APPENDIX E: EXAMPLES

In this appendix we shall give examples of some facts that are stated in the previous chapters. Many of the examples that will be discussed here are based on spectral realisations of delay equations. The reason for using this realisation is that on the one hand we have that these transfer functions are of a simple form and on the other hand we have from chapter IV a complete characterization of all controlled invariant subspaces. So we have the best of both worlds.

In the first section we shall recapitulate the theory of spectral realisations for delay systems as presented in Curtain & Zwart [10] and [12]. In the subsequent sections we shall give many (counter) examples of which we shall present a list.

Fact	Example
$\mathcal{V}^*(K)$ need not necessarily exist	E.9
$\mathcal{V}^*(K)$ may exists, and be unequal to $\mathcal{V}_{\Sigma}(K)$	E.10
$\mathcal{V}^*(K)$ may exists, and be unequal to $\mathcal{V}_{ol}(K)$	E.11
The closed sum of two controlled invariant subspaces need not remain controlled invariant	E.15

List of examples.

Section E.1: Spectral Realisations of Delay Equations.

The delay transfer functions that will be considered are of the form

$$(e.1) \qquad f(s) = \frac{p(s,e^{-s})}{q(s,e^{-s})}$$

where $p(x,y)$ and $q(x,y)$ are polynomials in two variables. Furthermore we shall assume that the lowest power of y in $q(x,y)$ is zero. With a polynomial in two variables we define the distribution diagram.

Definition E.1: Distribution Diagram

Let $q(s,e^{-s})$ by a polynomial in two variables, then we can rewrite it in the following form

$$q(s,e^{-s}) = \sum_{i=0}^{n} q_i(s)e^{-\beta_i s} \qquad (e.2)$$

where $q_i(s)$ is a polynomial of degree m_i and $0=\beta_0<\beta_1<..<\beta_n$.
With the points (β_i,m_i) we can define the distribution diagram of $q(.,.)$; this is the polygonal line L which is the upper boundary part of the convex hull of the points $(0,0)$, (β_i,m_i), $(\beta_n,0)$.

Since this definition is rather technical we shall give a simple example to illustrate it.

Example E.2.

Consider the function $q(s,e^{-s}):=s(s+e^{-s})$. Then $(\beta_0,m_0)=(0,2)$ and $(\beta_1,m_1)=(1,1)$. So the distribution diagram has the following form.

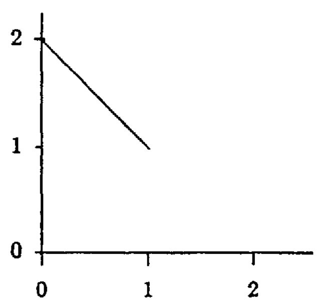

figure e.1.

□

The distribution diagram is closely related with the asymptotic distribution of the zeros of the function $q(s,e^{-s})$. To every slope of the distribution diagram there are infinitely many zeros of $q(s,e^{-s})$.

Lemma E.3.

Assume that there is no horizontal slope in the distribution diagram of the function $q(s,e^{-s})$ and assume further that on every segment of L there lie exactly two points. Let L_i be a line segment of the distribution diagram L with left end point (β_l,m_l) and right end point (β_r,m_r). So L has a kink at these points. Then the asymptotic distribution of the zeros related with this segment is given by $z_{n,j} = \mathcal{R}e(z_{n,j}) + i\mathcal{I}m(z_{n,j})$, where

$$(e.3) \quad \begin{cases} \mathcal{R}e(z_{n,j}) = m\left[log(|w_j|) - log(|2\pi\, n\, m + m\, arg(w_j) \mp m\pi/2|)\right] \\ \\ \mathcal{I}m(z_{n,j}) = m\left[2\pi\, n + arg(w_j) \mp \pi/2\right]; \quad n \in \mathbb{N},\ 0 < j \leq |m_l - m_r| \text{ and} \end{cases}$$

$m = \dfrac{m_l - m_r}{\beta_r - \beta_l}$; and w_j is j'th complex $m_l - m_r$'th root of $\dfrac{-p_{r,m_r}}{p_{l,m_l}}$; p_{i,m_i} is the leading coefficient in $p_i(s)$.

Proof:

See Bellman and Cooke [3, th. 12.10.]. □

The next lemma will play a key role in obtaining spectral factorizations of delay transfer functions. Before we can state this lemma we need the concept of vertical distance.

Definition E.4: Vertical Distance.

Let L denote the distribution diagram of the function $q(s, e^{-s})$. By the vertical distance of the pair (α, d) to the distribution diagram L we shall denote the number $\tau \in \mathbb{R}$ such that $(\alpha, d+\tau) \in L$. When such a number does not exist, then the vertical distance is by definition minus infinity.

Remark:

By the definition of L we have that τ is well defined.

For example if L is the distribution diagram of $q(s, e^{-s})$, where $q(.)$ is as in example E.2, then the vertical distance of $(0,2)$ to L is 0 and the vertical distance of $(2,0)$ is $-\infty$.

With the concept of vertical distance we can state the main result of Curtain and Zwart [10].

Lemma E.5.

Let $q(s,e^{-s})$ be the same as in $(e.2)$ and let L denote the distribution diagram of q. Suppose that on each segment of L lie exactly two points (β_i,m_i).

Let $p(s,e^{-s})=\sum_{i=1}^{n_p} p_i(s)e^{-\alpha_i s}$ with $p_i(s)$ a polynomial of degree d_i, and assume that p and q are coprime. If the vertical distance of the points (α_i,d_i) to L is larger than zero for $1\le i\le n_p$, then $f(s)=\dfrac{p(s,e^{-s})}{q(s,e^{-s})}$ has a partial fraction expansion, given by

$$(e.4)\qquad f(s)=\sum_{j=m+1}^{\infty}\frac{p(z_j)\,/\,q'(z_j)}{(s-z_j)}+\sum_{j=0}^{m}\sum_{i=0}^{n_j}\frac{c_{i,j}}{(s-z_j)^i},$$

where z_j are the zeros of $q(s)$ with multiplicity n_j, and this sum is uniformly convergent in s in any compact subset of $\mathbb{C}/\{z_j\}$.

Furthermore we have that $|p(z_j)/q'(z_j)|\le C|z_j|^{-\tau}$, where C is independent of z_j and τ is the minimal vertical distance of the points (α_i,d_i) to L.

Proof:

See Curtain and Zwart [10, p.73 and 74]. □

This theorem enables us to make spectral realisations of delay transfer functions. In order to improve the readability we shall list the required properties of $f(s)=\dfrac{p(s,e^{-s})}{q(s,e^{-s})}$.

Definition E.6: Property $\mathcal{P}$.

Let $q(s,e^{-s})=\sum_{i=0}^{n} q_i(s)e^{-\beta_i s}$, where $q_i(s)$ is a polynomial of degree m_i and $0=\beta_0<\beta_1<\ldots<\beta_n$ and let L denote the distribution diagram of q.

Let $p(s,e^{-s})=\sum_{i=1}^{n_p} p_i(s)e^{-\alpha_i s}$ with $p_i(s)$ a polynomial of degree d_i, and assume that p and q are coprime i.e. no common zeros. Then $f(s)=\dfrac{p(s,e^{-s})}{q(s,e^{-s})}$ satisfies property $\mathcal{P}$ if it satisfies $\mathcal{P}1$, $\mathcal{P}2$ and $\mathcal{P}3$, where

$\mathcal{P}1$. On each segment of L lie exactly two points (β_i,m_i).

$\mathcal{P}2$. If (β_l,m_l) and (β_r,m_r) are two succeeding points on L with $\beta_l<\beta_r$, then $m_l-m_r=1$.

$\mathcal{P}3$. Let τ_i denote the vertical distance of the point (α_i,d_i) to L. Then the minimum of these τ_i's is larger than one.

Note that $P2$ implies that there are no horizontal segments in L, and furthermore it implies that to every segment of L there is one string of zeros, see (e.3).
With the property P we can make a spectral realisation. In order to shorten the notation we shall assume that q has no multiple zeros. We stress that this assumption is made only to improve the readability. The same results as stated in the next theorem holds if this condition would be omitted.

Theorem E.7.

If $f(s)=\dfrac{p(s,e^{-s})}{q(s,e^{-s})}$ has property P and assume that all the poles of f have multiplicity one, (see the remark above), then there exists a spectral realisation (D,A,B) with state space ℓ^2 such that $f(s)=D(s-A)^{-1}B$, where

$$Ax=\sum_{n=1}^{\infty} z_n<x,e_n>e_n;$$

$$\mathrm{D}(A)=\{x\in\ell^2|\sum_n |z_n|^2|<x,e_n>|^2<\infty\};\ z_n \textit{ are the zeros of } q.$$

$$Bu=bu \textit{ and } Dx=<x,d>;\ b=d=\{\left(p(z_n)/q'(z_n)\right)^{1/2}\}$$

Furthermore this realisation has the following properties from chapter IV:

($\Delta 1$) The generator A is a discrete spectral operator.

($\Delta 2$) $b_i:=<b,e_i>\neq 0$, for all $i\geq 1$.

($\Delta 3$) For all $i\in\mathbb{N}$, $De_i\neq 0$.

($\Delta 4$) $\inf_{i\neq j}|z_i-z_j|=\delta>0$,

($\Delta 5$) $\sup_{i\in\mathbb{N}}\sum_{\substack{j=1\\ i\neq j}}^{\infty}\left|\dfrac{1}{z_i-z_j}\right|^2<\infty.$

We remark that $\sigma(A)=\{z_j\}$.

Proof:

From property $P3$ and lemma E.5 we have that $|p(z_n)/q'(z_n)|<Cn^{-\tau}$ with $\tau>1$. So b and d are elements of ℓ^2. That (D,A,B) is a spectral realisation of $f(s)$ follows now easily from lemma E.5. Now we shall prove that this realisation satisfies $\Delta 1$ up to $\Delta 5$.

The resolvent of A is given by

$$(\lambda-A)^{-1}=\sum_{n\in\mathbb{N}}\frac{1}{\lambda-z_n}<\cdot,e_n>e_n \qquad (e.5)$$

and from (e.3) we have that $|z_n|<c(n)$. Thus the resolvent is compact, and

so we have proved Δ1. Δ2 and Δ3 follow easily from the definition of b and d and the condition that p and q are coprime.

Furthermore we have from (e.3) that condition Δ4 holds and we have that fixed $i \in \mathbb{N}$ $|z_i - z_j| > c*j;\ i \neq j$ with $c>0$ independent of i. So Δ5 is also satisfied

□

Remark:

The properties Δ1, Δ4 and Δ5 are properties of the system matrix. So if b and d is another pair such that $f(s) = <(s-A)^{-1}b, d>$ and b satisfies Δ2 and d satisfies Δ3, then this realisation satisfies also Δ1 up to Δ5.

We shall illustrate this theorem by the following easy example.

Example E.8.

Consider the transfer function $f(s) = \dfrac{1}{4s(-4s+e^{-4s})}$. Then by theorem E.7 we have that the spectral realisation of f is given by

$$Ax = \sum_{n=1}^{\infty} z_n <x, e_n> e_n;$$

$$D(A) = \{x \in \ell^2 | \sum_n |z_n|^2 | <x, e_n> |^2 < \infty\};$$ z_n, $n>1$ are the zeros of $(-4s+e^{-4s})$;

$z_1 = 0$ and

$$b = d = \{1/2, 1/\left(16z_2(-1-4z_2)\right)^{1/2}, .., 1/\left(16z_j(-1-4z_j)\right)^{1/2}, ..\}.$$

Furthermore this realisation satisfies Δ1 up to Δ5, and so we have from theorem IV.16 a complete characterization of all controlled invariant subspaces in the kernel of D. Since the transfer function has no zeros we have from theorem IV.16 that the unique controlled invariant subspace in the kernel of D is the zero subspace. So we have for this realisation that $V^*(Ker\, D)$ exists and it is equal to the zero subspace. Note that we do not state anything about the more usual realisation in the space M^2.

□

In the next section we shall use these spectral realisations of delay equations in order to obtain counter examples for properties that hold if the state space is finite dimensional but do not alway hold for the infinite dimensional case.

Section E.2: The Relation between $V^*(K)$, $V_{ol}(K)$ and $V_\Sigma(K)$

The next example will show that $\mathcal{V}^*(K)$ need not exist.

Example E.9.

Let $\mathcal{H}$ be $L^2(0,1)$ and A is the "heat operator", $A = \frac{d^2}{dz^2}$ with domain, $D(A) = \{x \in L^2(0,1) | x'' \in L^2(0,1);\ x(0) = x(1) = 0\}$,

$$B := b(z) = \begin{cases} 0 & ;\ z \in [0, \frac{1}{2}] \\ \sin(2\pi z); & z \in [\frac{1}{2}, 1] \end{cases}$$

$$K = \{x \in L^2(0,1) |\ x(z) = 0;\ a.e.\ on\ [0, \tfrac{1}{2}]\}.$$

Notice that $K \supset Im\,B$. So $B^0(K) = \{0\}$, and from lemma II.19 we may conclude that, if K is controlled (or equivalently frequency) invariant, then every x in K has an unique (ξ, ω) representation with $B\omega(s) \equiv 0$, which means that $(s-A)^{-1}K \subset K$. Thus K would be $T_A(t)$-invariant, but this is in contradiction with example I.6. So K cannot be controlled invariant.
Define for $n > 1$

$$e_n(z) = \begin{cases} 0 & z \in [0, \frac{1}{2}] \\ \dfrac{1}{4\pi^2 n(-1)^n(n^2-1)} \{\sin(2\pi n\, z) + n(-1)^n \sin(2\pi\, z)\}; & z \in [\frac{1}{2}, 1] \end{cases},$$

and μ_n in $\mathbb{C}$ by $\mu_n = -4\pi^2 n^2$.

Then 1) $\underset{i=2..n}{\text{span}}\{e_i\}$ is controlled invariant for all $n \in \mathbb{N}$

2) $\overline{\underset{i \in \mathbb{N}/\{1\}}{\text{span}}}\{e_i\}$ is K.

Proof of 1):

By a simple calculation one can show that $(A - \mu_n)e_n = b$. So

$$(e.6) \qquad (s-A)e_n = -b + (s-\mu_n)e_n \ or\ e_n = (s-A)\frac{e_n}{s-\mu_n} - b\frac{1}{\mu_n - s}.$$

Thus span$\{e_n\}$ is controlled invariant. By lemma III.14 we have that $\underset{i=2..n}{\text{span}}\{e_i\}$ is controlled invariant for all $n \in \mathbb{N}$.

Proof of 2):

If $x \in K$, then $x = \begin{cases} 0 \text{ on } [0, \frac{1}{2}] \\ \tilde{x} \text{ on } [\frac{1}{2}, 1] \end{cases}$, with $\tilde{x} \in L^2(\frac{1}{2}, 1)$. So

$$\tilde{x}(z) = \sum_{n=1}^{\infty} <\tilde{x}(z), 2sin(2\pi n\, z)>_{L^2(\frac{1}{2},1)} 2sin(2\pi n\, z) \;\; and$$

$$\|\tilde{x}\|^2 = \sum_{n=1}^{\infty} |<\tilde{x}(z), 2sin(2\pi n\, z)>_{L^2(\frac{1}{2},1)}|^2 < \infty$$

Let $x \perp e_n \;\forall\; n>1 \Rightarrow <x,e_n>_{L^2(\frac{1}{2},1)} = 0 \quad \Rightarrow$

$<\tilde{x}(z), sin(2\pi n\, z) + n(-1)^n sin(2\pi\, z)>_{L^2(\frac{1}{2},1)} = 0$. So

$$(e.7) \qquad <\tilde{x}, sin(2\pi n\, z)>_{L^2(\frac{1}{2},1)} = -n(-1)^n <\tilde{x}, sin(2\pi\, z)>_{L^2(\frac{1}{2},1)} \;\forall\; n>1$$

$$(e.8) \qquad \infty > \|\tilde{x}\|^2 \geq \sum_{n=2}^{\infty} |<\tilde{x}, sin(2\pi n\, z)>_{L^2(\frac{1}{2},1)}|^2 = \sum_{n=2}^{\infty} n^2 |<\tilde{x}, sin(2\pi\, z)>_{L^2(\frac{1}{2},1)}|^2$$

(e.8) implies that $<\tilde{x}, sin(2\pi\, z)>_{L^2(\frac{1}{2},1)} = 0$, and with (e.7) we have that $<\tilde{x}, sin(2\pi n\, z)>_{L^2(\frac{1}{2},1)} = 0$, for all n in $\mathbb{N}/\{0\}$, so $\tilde{x} = 0$.

Thus $x \equiv 0$ is the only vector in K perpendicular on all e_n, so $\overline{\text{span}}_{i \in \mathbb{N}/\{1\}} \{e_i\}$ is K.

If $\mathcal{V}^*(K)$ were to exist, then it would necessarily be closed, contained in K and by 1) it must contain $\text{span}_{i=2..n} \{e_i\}$, for all $n \in \mathbb{N}$. This together with 2) would imply that $\mathcal{V}^*(K)$ equals K. This contradicts the fact that K is not controlled invariant. Thus $\mathcal{V}^*(K)$ cannot exist in this example. □

The next example will show that it is possible that $\mathcal{V}^*(K)$ exists but it is unequal to $\mathcal{V}_{\Sigma}(K)$. Note that theorem III.12 implies that $\mathcal{V}_{\Sigma}(K)$ cannot be closed then.

Example E.10.

In this example we shall study the delay transfer function as introduced in example E.8. So

$$(e.9) \qquad f(s) = \frac{1}{4s(-4s + e^{-4s})}$$

From example E.8 we have that this system has a spectral realisation (D,A,B) which satisfies the conditions $\Delta 1$ up to $\Delta 5$ and we have that D is given by $D=\langle .,d\rangle$ where $d=\{1/2,1/\left(16z_2(-1-4z_2)\right)^{1/2},.,1/\left(16z_j(-1-4z_j)\right)^{1/2},.\}$; z_j, $j\geq 2$ is the j–1 th zero of $-4s+e^{-4s}$. Furthermore we have proved that $\mathcal{V}^*(Ker\,D)$ is the zero subspace and thus by theorem III.3 the DDP is only solvable if $E=0$, i.e. no disturbances.

Now we shall show that there exists a bounded operator E and a strictly proper $\mathrm{U}(s)\in\left[\mathcal{L}(\mathbb{C},\mathbb{C})\right]_{-1}(s)$ such that

$$D(s-A)^{-1}B\mathrm{U}(s)=D(s-A)^{-1}E. \tag{e.10}$$

The existence of such a disturbance input operator will prove two facts. Firstly; since the D.D.P. is non solvable, but the meromorphic matrix equations (3.13), (3.14), and (3.15) are solvable, we have that the solvability of these equations is in general not equivalent to the solvability of DDP. So the condition that $\mathcal{V}_{\Sigma}(Ker\,D)$ is closed can not be omitted in theorem III.10. It easy to prove that the solvability of (3.13), (3.14) or (3.15) is always a necessary condition for the solvability of DDP.

Secondly, we have that $\mathcal{V}^*(Ker\,D)$ can exists, and it is unequal to $\mathcal{V}_{\Sigma}(Ker\,D)$. Namely from equation (e.10) we have that Eq is contained in $\mathcal{V}_{\Sigma}(Ker\,D)$, since

$$D(s-A)^{-1}Eq=D(s-A)^{-1}B\mathrm{U}(s)q,\ \ so \tag{e.11}$$

$$(s-A)^{-1}Eq=\xi(s)+(s-A)^{-1}B\mathrm{U}(s)q,\ \ with\ \ \xi(s)\in Ker\,D.$$

Thus

$$Eq=(s-A)\xi(s)+B\mathrm{U}(s)q \tag{e.12}$$

and by definition this shows that $Eq\in\mathcal{V}_{\Sigma}(Ker\,D)$, and thus $\{0\}\neq\mathcal{V}_{\Sigma}(Ker\,D)$. Now we shall define this operator E.

Let $E:\mathbb{C}\mapsto\ell^2$ be defined by $Eq=eq$, where

$$e=\{1/2,1/\left[2\left(4z_2\right)^{1/4}\left(-1-4z_2\right)^{1/2}\right],..,1/\left[2\left(z_j\right)^{1/4}\left(-1-4z_j\right)^{1/2}\right],..\} \tag{e.13}$$

From (e.3) we have that the j th component of e is of the order $j^{-\frac{3}{4}}$, so $e\in\ell^2$, and thus E is a bounded operator.

Furthermore it is from lemma E.5 easy to see that (D,A,E) is a realisation of the transfer function

$$D(s-A)^{-1}E = \frac{e^{-s}}{4s(-4s+e^{-4s})} \tag{e.14}$$

If we define U(s) as e^{-s}, then the operator E and the function U(s) satisfies equation (e.10) and so we have constructed the counter example. □

Similar to this example we shall construct an example such that $V^*(Ker\,D)$ exists, but it is unequal to the largest open loop invariant subspace in the kernel of D, $V_{ol}(Ker\,D)$.

Example E.11.

Again we shall consider the spectral realisation of the delay transfer function $f(s) = \frac{1}{4s(-4s+e^{-4s})}$. By example E.8 we have that the spectral realisation of f is given by

$$Ax = \sum_{n=1}^{\infty} z_n < x, e_n > e_n;$$

$\mathrm{D}(A) = \{x \in \ell^2 | \sum_n |z_n|^2 | < x, e_n > |^2 < \infty\}$; z_n, $n>1$ are the zeros of $(-4s+e^{-4s})$;

$z_1 = 0$ and

$$b = d = \{1/2, 1/\left(16z_2(-1-4z_2)\right)^{1/2}, .., 1/\left(16z_j(-1-4z_j)\right)^{1/2}, ..\}.$$

Now we shall construct an initial value $x_0 \in Ker\,D$ such that there exists a continuous input function $u(t)$ such that the solution of $\dot{x}(t) = Ax(t) + bu(t)$; $x(0) = x_0$ remains in the kernel of D. As x_0 we take

$$x_0 = \Big\{(1+2a)/2, \left(-(4z_2)^{1/4} - 2a(z_2)^{1/2} - a)\right) / \left((z_2^2-1)\left(16z_2(-1-4z_2)\right)^{1/2}\right), . \tag{e.14}$$

$$., \left(-(4z_j)^{1/4} - 2a(z_j)^{1/2} - a)\right) / \left((z_j^2-1)\left(16z_j(-1-4z_j)\right)^{1/2}\right), \ldots\Big\}$$

where

$$a = \frac{sinh(-1)}{sinh(2)} \tag{e.15}$$

and with $u(t)$ defined as

$$(e.16)\quad u(t)=\begin{cases} a\,sinh(t) & ;0\le t\le 1\\ sinh(t-1)+a\,sinh(t) & ;1\le t\le 2\\ 0 & ;t\ge 2\end{cases}$$

we claim that this input will keep x_0 in the kernel of D.

So we have to prove that

$$(e.17)\qquad 0=Dx(t)=DT_A(t)x_0+D\int_0^t T_A(t-s)Bu(s)ds;\ \forall t\ge 0.$$

First we remark that, by the definition of a, $u(.)$ is a continuous function on $[0,\infty)$. Furthermore the Laplace transform of $u(.)$ is given by

$$(e.18)\qquad \mathcal{L}(u)(s):=\omega(s)=\frac{e^{-s}+a\,e^{-2s}+a}{s^2-1}$$

Taking the Laplace transform of (e.17) yields

$$(e.19)\qquad 0=D(s-A)^{-1}x_0+D(s-A)^{-1}B\omega(s)$$

So from our realisation it thus remains to show that

$$(e.20)\qquad D(s-A)^{-1}x_0=-\frac{1}{4s(-4s+e^{-4s})}\frac{e^{-s}+a\,e^{-2s}+a}{s^2-1}.$$

Or, equivalently, we must prove that (D,A,x_0) is a realisation of

$$(e.21)\qquad g_a(s)=-\frac{e^{-s}+a\,e^{-2s}+a}{4s(s^2-1)(-4s+e^{-4s})}.$$

This result follows directly from the next lemma. Thus $x_0\in V_{ol}(Ker\,D)$, and so we have constructed our example.

□

Lemma E.12.

Consider the function $g_a(s)=\dfrac{-e^{-s}-a\,e^{-2s}-a}{4s(s^2-1)(-4s+e^{-4s})}$, with $a=\dfrac{sinh(-1)}{sinh(2)}$. Then on $\mathbb{C}$ the following equality holds: $g_a(s)=D(s-A)^{-1}x_0$, where D, A are as defined in example E.8 and x_0 is defined by (e.14).

Proof:

Since the numerator and the denominator have by the special value of a, common zeros we can not apply theorem E.7 directly. However if we define the sequence $\{a_n\}\subset\mathbb{R}$ such that $a_n\to a$ and $-e^{-s}-a_ne^{-2s}-a_n$ has no common zeros with the denominator of $g_a(.)$. Then we can construct a spectral realisation of

$$(e.22)\qquad g_{a_n}(s) := \frac{-e^{-s} - a_n e^{-2s} - a_n}{4s(s^2-1)(-4s+e^{-4s})}.$$

As state space for these realisation we choose $\ell^2(\{-1,0,1..\})$. This space is obviously isometric isomorph with $\ell^2(\mathbb{N})$, but we shall need this space for notational convenience.

From theorem E.7 we have that a realisation of $g_{a_n}(s)$ is given by

$$A^e x = \sum_{n=-1}^{\infty} z_n < x, e_n > e_n;$$

$$D(A) = \{x \in \ell^2(\{-1,0,..\}) | \sum_{-1}^{\infty} |z_n|^2 | < x, e_n > |^2 < \infty\};$$

$$z_{-1} = -1, z_0 = 1,\ z_1 = 0 \text{ and } z_n,\ n \geq 2 \text{ are the zeros of } (-4s + e^{-4s});$$

$$d^e = \{1, 1, 1/2, 1/\left(16z_2(-1-4z_2)\right)^{1/2}, .., 1/\left(16z_j(-1-4z_j)\right)^{1/2}, ..\}.$$

$$x^n = \left\{\frac{(e + a_n e^2 + a_n)}{8(4+e^4)}, \frac{(e^{-1} + a_n e^{-2} + a_n)}{8(4-e^{-4})}, \frac{(1+2a_n)}{2}, \frac{-(4z_2)^{1/4} - 2a_n(z_2)^{1/2} - a_n}{4(z_2^2-1)\left(z_2(-1-4z_2)\right)^{1/2}}, .. \right.$$
$$\left. .., \frac{-(4z_j)^{1/4} - 2a_n(z_j)^{1/2} - a_n}{(z_j^2-1)\left(16z_j(-1-4z_j)\right)^{1/2}}, .. \right\}$$

If n converges to infinity, then $g_{a_n}(s)$ converges for fixed $s \in \mathbb{C}$ to $-g_a(s)$. Furthermore it is easy to see that x^n converges to $x_0^e := \{0,0,x_0\}$ in the norm of $\ell^2(\{-1,0,1,..\})$. So for fixed $s \in \mathbb{C}$ $D^e(s-A^e)^{-1}x^n$ converges to $D^e(s-A^e)^{-1}x_0^e = D(s-A)^{-1}x_0$. So for fixed $s \in \mathbb{C}$ we have that $g_a(s) = D(s-A)^{-1}x_0$. □

Section E.3: On the Sum of two Controlled Invariant Subspaces

In this section we shall show by means of a counter example that the closure of the sum of two controlled invariant subspaces is not necessarily controlled invariant. In this example we shall use the spectral realisation of a delay equation as derived in section E.1. The delay equation that will be considered is given by

$$(e.23)\qquad F(s) = \frac{(se^{-s} + s^2 + 1)(se^{-s} + s^2 - 1)}{(s^4-1)(e^{-s}+s)(e^{-2s}+s)}$$

We shall prove that the spectral realisation of this equation satisfies the

conditions of chapter 4. Furthermore we shall prove that the subspace associated with the zeros of $se^{-s}+s^2+1$ as well as the subspace associated with the zeros of $se^{-s}+s^2-1$ is controlled invariant, (see theorem IV.16). However the sum of these subspaces will not remain controlled invariant, since to every pole of $F(s)$ associated with $(e^{-s}+s)$ there are two zeros of the system, see theorem IV.16.
Before we can prove our example we need some results on the distribution of the poles and zeros of $F(s)$.

Lemma E.13.

Let $h(.)$ and $\eta(.)$ satisfies on $\mathbb{C}$ the following relation

$$h(s)=h(s_0)+(s-s_0)h'(s_0)+(s-s_0)\eta(s). \qquad (e.24)$$

where $h'(s_0)\neq 0$.
Assume further that on the circle $C:=\{z:\ |z-s_0|=2\frac{|h(s_0)|}{|h'(s_0)|}\}$ the following inequality holds $|\eta(z)|<\frac{1}{2}|h'(s_0)|$. Then $h(.)$ has exactly one zero inside C.

Proof:

Define $f(s):=h(s)-(s-s_0)h'(s_0)$ and $g(s):=(s-s_0)h'(s_0)$. Then for $z\in C$ we have that

$$|g(z)|=|z-s_0|\,|h'(s_0)|=\tfrac{1}{2}|z-s_0|\,|h'(s_0)|+\tfrac{1}{2}|z-s_0|\,|h'(s_0)|=$$

$$=|h(s_0)|+\tfrac{1}{2}|z-s_0|\,|h'(s_0)|>|h(s_0)|+|z-s_0|\,|\eta(z)|>$$

$$|h(s_0)+(z-s_0)\eta(z)|=|f(z)|,$$ by $(e.24)$ and the definition of $f(.)$.

Now with Rouché theorem, Rudin [31] we have that $g(.)$ and $f+g=h$ have the same number of zeros inside C. So $h(.)$ has exactly one zero inside C. □

From this lemma we obtain the implication concerning the distance between the zeros and poles of $F(s)$.

Corollary E.14.

Let s_0 be a zero of $e^{-s}+s$ with norm sufficiently large. Then inside a circle with centre s_0 and radius $\frac{2}{|s_0(s_0+1)|}$ there is exactly one zero of $se^{-s}+s^2+1$.

Proof:

Defining $h(s):=se^{-s}+s^2+1$ and $\eta(s):=\dfrac{se^{-s}+s^2-(s-s_0)s_0(s_0+1)}{(s-s_0)}$ gives that this pair satisfies $(e.6)$. Furthermore we have from $(e.6)$ and the fact that h and η are entire function that

$$(e.25)\qquad \eta(s)=\sum_{k=2}^{\infty}\frac{h^{(k)}(s_0)}{k!}(s-s_0)^{k-1};\ s\in\mathbb{C}.$$

Calculating $h^{(n)}(s_0)$ gives; $h(s_0)=1$; $h'(s_0)=s_0(s_0+1)$; $h''(s_0)=-s_0^2+2s_0+2$ *and* $h^{(n)}(s_0)=n(-1)^n s_0+(-1)^{n+1}s_0^2$;$n>2$. So

$$(e.26)\qquad \eta(s)=\sum_{k=2}^{\infty}\frac{(-1)^{k+1}s_0^2}{k!}(s-s_0)^{k-1}+\sum_{k=2}^{\infty}\frac{k(-1)^k s_0}{k!}(s-s_0)^{k-1}+(s-s_0)=$$

$$s_0^2\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}(s-s_0)^k+s_0\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}(s-s_0)^k+(s-s_0).$$

With equation $(e.26)$ we have that

$$(e.27)\qquad |\eta(s)|\le|s_0|^2\sum_{k=1}^{\infty}\frac{|s-s_0|^k}{(k+1)!}+|s_0|\sum_{k=1}^{\infty}\frac{|s-s_0|^k}{k!}+|s-s_0|=$$

$$=|s-s_0||s_0|^2\sum_{k=0}^{\infty}\frac{|s-s_0|^k}{(k+2)!}+|s-s_0||s_0|\sum_{k=1}^{\infty}\frac{|s-s_0|^k}{(k+1)!}+|s-s_0|<$$

$$<|s-s_0|\left\{\tfrac{1}{2}|s_0|^2e^{|s-s_0|}+|s_0|e^{|s-s_0|}+1\right\}.$$

So on the circle $|z-s_0|=2\dfrac{|h(s_0)|}{|h'(s_0)|}=\dfrac{2}{|s_0(s_0+1)|}$ we have that

$$(e.28)\qquad |\eta(z)|<\frac{1}{|s_0(s_0+1)|}\left\{|s_0|^2e^{\frac{2}{|s_0(s_0+1)|}}+2|s_0|e^{\frac{2}{|s_0(s_0+1)|}}+2\right\}.$$

and for $|s_0|$ sufficiently large this is smaller than $\tfrac{1}{2}|s_0(s_0+1)|=\tfrac{1}{2}|h'(s_0)|$. So lemma E.13 gives the desired result. □

Remark:

With a similar proof one can show that the zeros of the function $se^{-s}+s^2-1$ have the same property.

Now we have proved the necessary lemmas to give a counter example for the fact that the closure of the sum of two controlled invariant subspace is not necessarily controlled invariant.

Example E.15.

Consider the delay transfer function

$$F(s)=\frac{(se^{-s}+s^2+1)(se^{-s}+s^2-1)}{(s^4-1)(e^{-s}+s)(e^{-2s}+s)}$$

We shall begin by constructing a spectral realisation of this transfer function. Let $\{z_j;j>0\}$ denote the zeros of $e^{-s}+s$, and $\{z_j;j<-3\}$ denote the zeros of $e^{-2s}+s$, and $z_0=1$, $z_{-1}=-1$, $z_{-2}=i$, $z_{-3}=-i$. So $\{z_j;\ j\in\mathbf{Z}\}$ are the poles of $F(s)$.

The spectral realisation of $F(s)$ will be constructed on $\ell^2(\mathbf{Z})$. Since this space is isometrically isomorph with $\ell^2(\mathbf{N})$ we have that a realisation on $\ell^2(\mathbf{Z})$ gives by renumbering of the indices also a realisation on $\ell^2(\mathbf{N})$. So the realisation on $\ell^2(\mathbf{Z})$ has similar properties as the realisation on $\ell^2(\mathbf{N})$ i.e. if the transfer function satisfies the conditions in theorem E.7., then the realisation on $\ell^2(\mathbf{Z})$ will satisfy $\Delta1,..,\Delta5$.

On $\ell^2(\mathbf{Z})$ we have the following realisation of $F(s)$;

$$Ax=\sum_{n\in\mathbf{Z}} z_n<x,e_n>e_n;$$

$$\mathrm{D}(A)=\{x\in\ell^2(\mathbf{Z})|\ \sum_{n\in\mathbf{Z}}|z_n|^2|<x,e_n>|^2<\infty\};$$

z_n, $j<-3$ are the zeros of $e^{-2s}+s$;

$z_{-3}=-i$, $z_{-2}=i$, $z_{-1}=-1$, $z_0=1$, and

z_n, $n\geq 1$ are the zeros of $(s+e^{-s})$;

$$b=\{b_j\}_{j\in\mathbf{Z}};\qquad b_j=\frac{1}{z_j+1};\ j\in\mathbf{N},$$

$$b_j=1;\ j=-3,..,0$$

$$b_j=\frac{1}{2z_j+1};\ j<-3.$$

$$D = <.,d> ;\ d = \{d_j\};\ d_j = \frac{-1}{(z_j^4-1)(z_j^2+z_j)}\ j \geq 1;$$

$$d_0 = \frac{(e^{-1}+2)(e^{-1})}{4(e^{-1}+1)(e^{-2}+1)},\ d_{-1} = \frac{(-e^1+2)(-e^1)}{-4(e^1-1)(e^2-1)},$$

$$d_{-2} = \frac{(ie^{-i})(ie^{-i}-2)}{4i(e^{-i}+i)(e^{-2i}+i)},\ d_{-3} = \frac{(-ie^i)(-ie^i-2)}{-4i(e^i-i)(e^{2i}-i)},$$

$$d_j = \frac{z_j^2\left[(-z_j)^{1/2}+z_j\right]^2-1}{(z_j^4-1)\left[(-z_j)^{1/2}+z_j\right]},\ j < -3.$$

From theorem E.7 we have that (D,A,b) is a spectral realisation of $F(.)$ and furthermore it has the properties $\Delta 1,..,\Delta 5$ from chapter IV. So we have in this realisation a complete characterization of all controlled invariant subspaces.

Now we shall construct two subspaces which are controlled invariant, but whose closed sum is no longer controlled invariant. From chapter IV we have that controlled invariant subspaces are closely related to the zeros of the transfer function. Corollary E.14 gives that outside a sufficiently large circle Γ there is with every zero of $(e^{-s}+s)$ exactly one zero of $se^{-s}+s^2+1$ and also one zero of $se^{-s}+s^2-1$. Let $\{\mu_j\}_{j\in\mathbb{N}}$ and $\{v_j\}_{j\in\mathbb{N}}$ denote these zeros of respectively $se^{-s}+s^2+1$ and $se^{-s}+s^2-1$. With these zeros we define the following closed subspaces

$$(e.29)\qquad V_1 = \overline{span}_{j\in\mathbb{N}}\{(\mu_j - A)^{-1}b\}$$

$$(e.30)\qquad V_2 = \overline{span}_{j\in\mathbb{N}}\{(v_j - A)^{-1}b\}$$

Now we shall show that V_1 and V_2 are controlled invariant. This we shall only do for V_1, the proof for V_2 is similar.

By definition IV.12 we have that V_1 is of the form (4.9). So V_1 satisfies condition $a)$ of theorem IV.16, furthermore it also satisfies condition $b)$. Thus it remains to prove the existence of a subsequence n_j such that $\sum_{j\in\mathbb{N}}\left[\frac{z_{n_j}-\mu_j}{b_{n_j}}\right]^2 < \infty$. As the sequence of poles we take those poles that are closest to the zeros. Then we have by definition of μ_j that this pole is z_{j+k}, k is the number of zeros of $e^{-s}+s$ inside the circle Γ and from corollary E.14 we have that $|z_{n_j}-\mu_j| = |z_{j+k}-\mu_j| < \frac{2}{|z_{j+k}(z_{j+k}+1)|}$.

So $\sum_{j\in\mathbb{N}}\left|\frac{z_{n_j}-\mu_j}{b_{n_j}}\right|^2=\sum_{j\in\mathbb{N}}\left|\frac{z_{j+k}-\mu_j}{b_{j+k}}\right|^2\leq\sum_{j\in\mathbb{N}}\left|\frac{\frac{2}{|z_{j+k}(z_{j+k}+1)|}}{\frac{1}{z_{j+k}+1}}\right|^2=$

$$\sum_{j\in\mathbb{N}}\left|\frac{2}{z_{j+k}}\right|^2<\infty, \text{ by } (e.3).$$

Thus V_1 is controlled invariant.

Now we shall prove that $\overline{V_1+V_2}$ is not controlled invariant.

As a consequence of theorem IV.16, Δ4 and the fact that b_{n_j} converges to zero if j goes to infinity, we have that outside a sufficiently large circle to every pole of $F(s)$ there is one or none zero of the controlled invariant subspace. So if $V:=\overline{V_1+V_2}$ were to be controlled invariant, then there would exists to the poles z_j; $j\in\mathbb{N}$, for j sufficiently large, one or no zero. However we have that to every such pole there exists a zero of $se^{-s}+s^2+1$ and one of $se^{-s}+s^2-1$. Thus V cannot be controlled invariant. □

CONCLUSIONS

In this monograph we have presented a fundamental treatment of the geometric theory for infinite dimensional systems in Banach spaces. We have proved that for closed subspaces the concepts of open loop invariance, closed loop invariance and frequency invariance are equivalent. The condition that the subspace must be closed is really essential; there are examples of open loop invariant subspaces which are not closed loop invariant, and the same holds for frequency invariance. There is another fundamental difference between the concepts of frequency invariance, open loop invariance and the concept of closed loop invariance. Namely the largest frequency invariant subspace contained in a given closed subspace K, $\mathcal{V}_{\Sigma}(K)$, and the largest open loop invariant subspace contained in K, $\mathcal{V}_{ol}(K)$ both exist, but in general there may not exist a largest closed loop invariant subspace contained in K, $\mathcal{V}^{*}(K)$. There can concur even stranger things. It has been shown by means of an example that it is possible that both the largest closed loop invariant and the largest frequency invariant subspace exist, but are unequal. This is very unsatisfactory, since an easy test for the solvability of the DDP in terms of the transfer functions from the input to the measurement and from the disturbance to the measurement, which was valid in the finite dimensional case, loses it validity for infinite dimensional systems. This test is essentially based on the concept of (ξ,ω)-representation. This concept, although introduced by Hautus [19] for finite dimensional systems, is of great use for the infinite dimensional case. For example with this concept we proved the equivalence between open loop and closed loop invariance, but it can also be used to obtain strong results in the theory of stabilizability, Zwart [49].

As we have seen, if the largest frequency invariant subspace, $\mathcal{V}_{\Sigma}(K)$, equals the largest closed loop invariant subspace, $\mathcal{V}^{*}(K)$, then the theorems form a complete analogy to the finite dimensional case. So the equality between $\mathcal{V}_{\Sigma}(K)$ and $\mathcal{V}^{*}(K)$ is essential. For applications it is however very hard to check whether $\mathcal{V}_{\Sigma}(K)$ equals $\mathcal{V}^{*}(K)$. In chapter four we have focussed on the existence of $\mathcal{V}^{*}(K)$ for the class of discrete spectral systems on a Hilbert space. There we proved that the existence of this subspace is equivalent to a pole placement problem, where the new poles must be placed on the zeros of the system. However since the existence of $\mathcal{V}^{*}(K)$ does not imply that this subspace is equal to $\mathcal{V}_{\Sigma}(K)$, it is again hard to show that the DDP is solvable if this pole placement problem is solvable. From all these results we see that there is a gap between the nice mathematical theory and the applications. One could think of several ways to investigate the solvability of the DDP in practical applications.

One way of attacking this problem would be to approximate the given system by a finite dimensional one, for which one solves the disturbance decoupling problem, and then to use this solution as an approximate solution for the general system. By the non-robustness of the disturbance decoupling problem it is not clear that this approach should work. The non-robustness of the DDP can easily be seen from the following remark. For every positive number ε there exists a D_ε such that $\|D-D_\varepsilon\|<\varepsilon$ and $D_\varepsilon E\neq 0$. So by making only a small change in norm one can violate the necessary condition for the solvability of the DDP.

A better approach to the general solvability of the DDP would be to investigate its almost version as in Trentelman [39] in the hope that this solution will be more robust. In the infinite dimensional case we also covered the special case that the generator A is bounded. For continuous time systems the condition that A is bounded is too strong to include interesting examples. However if one considers discrete time systems, then most generators will be bounded. It can easily be deduced from the results in this monograph that for this case the geometric theory is completely analogous to the finite dimensional case. One can state this as "the geometric theory is a theory for bounded operators". The "almost" version of the DDP for continuous time systems is still completely open, however.

So we still have many open problems if we want to solve the DDP for applications. However since we are working in an infinite dimensional state space the assumption in the DDP that the whole state is measured is not very reasonable. In most applications the output will be a part of the state, and this brings us to the problem of DDPM or DDPMS if we also want to guarrantee internal stability. For a system defined in a Hilbert space we have derived necessary and sufficient conditions in terms of system-invariance subspaces for the solvability of the DDPM and the DDPMS with a finite dimensional compensator In these conditions we did not needed the condition that $\mathcal{V}_{\Sigma}(K)$ equals $\mathcal{V}^*(K)$, but this only holds for the case that the to-be-constructed compensator is finite dimensional. Furthermore these results are (still) not applicable in a general Banach space.

Concluding we can see that at this moment it is not clear what one can do about general disturbance decoupling problems for systems for which the equality between $\mathcal{V}_{\Sigma}(K)$ and $\mathcal{V}^{*}(K)$ is unknown.

The equality between $\mathcal{V}_{\Sigma}(K)$ and $\mathcal{V}^{*}(K)$ holds if $\mathcal{V}_{\Sigma}(K)$ is closed. From examples we have seen that it is possible that $\mathcal{V}_{\Sigma}(K)$ is not closed, but this property is realisation dependent. This motivates the following (still) open problem: under what conditions does the transfer function $f(s)$ have a realisation (D,A,B) such that $f(s)=D(s-A)^{-1}B$ and $\mathcal{V}_{\Sigma}(Ker\, D)$ is closed. In order to solve this one would probably need the concepts of open loop invariance, closed loop invariance and frequency invariance for more general systems, for instance with an unbounded input operator B. However there is reason to believe that the following conjecture will hold; let (A,B) be a general systems with possibly unbounded input operator B, and let the feedback laws F be elements of some class of linear operators such that $A+BF$ generates a C_0-semigroup, then the largest closed loop invariant subspace, with respect to our class of feedback operators, does not necessarily exist.

So geometric theory for infinite dimensional systems has brought and will continue to produce some nice mathematical and system theoretic results, but it will be very hard to apply these results to applications.

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LIST OF ALL INVARIANCE CONCEPTS AND THEIR RELATION

Definitions:

A–invariant	$A(V \cap D(A)) \subset V$.
(A,B)–invariant	$A(V \cap D(A)) \subset V + Im\, B$.
closed loop invariant	there exists a bounded F such that $T_{A+BF}(t)V \subset V$ $\forall t \geq 0$.
conditioned invariant	there exists a bounded G such that $T_{A+GC}(t)V \subset V$ $\forall t \geq 0$.
controlled invariant	closed loop invariant = open loop invariant = frequency invariant for closed subspaces V.
feedback invariant	there exists an A–bounded F such that $(A+BF)(V \cap D(A)) \subset V$.
frequency invariant	For every $x_0 \in V$ there exist $\xi(.) \in D(A)(s)$ and $\omega(.) \in \mathcal{U}_{-1}(s)$ such that $x_0 = (s-A)\xi(s) - B\omega(s)$.
generator invariant	A–invariant.
open loop invariant	For every $x_0 \in V$ there exists a continuous $u(.)$ such that the solution of $\dot{x}(.) = Ax(.) + Bu(.)$; $x(0) = x_0$ remains in V.
semigroup invariant	$T_A(t)$–invariant.
$T_A(t)$–invariant	$T_A(t)V \subset V$; $\forall t \geq 0$.

Relations:

For a closed subspace V we have that the following concepts are equivalent:

i) closed loop invariance.

ii) open loop invariance.

iii) frequency invariance.

iv) controlled invariance.

Furthermore the following two concepts are also equivalent for a closed subspace V.

v) (A,B)–invariance.

vi) feedback invariance.

Note: For a closed subspace V *i)* always implies *v)*, but the converse does not hold in general.

NOTATION

A	The system operator, which is assumed to generate the C_0–semigroup $T_A(t)$.
B	The input operator, which is assumed to be bounded and to have finite dimensional range.sional range.
$B^0(V)$	A subspace of $Im\,B$ which has zero intersection with V and is of maximal dimension under this restriction. restriction.
$B^1(V)$	$Im\,B \cap V$.
C	An output operator, which is assumed to be bounded.
D	An output operator, which is assumed to be bounded.
$D(A)$	The domain of the operator A.
E	The disturbance input operator, which is assumed to be bounded.
F	A feedback operator from the state space to the input space.
G	A feedback operator from an output space to the state space.
$H^2(V)$	The space of Hardy functions with values in V. V is assumed to be a closed subspace of a Hilbert space.
$\mathcal{H}$	The state space, if this state space is a Hilbert space.
i_V	The inclusion map, i.e. if $V \subset \mathcal{X}$, then $i_V : V \mapsto \mathcal{X}$ and $i_V(x) = x$.
$Im\,B$	The image of the operator B.
$\mathcal{I}m(z)$	The imaginary part of the complex number z.
K	This is the feed through term in the compensator, or an arbitrary closed subset of the state space. The meaning will be clear from the context.
$Ker\,D$	The kernel of the operator D.
L	The output operator from the compensator, or the distribution diagram (only in appendix E).
$L^2([a,b);Q)$	The space of all square integrable functions from $[a,b)$ to Q.
$\mathcal{L}$	The Laplace transform.
$\mathcal{L}(\mathcal{X})$	The space consisting of all bounded linear operators from $\mathcal{X}$ to $\mathcal{X}$.
$\mathcal{L}(\mathcal{X},\mathcal{Z})$	The space consisting of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Z}$.
M	The input operator in the compensator.

N	The system operator for the compensator, which is assumed to generate a C_0-semigroup.
(N,M,L,K)	The compensator.
P	A projection operator.
P_V	The projection operator on V.
Q	An arbitrary operator.
$\mathcal{Q}$	The disturbance input space.
$\mathcal{R}e(z)$	The real part of the complex number z.
S	A subspace of the state space.
$S^*(Im\,E)$	The smallest conditioned invariant subspace that containes $Im\,E$
$T_A(t)$	The semigroup generated by A.
$\mathcal{U}$	The controlled input space.
V	A subspace of the state space.
$\mathcal{V}^*(K)$	The largest controlled invariant subspace contained in the closed subspace K.
$\mathcal{V}_{ol}(K)$	The largest open loop invariant subspace contained in the closed subspace K.
$\mathcal{V}_{\Sigma}(K)$	The largest frequency invariant subspace contained in the closed subspace K.
$\mathcal{V}_{(A,B)}(K)$	$\mathcal{V}_{\Sigma}(K)$ for the system (A,B).
W	The state space of the compensator.
$\mathcal{X}$	The state space of the system if this state space is an arbitrary Banach space.
$\mathcal{Y}$	The output space.
$Y(s)$	The space consisting of all function $f(.)$ from $\mathbb{R}$ to Y which are continuous on some interval of the from $[r,\infty)$.
$Y_{-1}(s)$	$\{f \in Y(s) \mid \lim_{s\to\infty} sf(s) \text{ exists}\}$
$\mathcal{Z}$	The output space of the to be controlled output.
Ξ_s	All states that can be reached at frequency s, under the condition that the state trajectory has to stay in V.
Ξ	Ξ_s for s sufficiently large.
$*$	The * will denote the Hilbert space dual of an operator. The ' will denote the Banach space dual.
$\mathbb{1}_{[a,b]}$	The indicator function of the interval $[a,b]$.