Appendix A

Averaging quasilinear systems with several fast variables

A.1 Linear equation with coefficients given by asymptotical series

Let us consider a system

$$\dot{y} = A(x)y + f(x,t) \quad (\dot{y} = dy/dt), \tag{A.1.1}$$

where y denotes the sought vector of dimension m, A is a $m \times m$ matrix, x is a vector of dimension n which is a function of fast time t, slow time $\tau = \varepsilon t$ and parameter ε. Vector x is assumed to be given by the asymptotic series

$$x \sim \xi(\tau) + \varepsilon u_1(\tau,t) + \varepsilon^2 u_2(\tau,t) + \dots \tag{A.1.2}$$

with the prescribed coefficients $\xi(\tau)$ and $u_i(\tau,t)$.

By definition of the asymptotic series for $t \in T_\varepsilon$, $T_\varepsilon = [t_0, t_0 + L/\varepsilon]$, $0 < \varepsilon \le \varepsilon_0$, the following inequality $|x - x^{(k-1)}| \le \varepsilon^k C_k$ holds where $C_k = \text{const}$ is independent of ε and

$$x^{(k-1)} = \xi(\tau) + \varepsilon u_1(\tau,t) + \dots + \varepsilon^{(k-1)} u_k(\tau,t). \tag{A.1.3}$$

In what follows we make the following assumptions: A and f are analytical functions of x provided that x belongs to a certain region G and all values of x $\left(x^{(k)} \in G\right)$, ξ and u_i are infinitely differentiable with respect to τ in $\tau_0 \le \tau \le \tau_0 + L$. Function x is assumed to be continuously differentiable with respect to t, and the inequality $|\dot{x}| \le \varepsilon M = \text{const}$ is valid for $x \in G$, $t \in T_\varepsilon$, $0 < \varepsilon \le \varepsilon_0$; $|f| \le F = \text{const}$ for $x \in G$, $t_0 \le t < \infty$; $|u_i| \le U_i =$

const for $\tau_0 \le \tau \le \tau_0 + L$, $t_0 \le t < \infty$. Analogous inequalities are valid for all derivatives of f and u_i with respect to x or τ. The roots $\lambda_\nu(x)$, $\nu = 1, \cdots, m$, of the equation

$$|A(x) - \lambda E| = 0 \tag{A.1.4}$$

satisfy the condition $\operatorname{Re}\lambda_\nu(x) < -\beta$, $\beta = \text{const} > 0$ for all $x \in G$. Of special interest is the case when f and u_i are periodic or quasi-periodic functions of t having the same period (or frequency basis) which is independent of x or τ. Under a quasi-periodic function we understand a linear combination of $\cos\omega_i t$ and $\sin\omega_i t$, $i = 1, \ldots, N$, when at least two numbers ω_k, ω_j are incommensurable.

Let $U(t, t_1, \varepsilon)$ be the fundamental matrix of the solutions of the homogeneous equation $\dot{v} = A(x)v$ such that $U(t_1, t_1, \varepsilon) = E$, where E is the identity matrix and $t_1 \in T_\varepsilon$. In what follows we use the following

Lemma 1. *Under the above assumptions there exists such ε_* that for $0 < \varepsilon \le \varepsilon_*$ the resolving matrix $U(t, \theta, \varepsilon)$ for $t_0 \le \theta \le t$, $t \in T_\varepsilon$, fulfils the condition*

$$\|U(t, \theta, \varepsilon)\| \le N e^{-\gamma(t-\theta)}, \tag{A.1.5}$$

where the constants γ, $N > 0$ are independent of ε.

This lemma presents Coppel's theorem [128] however the inequality $|\dot{x}| \le \varepsilon M$ holds in the interval $t \in T_\varepsilon$, that is, inequality (A.1.5) is valid in the above interval rather than in the infinite interval, as in Coppel's theorem. In the following we assume that $\varepsilon \le \varepsilon_*$.

Let us describe the process of construction of a formal series defining a partial solution φ of systems (A.1.1)

$$\varphi \sim \varphi_0(\tau, t) + \varepsilon\varphi_1(\tau, t) + \ldots . \tag{A.1.6}$$

Functions $A(x)$ and $f(x,t)$ are represented by asymptotic series (for matrices, the difference $A - A^{(k)}$ is estimated by means of the norm)

$$A(x) \sim A(\xi) + \varepsilon A_1(\tau, t) + \ldots, \; f(x,t) \sim f(\xi, t) + \varepsilon f_1(\tau, t) + \ldots, \tag{A.1.7}$$

which can be obtained by substitution of eq. (A.1.2) into the expansion of functions A and f in the neighbourhood of point $x = \xi$, while the series coefficients are infinitely differentiable with respect to τ. Substituting eqs. (A.1.2) and (A.1.7) into (A.1.1) and equating the coefficients at the powers of ε we obtain the system

$$\frac{\partial\varphi_i}{\partial t} = A(\xi)\varphi_i + g_i, \quad g_i = f_i + A_i\varphi_0 + \cdots + A_1\varphi_{i-1} - \frac{\partial\varphi_{i-1}}{\partial\tau}, \quad g_0 = f_0. \tag{A.1.8}$$

By virtue of the infinite differentiability of A_i,f_i and ξ with respect to τ, functions g_i are determined for all k as functions of t, τ, φ_0 etc. Equations (A.1.8) are successively integrable for ξ, $\tau = \text{const}$. When $\varphi_0, \varphi_1, \cdots, \varphi_{i-1}$ have been found, function g_i is a known function of t and τ.

Let φ_0 be some partial solution of the first equation in (A.1.8). Since function g_0 is bounded at $t_0 \leq t < \infty$, and $\operatorname{Re}\lambda_\nu < 0$ function φ_0 is also bounded in $t_0 \leq t < \infty$, $\xi \in G$. Let us denote $\psi_0 = \partial\varphi_0/\partial\tau$. From the first equation in (A.1.8) we obtain

$$\frac{\partial\psi_0}{\partial t} = A(\xi)\psi_0 + \frac{\partial g_0}{\partial\tau} + \left(\frac{\partial}{\partial\tau}A(\xi)\right)\varphi_0. \tag{A.1.9}$$

Under the made assumptions "the inhomogeneous part" of eq. (A.1.9) is bounded in $t \geq t_0$ therefore function ψ_0 is also bounded. Hence function g_1, and in turn φ_1 are bounded, too. Having chosen φ_1, we find functions g_2, φ_2 etc., all of them being bounded for $x \in G$, $\tau_0 \leq \tau \leq \tau + L$, $t_0 \leq t < \infty$.

When f and u_i are periodic or quasi-periodic functions of t having the same period (or frequency basis) which is independent of x and τ, any periodic or quasi-periodic solution of the first equation in (A.1.8) can be understood as φ_0. Then function g_1 will be periodic or quasi-periodic, in a similar manner one can take φ_1, φ_2 etc.

Theorem 1. *A formal series (A.1.6) asymptotically converges to a partial solution of equation (A.1.1).*

Let $\varphi_k(t_0)$ denote the value $\varphi_k(t)$ for $t = t_0$, and $\varphi(t_0,\varepsilon)$ denote one of the functions ε such that

$$\varphi(t_0,\varepsilon) \sim \varphi_0(t_0) + \varepsilon\varphi_1(t_0) + \ldots . \tag{A.1.10}$$

Let us consider a partial solution $\varphi(t,\varepsilon)$ of system (A.1.1) coinciding with $\varphi(t_0,\varepsilon)$ at $t = t_0$ and denote $d_k = \varphi(t,\varepsilon) - \varphi^{(k)}(t,\varepsilon)$. The function $\varphi^{(k)} = \varphi_0 + \varepsilon\varphi_1 + \cdots + \varepsilon^k\varphi_k$ satisfies the equation

$$\begin{aligned}\dot{\varphi}^{(k)} = A^{(k)}\varphi^{(k)} + f^{(k)} + \varepsilon^{k+1}\Big[\frac{\partial\varphi_k}{\partial\tau} - \\ -A_1\varphi_k - \cdots - A_k\varphi_1 - \varepsilon(A_2\varphi_k + \cdots + A_k\varphi_2)\ldots\Big].\end{aligned} \tag{A.1.11}$$

For d_k we have the following equation

$$\begin{aligned}\dot{d}_k = A(x)d_k + (f - f^{(k)}) + (A - A^{(k)})\varphi^{(k)} + \varepsilon^{k+1}\Big[\frac{\partial\varphi_k}{\partial\tau} + A_1\varphi_k + \ldots \\ \cdots + A_k\varphi_1 + \varepsilon(A_2\varphi_k + \cdots + A_k\varphi_2) + \cdots + \varepsilon^{k-1}A_k\varphi_k\Big]\end{aligned} \tag{A.1.12}$$

and the initial conditions $d_k(t_0) = \varphi(t_0,\varepsilon) - \varphi^{(k)}(t_0,\varepsilon)$. The solution of eq. (A.1.12) can be written in the form

$$d_k = U\left(\varphi(t_0,\varepsilon) - \varphi^{(k)}(t_0,\varepsilon)\right) + \int_{t_0}^{t} U\rho_k d\theta, \tag{A.1.13}$$

where ρ_k designates the sum of all terms in the right hand side of eq. (A.1.12) beginning with $f - f^{(k)}$. It follows from eq. (A.1.13) that

$$|d_k| \leq \|U\| |\varphi(t_0, \varepsilon) - \varphi^{(k)}(t_0, \varepsilon)| + \int_{t_0}^{t} \|U\| |\rho_k| d\theta. \tag{A.1.14}$$

Taking into account eq. (A.1.5) we obtain

$$|d_k| \leq N_0 |\varphi(t_0, \varepsilon) - \varphi^{(k)}(t_0, \varepsilon)| e^{\gamma(t-t_0)} + N \int_{t_0}^{t} e^{-\gamma(t-\theta)} |\rho_k| d\theta. \tag{A.1.15}$$

The definition (A1.1.3) of the asymptotic series yields

$$|\varphi(t_0, \varepsilon) - \varphi^{(k)}(t_0, \varepsilon)| \leq \varepsilon^{k+1} C_{k+1}, \qquad |f - f^{(k)}| \leq \varepsilon^{k+1} C'_{k+1},$$

where the constants C_{k+1}, C'_{k+1} are independent of ε.

Since functions $\varphi_k(t, \tau)$ are bounded (see above) for $t_0 \leq t < \infty$, $\tau_0 \leq \tau \leq \tau_0 + L$ we have $|\varphi^{(k)}| \leq \Phi^{(k)} = \sum_i \varepsilon_0^i \Phi_i$ where Φ_i are independent of ε. The estimates for all values in square brackets in eq. (A.1.12) are obtained by analogy. Then we have

$$|(A - A^{(k)})\varphi^{(k)}| \leq \|A - A^{(k)}\| |\varphi^{(k)}| \leq \varepsilon^{k+1} \Phi'_{k+1}.$$

Collecting the estimates of values entering in the expression for ρ_k, we obtain $|\rho_k| \leq \varepsilon^{k+1} R_{k+1}$ and

$$|d_k| \leq \varepsilon^{k+1} \left| N_0 C_{k+1} e^{-\gamma(t-t_0)} + \frac{N R_{k+1}}{\gamma} (1 - e^{\gamma(t-t_0)}) \right| \leq \varepsilon^{k+1} D_{k+1}, \tag{A.1.16}$$

where D_{k+1} is independent of ε. This proves the asymptotic convergence of series (A.1.6) to the partial solution of equation (A.1.1).

In order to construct the general integral of the homogeneous system we look for the fundamental matrix $U(t, \tau, \varepsilon)$ in the following form

$$U(t, \tau, \varepsilon) = U_0(t, \tau) + \varepsilon U_1(t, \tau) + \ldots, \tag{A.1.17}$$

where U_i satisfies the matrix equations

$$\frac{\partial U_0}{\partial t} = A U_0, \quad \frac{\partial U_1}{\partial t} = A U_1 + A_1 U_0 - \frac{\partial U_0}{\partial \tau} \tag{A.1.18}$$

etc. and the initial conditions $U_0(t_0, \tau) = E$, $U_i(t_0, \tau) = 0$, $i = 1, 2, \ldots$. One determines U_0 from the first equation in (A.1.18). Substituting matrix U_0 into the second equation in eq. (A.1.18) we determine U_1 and so on.

Theorem 2. *Sequence (A.1.17) asymptotically converges to the fundamental matrix of the homogeneous system.*

It is evident that derivatives $\partial U_i/\partial\tau$, $i=0,1,\ldots$, are bounded by norm by the functions $C_i t^k e^{-\gamma t}$, $\gamma>0$, where k is integer and $C_i=\text{const}$. In addition to this, all A_1, A_2, ... are bounded. The convergence is proved similar to the proof of theorem 1.

The introduced matrices contain secular terms which are products of exponential functions and powers of t. For this reason, in the case when the roots λ_ν are real-valued, negative and simple (or a multiple root with simple divisors) whereas u_i are periodic or quasi-periodic functions (with period or frequency basis independent of τ) another algorithm is preferable. We will seek the partial solutions y_ν of the homogeneous system in the form $y_\nu=\zeta_\nu z_\nu$ where ζ_ν is a scalar functions and z_ν is a vector function, both being governed by the following differential equations

$$\dot{\zeta}_\nu=\mu_\nu\zeta_\nu \quad \dot{z}_\nu+\mu_\nu z_\nu=Az_\nu. \tag{A.1.19}$$

We look for the asymptotic series for z_ν and μ_ν in the form

$$z_\nu\sim z_{\nu 0}(\tau)+\varepsilon z_{\nu 1}(\tau,t)+\ldots,\qquad \mu_\nu\sim\mu_{\nu 0}(\tau)+\varepsilon\mu_{\nu 1}(\tau)+\ldots. \tag{A.1.20}$$

Inserting eqs. (A.1.20) and (A.1.7) into eq. (A.1.19) and equating the terms with the equal powers of ε we obtain the system

$$\begin{aligned} &A(\xi)z_{\nu 0}-\mu_{\nu 0}z_{\nu 0}=0,\\ &\quad\frac{\partial z_{\nu 1}}{\partial t}=\left(A(\xi)-\mu_{\nu 0}E\right)z_{\nu 1}+h_{\nu 1},\quad h_{\nu 1}=A_1 z_{\nu 0}-\frac{\partial z_{\nu 0}}{\partial\tau}-\mu_{\nu 1}z_{\nu 0} \end{aligned} \tag{A.1.21}$$

and so on. Let us assume that $\mu_{\nu 0}(\xi)$ is equal to one of the roots of the equation

$$|A(\xi)-\mu E|=0. \tag{A.1.22}$$

Then $z_{\nu 0}$ is one of eigenvectors of matrix $A(\xi)$. It is determined up to a scalar factor which is an arbitrary infinitely differentiable function of τ, a specific form of this factor being out of interest. The equations for $z_{\nu 1}$, $z_{\nu 2}$ etc. are integrated at ξ, $\tau=\text{const}$.

The roots of the characteristic equation

$$|A(\xi)-(\mu_{\nu 0}+\mu)E|=0 \tag{A.1.23}$$

are the differences $\mu_\alpha-\mu_{\nu 0}$, where μ_α is a root of eq. (A.1.22). All these differences are real-valued, only one among them being identically equal to zero. Let us choose $\mu_{\nu i}$ by means of the following condition

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_{t_0}^{t_0+T}(h_{\nu k},z^*_{\nu 0})dt=0, \tag{A.1.24}$$

where $z_{\nu 0}^*$ denotes the eigenvector of matrix $A^* - \mu_{\nu 0} E$ corresponding to zero root of the characteristic equation. Equalities (A.1.24) are simple linear equations allowing one to determine successively $\mu_{\nu 1}$, $\mu_{\nu 2}$ ... These equations are resolvable since the coefficient at $\mu_{\nu k}$ in $k - th$ equation is equal to $(z_{\nu 0}, z_{\nu 0}^*)$ and does not vanish.

If u_j are periodic, then function $h_{\nu i}$ will be also periodic provided that $z_{\nu 1}$, ..., $z_{\nu i-1}$ are periodic functions. Under the condition (A.1.24) which can be replaced by the equality that mean value averaged over the period is equal to zero, function $z_{\nu i}$ can be understood as a periodic one. But $h_{\nu i}$ is periodic, thus all $z_{\nu i}$ can be taken as being periodic.

Functions $h_{\nu i}$ are polynomials of u_j whereas $h_{\nu i}$ is a linear form in $z_{\nu 1}$, ..., $z_{\nu i-1}$. For this reason, for quasi-periodic functions u_j, $z_{\nu 1}$, ..., $z_{\nu i-1}$ the function $h_{\nu i}$ will be quasi-periodic and under condition (A.1.24) all $z_{\nu i}$ can be chosen as quasi-periodic.

Taking periodic or quasi-periodic functions $z_{\nu i}$ and having found some number of values $\mu_{\nu i}$, we can calculate the corresponding approximations to ζ_ν

$$\zeta_\nu^{(k)} = \exp\left(\int_{t_0}^{t} \mu_\nu^{(k)} \xi(\varepsilon\sigma) d\sigma\right). \tag{A.1.25}$$

The sequence $y_\nu^{(k)} = \zeta_\nu^{(k)} z_\nu^{(k)}$ converges asymptotically to a partial solution of the homogeneous system. For different ν the solutions are linearly independent and the general integral can be constructed with the help of these solutions.

A.2 The case when function $\xi(\tau, \varepsilon)$ is given by an asymptotic sequence

In the previous analysis $\xi(\tau)$ and $u_i(\tau, t)$ were considered as prescribed functions of τ and t. Let us now consider the case when u_i are given functions of ξ and t, analytical with respect to ξ while function ξ is assumed to be given by an asymptotic sequence $\xi^{(i)}(\tau, \varepsilon)$, $i = 0, 1, \dots$, and the derivatives of functions $\xi^{(i)}$ are expressed in terms of these functions by the relationships

$$\frac{d\xi^{(i)}}{dt} = \varepsilon \Xi_1(\xi^{(i)}) + \cdots + \varepsilon^{i+1} \Xi_{i+1}(\xi^{(i)}), \tag{A.2.1}$$

where functions Ξ_1, Ξ_2, ... are analytical. As for $\xi(\tau, \varepsilon)$ we assume that for $0 < \varepsilon \le \varepsilon_0$, $\tau_0 \le \tau \le \tau_0 + L$ there exists the derivative $\partial\xi/\partial\tau$ which is the limit of the asymptotically converging sequence $\partial\xi^{(k)}/\partial\tau$.

In this case the approximations $y^{(k)}$ can be expressed in terms of $\xi^{(k)}(\tau)$ and $\Xi_i(\xi^{(i)})$. We seek $\varphi^{(k)}$ in the following form

$$\varphi^{(k)} = \varphi_0(\xi^{(k)}, t) + \varepsilon\varphi_1(\xi^{(k)}, t) + \cdots + \varepsilon^k\varphi_k(\xi^{(k)}, t), \tag{A.2.2}$$

and instead of $d\varphi^{(k)}/dt$ we use the sum

$$\frac{\partial\varphi_0}{\partial t} + \varepsilon\left[\frac{\partial\varphi_0}{\partial\xi^{(k)}}\Xi_1 + \frac{\partial\varphi_1}{\partial t}\right] + \varepsilon^2\left[\frac{\partial\varphi_0}{\partial\xi^{(k)}}\Xi_2 + \frac{\partial\varphi_1}{\partial\xi^{(k)}}\Xi_1 + \frac{\partial\varphi_2}{\partial t}\right] + \dots . \tag{A.2.3}$$

The approximation to x up to the term of ε^{k+1} is given by

$$x^{(k)} = \xi + \varepsilon u_1(\xi, t) + \cdots + \varepsilon^k u_k(\xi, t). \tag{A.2.4}$$

Since $\xi^{(k)}$ approximates ξ with the accuracy $O(\varepsilon^{k+1})$ then

$$x^{(k)} = \xi^{(k)} + \varepsilon u_1(\xi^{(k)}, t) + \cdots + \varepsilon^k u_k(\xi^{(k)}, t) \tag{A.2.5}$$

approximates $x(t, \varepsilon)$ with the same accuracy. Similar to eq. (A.1.7), we can replace ξ by $\xi^{(k)}$, to have

$$\begin{aligned} A^{(k)} &= A(\xi^{(k)}) + \varepsilon A_1(\xi^{(k)}, t) + \cdots + \varepsilon^k A_k(\xi^{(k)}, t), \\ f^{(k)} &= f(\xi^{(k)}) + \varepsilon f_1(\xi^{(k)}, t) + \cdots + \varepsilon^k f_k(\xi^{(k)}, t). \end{aligned} \tag{A.2.6}$$

Let us insert these approximations into eq. (A.1.1) and equate the coefficients in front of the powers of ε. We arrive at the system

$$\begin{aligned} \frac{\partial\varphi_0}{\partial t} &= A(\xi^{(k)})\varphi_0 + g_0, \quad g_0 = f(\xi^{(k)}, t), \\ \frac{\partial\varphi_1}{\partial t} &= A(\xi^{(k)})\varphi_1 + g_1, \quad g_1 = f(\xi^{(k)}, t) + A_1\varphi_0 - \frac{\partial\varphi_0}{\partial\xi^{(k)}}\Xi_1, \quad \dots \end{aligned} \tag{A.2.7}$$

To prove the asymptotic convergence of sequence $\varphi^{(k)}$ to the solution of equation (A.1.1) we construct an equation for $\varphi^{(k)}$. We multiply equations (A.2.7) by the corresponding degree of ε and sum the results, then we obtain

$$\begin{aligned} \dot{\varphi}^{(k)} &= A^{(k)}\varphi^{(k)} + f^{(k)} + \varepsilon^{k+1}[A_1\varphi_k + \cdots + A_k\varphi_1 + \varepsilon(A_2\varphi_k + \cdots + \\ &\quad + A_k\varphi_2) + \cdots + \varepsilon^{k-1}A_k\varphi_k + \frac{\partial\varphi_k}{\partial\xi^{(k)}}(\Xi_1 + \cdots + \varepsilon\Xi_{k+1})], \end{aligned} \tag{A.2.8}$$

that differs from eq. (A.1.8) in the terms of order ε^{k+1} and higher, as well as that the argument of y, $A^{(k)}$, $f^{(k)}$ etc. is $\xi^{(k)}$ rather than ξ. It is however insignificant and therefore the convergence is proved by analogy with Theorem 1.

The fundamental matrix U is obtained from eq. (A.1.18) where $\partial U_0/\partial\tau$ is replaced by $(\partial U_0/\partial\xi^{(k)})\Xi_1$ etc. The asymptotic convergence is proved as above.

Finally, the $k-th$ approximation to $y=\varphi+UC$ is as follows

$$y^{(k)}=\varphi_0+\varepsilon\varphi_1+\dots+(U_0+\varepsilon U_1+\dots)C=y_0+\varepsilon y_1+\dots, \qquad (A.2.9)$$

where C denotes the vector of the arbitrary constants determined from the initial conditions, whereas φ_0, φ_1, U_0, U_1, ... are prescribed functions of $\xi^{(k)}$, Ξ_1 etc.

Having determined $y^{(k)}$ we can also take $x^{(k)}$ in the following form

$$x^{(k)}=\xi^{(k)}+\varepsilon u_1(\xi^{(k-1)},t)+\dots+\varepsilon^k u_k(\xi^0,t). \qquad (A.2.10)$$

Analogously, we can write formulae for $A^{(k)}$and $f^{(k)}$. The argument $\xi^{(k)}$ in eq. (A.2.7) should be replaced by the lowest approximations, namely in the equation for $\varphi^{(i)}$ it is necessary to replace $\xi^{(k)}$ by $\xi^{(k-i)}$. The remaining equations will be not changed and the obtained solution approximates the exact solution with the accuracy $O(\varepsilon^{k+1})$.

A.3 Separating slow variables in quasi-linear systems

Let us consider the system

$$\begin{aligned}\dot{x}&=\varepsilon X(x,y,t,\varepsilon)=\varepsilon X_1(x,y,t)+\varepsilon^2 X_2(x,y,t)+\dots,\\ \dot{y}&=A(x)y+f(x,t).\end{aligned} \qquad (A.3.1)$$

The quasi-linear system (A.3.1) presents a particular case of systems with many fast variables studied by V.M. Volosov [128] who indicated a replacement of variables allowing separation of slow and fast variables. However the fact that system (A.3.1) is quasi-linear suggests a method of separation different from Volosov's method. The present method relies on combination of the averaging method with the method of asymptotic integration of linear equations explained in Sections A1 and A2.

Let us assume that $X(x,y,t,\varepsilon)$ is an analytical function of x, y, ε for $x\in G$, $y\in G_1$, $0\le\varepsilon\le\varepsilon_0$, and $A(x)$ and $f(x,t)$ are analytical functions of x. The other assumptions will be listed in what follows.

We look the $k-th$ approximation to $x(t,\varepsilon)$ ($k=0,1,\dots$) in the following form

$$x^{(k)}(t,\varepsilon)=\xi^{(k)}+\varepsilon u_1(\xi^{(k)},t)+\dots+\varepsilon^k u_k(\xi^{(k)},t), \qquad (A.3.2)$$

where $\xi^{(k)}(\tau,\varepsilon)$ satisfies the equation

$$\frac{d\xi^{(k)}}{dt}=\varepsilon\Xi_1(\xi^{(k)})+\dots+\varepsilon^{k+1}\Xi_{k+1}(\xi^{(k)}), \qquad (A.3.3)$$

and express $k-th$ approximation to $y(t,\varepsilon)$ in terms of $\xi^{(k)}$ and Ξ_i with the help of relationships (A.2.2).

We replace $X_i(x^{(k)}, y^{(k)}, t)$, $i = 1, \ldots, k+1$, by the sums

$$X_i(x^{(k)}, y^{(k)}, t) = X_i(\xi^{(k)} + \varepsilon u_1 + \ldots, y_0 + \varepsilon y_1 + \ldots, t) \tag{A.3.4}$$
$$= X_i(\xi^{(k)}, y_0, t) + \varepsilon \left[\frac{\partial X_i}{\partial x} u_1 + \frac{\partial X_i}{\partial y} y_1 \right] + \cdots + \varepsilon^{k+1} [\ldots],$$

keeping in each sum the terms with powers of ε not higher than $k - i + 1$ (the derivatives in eq. (A.3.4) are calculated at $x = \xi^{(k)}$, $y = y_0$).

Let us take a derivative in eq. (A.3.2), replace the derivative of $\xi^{(k)}$ by means of eq. (A.3.3) and keep the terms with powers of ε not higher than $k+1$. We replace $\dot{x}$ in the first equation in (A.3.1) by the obtained expressions for the derivative, omit the terms beginning with $O(\varepsilon^{k+2})$ in the right hand side of this equation and replace X_i according to eq. (A.3.4). Then we obtain the following formal equality

$$\varepsilon \Xi_1 + \cdots + \varepsilon^k \Xi_k + \varepsilon \frac{\partial u_1}{\partial t} + \varepsilon \frac{\partial u_1}{\partial \xi^{(k)}} (\varepsilon \Xi_1 + \cdots + \varepsilon^{k-1} \Xi_{k1}) + \varepsilon \frac{\partial u_k}{\partial t} =$$
$$= \varepsilon X_1(\xi^{(k)}, y_0, t) + \varepsilon^2 \left[\frac{\partial X_1}{\partial x} u_1 + \frac{\partial X_1}{\partial y} y_1 \right] + \cdots + \varepsilon^{k+1} [\ldots]. \tag{A.3.5}$$

In equation (A.3.5) we equate the terms at the powers of ε and obtain

$$\frac{\partial u_1}{\partial t} = X_1(\xi^{(k)}, y_0(\xi^{(k)}, t), t) - \Xi_1, \tag{A.3.6}$$
$$\frac{\partial u_2}{\partial t} = X_2(\xi^{(k)}, y_0(\xi^{(k)}, t), t) + \frac{\partial X_1}{\partial x} u_1 + \frac{\partial X_1}{\partial y} y_1 - \frac{\partial u_1}{\partial \xi^{(k)}} \Xi_1 - \Xi_2$$

etc. Altogether we obtain $k+1$ equations.

It is typical for the asymptotic methods that u_i are required to be bounded functions of t. To this end, it is necessary and sufficient that the following mean values

$$\langle W_i \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} W_i(\xi, t) dt, \qquad i = 1, \ldots, k+1 \tag{A.3.7}$$

exist, where W_i is the sum of all terms in the right hand sides of eq. (A.3.6), except for Ξ_i, and the integrals

$$\int_{t_0}^{t} [W_i(\xi, \theta) - \langle W_i \rangle] d\theta, \qquad i = 1, \ldots, k+1 \tag{A.3.8}$$

should be bounded functions of time.

We point out two important cases in which these conditions are satisfied.

1. Functions X_i and f are periodic functions of t with period T which is independent of x. The mean values (A.3.7) are assumed to be independent of C.

2. Functions X_i and f are quasi-periodic functions of t with a frequency basis independent of x, and X_i are the polynomials in the components of y. In this case y_0 and W_1 are sums of the terms with attenuating exponential functions and a quasi-periodic function. Therefore for $\Xi_1 = \langle W_1 \rangle$ function u_1 has the form of W_1. But u_1 appears linearly in eq. (A.1.8) (in terms of A_1 and f_1) therefore φ_1 and y_1 are sums of attenuating functions and a quasi-periodic function. Function W_2 is the same character. In general, since u_i appear in eq. (A.1.8) in the form of u_i^m, m being positive and integer, all y_i are the sums of exponentially attenuating and quasi-periodic functions. Therefore under the conditions

$$\Xi_i(\xi) = \langle W_i(\xi, t) \rangle, \qquad i = 1, 2, \ldots, \tag{A.3.9}$$

u_i will have the form of W_i, and, thus, they are bounded.

The second case occurs, for example, in mechanics and the theory of electromechanical systems. Then X_i are quadratic forms in the components of y.

Provided that the integral (A.3.8) is bounded, one determines Ξ_i from eq. (A.3.9) whereas u_i is found by quadratures in terms of W_i up to an arbitrary function of ξ.

Finally we arrive at the following algorithm.

We take the initial conditions for $\xi(\tau)$ in the form of $\xi(\tau_0) = x(t_0)$ and choose $u_i(\xi(\tau), t)$ such that $u_i(\xi(\tau_0), t_0) = 0$. Then we find the solution of the first equation (A.2.7) under the condition $\varphi_0(t_0) = y(t_0)$. This condition can be satisfied since the value of $\xi(\tau_0)$ is known. Then we introduce the obtained function $\varphi_0(t) = y_0$ in eq. (A.3.6), average the result according to eq. (A.3.7) and find $\Xi_1(\xi)$. Integrating the first equation in eq. (A.3.6) we find u_1 by choosing the integration function of ξ such that $u_1(\xi(\tau_0), t_0) = 0$. Next, from the second equation in (A.2.7) we find $\varphi_1 = y_1$ under the condition $\varphi_1(t_0) = 0$. Introducing the result into the second equation in eq. (A.3.6) and constructing W_2 we find function Ξ_2 with the help of eq. (A.3.7). Then integrating the second equation in eq. (A.3.6) we find u_2 by virtue of $u_2(\xi(\tau_0), t_0) = 0$ and so on.

The choice of the conditions $\xi(\tau_0) = x(t_0)$, $u_i(\xi(\tau_0), t_0) = 0$, $\varphi_0(t_0) = y(t_0)$, $\varphi_i(t_0) = 0$ is not necessary. The arbitrary functions in u_i can be chosen from another consideration. Then the starting values $\xi(\tau_0)$, $\varphi_i(t_0)$ are found in terms of the initial conditions $x(t_0)$, $y(t_0)$ and the initial values $u_i(\xi(\tau_0), t_0)$.

In what follows we use the theorem [130] on higher approximations for the systems in standard form

$$\dot{x} = \varepsilon X(x, t, \varepsilon), \tag{A.3.10}$$

which states the following.

Let X have a sufficient number of the derivatives and $|X| \leq M = \text{const}$ for $x \in G$, $t \in T_\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$. Let us assume that we have the initial conditions for eq. (A.3.10) and the improved $k - th$ approximation $x^{(k)} = \xi + \varepsilon u_1(\xi, t) + \cdots + \varepsilon^k u_k(\xi, t)$, where ξ is the solution of equation in the form (A.3.3). Let u_i, $i = 1, \ldots, k$, be bounded function of time satisfying the above initial conditions. Let ξ remains in G for $t \in T_\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$. Then there exists such $\varepsilon_1 \leq \varepsilon_0$ that for $0 < \varepsilon \leq \varepsilon_1$ and the same initial conditions, equation (A.3.10) admits a solution remaining in G for $t \in T_{\varepsilon_1}$ and satisfying the inequality $|x - x^{(k)}| \leq \varepsilon^{k+1} \cdot \text{const}$.

We will also use the following

Lemma 2. *Let us consider two systems*

$$\dot{x}_1 = \varepsilon X_1(x_1, t, \varepsilon), \qquad \dot{x}_2 = \varepsilon X_2(x_2, t, \varepsilon), \tag{A.3.11}$$

where $|X_1(x, t, \varepsilon) - X_2(x, t, \varepsilon)| \leq \varepsilon^{k+1} K$, $K = \text{const}$ *in* $x \in G$, $t \in T_\varepsilon$. *Then under the assumption that the condition of the previous theorem is valid , we have* $|x_1 - x_2| \leq \varepsilon^{k+1} \cdot \text{const}$, $\left| x_1 - x_2^{(k)} \right| \leq \varepsilon^{k+1} \cdot \text{const}$ *for* $t \in T_\varepsilon$ *and the same initial conditions.*

The proof follows from the inequalities

$$|x_1 - x_2| \leq \varepsilon \int_{t_0}^{t} |X_1(x_1, t, \varepsilon) - X_2(x_2, t, \varepsilon)| dt \leq \varepsilon \int_{t_0}^{t} |X_1(x_1, t, \varepsilon) -$$
$$- X_1(x_2, t, \varepsilon)| + |X_1(x_2, t, \varepsilon) - X_2(x_2, t, \varepsilon)| dt \leq$$
$$\leq \varepsilon C \int_{t_0}^{t} |x_1 - x_2| dt + \varepsilon^{k+1} K \leq \varepsilon^{k+1} K e^{CL},$$
$$|x_1 - x_2^{(k)}| \leq |x_1 - x_2| + |x_2 - x_2^{(k)}| \leq \varepsilon^{k+1} \cdot \text{const}\,. \tag{A.3.12}$$

Here we used the lemma condition, Lipschits condition for X_1, inequality (A.3.12) and the theorem on systems in standard form.

Theorem 3. *Let for* $t \in T_\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$ *the curves* $\xi^{(0)}(\tau)$, $y_0(t, \tau)$ *satisfy* $\xi^{(0)}(\tau_0) = x(t_0)$, $y_0(t_0, \tau_0) = y_0(t_0)$ *and remain, along with their vicinities, in* G *and* G_1, *respectively. Then there exists such* $\varepsilon_1 \leq \varepsilon_0$ *that for* $0 < \varepsilon \leq \varepsilon_1$ *the solution* $x(t, \varepsilon)$, $y(t, \varepsilon)$ *of system (A.3.1) becomes* $x(t_0)$, $y(t_0)$ *at* $t = t_0$ *and remains in* G, G_1 *for* $t \in T_\varepsilon$, *i.e.* $x \in G$, $y \in G_1$. *The approximations* $x^{(k)}$, $y^{(k)}$ *obtained under the conditions* $x^{(k)}(t_0, \varepsilon) = x(t_0, \varepsilon)$, $y^{(k)}(t_0, \varepsilon) = y(t_0, \varepsilon)$ *remain in* G, G_1 *for* $t \in T_\varepsilon$ *and satisfy the inequalities* $|x - x^{(k)}| \leq \varepsilon^{k+1} \cdot \text{const}$, $|y - y^{(k)}| \leq \varepsilon^{k+1} \cdot \text{const}$, *where the constants depend only on* k.

Let functions u_i, y_i, Ξ_i, $i = 1, \dots, k$, be constructed. We resolve eq. (A.3.2) for $\xi^{(k)}$, to obtain

$$\xi^{(k)} = x^{(k)} + \varepsilon v_1(x^{(k)}, t) + \cdots + \varepsilon^k v_k(x^{(k)}, t) + \varepsilon^{k+1} \delta\xi_k,$$
$$v_1 = -u_1(x^{(k)}, t), \quad v_2 = -u_2(x^{(k)}, t) + \left(\frac{\partial u_1}{\partial \xi^{(k)}}\right) u_1(\xi^{(k)}, t), \dots, \tag{A.3.13}$$

which is possible only for sufficiently small ε. Let us keep in eq. (A.3.13) the terms up to ε^k included and substitute them in the expression for $y^{(k)}(\xi^{(k)}, t, \varepsilon)$ and keep the terms of the same order. Then we obtain

$$y_*^{(k)} = y_0(x^{(k)}, t) + \varepsilon y_{*1}(x^{(k)}, t) + \cdots + \varepsilon^k y_{*k}(x^{(k)}, t). \tag{A.3.14}$$

Now we introduce eq. (A.3.14) into the first equation in (A.3.1) and arrive at the system in standard form

$$\dot{x}_{*k} = \varepsilon X(x_{*k}, y_*^{(k)}(x_{*k}, t, \varepsilon), t, \varepsilon). \tag{A.3.15}$$

A function satisfying this formally constructed system is denoted as x_{*k}. We consider $i - th$ $(i \leq k)$ approximation to the solution of this system obtained by the averaging method

$$x_{*k}^{(i)} = \xi_*^{(i)} + \varepsilon u_{*1}(\xi_*^{(i)}, t) + \cdots + \varepsilon^i u_{*i}(\xi_*^{(i)}, t),$$
$$\dot{\xi}_*^{(i)} = \varepsilon \Xi_{*1}(\xi_*^{(i)}) + \cdots + \varepsilon^{i+1} \Xi_{*i+1}(\xi_*^{(i)}), \tag{A.3.16}$$

whose solution being independent of k.

Let us prove that functions $u_{*j}(\xi, t)$, $\Xi_{*j}(\xi)$ coincide with $u_j(\xi, t)$, $\Xi_j(\xi)$ in eqs. (A.3.2) and (A.3.3). Let us assume that this is correct for u_{*1}, Ξ_{*1}, ..., u_{*i-1}, Ξ_{*i1}. Substituting $x_{*k}^{(i)}$ from eq. (A.3.16) in the expression $y_*^{(k)}$, instead of $x^{(k)}$ we obtain a function of $\xi_*^{(i)}$, t, ε, whose coefficients of the expansion in terms of ε up to ε^{i-1} included are coincident with the coefficients $y^{(k)}(\xi^{(k)}, t, \varepsilon)$ provided that $\xi^{(k)}$ is replaced $\xi_*^{(k)}$ in the latter expression. Therefore function W_{*i} in the equation

$$\frac{\partial u_{*i}}{\partial t} = W_{*i} - \Xi_{*i} \tag{A.3.17}$$

determined by functions u_{*1}, ..., u_{*i-1} is coincident with W_i after replacing the argument $\xi_*^{(i)}$ by $\xi^{(k)}$. Hence, Ξ_{*i} and Ξ_i, coincide and under an appropriate choice of the arbitrary functions ξ after integration u_{*i} and u_i will be coincident, too.

Let us consider the first approximation. We have

$$\dot{x}_{*k} = \varepsilon X_1(x_{*k}, y_0(x_{*k}, t), t) + \dots; \qquad W_{*1} = X_1(\xi_*^{(i)}, y_0(\xi_*^{(i)}, t), t),$$

that is, u_{*1} coincides with u_1 after replacing the arguments. By induction we can conclude that all u_{*j}, Ξ_{*j} coincide with the corresponding u_j, Ξ_j up to $j = k + 1$.

Let us now assume that there exists such ε_1 and such initial conditions that for $0 < \varepsilon \le \varepsilon_1$ the solution of system (A.3.1) and all $i{-}th$, $i = 0, 1, \dots,$ approximations satisfying the initial conditions remain in G, G_1 for $t \in T_\varepsilon$.

We shall consider the solution and approximation to it under these initial conditions.

Function $y_0(x, t)$ satisfies the equation

$$\dot{y}_0 = A(x)y_0 + f(x, t) + \varepsilon \frac{\partial y_0}{\partial x}(X_1 + \varepsilon X_2 + \dots). \tag{A.3.18}$$

The coefficient at ε in eq. (A.3.18) is bounded under the considered values of the variables. By analogy with Section A.1 we can show that $|y_0(x, t) - y| \le \varepsilon \cdot \text{const}$.

Let us now consider the equation

$$\dot{x}_{*0} = \varepsilon X_1(x_{*0}, y_0(x_{*0}, t), t). \tag{A.3.19}$$

From the estimate $|y - y_0| \le \varepsilon \cdot \text{const}$ and the boundedness of function $X_2, \dots$ we conclude by virtue of Lemma 2 that $|x - x_{*0}| \le \varepsilon \cdot \text{const}$ and $|x - \xi^{(0)}| \le \varepsilon \cdot \text{const}$.

Let us assume that

$$|x - x^{(k-1)}(\xi^{(k-1)}, t, \varepsilon)| \le \varepsilon^k \cdot \text{const}, \quad |y - y^{(k-1)}(\xi^{(k-1)}, t, \varepsilon)| \le \varepsilon^k \cdot \text{const}$$

and consider the function $\xi^{(k-1)}(x^{(k1)}, t, \varepsilon)$ obtained from eq. (A.3.2) by replacing k by $k - 1$. According to the theorem on implicit functions this function exists for sufficiently small ε (say for $\varepsilon \le \varepsilon_1$) and is an analytical function of parameter ε and bounded functions of t. Hence

$$\xi^{(k-1)} = x^{(k-1)} + \varepsilon v_1 + \dots + \varepsilon^{k-1} v_{k-1} + \varepsilon^k \delta\xi_{k-1},$$

where $\left|\delta\xi_{k-1}\right|$ is a bounded function in the domain under consideration. Therefore we have $|y^{(k-1)}(\xi^{(k-1)}, t, \varepsilon) - y_*^{(k-1)}(x^{(k1)}, t, \varepsilon)| \le \varepsilon^k \cdot \text{const}$ and $|y - y_*^{(k-1)}(x^{(k-1)}, t, \varepsilon)| \le \varepsilon^k \cdot \text{const}$. Hence

$$|X(x, y_*^{(k-1)}(x, t, \varepsilon), t, \varepsilon) - X(x, y, t, \varepsilon)| \le \varepsilon^k \cdot \text{const}$$

and system (A.3.1) may be rewritten in the following form

$$\dot{x} = \varepsilon X(x, y_*^{(k1)}(x, t, \varepsilon), t, \varepsilon) + \varepsilon^{k+1} \delta X_k, \tag{A.3.20}$$

where δX_k is a bounded function. By virtue of the theorem on the systems in standard form there exists a function $\xi^{(k)}(\tau, \varepsilon)$ such that

$$x^{(k)} = \xi^{(k)} + \varepsilon u_1(\xi^{(k)}, t) + \dots + \varepsilon^k u_k(\xi^{(k)}, t), \tag{A.3.21}$$

approximates x with the accuracy $O(\varepsilon^{k+1})$, while $u_1, \dots, u_k$ are independent of the terms of the order ε^{k+1} in eq. (A.3.20). Thus under this choice $\xi^{(k)}$ $x = x^{(k)} + \varepsilon^{k+1}\delta x_k$ and $y = y^{(k)}(\xi^{(k)}, t, \varepsilon) + \varepsilon^{k+1}\delta y_k$, where $|\delta x_k|$ and $|\delta y_k|$ are bounded.

All $\xi^{(k)}(\tau, \varepsilon)$ satisfying this condition are expressed in terms of x by the relationship

$$\xi^{(k)} = x + \varepsilon v_1(x,t) + \cdots + \varepsilon^k v_k(x,t) + \varepsilon^{k+1}\delta\xi_k, \tag{A.3.22}$$

where the latter term in the right hand side of eq. (A.3.22) is also bounded. Let us construct function $y_*^{(k)}(x,t,\varepsilon)$ by using eq. (A.3.22). For this function we have $|y - y_*^{(k)}(x,t,\varepsilon)| \le \varepsilon^{k+1} \cdot \text{const}$. Therefore the first equation in (A.3.1) can be rewritten in the following form

$$\dot{x} = \varepsilon X(x, y_*^{(k)}(x,t,\varepsilon), t, \varepsilon) + \varepsilon^{k+2}\delta X_{k+1}, \tag{A.3.23}$$

where δX_{k+1} is bounded. Omitting the last term in eq. (A.3.23) we arrive at the system whose solution, under these initial conditions, approximates the exact solution with the accuracy $O(\varepsilon^{k+1})$. After replacing x by x_{*k} this system coincides with that in eq. (A.3.15), and $k-th$ approximation to its solution coincides with $x^{(k)}$. Thus, provided that $(k-1)-th$ approximation approximates the exact solution with accuracy $O(\varepsilon^k)$, then $k-th$ approximation approximates it with accuracy $O(\varepsilon^{k+1})$. However for $x^{(0)} = \xi^{(0)}$, $y^{(0)} = y_0(\xi^{(0)}, t)$ the approximation has been derived above, thus the approximation of order $k+1$ is proved for all k.

It remains to show that for sufficiently small values of ε, $\varepsilon > 0$ functions $x \in G$ and $y \in G_1$, provided that $\xi^{(0)}$, y_0 remain in G, G_1 along with some vicinities. We have

$$(\xi^{(k)} - \xi^{(0)})^{\cdot} = \varepsilon(\Xi_1(\xi^{(k)}) - \Xi_1(\xi^{(0)})) + \cdots + \varepsilon^{k+1}\Xi_{k+1}(\xi^{(k)}) \tag{A.3.24}$$

or after the integration

$$|\xi^{(k)} - \xi^{(0)}| \le \varepsilon \int_{t_0}^{t} |\Xi_1(\xi^{(k)}) - \Xi_1(\xi^{(0)})| dt + \varepsilon C_1 \le \tag{A.3.25}$$

$$\le \varepsilon C_2 \int_{t_0}^{t} |\xi^{(k)} - \xi^{(0)}| dt + \varepsilon C_1 \le \varepsilon C_1 e^{C_2 L}, \qquad C_1, C_2 = \text{const}\,.$$

Here we used the inequality of eq. (A.3.12). Hence for a sufficiently small ε function $\xi^{(k)} \in G$ for $t \in T_\varepsilon$. Since u_i are bounded functions of time which are differentiable with respect to $\xi^{(k)}$ then $y^{(k)} \in G_1$ for sufficiently small ε and $x^{(k)} \in G, y^{(k)} \in G_1$. Finally, the proved inequalities for $x - x^{(k)}$, $x - x_*$ suggest that x, $x_* \in G$, $y \in G_1$ for $t \in T_\varepsilon$ and sufficiently small values of ε.

A.4 Remarks

In practice it is often sufficient to construct only the equations for slow variables. Then for periodic or quasi-periodic X_i and f it is sufficient to find only periodic or quasi-periodic partial solution for y, while u_i will be periodic or quasi-periodic.

When $X_i(x, y, t)$ and $f(x, t)$ are periodic or quasi-periodic functions and λ_ν are simple and real-valued then one can use the algorithm of Section A.1 and obtain the approximations having only periodic or quasi-periodic functions and exponential functions rather than the secular terms. To this end we should use the following representation

$$y^{(k)} = y_0 + \varepsilon\delta y_1 + \cdots + \varepsilon^k \delta y_k, \qquad \delta y_i = \left(y^{(i)} - y^{(i-1)}\right)/\varepsilon^i, \qquad \text{(A.4.1)}$$

where the fractions are of the order of one, and to replace everywhere y_i by $(y^{(i)} - y^{(i-1)})/\varepsilon^i$.

The method substantiated above for the quasilinear systems of general form was originally proposed by K.Sh. Khodzhaev for the system describing slow motions in electromechanical systems and considered in [124]. In this case, in the first approximation the equations for slow variables describe the conservative system, and the higher approximations are required for a qualitative assessment of the motion. The corresponding computations were carried out by M.M. Vetyukov and K.Sh. Khodzhaev with the help of Volosov's method [124] and with the help of the above method. The approximations obtained by both methods were proved to be coincident however the computations by the suggested method turned out to be simpler.

Having studied the case in which f and X_i are periodic functions of t, R.F. Nagaev proposed to construct, instead of functions $\varphi^{(k)}(\xi, t, \varepsilon)$, the function $\varphi^{(k)}(x, t, \varepsilon)$ which are periodic with respect to t by replacing $\dot{x}$ by the expansion εX. Then $\varphi_0(x, t)$ is obtained from $\varphi_0(\xi, t)$ if ξ is replaced by x. For $\varphi_1(x, t)$ one obtains the equation

$$\frac{\partial \varphi_1}{\partial t} = A(x)\varphi_1 + f(x, t) - \frac{\partial \varphi_0}{\partial x} X_1(x, \varphi_0, t) \qquad \text{(A.4.2)}$$

and so on. Having found $\varphi^{(k)}$ and replaced y in first equation (A.3.1) by $\varphi^{(k)}$ one arrives at the system in standard form. It can be shown that equations (A.3.3) and (A.3.6) are obtained after the averaging. Analogously the general solution $y^{(k)}(x, t, \varepsilon)$ can be constructed and there is no need to consider only the periodic functions f and X_i.

Appendix B

Systems integrable in the first approximation of the averaging method

The higher approximations of the averaged equations obtained by the Krylov-Bogolyubov method are known to be not uniquely constructed since the corresponding replacement is performed up to some arbitrary functions of slow variables. It is natural to try to choose these functions such that the averaged equations are simplified. As shown in the present Section, such a possibility appears, in particular, in the case in which the general integral of the equation of the first approximation is known. In this case the choice of these arbitrary functions allows one to ensure that all highest terms of the averaged equations turn to zero, that is, for any approximation the averaged equations coincide with the equations of the first approximation.

Let us consider the system in standard form

$$\dot{x} = \varepsilon X(x, t, \varepsilon) = X_0(x, t) + \varepsilon X_1(x, t) + \dots , \tag{B.1}$$

where x and X are $n \times 1$ column-vectors. Function $X(x, t, \varepsilon)$ is assumed to be periodic function of the explicit time t with a given period independent of x and ε, have all partial derivatives and these derivatives are uniformly bounded for $|\varepsilon| < \varepsilon_0$ and x from a certain region. The approximate solution x_m of $m - th$ approximation of the averaging method is defined by the relationship

$$x_m = \xi + \varepsilon u_1(\xi, t) + \varepsilon^2 u_2(\xi, t) + \dots + \varepsilon^{m-1} u_{m-1}(\xi, t). \tag{B.2}$$

Here functions $u_1(\xi, t), \dots, u_{m-1}(\xi, t)$ are periodic with respect to t. The slow variables ξ are dependent on the number of approximation m, however for the sake of simplicity we will write ξ instead of ξ_m.

Generally speaking, the system for ξ is given by

$$\dot{\xi} = \varepsilon \Xi_0(\xi) + \varepsilon^2 \Xi_1(\xi) + \ldots + \varepsilon^m \Xi_{m-1}(\xi). \tag{B.3}$$

Functions $u_1, \ldots, u_{m-1}$ and, respectively, functions $\Xi_1, \ldots, \Xi_{m-1}$ are determined not uniquely since any arbitrary function of ξ can be added to u_i, however it does not affect the approximation of the accurate solutions x by the approximate solutions x_m, cf. [83].

Let us assume that the general integral of the averaged system of the first approximation

$$\dot{\xi} = \varepsilon \Xi_0(\xi) \tag{B.4}$$

be obtained. Let us show that in this case one can determine the arbitrary corrections to u_i such that all functions $\Xi_1, \ldots, \Xi_{m-1}$ are identically equal to zero. Then the averaged equations of any approximation coincide with the equation of the first approximation and can be integrated.

Let us consider the following relationships for determining functions u_1, u_2, Ξ_0, Ξ_1

$$\begin{aligned}
\Xi_0(\xi) + \frac{\partial u_1(\xi,t)}{\partial t} &= X_0(\xi,t), \\
\Xi_1(\xi) + \frac{\partial u_1(\xi,t)}{\partial t}\Xi_0(\xi) + \frac{\partial u_2(\xi,t)}{\partial t} &= \frac{\partial X_0(\xi,t)}{\partial \xi} u_1(\xi,t) + X_1(\xi,t).
\end{aligned} \tag{B.5}$$

Function $\Xi_0(\xi)$ is determined uniquely

$$\Xi_0(\xi) = \langle X_0(\xi,t) \rangle, \tag{B.6}$$

where the broken brackets denote time-averaging over the period. The expression for u_1 can be set in the form

$$u_1 = \int (X_0(\xi,t) - \Xi_0)dt + u_{c1}(\xi) = u_{\nu 1}(\xi,t) + u_{c1}(\xi). \tag{B.7}$$

Here and in what follows index ν denote the quadratures whose mean value over the period is equal to zero and u_{c1} is yet an arbitrary function of ξ.

Function Ξ_1 is determined from the condition of periodicity of u_2

$$\Xi_1(\xi) = \left\langle -\frac{\partial u_1(\xi,t)}{\partial \xi}\Xi_0 + \frac{\partial X_0(\xi,t)}{\partial \xi} u_1 + X_1(\xi,t) \right\rangle. \tag{B.8}$$

Introducing eq. (B.7) into eq. (B.8) we obtain

$$\Xi_1 = -\frac{\partial u_{c1}}{\partial \xi}\Xi_0 = \frac{\partial \Xi_0}{\partial \xi} u_{c1} + G_1(\xi), \tag{B.9}$$

where

$$G_1(\xi) = \left\langle \frac{\partial X_0}{\partial \xi} u_{\nu 1} + X_1 \right\rangle. \tag{B.10}$$

Let us select function u_{c1} such that $\Xi_1(\xi) \equiv 0$. This requirement leads to the equation for u_{c1}

$$\frac{\partial u_{c1}}{\partial \xi}\Xi_0 = \frac{\partial \Xi_0}{\partial \xi} u_{c1} + G_1(\xi). \tag{B.11}$$

Strictly speaking, for the vector-function u_{c1} we obtained a system of differential equations in partial derivatives of the first order with the identical principal part [26].

The characteristic system for eq. (B.11) is given by the following system, cf. [26]

$$\begin{aligned} \frac{\partial \xi^{(k)}}{\partial \xi^{(n)}} &= \frac{\Xi_0^{(k)}}{\Xi_0(n)}, \quad k = 1, \dots, n-1, \\ \frac{\partial u_{c1}}{\partial \xi^{(n)}} &= \frac{\partial \Xi_0}{\partial \xi} \frac{u_{c1}}{\Xi_0^{(n)}} + \frac{G_1}{\Xi_0^{(n)}}. \end{aligned} \tag{B.12}$$

Index k in the first group of equations in eq. (B.12) denotes $k - th$ component of the column-vectors $(\xi^{(1)}, \dots, \xi^{(n)})^{\mathrm{T}}$ and $\Xi_0 = (\Xi_0^{(1)}, \dots, \Xi_0^{(n)})^{\mathrm{T}}$. The unknowns in eq. (B.12) are $n-1$ functions $\xi^{(1)}(\xi^{(n)}), \dots, \xi^{(n-1)}(\xi^{(n)})$ and n components of vector $u_{c1}(\xi^{(n)})$. Any other component of vector ξ can be taken as the argument in system (B.12).

Provided that the general integral of equations (B.4) is prescribed in the form $F(\xi, \tau) = C$, where $\tau = \varepsilon t$, we can find the general integral of the first group of equations (B.12). To this end, the slow time τ should be eliminated from n relationships $F(\xi, \tau) = C$ and we obtain $n-1$ relationships of the form $F_*(\xi) = C_*$, where C_* is a $(n-1) \times 1$ column-vector. Another constant contained in the general integral of system (B.4) is removed under elimination of τ since system (B.4) is autonomous and one constant is contained additively with τ. Here we solve the problem without the general integral of equations (B.4) as it is sufficient to know their $n-1$ autonomous integrals.

The second group of equations in eq. (B.12) is also integrated since it is obtained from the system of equations in variations for equation (B.4). In order to obtain the equations of this group it is necessary to add the inhomogeneity $G_1(\xi)$ to the variational equations and take the argument $\xi^{(n)}$ instead of time.

Let us consider the inhomogeneous system with the argument τ and the unknown variable u_{c1}

$$\frac{du_{c1}}{d\tau} = \frac{\partial \Xi_0(\xi)}{\partial \xi} u_{c1} + G_1(\xi). \tag{B.13}$$

Its general solution is given by

$$u_{c1} = \frac{\partial \xi(\tau + C^{(n)}, C_*)}{\partial C} A + \frac{\partial \xi(\tau + C^{(n)}, C_*)}{\partial C} \times$$
$$\times \int\limits_{\tau_0}^{\tau + C^{(n)}} \left(\frac{\partial \xi(\vartheta + C^{(n)}, C_*)}{\partial C} \right)^{-1} G_1(\xi(\vartheta + C^{(n)}, C_*)) d(\vartheta + C^{(n)}). \quad \text{(B.14)}$$

Here A denotes $n \times 1$ column-vector of the arbitrary constants, and functions $\xi(\tau + C^{(n)}, C_*)$ are obtained by "inversion" of the general integral $F(\xi, \tau) = C$, where $C^{(n)}$ denotes the constant appearing in the solution of system (B.4) additively with τ.

In order to obtain the general solution of the second group of equations (B.12), it is necessary to substitute $\tau + C^{(n)} = f(\xi^{(n)}, C_*)$ into eq. (B.14). This substitution is determined in terms of the dependences

$$\xi^{(1)}(\xi^{(n)}, C_*), \ldots, \xi^{(n-1)}(\xi^{(n)}, C_*), \tau + C^{(n)} = f(\xi^{(n)}, C_*)$$

derived from n equations $F(\xi, \tau) = C$.

Now we can write down the general integral of system (B.12). Instead of eq. (B.14) we have

$$\left(\frac{\partial \xi(\tau + C^{(n)}, C_*)}{\partial C} \right)^{-1} u_{c1} - \int\limits_{\tau_0}^{\tau + C^{(n)}} \left(\frac{\partial \xi(\vartheta + C^{(n)}, C_*)}{\partial C} \right)^{-1} \times$$
$$\times G_1(\xi(\vartheta + C^{(n)}, C_*)) d(\vartheta + C^{(n)}|_{\tau + C^{(n)} = f(\xi^{(n)}, C_*)} = A. \quad \text{(B.15)}$$

Then one should replace C_* on $F_*(\xi)$ in eq. (B.15). Then we arrive at the relationship of the form $H_1(u_{c1}, \xi) = A$. Along with the relationship $F_*(\xi) = C$ it comprises the general integral of system (B.12).

The general solution of system (B.11) is now obtained in the following way. We take n arbitrary differentiable functions of $2n-1$ arguments (these functions can be viewed as the components of vector Ψ) and write down n functional equations

$$\Psi(F_*(\xi), H_1(u_{c1}, \xi)) = 0. \quad \text{(B.16)}$$

The vector-function Ψ should satisfy the condition that equations (B.16) are resolvable for u_{c1}. Then the dependence $u_{c1}(\xi)$ ensuring the equality $\Xi_1 = 0$ is found from eq. (B.16).

It is evident that when we use this method the "degree of arbitrariness" in the choice of functions u_{c1} is less than in the general case, however it is not clear how it is possible to decrease this "degree".

For the estimation of functions $u_{c2}, \ldots, u_{cm-1}$ we obtain the following equations

$$\frac{\partial u_{ci}}{\partial \xi} \Xi_0(\xi) = \frac{\partial \Xi_0}{\partial \xi} u_{ci} + G_i(\xi), \quad \text{(B.17)}$$

which differ from equation (B.11) in the form of the “inhomogeneous” part of G_i, function G being a prescribed function of ξ at $i-th$ stage of the investigation. For this reason, u_{ci} will be determined in the same manner as u_{c1}, with the only difference that G_1 is replaced by G_i.

Instead of relationship (B.2) we can look for an approximate solution in the following form

$$x_m = \xi + \varepsilon u_1(\xi,\tau,t) + \ldots + \varepsilon^{m-1} u_{m-1}(\xi,\tau,t). \tag{B.18}$$

In general case, if we do not try to eliminate functions $\Xi_1,\ldots,\Xi_{m-1}$ these replacements depending explicitly upon the slow time τ, are clearly not rational, because the averaged equations for the higher approximations are not autonomous and more difficult than those under standard replacement. When the suggested method is used, the non-autonomous (with respect to τ) replacement leads even to simpler relationships than the autonomous one.

Provided that the replacement of variables with the explicit slow time τ is used, the averaged system takes the form

$$\begin{aligned} &\dot{\xi} = \varepsilon\Xi_0(\xi) + \varepsilon^2\Xi_1(\xi,\tau) + \ldots + \varepsilon^{m-1}\Xi_{m-1}(\xi,\tau),\\ &\dot{\tau} = \varepsilon\,. \end{aligned} \tag{B.19}$$

As above, function $\Xi_0(\xi)$ is defined by relationship (B.6). Let us take u_1 in the following form $u_1 = u_{\nu 1}(\xi,t) + u_{c1}(\xi,\tau)$ where function $u_{\nu 1}$ is given by eq. (B.7). For $u_2(\xi,\tau,t)$ we obtain the equation

$$\begin{aligned} \Xi_1(\xi,\tau) + \frac{\partial u_1(\xi,\tau,t)}{\partial\xi}\Xi_0(\xi) + \frac{\partial u_1}{\partial\tau} + \frac{\partial u_2(\xi,\tau,t)}{\partial t} &=\\ = \frac{\partial X_0(\xi,t)}{\partial\xi} u_1(\xi,\tau,t) - X_1(\xi,t). \end{aligned} \tag{B.20}$$

The condition of periodicity of u_2 with respect to t yields

$$\Xi_1 = \left\langle -\frac{\partial u_1}{\partial\xi}\Xi_0 - \frac{\partial u_1}{\partial t} + \frac{\partial X_0}{\partial\xi}u_1 + X_1 \right\rangle. \tag{B.21}$$

Let us require that $\Xi_1 \equiv 0$, then we arrive at the equation for u_{c1}, analogous to eq. (B.11)

$$\frac{\partial u_{c1}}{\partial\xi}\Xi_0 + \frac{\partial u_{c1}}{\partial\tau} = \frac{\partial\Xi_0}{\partial\xi}u_{c1} + G_1(\xi), \tag{B.22}$$

where

$$G_1 = \left\langle \frac{\partial X_0(\xi,t)}{\partial\xi} u_{\nu 1} + X_1(\xi,t) \right\rangle. \tag{B.23}$$

Under this assumption, the corresponding characteristic system takes the following form

$$\frac{d\xi}{d\tau} = \Xi_0(\xi),$$
$$\frac{du_{c1}}{d\tau} = \frac{\partial \Xi_0(\xi)}{\partial \xi} u_{c1} + G_1(\xi)\,. \tag{B.24}$$

Basically speaking, the general integral $F(\xi, \tau) = C$ of system (B.4) can deliver the dependences $\xi(\tau, C)$. For this reason, the solution of the second equation in eq. (B.24) is as follows

$$u_{c1} = \frac{\partial \xi(\tau, C)}{\partial C} A + \frac{\partial \xi(\tau, C)}{\partial C} \int\limits_{\tau_0}^{\tau} \left(\frac{\partial \xi(\vartheta, C)}{\partial C} \right)^{-1} G(\xi(\vartheta, C)) d\vartheta. \tag{B.25}$$

From this equation we determine the general integral analogous to that in eq. (B.15). Replacing in this integral C by $F(\xi, \tau)$ we arrive at the relationship of the form $H(u_{c1}, \xi, \tau) = A$. By analogy with the previous analysis we introduce n arbitrary differentiable functions which form vector Ψ and construct the equations

$$\Psi\left(F(\xi, \tau), H(u_{c1}, \xi, t)\right) = 0. \tag{B.26}$$

Provided that functions Ψ are chosen in such a way that equations (B.26) are resolvable for u_{c1}, from eq. (B.26) one can obtain the dependences $u_{c1}(\xi, \tau)$ guaranteeing the equality $\Xi_1 \equiv 0$.

For the following functions $u_{c2}, \ldots, u_{cm-1}$ we obtain the equations of the form of eq. (B.22), where however functions G_i are dependent on ξ and τ. Functions $u_{\nu 2}, \ldots, u_{\nu m-1}$ will also depend upon τ.

The suggested method can be applied to the systems with a single fast phase

$$\dot{x} = \varepsilon X(x, \varphi, \varepsilon),$$
$$\dot{\varphi} = \omega(x) + \varepsilon \Phi(x, \varphi, \varepsilon), \quad \omega > 0. \tag{B.27}$$

Indeed, by introduced a new argument φ we arrive at the system in the standard form.

Finally, this method can be also utilised for averaging the systems with many fast variables different from the phases [128], that is,

$$\dot{x} = \varepsilon X(x, y, t, \varepsilon),$$
$$\dot{y} = Y(x, y, t, \varepsilon)\,. \tag{B.28}$$

Specifically, the following quasi-linear system presents some interest

$$\dot{x} = \varepsilon X(x, y, t, \varepsilon),$$
$$\dot{y} = A(x) y + F(x, t, \varepsilon)\,. \tag{B.29}$$

In this case we can use the expansions described in [96]. An alternative approach is to eliminate the fast variables by determining the coefficients of the following expansion

$$y = y_0(t, x) + \varepsilon y_1(t, x) + \dots \tag{B.30}$$

When the expression in eq. (B.30) is differentiated, the derivatives $\dot{x}$ are substituted by means of the first equation in (B.29). After substitution of eq. (B.30) into the first equation (B.29) we obtain a system in the standard form.

Appendix C

Higher approximations of the averaging method for systems with discontinuous variables

Let us consider a system whose state is characterised by n-dimensional vector x which is a discontinuous function of time. Let the system be described by the equations in standard form

$$\dot{x} = \varepsilon X(x, t, \varepsilon) \tag{C.1}$$

within the time intervals between the discontinuities. When the phase point in the extended phase space x, t reaches the surface $F(x, t, \varepsilon)$ the unknown variables x becomes discontinuous. Let t_j denote the time instant of the discontinuity. The relation between the value of variable x before and after the discontinuity is described by the formula

$$x(t_j + 0) - x(t_j - 0) = \varepsilon\Delta\left(x(t_j - 0), t_j, \varepsilon\right). \tag{C.2}$$

We assume that function Δ given on the surface $F(x, t\varepsilon)$ is such that the phase point crosses the surface of discontinuity, i.e. it transits from the subspace where $F(x, t, \varepsilon) > 0$ (or $F(x, t, \varepsilon) < 0$) into the subspace, where $F(x, t, \varepsilon) < 0$ (or $F(x, t, \varepsilon) > 0$). The sliding regimes with repetitive crossings, which occur in the small time interval of the order of ε, are not possible. It is also assumed that for $|\varepsilon| < \varepsilon_0$ functions X, F and Δ are continuous with respect to t and determined in certain regions of the corresponding spaces, have continuous derivatives with respect to x, ε of the necessary order and uniformly bounded in these regions along with the above-mentioned derivatives.

The objective of the forthcoming analysis is to substantiate and apply the analogue of the averaging method developed for the systems in standard form to the systems under consideration at any approximation.

The first approximation of the averaging method for the systems with discontinuous variables was proposed and substantiated in the works by A.M. Samoilenko (see for example [101]). In these works the substantiation of the method is presented under weaker requirements (than those in the present Section) to smoothness of the functions, in particular, the functions in the right hand sides of the equation of motions between the discontinuities. Under these conditions the substantiation of the averaging method in the first [3] and higher [130] approximations turns out to be essentially more difficult even for systems with continuous variables than in the case when there is a required number of bounded derivatives (in the latter case the substantiation of the averaging method at the first approximation is suggested, for example, in [5]). Hence under the assumptions of the present work it would be essentially simpler to substantiate the method for the systems with discontinuous variables at the first approximation than in [3] and [130]

According to the suggested method, $m-th$ approximation is sought in the following form

$$x_m = \xi_m + \varepsilon u_1^{(m)}(\xi_m, t, \varepsilon) + \ldots + \varepsilon^{m-1} u_{m1}^{(m)}(\xi_m, t, \varepsilon), \qquad (C.3)$$

where $u_r^{(m)}$ is a $2\pi/\omega-$periodic function of time having discontinuities at $t = t_{im}, i = 1, \ldots, h$. Function ξ_m is continuous and one can construct its equation in the form of a "standard" averaging method

$$\dot{\xi}_m = \varepsilon \Xi_0(\xi_m) + \ldots + \varepsilon^m \Xi_{m-1}(\xi_m). \qquad (C.4)$$

The time instants of discontinuities are defined in the form of following expansion

$$t_{im} = t_{i0}(\xi_m) + \varepsilon \Delta t_{i1}(\xi_m) + \ldots + \varepsilon^{m-1} \Delta t_{im-1}(\xi_m). \qquad (C.5)$$

For determining t_{im} we use the equation $F(x, t, \varepsilon) = 0$. For the sake of simplicity we assume in what follows that for all x, ε from the region under consideration this equation in the half-interval $0 \leq t < 2\pi/\omega$ has the same number h of the simple roots $t_i(x, \varepsilon)$.

Because functions $u_r^{(m)}$ are $2\pi/\omega-$periodic with respect to time it is sufficient to consider only the roots from the indicated half-interval.

The approximate value of the instants of discontinuity can be sought in parallel with functions u_r. In order to keep the description of the method down it is more convenient to express the expansion coefficients (C.5) and the right hand sides of equations (C.4) in terms of $t_{im}(\xi_m, \varepsilon)$ rather than determine these functions.

Decompositions (C.5) will be found in the end of calculations.

Let us denote the coefficients in eq. (C.3) and the second terms of equations in (C.4) expressed in terms of the yet unknown functions $t_{im}(\xi_m, \varepsilon)$ respectively through $u^{(r)}(\xi_m, t, \varepsilon)$ and $\Xi^{(r)}(\xi_m, \varepsilon)$, functions $u^{(r)}$ and $\Xi^{(r)}$

being dependent on ε only in terms of $t_{1m}, \ldots, t_{hm}$ (there is no need to note such a dependence on m by an index). Hence if we wish to find, for example, a $(m+1)-th$ approximation, one can utilise functions $u^{(r)}$, found under the calculation of $m-th$ approximation by replacing the index of ξ_m and t_{im} by $m+1$. The same is valid for functions $\Xi^{(r)}$ and Ξ_r, but $u_r^{(m)}$, whose discontinuity times are assumed to be computed according to eq. (C.5). Instead of (C.3), (C.4) we obtain

$$x_m = \xi_m + \varepsilon u^{(1)}(\xi_m, t, \varepsilon) + \ldots + \varepsilon^{m-1} u^{(m-1)}(\xi_m, t, \varepsilon) \tag{C.6}$$

$$\dot{\xi}_m = \varepsilon \Xi^{(0)}(\xi_m, \varepsilon) + \ldots + \varepsilon^{m-1} \Xi^{(m-1)}(\xi_m, \varepsilon). \tag{C.7}$$

Functions Ξ_r in eq. (A2.4) are obtained by a "re-expansion" of the right hand sides of eq. (C.7) in terms of ε after the expansion (C.5) has been found and inserted into eq. (C.7), and u_r has been obtained from $u^{(r)}$ by substitution of eq. (C.5).

For computation of functions $u^{(r)}$ within the time intervals between the discontinuities one should substitute (C.6) in eq. (C.1), differentiate $u^{(r)}$ with respect to t taking into account that ξ_m are functions of time, replace $\dot{\xi}_m$ by the right hand side of eq. (C.7), expand X in a power series in terms of ε by using Taylor's formula with the remainder of order of ε^m and equate the coefficients at the identical powers of ε in the left and right hand sides.

While the expressions $(\partial u^{(r)}/\partial \xi_m)\dot{\xi}_m$ are calculated one should take into account dependence $u^{(r)}$ on ξ_m in terms of $t_{1m}, \ldots, t_{hr}$. However for balance of terms with the identical powers of ε one should take into account only the powers of ε in front of functions $\Xi^{(r)}, u^{(r)}$ and their derivatives, however these functions are not expanded in power series in terms of ε. As a result one obtains equations for $u^{(r)}$ coinciding formally with those of the "standard" method of averaging.

Having substituted eq. (C.6) into eq. (C.2) and using Taylor's formula for function Δ we obtain the relationships which determine the discontinuities of functions $u^{(r)}$. These relationships have the form

$$\begin{aligned} \Delta_{i1} &= u^{(1)}(\xi_m, t_{im}+0, \varepsilon) - u^{(1)}(\xi_m, t_{im}-0, \varepsilon) = \Delta(\xi_m, t_{im}, 0), \\ \Delta_{i2} &= u^{(2)}(\xi_m, t_{im}+0, \varepsilon) - u^{(2)}(\xi_m, t_{im}-0, \varepsilon) = \\ &= \left(\frac{\partial \Delta}{\partial x}\right)_i u^{(1)}(\xi_m, t_{im}-0, \varepsilon) + \left(\frac{\partial \Delta}{\partial \varepsilon}\right)_i \end{aligned} \tag{C.8}$$

and so on. In eq. (C.8) derivations of function Δ are calculated for $x = \xi_m, \varepsilon = 0, t = t_{im}$. The dependence of $t_{1m}, \ldots, t_{im}$ on ε are not taken into account in the expansion in terms of ε.

It is essential that the discontinuities of functions u_1 can be found from eq. (C.8) without knowledge of this function. Let us proceed to equation for $u^{(1)}$ in the time interval between the discontinuities

$$\Xi^{(0)}(\xi_m, \varepsilon) + \frac{\partial u^{(1)}}{\partial t} = X(\xi_m, t, 0). \tag{C.9}$$

From the requirement of $2\pi/\omega-$periodicity of the discontinuous function $u^{(1)}$ it is possible to determine $\Xi^{(0)}(\xi_m, \varepsilon)$

$$\Xi^{(0)} = \langle X(\xi_m, t, 0)\rangle + \frac{\omega}{2\pi}\sum_{i=1}^{h}\Delta_{i1},$$

$$\Delta X(\xi_m, t, 0)\rangle = \frac{\omega}{2\pi}\int\limits_0^{2\pi/\omega} X(\xi_m, t, 0)dt. \qquad \text{(C.10)}$$

The mean value $\langle X(\xi_m, t, 0)\rangle$ is also dependent on ε since it is calculated for function having discontinuities at $t_{im}(\xi_m, \varepsilon), \ldots, t_{hm}(\xi_m, \varepsilon)$. Then from eq. (C.9) and the first relationship (C.8) we find

$$u^{(1)} = \int\limits_0^t (X(\xi_m, t, 0) - \Xi^{(0)})dt + \sum_{i=1}^{h}\Delta_{i1}\sigma(t - t_{im}). \qquad \text{(C.11)}$$

Here σ is Heaviside function, $\sigma(t - t_{im}) = 0, t < t_{im}, \sigma(t - t_{im}) = 1, t > t_{im}$ and values σ at $t = t_{im}$ are not considered. Typical for the averaging method is that an arbitrary function of ξ_m can be added to the right hand side of eq. (C.11) (for brevity, such functions are omitted here and in what follows). Having obtained function $u^{(1)}$ one can determine the discontinuities of functions $u^{(2)}$ from the second relationship (C.8) and write down the equation for $u^{(2)}$ between the discontinuities. From the condition of $2\pi/\omega-$periodicity of $u^{(2)}$ we obtain the following expression for $\Xi^{(1)}$

$$\Xi^{(1)} = \left\langle \left(\frac{\partial X}{\partial x}\right)_0 + \left(\frac{\partial X}{\partial \varepsilon}\right)_0 - \frac{\partial u^{(1)}}{\partial \xi_m}\Xi^{(0)}\right\rangle + \frac{\omega}{2\pi}\sum_{i=1}^{h}\Delta_{i2}. \qquad \text{(C.12)}$$

Then it is possible to find $u^{(2)}$

$$u^{(2)} = \int\limits_0^t \left[\left(\frac{\partial X}{\partial x}\right)_0 u^{(1)} + \left(\frac{\partial X}{\partial \varepsilon}\right)_0 - \frac{\partial u^{(1)}}{\partial \xi_m}\Xi^{(0)} - \Xi^{(1)}\right] dt + $$
$$+ \sum_{i=1}^{h}\Delta_{i2}\sigma(t - t_{im}). \qquad \text{(C.13)}$$

While computing $\partial u^{(1)}/\partial \xi_m$ and in the similar cases one should take into account that u_1 is dependent on ξ_m in terms of $t_{im}(\xi_m, \varepsilon)$. This derivative is a function defined everywhere in the region under consideration, except for the time instants $t = t_{im}$ and is $2\pi/\omega-$periodic with respect to t.

Continuing the process we can calculate the discontinuities of functions $u^{(3)}$ in the time intervals between the discontinuities, determine $\Xi^{(2)}$ from the conditions of $2\pi/\omega-$periodicity of $u^{(3)}$, then determine $u^{(3)}$ and so on. By analogy with the "standard" averaging method in order to estimate function $\Xi^{(m-1)}$ it is necessary to add the term $\varepsilon^m u^{(m)}(\xi_m, t, \varepsilon)$ in eq. (C.3), find the discontinuities Δ_{im} of functions $u^{(m)}$ and write down the equation for $u^{(m)}$ between the discontinuities. There is no need to calculate function $u^{(m)}$ itself since it is required only for construction of the improved $m-th$ approximation.

Next one should substitute eq. (C.3) with the already obtained functions $u^{(r)}(\xi_m, t)$ in equation $F(x, t, \varepsilon) = 0$ and use eq. (C.5). Then we have

$$F(\xi_m + \varepsilon u^{(1)}(\xi_m, t_{im} - 0, \varepsilon) + ... + \varepsilon^{(m-1)} u_{m-1}(\xi_m, t_{im} - 0, \varepsilon), t_{im}, \varepsilon) = 0. \tag{C.14}$$

The values $u^{(r)}(\xi_m, t_{im} - 0, \varepsilon) = u^{(r)}(\xi_m, t_{i0} + \ldots + \varepsilon^{m1}\Delta t_{i,m-1} - 0, \varepsilon)$ are dependent on $t_{10}, \ldots, t_{h0}, \ldots, \Delta th, m-1$ and their derivative with respect to ξ_m. They can be expanded in a series in terms of ε by keeping the necessary number of conditions. The same should be done for functions $\Xi^{(r)}(\xi_m, \varepsilon)$. Then function F of the indicated arguments can be expanded, too. Equating the coefficients at the powers of ε to zero we arrive at the equations for t_{i0}, Δ_{i1} and so on.

Considering the non-small terms we obtain

$$F(\xi_m, t_{i0}, 0) = 0, \tag{C.15}$$

yielding h roots $t_{i0}(\xi_m), i = 1, \ldots, h$.

Let us consider the first order terms. They contain the following expression

$$u^{(1)}(\xi_m, t_{im} - 0, \varepsilon) = \frac{\omega}{2\pi} \int\limits_0^{t_{i0}+\ldots+\varepsilon^{m-1}\Delta t_{i,m-1}} (X(\xi_m, t, 0) - \Xi^{(0)})dt + \sum_{j=1}^{i-1} \Delta_{j1} \tag{C.16}$$

which should be expanded in series in terms of ε. This expansion is written in the following form

$$u^{(1)}(\xi_m, t_{im} - 0) = u_{1i,0} + \varepsilon u_{1i,1} + \ldots + \varepsilon^{m-1} u_{1i,m1} + \ldots, \tag{C.17}$$

where dots denote the remainder in Taylor's formula. The first terms of the expansion is as follows

$$u_{1i,0}(\xi_m, t_{i0}) = \int\limits_t^{t_{i0}} (X - \Xi_0^{(0)})dt + \sum_{j=1}^{h} \Delta(\xi_m, t_{j0}, 0), \tag{C.18}$$

where

$$\Xi_0^{(0)} = \frac{\omega}{2\pi} \int\limits_0^{2\pi/\omega} X(\xi_m, t, 0)dt + \frac{\omega}{2\pi} \sum_{j=1}^{h} \Delta(\xi_m, t_{i0}, 0). \tag{C.19}$$

For estimation Δt_{i1} we have the linear equations

$$\left(\frac{\partial F}{\partial x}\right)_{i0} u_{i1,0} + \left(\frac{\partial F}{\partial t}\right)_{i0} \Delta t_{i1} + \left(\frac{\partial F}{\partial \varepsilon}\right)_{i0} = 0, \quad i = 1, \dots, h. \tag{C.20}$$

The derivatives with index "$i0$" are calculated at $x = \xi_m, t = t_{i0}, \varepsilon = 0$.

In order to show the structure of quantities obtained at the expansion in terms of powers of ε we will consider an additional equation for Δt_{i2}. We obtain

$$\frac{1}{2}\left[u_{i1,0}^{\mathrm{T}} \left(\frac{\partial^2 F}{\partial x^2}\right)_{i0} u_{1i,0} + \left(\frac{\partial^2 F}{\partial t^2}\right)_{i0} \Delta t_{i1}^2 + \left(\frac{\partial^2 F}{\partial \varepsilon^2}\right)_{i0}\right] + \\ + \left(\frac{\partial^2 F}{\partial x \partial t}\right)_{i0} u_{1i,0} \Delta t_{1i} + \left(\frac{\partial^2 F}{\partial t \partial \varepsilon}\right)_{i0} \Delta t_{i1} + \left(\frac{\partial^2 F}{\partial x \partial \varepsilon}\right)_{i0} u_{1i,0} + \\ + \left(\frac{\partial F}{\partial x}\right)_{i0} u_{1i,1} + \left(\frac{\partial F}{\partial t}\right)_{i0} \Delta t_{i2} = 0. \tag{C.21}$$

The expression for $u_{1i,1}$ is of interest

$$u_{1i,1} = -\Xi_1^{(0)} t_{i0} + \left(X(\xi_m, t_{(i0)}, 0) - \Xi_0^{(0)}\right) \Delta t_{i1} + \sum_{j=1}^{i-1} \left(\frac{\partial \Delta}{\partial t}\right)_{i0} \Delta t_{i1},$$

$$\Xi_1^{(0)} = \frac{\omega}{2\pi} \sum_{i=1}^{h} \left(\frac{\partial \Delta}{\partial t}\right)_{i0} \Delta t_{i1}. \tag{C.22}$$

In the following approximation it is necessary to expand the repeated integrals of the discontinuous functions in series in powers of ε however it is necessary to take into account that ε appears also in the integration limits. As one can see from eqs. (C.20) and (C.21) it is essential that the absolute term and the coefficient at Δt_{ir} in the linear equation for Δt_{ir} are known provided that $\Delta t_{i1}, \dots, \Delta t_{ir-1}$ are calculated.

Corrections to the time instants of discontinuity can be sought alternatively. Let us consider the equation $F(x, t, \varepsilon) = 0$. It has h solutions $t^{(i)}(x, \varepsilon)$. Substituting x from eq. (C.7) in these solutions we obtain that $t_{im}(\xi_m, \varepsilon) = t^{(i)}(\xi_m + \varepsilon u^{(1)} + \dots + \varepsilon^{m-1} u^{(m-1)}, \varepsilon)$. Using eq. (C.5) we find Δt_{ir} which will be expressed in terms of functions of $t^{(i)}$ and their derivatives.

The described computations yield the expansions of functions $\Xi^{(r)}(\xi_m, \varepsilon)$. Inserting these expansions into eq. (C.7) and keeping the required powers of ε we obtain eq. (C.4). In the first approximation it is necessary to find only values $t_{i0}, \Delta(\xi_m, t_{i0}, 0)$ and function $\Xi = \Xi_0^{(0)}$. Up to the designations we obtain the results reported e.g. in [101].

Under computation of the discontinuities of solution x_m

$$x_m(t_i + 0) - x_m(t_i - 0) = \varepsilon \Delta(\xi_m, t_{im}, 0) + \varepsilon^2 \Delta_{i2} + \dots + \varepsilon^{m-1} \Delta_{im1} \tag{C.23}$$

we obtain the terms with the "redundant" powers of ε because $t_{im}, \dots, t_{hm}$ are also functions of ε. These "redundant" powers of ε should be eliminated by a re-expansion with the help of eq. (C.5).

As a result all values which are independent of the explicitly entering time t will be presented by their expansions in terms of ε. Only functions u_r will be not decomposed.

Let the initial condition $x(t_0)$ be given for the original problem. From eq. (C.3) we can find the corresponding initial condition for ξ_m. Let the solution $\xi = f(t)$ of equation (C.4) be found for these initial conditions. The approximate solution of the original problem is obtained provided that $\xi_m = f(t)$ is substitute into eq. (C.3). Then the argument of the Heaviside function in the expressions for u_r is the difference $t - t_{im}\left(f(t), \varepsilon\right)$. Hence we can calculate the time instances of the discontinuities $t_m^{(ik)}$ as solutions of the equations $t_m^{(ik)} = T_{im}\left(f(t_m^{(ik)}), \varepsilon\right)$. Under $T_{im}(\xi_m, \varepsilon)$ we understand an infinite-valued function which is obtained if one takes h dependences $t_{im}(\xi_m, \varepsilon)$ in the half-interval $0 \le t < 2\pi/\omega$ and continues them $2\pi/\omega-$periodically along axis T_{im}. Index k takes values $k = k_1, k_1 + 1, \dots$, where k_1 is dependent on t_0. It is also possible to find values ξ_m at discontinuities from the equation $\xi_m^{(ik)} = f\left(T_{im}(\xi_m^{(ik)}, \varepsilon)\right)$ and then find $t_m^{(ik)}$.

It is also possible to use a variant of the method when the corrections to the time instants of discontinuity Δ_{ik} are sought together with functions $u_r^{(k)}, k = 1, \dots, m$, appearing in the required approximations. Let us assume that t_{i0} can be found without knowledge of functions $u_{(k)}$. If we know t_{i0} we can found $u_1^{(1)}$ in the form of eq. (C.10) however with the discontinuities at $t = t_{i0}$. Having such functions we can find the corrections Δt_{i1} and construct function $u_1^{(2)}$ with discontinuities at $t = t_{i0} + \varepsilon\Delta_{i1}$. i.e. this variant of the method is concerned with the functions having discontinuities first at t_{i0} and then at $t_{i0} + \varepsilon\Delta_{i1}$ and so on.

Appendix D

On qualitative investigation of motion by the asymptotic methods of nonlinear mechanics

The general theorem on the asymptotic methods for the systems in standard form [83] or systems with many fast and slow variables [128] allows one to judge whether the exact and approximate solutions are close in a finite interval of time of the order of $1/\varepsilon$ where ε is a small parameter. For study of the features of solutions of these systems in an infinite time interval, one uses, firstly, the theorems on existence of the exact solutions of the original equation in a certain form (for example, quasi-periodic solutions) obtained by the method of integral manifolds [84], secondly, the theorems on approximation of the exact solutions by approximate ones in the infinite time interval. Banfi's theorem [31, 38] and its generalization to the systems with many fast variables [31] were proved to be useful among the theorems of the second group.

It should be mentioned that these results are applicable only under rather hard restrictions imposed on the solutions of the averaged system. These restrictions are uniform asymptotic stability in the case of Banfi's theorem, existence of a limit cycle for the equation of first approximation in the case of the theorem about quasi-periodic solutions etc. Specifically, one can not handle the case in which the averaged system is "indifferent" in the first approximation, for example, it is conservative whereas the attenuation and the limit cycle are exposed only in higher approximations. Then the approximation does not provide one with a correct approximate solution of the original equation in the infinite time interval. It can be seen already in the case of uniform exponential stability detected in higher approximations [96].

Similar situation is observed at study of slow motions of a rigid body in an alternating magnetic field. In the most interesting case of "pure mechanical" potential generalized forces, for example the moments of gravity force, the first approximation in the averaged equations of motion describes a conservative system whose motions which can be qualitatively changed by the forces of higher order of smallness. Therefore the second approximations are necessary for deriving the averaged equations describing various motions such as attenuating or divergent ones.

In that follows we suggest a simple method of purely qualitative investigation of motions with the help of the asymptotic methods of nonlinear mechanics under the circumstances when the exact solutions of the initial system and the approximate solutions obtained by the averaging method are close only in a finite time interval. The qualitative comparison of the accurate and approximate solutions will be carried out for the systems in standard form and quasi-linear systems with many fast variables.

Let $x(t)$ denote the sought $n \times 1$ column-vector governed by the system in standard form

$$\dot{x} = \varepsilon X(x, t, \varepsilon), \quad \varepsilon \geq 0 \tag{D.1}$$

and $x^{(m)}(\xi^{(m)}, t, \varepsilon)$ denote $m - th$ approximation to $x(t)$ obtained by the averaging method

$$x^{(m)} = \xi^{(m)} + \varepsilon u_1(\xi^{(m)}, t) + \ldots + \varepsilon^{m-1} u_{m-1}(\xi^{(m)}, t). \tag{D.2}$$

The equation for $\xi^{(m)}$ is given by

$$d\xi^{(m)}/dt = \varepsilon \Xi_1(\xi^{(m)}) + \ldots + \varepsilon^m \Xi_m(\xi^{(m)}) = \varepsilon \Xi^{(m)}(\xi^{(m)}, \varepsilon). \tag{D.3}$$

It is assumed that for $t \geq t_*, \varepsilon \leq \varepsilon_0$ and for x from a certain region D function X is continuous with respect to t and uniformly bounded along with the derivative of order m with respect to x and ε, and functions $\Xi_1, \ldots, \Xi_m$ and $u_1, \ldots, u_{m-1}$ are uniformly bounded along with the first order derivatives with respect to $\xi^{(m)}$ respectively for $\xi^{(m)} \in D$ and $\xi^{(m)} \in D$, $t \geq t_*$.

Let us consider the function $\xi_m(t, \varepsilon)$ defined by the relationship

$$x = \xi_m + \varepsilon u_1(\xi_m, t) + \ldots + \varepsilon^{m-1} u_{m-1}(\xi_m, t). \tag{D.4}$$

The results will be the same if we introduce $\xi_m(t, \varepsilon)$ by a similar relationship with the component $O(\varepsilon^m)$.

The following estimate of closeness of functions ξ_m and $\xi^{(m)}$ is known. Let $x(t_0) \in D_\alpha$ where region D_α is such that its α-vicinity, $\alpha = O(\varepsilon)$, coincides with D. We will find $\xi_m(t_0)$ with the help of eq. (D.4), then $\xi_m(t_0) \in D_\alpha$ for sufficiently small values of ε. Let the solution of eq. (D.3) for the initial condition $\xi^{(m)}(t_0) = \xi_m(t_0)$ remains in D_α within the time interval $t_0 \le t \le t_0 + T/\varepsilon$. Then for functions $\xi^{(m)}, \xi_m$ obeying the above initial conditions the relationship

$$|\xi_m(t) - \xi^{(m)}(t)| \le c_m \varepsilon^m, \quad t_0 \le t \le t_0 + T/\varepsilon$$

is valid for sufficiently small $\varepsilon \le \varepsilon_*$, constants c_m and T being independent of ε.

In what follows we point out some cases when some features of function ξ_m in an infinite time interval can be established by means of the analogous features of functions $\xi^{(m)}$.

Let us consider a bounded domain D_1 and its $\delta-$vicinity $D_\delta, \delta = d\varepsilon^{m-1}$, $d = \text{const} > 0$, where $D_\delta \subset D_\alpha$, and a single-valued scalar function $V(\xi, \varepsilon)$ defined in $D_\delta, \varepsilon \le \varepsilon_*$. It is assumed that $|\partial V/\partial \xi| \ne 0$ in $\xi \le D_\delta, \varepsilon \le \varepsilon_*$ and there exist

$$V_M = \sup_{\xi \in D_\delta, \varepsilon \le \varepsilon_*} V(\xi, \varepsilon), \quad F = \sup_{\xi \in D_\delta, \varepsilon \le \varepsilon_*} \left| \frac{\partial V}{\partial \xi} \right|. \tag{D.5}$$

Theorem 1. *Let us assume that under the above assumptions there exists such a function $V(\xi, \varepsilon)$ that for any solution $\xi^{(m)}(t)$ of eq. (D.3) remaining in D_δ within some time interval $t^{(0)} \le t \le t^{(0)} + T/\varepsilon$ the following inequality*

$$V(\xi^{(m)}(t^{(0)} + T/\varepsilon)) \ge V(\xi^{(m)}(t^{(0)})) + \varepsilon^{m-1} W_0 \tag{D.6}$$

holds for the same $W_0 > 0$ for all $t^{(0)} \ge t_*, \varepsilon \le \varepsilon_*$ and all $\xi^{(m)}(t)$.

Then there exists no solution $\xi_m(t)$ remaining in D_1 for all $t \ge t_*$.

Remarks. 1. Condition (D.6) is automatically satisfied if there exists a function $V(\xi, \varepsilon)$ in D_δ whose derivative satisfies the relationship $\dot{V} \ge \varepsilon^m w_0, w_0 = \text{const} > 0$, by virtue of eq. (D.3). Then it can be accepted that $W_0 = w_0 T$.

2. It immediately follows from condition (D.6) that there exists no solution $\xi^{(m)}(t)$ remaining in D_δ for all $t \ge t_*$. Indeed, let $\xi^{(m)}(t_0) \in D_\delta$. Let us consider the time interval $t_0 \le t \le t_0 + kT/\varepsilon$, where k is an integer. After this time interval function V gains an increment $V(\xi^{(m)}(t_0 + kT/\varepsilon)) - V(\xi^{(m)}(t_0)) \ge k\varepsilon^{m-1} W_0$ for any solution remaining in D_δ. As a result, for a sufficiently great k the value of function $V(\xi, \varepsilon)$ for the solution $\xi^{(m)}(t)$ exceeds V_M, which is impossible.

Proof. Let us compare the sequence of "approximate" solutions $\xi_j^{(m)}(t)$, $j = 0, 1, \dots$, defined by the conditions

$$\xi_0^{(m)}(t_0) = \xi_m(t_0),$$
$$\xi_1^{(m)}(t_0 + T/\varepsilon) = \xi_m(t_0 + T/\varepsilon), \dots,$$
$$\xi_j^{(m)}(t_0 + jT/\varepsilon) = \xi_m(t_0 + jT/\varepsilon)$$

with the “exact” solution $\xi_m(t), \xi_m(t_0) \in D_1$. Let us prove that it is not possible that $\xi_j^{(m)}(t) \in D_\delta$ for all arbitrary large j in the interval $t_0+jT/\varepsilon \leq t \leq t_0 + (j+1)T/\varepsilon$. Let $\xi_j^{(m)}(t) \in D_\delta, t_0 + jT/\varepsilon \leq t \leq t_0 + (j+1)T/\varepsilon$ for all j. Then $\xi_m(t_0 + jT/\varepsilon) \in D_\delta$ for all j. For the solution $\xi_0^{(m)}(t)$ at $t_0 \leq t \leq t_0 + T/\varepsilon$ function $V(\xi, \varepsilon)$ obtains the increment

$$V\left(\xi_0^{(m)}(t_0 + T/\varepsilon)\right) - V\left(\xi_0^{(m)}(t_0)\right) \geq \varepsilon^{m-1} W_0 \tag{D.7}$$

by virtue of the theorem conditions.

Let us now consider the value $V(\xi_m(t_0 + T/\varepsilon))$. According to eq. (D.5) we have

$$\left|V\left(\xi_m(t_0 + T/\varepsilon)\right) - V\left(\xi_0^{(m)}(t_0 + T/\varepsilon)\right)\right| \leq \\ \leq F\left|\xi_m(t_0 + T/\varepsilon) - \xi_0^{(m)}(t_0 + T/\varepsilon)\right| \leq F c_m \varepsilon^m. \tag{D.8}$$

Hence

$$V(\xi_m(t_0 + T/\varepsilon)) \geq V(\xi_0\,(t_0 + T/\varepsilon)) - \varepsilon^m F c_m. \tag{D.9}$$

Taking into account that $V(\xi_0^{(m)}(t_0) = V(\xi_m(t_0))$ and denoting $W = W_0(1 - \varepsilon_* F c_m / W_0)$ we obtain the following estimate from eqs. (D.7) and (D.9)

$$V(\xi_m(t_0 + T/\varepsilon)) - V(\xi_m(t_0)) \geq \varepsilon^{m-1} W. \tag{D.10}$$

By analogy with the increment $V(\xi_1^{(m)}(t))$ in the time interval $t_0+T/\varepsilon \leq t \leq t_0+2T/\varepsilon$ one can estimate $V(\xi_m(t_0+2T/\varepsilon)$. Analogously to eq. (D.10) we obtain

$$V(\xi_m(t_0 + 2T/\varepsilon)) - V(\xi_m(t_0 + T/\varepsilon)) \geq \varepsilon^{m-1} W.$$

From this equation and eq. (D.10) we have

$$V(\xi_m(t_0 + 2T/\varepsilon)) - V(\xi_m(t_0)) \geq 2\varepsilon^{m-1} W. \tag{D.11}$$

Using $j+1$ functions $\xi_0^{(m)}, \ldots, \xi_j^{(m)}$ yields

$$V(\xi_m(t_0 + (j+1)T/\varepsilon)) - V(\xi_m(t_0)) \geq (j+1)\varepsilon^{m-1} W. \tag{D.12}$$

As a result value $V(\xi_m(t_0+(j+1)T/\varepsilon))$ exceeds V_M for a sufficiently great j which is impossible.

Thus, there exists such k and $t_1, t - 0 + kT/\varepsilon \leq t_1 \leq t_0 + (k+1)T/\varepsilon$ that $\xi_k^{(m)}(t_1) \notin D_1$. In a similar manner one can show that $\xi_m(t)$ does not leave not only D_1 but also any $\theta\delta$-vicinity of D_1, where $\theta < 1$ and θ is independent of ε.

It is helpful to estimate time Δt which is needed for solution $\xi_m(t)$ to leave region D_1

$$\Delta t \leq \frac{V_M}{\varepsilon^m W} = \frac{T_1}{\varepsilon^m}.$$

Theorem 2. *Let function V satisfy the conditions of Theorems 1 and besides the derivative $\dot{V}$ computed by virtue of equation (D.3) is nonnegative everywhere in D_δ. Let region D_1 be bounded by the surfaces $V = C_1, V = C_2, C_2 > C_1$ and all surfaces of the family $V = C$ are closed. Then any solution $\xi_m(t), \xi_m(t_0) \in D_1$ after some time will cross the surface $V = C_2$ and leave region D_1 forever.*

Remark. Let surface $V = C$ be denoted as $S(C)$. It follows from the above features of the derivative $\partial V/\partial\xi$ that family $S(C)$ has no singularities in D_δ. Since the surfaces are closed they cover each another. Let us assume that for all $C' < C''$ the surface $S(C')$ covers $S(C'')$ and this enables using the expressions "a region outside $S(C_1)$", "a region inside $S(C_2)$" etc. The case when $S(C'')$ covers $S(C')$ is considered by analogy.

Proof. First we show that $\xi_m(t)$ will necessarily fall in $S(C_2)$. An "approximate" solution $\xi_0^{(m)}(t)$ can leave D_δ only by crossing $S(C_2)$. If $\xi_0^{(m)}(t)$ leaves D_δ in the time interval $t_0 \le t \le t_0 + T/\varepsilon$ then $\xi_m(t)$ leaves D_1 in the same time interval through $S(C_2)$ (if $\xi_0^{(m)}(t)$ lies near the boundary $S(C_1)$ then $\xi_m(t)$ can leave D_1 through this boundary however it is not important).

Let $\xi_0^{(m)}(t) \in D_\delta$ for $t_0 \le t \le t_0 + T/\varepsilon$. The point $\xi_0^{(m)}(t_0 + T/\varepsilon)$ lies in $S(C_1)$ on the distance ρ from it. For ρ we have an estimate (analogous to (D.8)) $\rho \ge \varepsilon^{m-1}W_0/F$. By virtue of (D.5) point $\xi_m(t_0 + T/\varepsilon)$ is also in $S(C_1)$ on the distance $\rho_1 = O(\rho)$ from it. Considering function $\xi_1^{(m)}(t)$ we obtain that point $\xi_m(t_0 + 2T/\varepsilon)$ is in $S(C_1)$ on the distance $\rho_2 \ge \rho_1 + \varepsilon^{m-1}W/F$ from the boundary. In the second interval the point $\xi_m(t)$ can not be outside $S(C_1)$. Then we obtain $\rho_3 \ge \rho_1 + 2\varepsilon^{m-1}W/F$ and so on. Thus point $\xi_m(t)$ which must leave D_1 according to Theorem 1, leaves D_1 through $S(C_2)$.

Let now $\xi_m(t_1)$ lies in $S(C_2)$. Let us assume that some number of solutions $\xi_j^{(m)}(t)$, $\xi_0^{(m)}(t_1)$ etc. remain in D_δ. Then already in the second time interval $t_1 + T/\varepsilon \le t \le t_1 + 2T/\varepsilon$ the solution $\xi_m(t)$ can not leave $S(C_2)$ just like the solution $\xi_m(t)$ could not leave $S(C_1)$ in the corresponding intervals. There remains the case in which solution $\xi_k^{(m)}(t)$ escapes D_δ. Let C_M denote the largest value of C for which $S(C)$ lies completely in D_δ, then $C_M - C_2 = O(\varepsilon^{m-1})$. In the time interval $t_1 + kT/\varepsilon \le t \le t_1 + (k+1)T/\varepsilon$ point $\xi_m(t)$ can lie outside $S(C_M)$ only on the distance $O(\varepsilon^m)$ from the boundary. It is now clear that $\xi_m(t)$ can not reach D_1 in the next interval irrespective whether $\xi_{k+1}^{(m)}(t)$ enters D_δ or escapes it.

From Theorem 2 follows

Theorem 3. *Let surface $S(C)$ collapse into a point $C \to C_*$. Assume that for any infinitesimally small $\eta > 0$ there exists such $\varepsilon(\eta)$ that for $\varepsilon \le \varepsilon(\eta) \le \varepsilon_*$ and $C_2 - C_* = \eta$ Theorem 2 holds true. Then for sufficiently small ε beginning with a certain $t = t(\eta)$ the solution $\xi_m(t)$ stays forever in the infinitesimally small $\eta-$vicinity of the point $V = C_*$.*

According to eq. (D.4), solution $x(t)$ of the initial system (D.1) for sufficiently great values of t remains in $(\eta+O(\varepsilon))-$vicinity of the point $V = C_*$. Let point $V = C_*$ correspond to the equilibrium position of a mechanical system. Then the vibrations described by the initial system (D.1) will qualitatively present the superposition of (D.4) of a slow evolutional motion tending "almost" to the equilibrium position and small fast vibrations. For sufficiently great t the motion reduces to the small fast vibrations about the mean position, which probably is slowly wandering in a small vicinity of the equilibrium position. Such vibrations slightly differ from the quasi-static ones.

In the case when surface $S(C'')$ covers $S(C')$ at $C'' > C'$ it follows from Theorem 2 that solution $\xi_m(t)$ leaves surface $S(C_2)$ forever. Provided that the condition of Theorem 2 are met for sufficiently large or even arbitrary large values of C_2 then $x(t)$ describes the superposition of small vibrations and the "outgoing" motion.

After the evident modifications in the formulations of Theorems 1-3 it is possible to rely on existence of decreasing functions V rather than increasing ones. For example, in Theorem 1 we can take the condition

$$V\left(\xi^{(m)}(t_0 + T/\varepsilon)\right) - V\left(\xi^{(m)}(t^{(0)})\right) \leq -\varepsilon^{m-1} W_0$$

assuming that $\inf V$ exists in D_δ.

Function V can be found in the case in which eq. (D.3) admits first integral $V(\xi, \varepsilon) = \text{const}$ at the first approximation or in the several lowest approximations and in the following approximation this integral vanishes and sign of $\dot{V}$ can be determined. Clearly, in this case $\dot{V} = O(\varepsilon^m)$, m denoting the number of approximation the integral vanishes for the first time. The simplest case is that when $V = \text{const}$ is the energy integral and the relationship $\dot{V} = 0$ is violated because of the dissipation in higher approximations.

The above theorems can be generalised on the systems different from the systems in standard form provided that the exact and approximate solutions are proved to be close in the interval T/ε and the determination of the approximate solution is reduced to integration of the autonomous system. This is the case of quasi-linear system with many fast variables different from the phases

$$\begin{aligned} \dot{x} &= \varepsilon X(x, y, t, \varepsilon), \\ \dot{y} &= A(x)y + f(x, t). \end{aligned} \tag{D.13}$$

Here x, y denote $n \times 1$ and $l \times 1$ vectors-columns and matrix $A(x)$ is such that its eigenvalues $\lambda_1(x), \ldots, \lambda_l(x)$ satisfy the condition $\operatorname{Re} \lambda_i \leq -\mu < 0$, $\mu = \text{const}$.

For system (D.13) we have [82, 96]

$$|\xi_m - \xi^{(m)}| \leq c_m \varepsilon^m, \quad |x - x_m| \leq c^{(m)} \varepsilon^m,$$
$$|x - x_m^{(m)}| \leq c_m^{(m)} \varepsilon^m, \quad |y - y^{(m-1)}(\xi^{(m)}, t, \varepsilon)| \leq b_m \varepsilon^m. \quad \text{(D.14)}$$

Here

$$y^{(m-1)}(\xi_m, t, \varepsilon) = \phi^{(m-1)}(x(\xi_m, t, \varepsilon), t, \varepsilon) = \sum_{i=0}^{m1} \varepsilon^i \phi_i(x, t, \varepsilon) \quad \text{(D.15)}$$

and functions $\phi_0, \phi_1, \ldots, \phi_{m-1}$ are determined from the linear equations

$$\dot{\phi}_0 = A\phi_0 + f,$$
$$\dot{\phi}_1 = A\phi_1 - \frac{\partial \phi_0}{\partial x} X(x, \phi_0, t, 0) \quad \text{(D.16)}$$

etc. which are integrated if $x = \text{const}$ and under the initial conditions $\phi_0(x(t_0), t_0) = y(t_0), \phi_i(x(t_0), t_0) = 0$. Next, x_m in eq. (D.14) denotes the solution of the system in standard form

$$\dot{x}_m = \varepsilon X(x_m, \phi^{(m-1)}(x_m, t, \varepsilon), t, \varepsilon), \quad \text{(D.17)}$$

where $x_m^{(m)}$ is $m - th$ approximation to x_m in the form of eq. (D.2) and $\xi_m, \xi^{(m)}$ are introduced for system (D.17) by analogy with (D.1). For $\xi^{(m)}$ one obtains the autonomous system in the form of (D.3), then $\xi^{(m)}$ and ξ_m can be qualitatively compared with the help of above theorems. The corresponding properties of the sought functions $x(t)$ and $y(t)$ can be proved, too.

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