502	486.8470
0.5330	0.9251	-0.0394	-6.3037	-0.7741	1.7674	-92.7067
0.0194	0.0514	-0.0214	-0.4832	-0.0876	0.2325	-6.7630
-0.6369	-1.5773	-0.0335	5.0642	0.7077	4.2199	88.4388
-0.0809	-0.1393	-0.0251	0.3697	0.0351	0.4344	6.8548
0.9873	-0.0332	-0.0002	0.0543	0.0131	0.1553	1.2203
-0.0404	-0.0116	-0.0149	0.0068	-0.0047	0.0430	0.2756
-0.0124	-0.0377	0.9989	-0.0132	0.0088	0.2716	0.4683
-0.0385	-0.3030	-0.0484	-0.0119	-0.0486	0.1003	0.2135
-0.0012	-0.0170	-0.0025	-0.0791	0.9943	0.2156	-0.6850
-0.0085	-0.0075	-0.0517	-0.6457	-0.1283	0.0545	-0.6439
0	0	0	0	0	0	0
0	0	0	0	0	0	0

B=

1.0e+06 *

0.0452	-0.0083	0.0034	-0.0005	0.0019
0.0131	1.0224	-0.0014	0.0510	0.0205
0.0037	-0.1947	0.0010	-0.0098	-0.0036
0.0005	-0.0142	0.0001	-0.0007	-0.0002
0.0089	0.1857	0.0005	0.0094	0.0042
0.0009	0.0144	0.0001	0.0007	0.0004
0.0003	0.0026	0.0000	0.0001	0.0001
0.0001	0.0006	0.0000	0.0000	0.0001
0.0006	0.0010	0.0000	0.0001	0.0001
0.0002	0.0004	0.0001	0.0000	0.0000
0.0005	-0.0014	0.0001	-0.0001	0.0000
0.0001	-0.0014	0.0002	-0.0001	-0.0003
0.0002	0	0	0	0
0	0.0002	0	0	0

C=

Columns 1 through 7

0.0100	0.0000	0.1489	1.2249	0.1583	1.2596	0.0407
0.0000	0.0100	-0.1235	-1.5542	0.1239	1.4991	0.0148

Columns 8 through 14

0.2379	0.0478	0.2110	0.0701	0.2915	0	0
0.0948	0.0019	0.0230	-0.0155	-0.1225	0	0

D=

0	0	0	0	0
0	0	0	0	0

D.3 State-Space Matrices for Design Example 3

1. Nominal Aircraft Plant

A=

0	0	1.1320	0	-1.0000
0	-0.0538	-0.1712	0	0.0705
0	0	0	1.0000	0
0	0.0485	0	-0.8556	-1.0130
0	-0.2909	0	1.0532	-0.6859

B=

0	0	0
-0.1200	1.0000	0
0	0	0
4.4190	0	-1.6650
1.5750	0	-0.0732

C=

1	0	0	0	0
0	1	0	0	0
0	0	1	0	0

D=

0	0	0
0	0	0
0	0	0

2. Controller for Design (1) - K_6

A=

1.0e+03 *

-0.1146	-0.0030	-0.0336	0	-0.0010
-0.0136	-0.0301	0.0069	0.0000	0.0005
-0.0347	0.0015	-0.3703	0.0010	0
-0.0712	0.0269	-3.5465	-0.0145	-0.0029
0.3405	0.0188	-1.0749	-0.0019	-0.0056

B=

1.0e+03 *

0.1146	0.0030	0.0347
0.0119	0.0180	-0.0061
0.0347	-0.0015	0.3703
0.1034	-0.0247	3.4411
-0.3191	-0.0182	1.0501

C=

-14.4891	-0.5998	14.6035	1.7076	3.5333
-0.0356	11.9282	0.8525	0.1936	-0.0152
-19.1328	-0.3213	-24.5604	-3.6594	8.2734

D=

0	0	0
0	0	0
0	0	0

3. Controller for Design (2) - K_7

```
A=

 Columns 1 through 7

-181.9097  106.9798   11.9018  191.6454  720.5882   32.2075  315.5145
  33.1798 -148.0843   19.9716 -450.1834 -237.7881    5.0502  117.0712
   0.2242    0.6350  -29.5511   -0.7973   -0.6308   -0.0031    0.1117
  -0.4844    3.6851   -0.3047   -8.3162   -3.2156    0.2453    0.6790
  -1.8054    1.9433   -0.5018   -3.6390   -5.5760   -0.2499   -0.3176
  -0.0382    0.0601   -0.2716   -0.1850   -0.2105   -0.1062   -0.0028
        0         0         0         0         0         0         0
        0         0         0         0         0         0         0
        0         0         0         0         0         0         0

 Columns 8 through 9

  12.5872  381.8756
   9.6405 -210.5051
  -8.4950    0.0588
  -0.0324   -0.8527
  -0.0421   -0.8143
  -0.0001   -0.0007
        0         0
        0         0
        0         0

B=

  1.0e+03 *

   7.5723    0.1510    9.1650
   2.8097    0.1157   -5.0521
   0.0027   -0.1019    0.0014
   0.0163   -0.0004   -0.0205
  -0.0076   -0.0005   -0.0195
  -0.0001    0.0000    0.0000
   0.0024         0         0
        0    0.0012         0
        0         0    0.0024

C=

 Columns 1 through 7

   0.1858   -1.2947    0.1808    0.7239    0.3055    0.0011         0
   0.0364   -0.0390   -2.0583    0.0394   -0.0136   -0.0019         0
  -0.8124    0.1899    0.0158    0.0135   -0.4994   -0.0010         0

 Columns 8 through 9

        0         0
        0         0
        0         0

D=

     0     0     0
     0     0     0
     0     0     0
```

REFERENCES

Al-Saggaf, U., and Franklin, G., (1988) 'Model reduction via balanced realizations: an extension and frequency weighting techniques', *IEEE Trans. Auto. Control*, 33:687-691.

Anderson, B. and Liu, Y. (1987) 'Controller reduction: concepts and approaches', *Plenary address, Proceedings of the American Control Conference, Minneapolis.*

Athans, M. (1986) 'A tutorial on the LQG/LTR method', *Proceedings of the American Control Conference, Seattle.*

Ball, J., and Ran, A. (1986) 'Hankel Norm Approximation of a rational matrix function in terms of its realization', in *Modelling, Identification and Robust Control,* (C.I. Byrnes and A. Lindquist, Eds.), North-Holland.

Bode, H. (1945) 'Network Analysis & Feedback Amplifier Design', *Van Nostrand, Princeton.*

Bucy, R. (1972) 'The Riccati equation and its bounds', *J. Comput. Systems Sci.*, 6:343-353.

Chen, M. and Desoer,C. (1982) 'Necessary and sufficient for robust stability of linear distributed feedback systems', *Int. J. Control*, 35:255-267.

Doyle, J. (1984) 'Lecture notes', *ONR/Honeywell Workshop on Advances in Multivariable Control, Honeywell Systems and Research Centre.*

Doyle, J. and Chu, C. (1986) 'Robust control of multivariable and large scale systems', *Final Technical Report, Honeywell Systems and Research Centre.*

Doyle, J., and Stein, G. (1979) 'Robustness with observers', *IEEE Trans. Auto. Control*, 24.4:607-611.

Doyle, J., and Stein, G. (1981) 'Multivariable feedback design: concepts for a classical/modern synthesis', *IEEE Trans. Auto. Control*, 26.1:4-16.

Doyle, J.,Glover, K., Khargonekar, P, Francis, B. (1988) 'State-space solutions to standard RH_2 and RH_∞ control problems', *Proceedings of the American Control Conference, Atlanta.*

Dym, H. (1988) 'J contractive matrix functions, reproducing kernel Hilbert spaces and interpolation', *CBMS Lecture Notes (draft), American Maths. Society.*

Enns, D. (1984) 'Model reduction for control system design', *PhD. Thesis, Stanford University.*

Foo, Y. and Postlethwaite, I. (1984) 'An H_∞ -minimax approach to the design of robust control systems', *Sys. and Contr. Letters.*, 5:81-88.

Francis, B. and Doyle, J. (1987) 'Linear control theory with an H_∞ optimality criterion.', *SIAM J. of Control & Opt.*, 25:815-844.

Francis, B. (1987) 'A Course in H_∞ Control Theory', *Springer Verlag, Berlin.*

Freudenberg, J., Looze, D. (1987) 'Frequency domain properties of scalar and multivariable feedback systems', *Springer Verlag, Berlin.*

Gantmacher, F. (1959) 'The Theory of Matrices', *Chelsea, New York.*

Glover, K. (1984) 'All optimal Hankel Norm Approximations of linear multivariable systems and their L_∞-error bounds', *Int. J. Control*, 39:1115-1193.

Glover, K. (1986a) 'Robust stabilization of linear multivariable systems: relations to approximation', *Int. J. Control*, 43:741-766.

Glover, K. (1986b) 'Multiplicative approximation of linear multivariable systems with L_∞ error bounds', *Proceedings of the American Control Conference, Seattle.*

Glover, K. (1987) 'Model reduction: a tutorial on Hankel Norm methods and lower bounds on L_2 errors', *10th IFAC Congress, Munich.*

Glover, K., and Doyle, J. (1988) 'State-space formulae for all stabilizing controllers that satisfy an H_∞ norm bound and relations to risk sensitivity' *Sys. and Contr. Letters, 11:167-172* .

Glover, K., and Mustafa, D. (1989) 'Derivation of the maximum entropy H_∞ controller and a state-space formula for its entropy', *Int. J. Control, to appear.*

Grimble, M. (1988) 'Optimal H_∞ multivariable robust controllers and the relationship to LQG design problems', *Int. J. Control*, 48.1:35-58.

Hinrichsen, D., and Pritchard, A. (1986) 'Stability radii of linear systems', *Sys. and Contr. Letters*, 7:1-10.

Horowitz, I. (1963) 'Synthesis of Feedback Systems', *Academic Press.*

Hung, Y.S. and MacFarlane, A.G.J. (1982) 'Multivariable feedback: a quasi-classical approach', *Springer Verlag, Berlin.*

Jonckheere, E. and Silverman, L.M. (1983) 'A new set of invariants for linear systems - applications to reduced order compensator design', *IEEE Transactions on Automatic Control.* 28.10:953-964.

Kalman, R., (1960) 'A new approach to linear filtering and prediction problems', *Journal of Basic Eng.*, 35-45.

Kasenally, E., and Limebeer, D. (1988) 'Closed formulae for a parametric mixed sensitivity problem', *(to appear Sys. and Contr. Letters).*

Khargonekar, P., Georgiou, T., Pascoal, A. (1987) 'On the robust stabilizability of linear time-invariant plants with unstructured uncertainty', *IEEE Trans. Auto. Control*, 32.3:201-207.

Khargonekar, P., Petersen, I., Zhou,K. (1987) 'Robust stabilization of uncertain systems and H_∞ optimal control', *(submitted for publication).*

Kimura, H. (1984) 'Robust stabilization for a class of transfer functions', *IEEE Trans. Auto. Control*, 29.9:788-793.

Kwakernaak, H. and Sivan, R. (1972) 'Linear Optimal Control Systems', *Wiley.*

Kwakernaak, H. (1985) 'Minimax frequency domain performance and robustness optimization of linear feedback systems', *IEEE Trans. Auto. Control*, 994-1004.

Latham, G. and Anderson, B. (1985) 'Frequency weighted optimal Hankel-Norm Approximation of stable transfer functions', *Sys. and Contr. Letters*, 5:229-236.

Limebeer, D. and Hung, S. (1987) 'An analysis of pole-zero cancellations in H_∞ optimal control problems of the first kind', *SIAM J. Control and Optimization*, 25.6:1457-1493.

Limebeer,D.J.N., Kasenally,E., (1986) 'H_∞ optimal control of a synchronous turbo-generator', *25th Conference on Decision and Control, Athens.*

Limebeer, D. and Halikias, G. (1988) 'A controller degree bound for H_∞ -optimal controllers of the second kind', *SIAM J. Control and Optimization*, 26.3:646-677.

Liu, L. and Anderson, B. (1986) 'Controller reduction via stable factorization and balancing', *Int. J. Control*, 507-531.

McFarlane, D., Glover, K., and Noton, M. (1988) 'Robust stabilization of a flexible space-platform: an H_∞ coprime factorization approach', *'Control 88',IEE Conference, Oxford.*

Maciejowski,J. (1989) 'Multivariable Feedback Design', (*to be published by Addison-Wesley*).

Meyer, D. (1987) 'Model reduction via fractional representations', PhD Thesis, *Stanford University.*

Meyer, D.G. (1988) 'A fractional approach to model reduction', *Proceedings 1988 American Control Conference, Atlanta.*

Meyer, D. and Franklin, G. (1987) 'A connection between normalized coprime factorizations and Linear Quadratic Regulator theory', *IEEE Trans. Auto. Control*, 32:227-228.

Moore B.C. (1981) 'Principal component analysis in linear systems: controllability, observability and model reduction', *IEEE Transactions on Automatic Control*, 26:17-32.

Nett, C., Jacobsen, C.A. and Balas, M.J. (1984) 'A connection between state-space and doubly coprime fractional representations.' *IEEE Transactions on Automatic Control*, 29:831-832.

Noton,M. (1987a) 'H_∞ control design for complex space structures', *Report TP9047 British Aerospace, Space and Communications Division, Bristol.*

Noton,M. (1987b) *Private communications.*

Ober, R. and McFarlane D. (1988) 'Balanced canonical forms for minimal systems: a normalized coprime factor approach', *(to appear in Linear Algebra and its Applications).*

Opdenacker, P. and Jonckheere, E. (1988) 'Characterization of passive systems through their closed-Loop LQG characteristic values', (*to appear IEEE Trans. Auto. Control).*

Pernebo, L. and Silverman, L.M., (1982), 'Model reduction via balanced state-space representations', *IEEE Transactions on Automatic Control*, 27:282-287.

Petersen, I. (1987b) 'A stabilization algorithm for a class of uncertain linear systems', *Sys. and Contr. Letters*, 8:351-357.

Postlethwaite,I., O'Young,S., Gu,D-W. (1987) 'H_∞ control system design: a critical assessment based on industrial applications', *IFAC 10th Conference Proceedings, Munich.*

Redheffer, M. (1960) 'On a certain linear fractional transformation', *J. Maths and Physics*, 39:269-286.

Rosenbrock, H. (1972) 'The stability of multivariable systems', *IEEE Trans. Auto. Control*, 17:105-107.

Rudin, W. (1986) 'Real and Complex Analysis', *McGraw-Hill, Singapore.*

Safonov, M. and Athans, M. (1977) 'Gain and phase margins of multiloop LQG regulators', *IEEE Trans. Auto. Control*, 22.2:173-179.

Safonov,M.,Chiang,R. (1988) 'CASCD using state-space L_∞ theory - a design example', *IEEE Trans. Auto. Control*, 33.5:477-479.

Safonov, M., Jonckheere, E., Verma, M., and Limebeer, D. (1987) 'Synthesis of positive real multivariable feedback systems', *Int. J. Control*, 45.3:817-842.

Safonov, M., Chiang, R. Limebeer, D. (1987) 'Hankel model reduction without balancing - a descriptor approach', in *Proc. IEEE Conf. Decision and Contr.*, Los Angeles.

Salehi,S. (1985) 'Application of adaptive observers to the control of flexible spacecraft', 10^{th} *IFAC Symposium, 'Automatic Control in Space', Toulouse.*

Stein, G., and Athans, M. (1987) 'The LQG/LTR procedure for multivariable feedback design', *IEEE Trans. Auto. Control*, 32.2.:105-114.

Tahk, M. and Speyer, J. (1987) 'Modelling of parameter variations and asymptotic LQG synthesis', *IEEE Trans. Auto. Control*, 32.9:793-801.

Verma, M., Helton, W. and Jonckheere, E. (1986) 'Robust stabilization of a family of plants with varying numbers of right half-plane poles', *Proceedings of the American Control Conference, Seattle, Washington*, 1827-1832.

Vidyasagar, M. (1984) 'The Graph Metric for unstable plants and Robustness estimates for feedback stability', *IEEE Trans. Auto. Control*, 29:403-417.

Vidyasagar, M. (1985) 'Control System Synthesis: A Coprime Factorization Approach.' *MIT Press.*

Vidyasagar, M. (1988) 'Normalized coprime factorizations for non strictly proper systems.' *IEEE Transactions on Automatic Control*, 33:300-301.

Vidyasagar, M. and Kimura, H. (1986) 'Robust controllers for uncertain linear multivariable systems', *Automatica*, 85-94.

Willems, J. (1971) 'The Analysis of Feedback Systems', *MIT Press.*

Willems, J. (1972) 'Dissipative dynamical systems, part II: linear systems with quadratic supply rates', *Arch. Rat. Mech. Analysis*, 45:352-393.

Yedavalli, R. (1985) 'Perturbation bounds for robust stability in linear state-space models', *Int. J. Control*, 42.6:1507-1517.

Zames,G. (1966) 'On input-output stability of time-varying nonlinear feedback systems, part 1', *IEEE Trans. Auto. Control*, 11.2:228-238.

Zames, G. (1981) 'Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms and approximate inverses', *IEEE Trans. Auto. Control*, 26.2:301-320.