PROBLEMS

1. [H. Hilden] Suppose G is a transitive subgroup of the permutation group Σ_n and π is a partition of n. A branched covering $p: M^3 \to S^3$, branched over the link L, given by $\rho: \pi_1(S^3 - L) \to \Sigma_n$ is of type G, π if im $\rho \subset G$ and meridians $\to$ π-cycles. (e.g. in Σ_7 the cycle (154)(32) is of type $3 + 2 + 1 + 1$).

   (a) Is every 3-manifold a G,π covering of S^3, branched over a knot? (Answer is yes for $n = 3$, $G = \Sigma_3$, $\pi = 2 + 1$; unknown for $n = 4$, $G = A_4$, $\pi = 3 + 1$.

   (b) Is there a G,π type covering $S^3 \to S^3$?

2. [J. Birman] Let F be a compact surface and $\mathcal{M}(F) = \mathrm{Diff}^+F/_{\mathrm{Iso}F}$ be its mapping class group. Theorem (Kerckhoff): If G is a finite subgroup, then it is realizable as a group of diffeomorphisms. Is there a subgroup $G \subset \mathcal{M}(F)$ which is not so realizable? (in particular, $G = \mathcal{M}(F)$).

3. [J. Hempel] Does there exist a complete hyperbolic 3-manifold of finite volume whose fundamental group has the finitely-generated intersection property?

4. [J. Hempel] Let α be a loop on the surface F. Is a lift of α simple in the covering $F_{\langle\alpha\rangle}$, corresponding to the normal closure of α ?

5. [J. Birman] Consider the mapping class $\mathcal{M}(F)$ of a surface $F \# T^2 - \{pt.\}$. Is $\mathcal{M}(F)$ linear (i.e. faithfully representable in a matrix group?) A candidate for positive answer is F = disk − n points, so that $\mathcal{M}(F)$ = the n-strand braid group.

6. [K. Millett] Immerse RP^2 in R^3 and project onto R^2. What is the nicest "fold locus" possible? (There must be a cusp.) Is this the simplest?

7. [E. Luft] If $|\pi_1(M^3)| = \infty$, is there a nontrivial normal subgroup?

8. [E. Luft] Is there a group with equal number (finite) of generators and relations, such that every proper subgroup is isomorphic with Z?

9. [E. Luft] Is there a group such that #gens = #rels. − 1 and each element is infinitely divisible?

10. [D. Gabai] Let M^3 be closed, oriented, with $H_2(M) \cong Z$. Let genus(M) = the smallest g such that there exists a surface S of genus g with $[S] \neq 0$

in $H_2(M)$. Given a knot K in S^3, is genus (K) = genus(M) where M is the manifold obtained by 0-framed surgery on K? (An affirmative answer would imply property R and the Poenaru conjecture that one can't get $S^2 \times S^1 \# N$ by such surgery.)

11. [D. Rolfsen] Is there a link of $S^2 \cup S^2$ in S^4 which cannot be separated by a homotopy which keeps the components disjoint?

12. [D. Rolfsen] Same for $S^2 \cup S^3$ in S^5. A related question is whether an embedding $S^3 \to S^2 \times R^3$ can be homotopically essential.*

13. [K. Millett] Does there exist an acyclic 3-manifold smoothly foliated by circles? What about R^3?

[Questions 14-27 were submitted by S.J. Lomonaco, Jr.].

Definition. An n-knot (S^{n+2}, kS^n) is a smooth imbedding $k:S^n \to S^{n+2}$ of the n-sphere into the (n+2)-sphere. The space $X = S^{n+2} - kS^n$ is called the complement of the knot. The fundamental group of X is denoted by π_1X. The second homotopy group (considered as a left $Z\pi_1X$-module) of X is denoted by π_2X. $\underline{\underline{k}}X$ denotes the first k-invariant of X $(\underline{\underline{k}}X \in H^3(\pi,X;\pi_2X))$.

14. Let X and X^1 be smooth 2-knot complements. Is X of the same homotopy type as X^1 iff X and X^1 have the same algebraic 2-type? (The algebraic 2-type of X is the triple consisting of π_1X, π_2X, and $\underline{\underline{k}}X$.) If so, this is a four dimensional analogue of the asphericity of classical knots. (see [1,§10].

15. Let X and X^1 be smooth n-knot complements $(n \geqslant 3)$. Is X of the same homotopy type as X^1 iff X and X^1 are of the same n-type. (see [1,§10], [2], [3].) (This is called the (n+1)-type in [3].)

16. (Problem 37 of Fox [4].) Are there any non-trivial smooth aspherical 2-knots? This appears to be a very difficult problem. The reader should note that a smooth 2-knot is aspherical iff the fundamental group π_1X of its complement X is infinite cyclic.

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* Such embeddings have been discovered recently by N. Habegger. They have even Hopf invariant -- odd Hopf invariant for an embedding of S^3 in $S^2 \times R^3$ is impossible.

16. (Problem 37 of Fox [4].) Are there any non-trivial smooth aspherical 2-knots? This appears to be a very difficult problem. The reader should note that a smooth 2-knot is aspherical iff the fundamental group $\pi_1 X$ of its complement X is infinite cyclic.

17. Let X be the complement of a smooth 2-knot. Is $\pi_2 X$ always finitely generated as a left $Z\pi_1 X$-module?

18. Let X be as in Problem 17. Is $\pi_2 X$, considered only as a group, always a free abelian group?

19. Let X be as in Problem 17. Is $\pi_2 X$ always finitely related as a left $Z\pi_1 X$-module?

20. Let X be as in Problem 17. If $\pi_2 X$ vanishes, then is X always aspherical?

21. Does there exist a 2-knot whose fundamental group has no non-trivial elements of finite order but whose k-invariant is non-zero? (see [1, Cor. 11,7].)

22. Do there exist two 2-knots with complements X and X^1 respectively such that $\pi_1 X \cong \pi_1 X^1$ and $\pi_2 X \cong \pi_2 X^1$ (as left $Z\pi_1 X$-modules) but with distinct k-invariants? ("$\cong$" denotes "isomorpshism.")

23. Let X be as in problem 17. Is $\underline{\underline{k}}X \neq 0$ iff $H_3 \pi_1 X \neq 0$? (see [1].)

24. Find an easy way to compute the k-invariant of a 2-knot complement.

25. When does a 2-knot diagram represent the trivial 2-knot. (see [1,§2] or [5] for definition of 2-knot diagram.)

26. When do two 2-knot diagrams represent the same 2- knot?

27. Construct a modern analogue of the classical 1- knot tables, i.e., construct a table of all 2-knot diagrams with ≤ 6 (or 7, or 8, ...) crossings and with $\leqslant 2$ (or 3 or 4, ...) hyperbolic points.

28. [S. Bleiler]. Is there a prime tangle such that distinct untangle attachments yield the unknot? Remarks: Such a tangle would be double branch covered by the exterior of a non-trivial knot in S which <u>fails</u> Property P. There are

non-prime tangles (e.g. an untangle) with distinct untangle attachments yielding the unknot, but the 2-fold br. cover is a solid torus, i.e. the exterior of the unknot.

REFERENCES FOR PROBLEMS

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