APPENDIX A ANALYTIC PROPERTIES OF THE SINGULAR VALUES OF A RATIONAL MATRIX

§A.1 Analytic Properties of the Characteristic Values

A rational matrix $G(s) \varepsilon \mathbb{R}(s)^{m \times m}$ has an associated set of frequency-dependent eigenvalues $\{g_i(s) \mid i=1,\ldots,m\}$, usually called the characteristic gains. If $G(s)$ has a right coprime MFD

$$G(s) = N(s)D(s)^{-1} \tag{A.1.1}$$

then these are determined from the equation

$$\Delta(s,g) := \det[gD(s) - N(s)] = 0$$

where $\Delta(s,g)$ is unique up to a possible constant for all possible right coprime factorizations of $G(s)$. $\Delta(s,g)$ can be factored into:

$$\Delta(s,g) = e(s)\Delta_1(s,g)\Delta_2(s,g)\cdots\Delta_q(s,g)c(g)$$

where $\{\Delta_i(s,g) \mid i=1,2,\ldots,q\}$ are the irreducible factors of $\Delta(s,g)$ dependent on both s and g, and the other factors represent factors independent of s ($c(g)$) or g ($e(s)$). If we assume, for simplicity of exposition,

$$e(s) = 1 \quad \text{and} \quad c(g) = 1$$

and take $\Delta(s,g)$ to be irreducible, and of the form

$$\Delta(s,g) = a_m(s)g^m + a_{m-1}(s)g^{m-1} + \cdots + a_o(s) \tag{A.1.2}$$

then this (implicitly) defines g as an <u>algebraic function</u> of s. For all s such that $a_m(s) \neq 0$, (A.1.2) defines an m-valued relation:

$$g(s) : \mathbb{C} - \{s \varepsilon \mathbb{C} \mid a_m(s) = 0\} \rightarrow \mathbb{C}$$

If $\Omega \subset \mathbb{C}$ is a domain which excludes all singular points (poles and branch points) then for $s \varepsilon \Omega$ the characteristic gain functions $\{g_i(s) \mid i=1,2,\cdots,m\}$ are (complex) analytic functions of s. In global terms these can be organized into a "single" analytic function $g(s)$ — the characteristic gain function — whose domain is a Riemann surface. For further information on the analytic properties of the characteristic gain functions and the background algebraic function theory, the reader may consult [POS1],[SMI1],[BLI].

§A.2 Analytic Properties of the Singular Values

Let $G(s) \in \mathbb{R}(s)^{m\times \ell}$. For notational simplicity, we assume $m > \ell$ throughout this section. Let $G(s)$ have a right coprime MFD given by

$$G(s) = N(s)D(s)^{-1} \tag{A.2.1}$$

Then the squares of the singular values, that is the eigenvalues of $G(s)^*G(s) = G(\bar{s})^T G(s)$, are determined by solving the equation

$$\det\left[\sigma^2 D(\bar{s})^T D(s) - N(\bar{s})^T N(s)\right] = 0$$

and one could attempt to investigate the analytic properties of the frequency-dependent singular values by considering a "squared-singular-value function" $\sigma^2(s,\bar{s})$ defined implicitly via the algebraic equation

$$\chi(s,\bar{s},\sigma^2) := \det\left[\sigma^2 D(\bar{s})^T D(s) - N(\bar{s})^T N(s)\right] = 0$$

This is not a very satisfactory approach, and, in order to explore the use of existing results for functions of several complex variables, it turns out to be better to consider the equation in three indeterminates s_1, s_2 and σ^2 given by

$$\chi(s_1,s_2,\sigma^2) := \det\left[\sigma^2 D(s_2)^T D(s_1) - N(s_2)^T N(s_1)\right] = 0 \tag{A.2.2}$$

This frees us from the awkwardness of handling the conjugate term, and the investigation proceeds first over the two complex variables s_1 and s_2, then specializes down to the required result by putting $s_1 = s$ and $s_2 = \bar{s}$. Before proceeding further it will be helpful to give some necessary background definitions and results.

§A.2.1 Analytic Sets, Pseudopolynomials and Real-Analytic Funtions

An analytic set is a subset of $\mathbb{C}^n$ defined locally by the zeros of a set of holomorphic functions. Let Ω be an open set in $\mathbb{C}^n$. Then $A \subset \Omega$ is called an analytic set (e.g. see [NAR, pp.50]) if $\forall a \in A$, $\exists$ a neighbourhood N of a and finitely many holomorphic functions $f_1, f_2, \ldots, f_p$ in N s.t.

$$N \cap A = \{ z \in N \mid f_1(z) = f_2(z) = \cdots = f_p(z) = 0 \}$$

Let $z := (z_1, \ldots, z_n) \in \mathbb{C}^n$. Then a function

$$f(z,\nu) := \nu^n + a_1(z)\nu^{n-1} + \cdots + a_n(z) \qquad \text{(A.2.3)}$$

is called a pseudopolynomial when the coefficients $\{a_i(z) \mid i = 1, \ldots, n\}$ are holomorphic functions of z on some open set $\Omega \subset \mathbb{C}^n$. If Ω is connected, then the set of all holomorphic functions on Ω is an integral domain. Moreover, one can consider factorizing the pseudopolynomial, regarded as a polynomial in ν, into a set of factors unique over this integral domain (e.g. see [GRA, Chapter 3 §6]). Furthermore $f(z,\nu)$ will have an associated pseudopolynomial discriminant function $D_\nu(f)$. At any z' for which $D_\nu(f)$ vanishes, $f(z',\nu)$ will have repeated roots in ν. The totality of these, defined as the zero set of the discriminant function, is an analytic set.

We can now state the key result we require (see [GRA, Chapter III Theorem 6.12]).

Theorem A.2.1

Let $\Omega \subset \mathbb{C}^n$ be a domain and let $f(z,\nu)$ be a pseudopolynomial in ν whose discriminant $D_\nu(f)$ does not vanish identically in z. Then the sets of zeros

$$Z(f) := \{ (z,\nu) \in \Omega \times \mathbb{C} \mid f(z,\nu) = 0 \}$$

$$Z(D_\nu(f)) := \{ z \in \Omega \mid D_\nu(f) = 0 \}$$

are both analytic sets.

For any $z° \in \Omega - Z(D_\nu(f))$ there exists an open neighbourhood of z°, N, and a set of holomorphic functions $\{\nu_1, \nu_2, \ldots, \nu_n\}$ on N, with $\nu_i(z) \neq \nu_j(z)$ for $i \neq j$ and $\forall z \in N$ s.t.

$$f(z,\nu) = (\nu - \nu_1(z))(\nu - \nu_2(z)) \cdots (\nu - \nu_n(z)) \qquad \forall z \in N$$

□

Finally we need to define the term real-analytic.

Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $f(x)$ be a real-valued function of $x = (x_1, \ldots, x_n) \in \Omega$. Then $f(x)$ is said to be real-analytic at $x° = (x_1°, \ldots, x_n°) \in \Omega$ if there exists a neighbourhood N of $x°$ and a set of real coefficients $c_\alpha \in \mathbb{R}$ s.t.

$$f(x) = \sum_{\alpha} c_\alpha (x - x°)^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of non-negative integers and we have used the standard notation that

$$(x - x°)^\alpha := (x_1 - x_1°)^{\alpha_1} \cdots (x_n - x_n°)^{\alpha_n}$$

If f can be expanded as a convergent Taylor series for every point in Ω, then $f(x)$ is said to be real-analytic in Ω. Real analytic functions possess derivatives of all orders and the principle of analytic continuation may be applied to them (e.g. see [NAR, pp.4] or [JOH, pp.65]).

§A.2.2 Real-Analyticity of Singular-Value Functions

We now proceed to demonstrate that locally the frequency-dependent singular values $\{\sigma_i(s,\bar{s}) \mid i = 1,2,\ldots,\ell\}$ of a rational matrix $G(s) \in \mathbb{R}(s)^{m\times\ell}$ are real-analytic functions of x and y, where $s = x + jy$. Let G(s) have a right coprime MFD given by (A.2.1) and consider the polynomial equation in the indeterminates s_1, s_2 and σ^2 given by (see (A.2.2)):

$$\det[\sigma^2 D(s_2)^T D(s_1) - N(s_2)^T N(s_1)] = 0 \qquad (A.2.4)$$

Expanding the determinant we get

$$p(s_2)p(s_1)\sigma^{2\ell} + \cdots + (-1)^\ell z(s_2)z(s_1) = 0 \qquad (A.2.5)$$

where $p(\cdot) := \det[D(\cdot)]$ and $z(\cdot) := \det[N(\cdot)]$. On dividing through (A.2.5) by the leading-term's coefficient, we get

$$f(s_1, s_2, \sigma^2) := \sigma^{2\ell} + \cdots + (-1)^{\ell} \frac{z(s_2)\, z(s_1)}{p(s_2)\, p(s_1)} = 0 \qquad \text{(A.2.6)}$$

which is a pseudopolynomial in σ^2.

Now let Ω be an open set given by

$$\Omega := \left(\mathbb{C} - \{\text{zeros of } p(\cdot)\}\right) \times \left(\mathbb{C} - \{\text{zeros of } p(\cdot)\}\right)$$

so that the coefficients of the pseudopolynomial $f(s_1, s_2, \sigma^2)$ are holomorphic in s_1 and s_2 over Ω. If we assume that the discriminant $D_\nu(f)$ (here ν stands for σ^2) of the pseudopolynomial $f(s_1, s_2, \sigma^2)$ does not vanish identically in Ω then Theorem A.2.1 above tells us that

$$Z(f) = \left\{ (s_1, s_2, \sigma^2) \in \Omega \times \mathbb{C} \mid f(s_1, s_2, \sigma^2) = 0 \right\}$$

and

$$Z(D_\nu(f)) = \left\{ (s_1, s_2) \in \Omega \mid D_\nu(f) = 0 \right\}$$

are analytic sets. Furthermore, for any (s_1°, s_2°) s.t.

$$(s_1^\circ, s_2^\circ) \in \Omega - Z(D_\nu(f))$$

there will exist a neighbourhood N of (s_1°, s_2°) on which $f(s_1, s_2, \sigma^2)$ factors <u>locally</u> into distinct linear factors:

$$f(s_1, s_2, \sigma^2) = \left(\sigma^2 - \sigma_1^2(s_1, s_2)\right) \cdots \left(\sigma^2 - \sigma_\ell^2(s_1, s_2)\right) \qquad \text{(A.2.7)}$$

where each term $\sigma_i^2(s_1, s_2)$ is holomorphic for all $(s_1, s_2) \in N$. Consequently, in the neighbourhood N, the holomorphic functions $\sigma_i^2(s_1, s_2)$ — which are the "solutions" of the equation (A.2.6) — can be expanded as a Taylor series. For notational simplicity, let us take the point (s_1°, s_2°) to be simply the point $(0,0)$. The Taylor series then take the form (e.g. see [GRA, Chapter I Theorem 3.8]):

$$\sigma_i^2(s_1, s_2) = \sum_{\alpha_1, \alpha_2} d_{i\alpha_1\alpha_2}\, s_1^{\alpha_1} s_2^{\alpha_2} \qquad i = 1, 2, \ldots, \ell \qquad \text{(A.2.8)}$$

where the coefficients $d_{i\alpha_1\alpha_2}$ are given by

$$d_{i\alpha_1\alpha_2} = \frac{1}{\alpha_1!\,\alpha_2!}\,\frac{\partial^{\alpha_1+\alpha_2}\,\sigma_i^2(s_1,s_2)}{\partial s_1^{\alpha_1}\,\partial s_2^{\alpha_2}} \tag{A.2.9}$$

We may now recover the squared-singular-value functions $\{\sigma_i^2(s,\bar{s})\,|\,i=1,2,\ldots,\ell\}$ of G(s) by making the substitutions $s_1 = s$ and $s_2 = \bar{s}$. If $s = x + jy$ then we can regard $\sigma_i^2(s,\bar{s})$ as a function of the real variables $(x,y) \in \mathbb{R}^2$. With a mild abuse of notation, we shall write $\sigma_i^2(x+jy, x-jy)$ as $\sigma_i^2(x,y)$. Accordingly, let us rewrite (A.2.8) as

$$\begin{aligned}\sigma_i^2(x,y) &= \sum_{\alpha_1,\alpha_2} d_{i\alpha_1\alpha_2}\,(x+jy)^{\alpha_1}\,(x-jy)^{\alpha_2} \\ &= \sum_{\beta_1,\beta_2} (e_{i\beta_1\beta_2} + jh_{i\beta_1\beta_2})\,x^{\beta_1}\,y^{\beta_2}\end{aligned}$$

where $e_{i\beta_1\beta_2}, h_{i\beta_1\beta_2} \in \mathbb{R}$.

In this last step we have simply made a rearrangement of the Taylor series so that the indices β_1, β_2 are associated with powers of the real variables x and y. Since $\sigma_i^2(x,y)$ must be real-valued it follows that

$$h_{i\beta_1\beta_2} = 0 \qquad i = 1,2,\ldots,\ell \quad \text{and} \quad \forall\, \beta_1, \beta_2 \geq 0$$

This implies that $\sigma_i^2(x,y)$ has a <u>real</u> Taylor series expansion

$$\sigma_i^2(x,y) = \sum_{\beta_1,\beta_2} e_{i\beta_1\beta_2}\,x^{\beta_1}\,y^{\beta_2} \tag{A.2.10}$$

which will converge in an appropriate neighbourhood of $(x,y) = (0,0)$. (Recall that we have taken $(s_1^\circ, s_2^\circ) = (s^\circ, \bar{s}^\circ) = (0,0)$.)

Since $\sigma_i^2(x,y)$ is real-analytic by definition and the (positive) square-root of a real-analytic function is a real-analytic function, it follows that $\sigma_i(x,y)$ is real-analytic. To summarize then, the singular-value functions $\sigma_i(x,y)$ of the rational matrix G(s) are real-analytic functions of (x,y), provided $s = x + jy$ is neither a pole nor a "branch point" at which some singular values are equal.

APPENDIX B PROOFS OF PROP 3.2.1, PROP 3.3.1, PROP 4.5.1 AND THEOREM 4.6.2

§B.1 Proof of Prop 3.2.1

We first note that for any $G \in \mathbb{C}^{m\times m}$ (e.g. see [STO])

$$\|G\|_2 \leq \|G\| \leq m^{1/2}\|G\|_2 \tag{B.1.1}$$

Using (3.1.9), (3.1.5) and (B.1.1), we have

$$\begin{aligned} MS(G) &\leq \left(\frac{m^3-m}{12}\right)^{1/4} \Delta(G) \\ &= \left(\frac{m^3-m}{12}\right)^{1/4} \frac{\|GG^*-G^*G\|^{1/2}}{\|G\|} \\ &\leq \left(\frac{m^3(m^2-1)}{12}\right)^{1/4} \frac{\|GG^*-G^*G\|_2^{1/2}}{\|G\|_2} \end{aligned} \tag{B.1.2}$$

Now suppose $Z\Gamma U^*$ is a QND of G (see §3.3) and that $m(G) = \|U^*Z-I\|_2 \leq \delta$. Then

$$\begin{aligned} \|GG^*-G^*G\|_2 &= \|Z\Gamma^2Z^*-U\Gamma^2U^*\|_2 \\ &= \|U^*Z\Gamma^2-\Gamma^2U^*Z\|_2 \\ &= \|(U^*Z-I)\Gamma^2-\Gamma^2(U^*Z-I)\|_2 \\ &\leq \|U^*Z-I\|_2\,\|\Gamma^2\|_2 + \|\Gamma^2\|_2\,\|U^*Z-I\|_2 \\ &= 2\,m(G)\,\|\Gamma\|_2^2 \end{aligned} \tag{B.1.3}$$

Substituting (B.1.3) into (B.1.2) and noting that $\|G\|_2 = \|\Gamma\|_2$, we have

$$\begin{aligned} MS(G) &\leq \left(\frac{m^3(m^2-1)}{3}\right)^{1/4} m(G)^{1/2} \\ &\leq \left(\frac{m^3(m^2-1)}{3}\right)^{1/4} \delta^{1/2} \end{aligned} \tag{B.1.4}$$

Defining $\varepsilon(\delta)$ to be the right hand side of (B.1.4) readily gives the required result (3.2.1) and (3.2.2). □

§B.2 Proof of Prop 3.3.1:

Let $D = \mathrm{diag}(e^{jd_1}, e^{jd_2})$. The squares of the singular values of $(V-D)$ are defined by the equation

$$\det[\nu I - (V-D)^*(V-D)] = 0$$

$$\Leftrightarrow \det[\nu I - (V\Theta^* - D\Theta^*)^*(V\Theta^* - D\Theta^*)] = 0 \tag{B.2.1}$$

Now $$(V\Theta^* - D\Theta^*)^*(V\Theta^* - D\Theta^*)$$

$$= 2I - (V\Theta^*)^*(D\Theta^*) - (D\Theta^*)^*(V\Theta^*)$$

$$= \begin{bmatrix} 2(1-\cos\delta_1\cos\phi) & e^{-j\delta}(e^{-j\delta_1} - e^{j\delta_2})\sin\phi \\ e^{j\delta}(e^{j\delta_1} - e^{-j\delta_2})\sin\phi & 2(1-\cos\delta_2\cos\phi) \end{bmatrix} \tag{B.2.2}$$

where $\delta_1 = d_1 - \theta_1 \;,\quad \delta_2 = d_2 - \theta_2$ (B.2.3)

Using (B.2.2) in (B.2.1) and after some simplifications, we have

$$\nu^2 - 4\nu\left(1 - \cos\frac{\delta_1+\delta_2}{2}\cos\frac{\delta_1-\delta_2}{2}\cos\phi\right)$$
$$+ 4\left(\cos\frac{\delta_1+\delta_2}{2} - \cos\frac{\delta_1-\delta_2}{2}\cos\phi\right)^2 = 0$$

The larger of the two roots of this equation is

$$\|V-D\|_2^2 = 2\left[\left(1 - \cos\frac{\delta_1+\delta_2}{2}\cos\frac{\delta_1-\delta_2}{2}\cos\phi\right) + \left|\sin\frac{\delta_1+\delta_2}{2}\right|\sqrt{1 - \cos^2\frac{\delta_1-\delta_2}{2}\cos^2\phi}\right]$$

$$\geq 2(1-\cos\phi) = 4\sin^2\left(\frac{\phi}{2}\right)$$

and equality holds if $\delta_1 = 0 = \delta_2$

$$\Leftrightarrow \quad d_1 = \theta_1 \;,\; d_2 = \theta_2$$

Hence $D = \Theta$ minimizes $\|V-D\|_2$ with the minimum given by

$$\|V-\Theta\|_2 = \left|2\sin\left(\frac{\phi}{2}\right)\right|$$ □

§B.3 Proof of Prop 4.5.1

For notational simplicity, we shall prove the case $m > \ell$ only. The following lemma will be needed.

Lemma B.3.1

Suppose $A \in \mathbb{C}^{m\times\ell}$, $B \in \mathbb{C}^{\ell\times m}$ $(\ell < m)$ are of full rank ℓ.

If

$$BA = \Lambda \tag{B.3.1}$$

$$AB = \Delta \tag{B.3.2}$$

where $\Lambda = \mathrm{diag}(\lambda_i)_{i=1}^{\ell}$, $\Delta = \mathrm{diag}(\delta_i)_{i=1}^{m}$ are diagonal marices whose diagonal elements are in descending order of magnitude, then:

(1) Λ is the $\ell\times\ell$ leading submatrix of Δ.

(2) If the diagonal elements of Λ are distinct, then both A, B are pseudo-diagonal (diagonal if $m = \ell$).

(3) If Λ has equal diagonal elements, then both A, B are pseudo-block-diagonal. The sizes of the diagonal blocks are compatible with the number of equal diagonal elements of Λ. Moreover, if $X \in \mathbb{C}^{t\times t}$ is the diagonal block of A corresponding to t $(< \ell)$ equal diagonal elements λ of Λ, then B has a corresponding diagonal block λX^{-1}.

Proof:

(1) Postmultiplying both sides of (B.3.2) by A and using (B.3.1), we have

$$A\Lambda = \Delta A$$

It follows that for any (i,j) s.t. $a_{ij} \neq 0$

$$a_{ij}\lambda_j = \delta_i a_{ij} \tag{B.3.3}$$

$$\lambda_j = \delta_i \tag{B.3.4}$$

Since A has full rank, it has at least one nonzero element in each column and so for each λ_j, $\exists\, \delta_i$ such that (B.3.4) holds, which implies that Λ and Δ have the same set of nonzero diagonal elements. If the diagonal elements are also ordered in decreasing order of magnitude, then (1) must be the case.

(2) From (B.3.3), we have

$$(\lambda_j - \delta_i) a_{ij} = 0$$

Now

$$\delta_i = \begin{cases} \lambda_i & i = 1, \ldots, \ell \quad \text{by part (1) above} \\ 0 & i = \ell+1, \ldots, m \quad \text{since AB has rank } \ell \end{cases}$$

If the λ_i's are distinct, then $\forall i \neq j$

$$(\lambda_j - \delta_i) \neq 0$$

so that we must have $a_{ij} = 0$. Hence A is pseudo-diagonal. A similar argument applies to B.

(3) The proof of (3) is a simple extension of that for (2) and the idea is illustrated by the following simple case.

Let $A \in \mathbb{C}^{4\times 3}$, $B \in \mathbb{C}^{3\times 4}$ and let

$$BA = \Lambda = \mathrm{diag}(\lambda, \lambda, \lambda') \qquad \text{where} \quad \lambda \neq \lambda'$$

$$AB = \Delta = \mathrm{diag}(\lambda, \lambda, \lambda', 0)$$

Then by an argument similar to that given in (2), we can show that A, B are of the form

$$A = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{X} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & x \\ 0 & 0 & 0 \end{array}\right] \qquad B = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{Y} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline 0 & 0 & y & 0 \end{array}\right] \tag{B.3.5}$$

where $X, Y \in \mathbb{C}^{2\times 2}$ are nonsingular and x, y are nonzero. It then follows that $Y = \lambda X^{-1}$. (B.3.6)

□

<u>Proof of Prop 4.5.1</u>:

<u>Necessity</u>:

If both GK and KG are normal, then each is unitarily similar to a diagonal matrix,

$$GK = W\Lambda_{GK}W^* \tag{B.3.7}$$

$$KG = V\Lambda_{KG}V^* \tag{B.3.8}$$

where $W \varepsilon \mathbb{C}^{m\times m}$ and $V \varepsilon \mathbb{C}^{\ell\times\ell}$ are unitary and we assume that the diagonal elements of $\Lambda_{GK}, \Lambda_{KG}$ are put in decreasing order of magnitude. From (B.3.7)

$$W^*GKW = \Lambda_{GK}$$

$$(W^*GV)(V^*KW) = \Lambda_{GK}$$

Similarly, from (B.3.8)

$$(V^*KW)(W^*GV) = \Lambda_{KG}$$

Putting $A = (W^*GV)$ and $B = (V^*KW)$, we can apply Lemma B.3.1 directly. We illustrate the idea behind the rest of the proof by assuming that A, B are of the form (B.3.5).

Now let X have an SVD

$$X = W_1\Sigma_1V_1^* \tag{B.3.9}$$

then

$$W^*GV = A = \left[\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{2}{*}{$W_1\Sigma_1V_1^*$}} & 0 \\ & & 0 \\ \hdashline 0 & 0 & x \\ 0 & 0 & 0 \end{array}\right]$$

so that

$$G = W \left[\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{2}{*}{W_1}} & 0 \\ & & 0 \\ \hdashline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \left[\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{2}{*}{Σ_1}} & 0 \\ & & 0 \\ \hdashline 0 & 0 & x \end{array}\right] \left[\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{2}{*}{V_1^*}} & 0 \\ & & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right] V^* \tag{B.3.10}$$

Also from (B.3.5), (B.3.6) and (B.3.9), we have

$$V^*KW = B = \left[\begin{array}{cc:cc} \multicolumn{2}{c:}{\multirow{2}{*}{$V_1(\lambda\Sigma_1^{-1})W_1^*$}} & 0 & 0 \\ & & 0 & 0 \\ \hdashline 0 & 0 & y & 0 \end{array}\right]$$

so that

$$K = V\left[\begin{array}{cc|c} \multicolumn{2}{c|}{V_1} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right]\left[\begin{array}{cc|c} \multicolumn{2}{c|}{\lambda\Sigma_1^{-1}} & 0 \\ & & 0 \\ \hline 0 & 0 & y \end{array}\right]\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{W_1^*} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right]W^* \qquad (B.3.11)$$

After possibly simple manipulations, we may assume that the right hand sides of (4.5.3) and (B.3.10) are identical decompositions in that

$$Z = W\left[\begin{array}{cc|c} \multicolumn{2}{c|}{W_1} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad \Gamma_G = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\Sigma_1} & 0 \\ & & 0 \\ \hline 0 & 0 & x \end{array}\right] \qquad U^* = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{V_1^*} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right]V^*$$

Hence (B.3.11) takes the form

$$K = U\Gamma_K Y^*$$

for some diagonal Γ_K, as required.

Sufficiency:

For the given G and K, we have

$$GK = Y\Gamma_G\Gamma_K Y^*$$

and

$$KG = U\Gamma_K\Gamma_G U^*$$

By Theorem 1.6.1, both of these are normal. □

§B.4 Proof of Theorem 4.6.2

(1) By definition

$$\begin{aligned} TPC(Q(s)) &= \lim_{\substack{\delta\to o \\ R\to\infty}} \sum_{i=1}^{m} \Delta_{j\delta}^{jR} \arg\gamma_{Qi}(s) \\ &= \lim_{\substack{\delta\to o \\ R\to\infty}} \Delta_{j\delta}^{jR} \arg\Big(\prod_{i=1}^{m}\gamma_{Qi}(s)\Big) \\ &= \lim_{\substack{\delta\to o \\ R\to\infty}} \Delta_{j\delta}^{jR} \arg\big(\det Q(s)\big) \end{aligned}$$

Now as s goes up the imaginary axis (along the Nyquist contour) between 0 and $j\infty$, each pole in $\mathbb{C}_+^*$ ($=\mathbb{C}_+-\{0\}$) and each zero in $\mathbb{C}_-^\circ$ contributes a net phase change of $\pi/2$ to $\det Q(s)$ while each pole in $\mathbb{C}_-^\circ$ and each zero in $\mathbb{C}_+^*$ contributes a net phase change of $-\pi/2$. Poles and zeros at $s=0$ or $s=\infty$ make no contribution. Hence

$$\mathrm{TPC}(Q(S)) = (P_R - P_L - Z_R + Z_L)\cdot\frac{\pi}{2} \tag{B.4.1}$$

where P_R, P_L and Z_R, Z_L denote the number of poles and zeros of $Q(s)$ in $\mathbb{C}_+^*$, $\mathbb{C}_-^\circ$. Since $Q(s)$ is full rank, we have (e.g. see [KAI, pp.460-461])

Total number of (finite + infinite) zeros of $Q(s)$

= Total number of (finite + infinite) poles of $Q(s)$

As $Q(s)$ has no infinite poles because it is proper, we have

$$0 = P_R + P_L + P_o - (Z_R + Z_L + Z_o + \#\mathrm{IZ}(Q(s))) \tag{B.4.2}$$

where P_o, Z_o denote the number of poles and zeros of $Q(s)$ at $s=0$. (B.4.1) and (B.4.2) together gives

$$\mathrm{TPC}(Q(s)) = [2(P_R - Z_R) + (P_o - Z_o) - \#\mathrm{IZ}(Q(s))]\cdot\frac{\pi}{2}$$

(2) Splitting $(P_R - Z_R)$ into two parts corresponding to $G(s)$ and $K(s)$, we have

$$\begin{aligned} P_R - Z_R = &\left(\#\mathrm{SMP}(G(s),\mathbb{C}_+^*) + \#\mathrm{SMP}(K(s),\mathbb{C}_+^*)\right) \\ &- \left(\#\mathrm{SMZ}(G(s),\mathbb{C}_+^*) + \#\mathrm{SMZ}(K(s),\mathbb{C}_+^*)\right) \end{aligned}$$

Likewise we split $(P_o - Z_o)$ and $\#\mathrm{IZ}(Q(s))$ and after some simple rearrangement, we get

$$\mathrm{TPC}(Q(s)) = \mathrm{TPC}(G(s)) + \mathrm{TPC}(K(s))$$

(3) Under the stated assumptions,

$$\mathrm{TPC}(K(s)) = \#\mathrm{SMP}(K(s),0)\cdot\frac{\pi}{2}$$

which together with (2), gives the required result. □

APPENDIX C THE SYSTEM AUTM

The system AUTM is a 2-input, 12-state, 2-output model of an automobile gas turbine. A state-space description is given in Table C.2. Fig.C.1 shows the Nyquist and Bode magnitude arrays (i.e., the arrays of diagrams consisting of the Nyquist and Bode magnitude plots of the individual elements of the system transfer function matrix).

The open-loop poles and finite/infinite zeros of the system are

Poles	Finite Zeros
-0.216	-0.0956
-0.217	-0.216
-0.218	$-0.233 \pm j1.08$
$-0.370 \pm j1.32$	$-0.923 \pm j1.57$
-0.931	-0.934
-0.933	-8.92
-0.934	-11.1
-8.00	
-9.06	**Infinite Zeros**
-11.0	one 1st order ∞-zero
-11.6	one 2nd order ∞-zero

The system is thus open-loop stable with no zeros in $\mathbb{C}_+$. AUTM has been studied in Examples 2.3.1, 3.2.1, 4.4.1, 5.4.1 and 6.3.1.

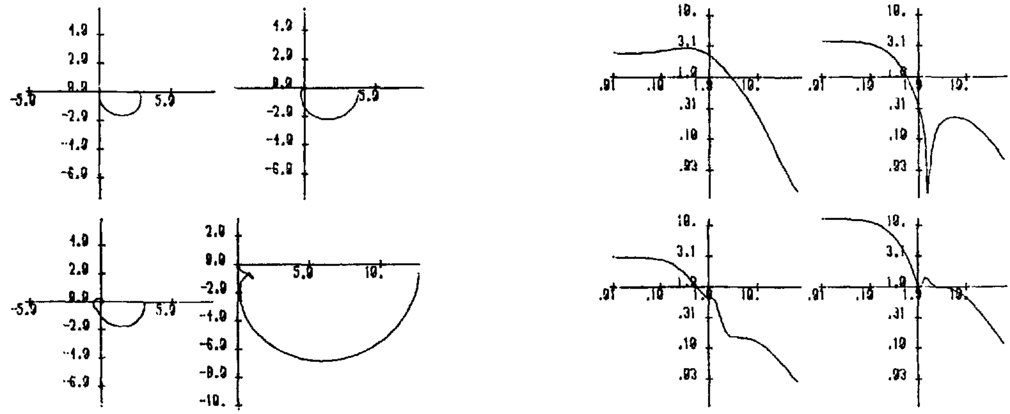

Fig.C.1 Nyquist and Bode magnitude arrays of the system AUTM.

$$A = \begin{bmatrix}
0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
-0.202 & -1.150 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & -2.360 & -13.60 & -12.80 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -1.620 & -9.400 & -9.150 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -188.0 & -111.6 & -116.4 & -20.80
\end{bmatrix}$$

$$B = \begin{bmatrix}
0.0000 & 1.0439 & 0.0000 & 0.0000 & -1.794 & 0.0000 & 0.0000 & 1.0439 & 0.0000 & 0.0000 & 0.0000 & -1.794 \\
0.0000 & 4.1486 & 0.0000 & 0.0000 & 2.6775 & 0.0000 & 0.0000 & 4.1486 & 0.0000 & 0.0000 & 0.0000 & 2.6775
\end{bmatrix}^T$$

$$C = \begin{bmatrix}
0.2640 & 0.8060 & -1.420 & -15.00 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 4.9000 & 2.1200 & 1.9500 & 9.3500 & 25.800 & 7.1400 & 0.0000
\end{bmatrix}$$

$$D = \begin{bmatrix}
0.0000 & 0.0000 \\
0.0000 & 0.0000
\end{bmatrix}$$

APPENDIX D THE SYSTEMS NSRE AND REAC

The system NSRE is a 2-input, 4-state, 3-output "non-square" model of a chemical reactor (see [MUN] and [MAC2]). A state-space description is given in Table D.2. The Nyquist and Bode magnitude arrays are shown in Fig.D.1.

NSRE has open-loop poles at $\{0.0622, 2.01, -5.06, -8.66\}$. The output variables of NSRE have already been arranged in such a way that the first two outputs may be taken as controlled variables. We shall refer to the square system, obtained by simply leaving out the last rows of the C and D matrices of NSRE, as REAC. REAC has finite zeros at $\{-1.19, -5.02\}$ and two 1st order ∞-zeros.

REAC has been considered in Example 3.5.2 and a design for the non-square system NSRE is given in Example 7.3.1.

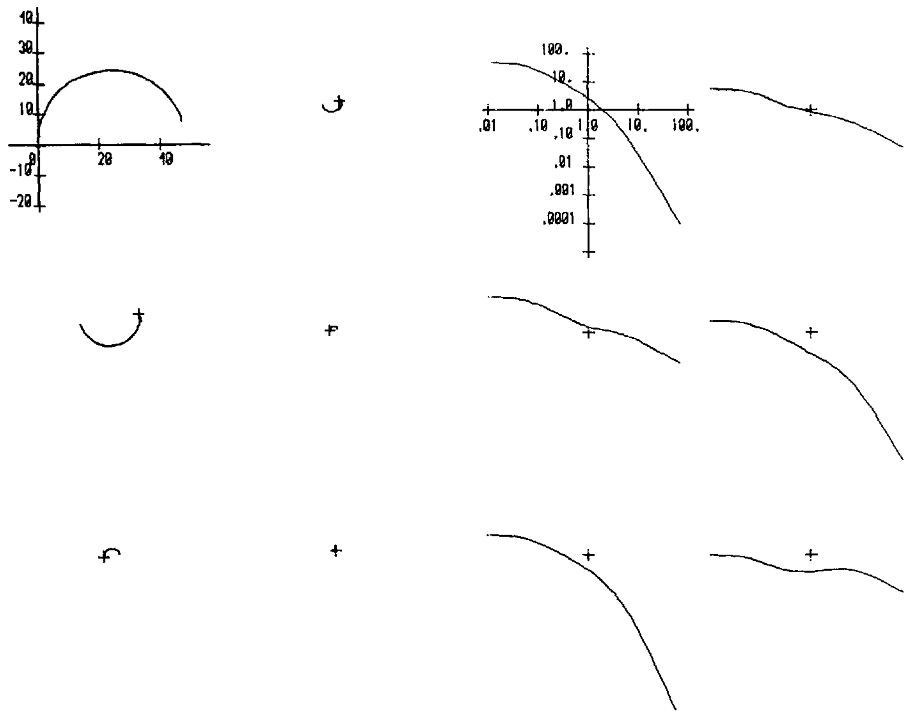

Fig.D.1 Nyquist and Bode magnitude arrays of the system NSRE.
The upper square square block corresponds to REAC.
(All elements are drawn to the same scale as the (1,1)-entry.)

$$A = \begin{bmatrix} 1.4000 & -0.208 & 6.7150 & -5.676 \\ -0.581 & -4.290 & 0.0000 & 0.6750 \\ 1.0670 & 4.2730 & -6.654 & 5.8930 \\ 0.0480 & 4.2730 & 1.3430 & -2.104 \end{bmatrix} \qquad B = \begin{bmatrix} 0.0000 & 0.0000 \\ 5.6790 & 0.0000 \\ 1.1360 & -3.146 \\ 1.1360 & 0.0000 \end{bmatrix}$$

$$C = \begin{bmatrix} 1.0000 & 0.0000 & 1.0000 & -1.000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & -1.000 \end{bmatrix} \qquad D = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}$$

Table D.2

APPENDIX E THE SYSTEM TGEN

The system TGEN is a 2-input, 10-state, 2-output state-space description of the dynamics (linearized at some nominal operating point) of a turbo-generator (see [LIM]). A listing of the state-space matrices is given in Table E.2 and the Nyquist and Bode magnitude arrays are shown in Fig.E.1.

The open-loop poles and finite/infinite zeros of the system are

Poles	Finite Zeros
-0.231	-1.25
$-0.351 \pm j6.34$	-11.1
-1.04	-20.0
-1.67	$23.0 \pm j425$
-10.0	
-10.7	**Infinite Zeros**
-17.7	one 2nd order ∞-zero
$-29.5 \pm j314$	one 3rd order ∞-zero

A design example for this system is given in §7.1.

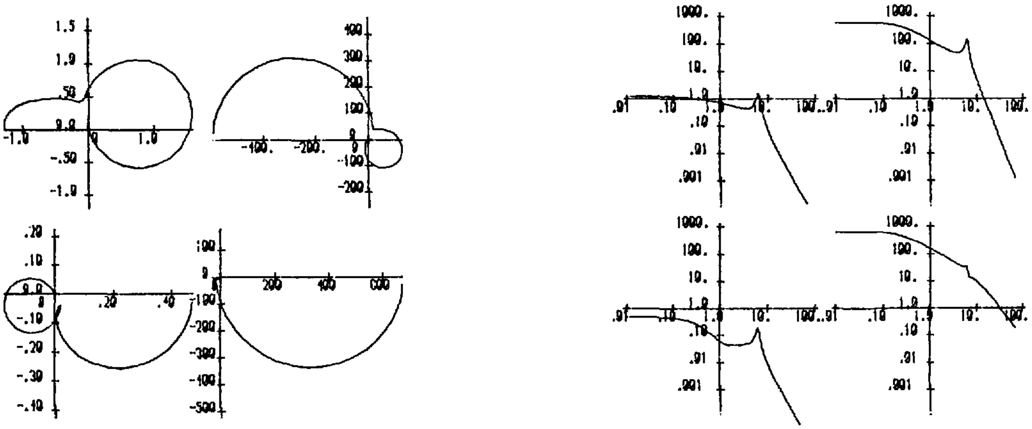

Fig.E.1 Nyquist and Bode magnitude arrays of the system TGEN

$$
A = \begin{bmatrix}
0.00000 & 1.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\
0.00000 & -.11323 & -.98109 & -11.847 & -11.847 & -63.080 & -34.339 & -34.339 & -27.645 & 0.00000 \\
324.121 & -1.1755 & -29.101 & 0.12722 & 2.83448 & -967.73 & -678.14 & -678.14 & 0.00000 & -129.29 \\
-127.30 & 0.46167 & 11.4294 & -1.0379 & 13.1237 & 380.079 & 266.341 & 266.341 & 0.00000 & 1054.85 \\
-186.05 & 0.67475 & 16.7045 & 0.86092 & -17.068 & 555.502 & 389.268 & 389.268 & 0.00000 & -874.92 \\
341.917 & 1.09173 & 1052.75 & 756.465 & 756.465 & -29.774 & 0.16507 & 3.27626 & 0.00000 & 0.00000 \\
-30.748 & -.09817 & -94.674 & -68.029 & -68.029 & 2.67753 & -2.6558 & 4.88497 & 0.00000 & 0.00000 \\
-302.36 & -.96543 & -930.96 & -668.95 & -668.95 & 26.3292 & 2.42028 & -9.5603 & 0.00000 & 0.00000 \\
0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & -1.6667 & 0.00000 \\
0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & -10.000
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 1.66667 & 0.00000 \\
0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 10.0000
\end{bmatrix}^T
$$

$$
C = \begin{bmatrix}
1.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\
-.49134 & 0.00000 & -.63203 & 0.00000 & 0.00000 & -.20743 & 0.00000 & 0.00000 & 0.00000 & 0.00000
\end{bmatrix}
$$

$$
D = \begin{bmatrix}
0.00000 & 0.00000 \\
0.00000 & 0.00000
\end{bmatrix}
$$

Table E.2

APPENDIX F THE SYSTEM AIRC

The system AIRC is a 3-input, 5-state, 5-output model for the vertical-plane dynamics (linearized at datum flight condition) of an aircraft. It is a re-scaled version of the example studied in [KOU] and a listing of the state-space description is given in Table F.2. The Nyquist and Bode magnitude arrays are shown in Fig.F.1. The 5 output measurements are in fact the states of the system, defined as:

x_1 = height error relative to ground or guidance aid, in m;
x_2 = forward speed, in m/sec;
x_3 = pitch angle, in degrees;
x_4 = rate of change of pitch angle, in degree/sec;
x_5 = vertical speed, in m/sec.

The first three states are the variables to be controlled. The inputs are:

u_1 = spoiler angle, in 10^{-1} degrees;
u_2 = forward acceleration due to engine thrust, in m/sec^2;
u_3 = elevator angle, in degrees.

AIRC has open loop poles at $\{0, -0.018 \pm j0.182, -0.78 \pm j1.03\}$, and being non-square, has no zeros. However, if we consider the transmittance from $[u_1\ u_2\ u_3]^T$ to $[x_1\ x_2\ x_3]^T$ (i.e., to the controlled variables), then this square system has zeros given by:

one 1st order ∞-zero
two 2nd order ∞-zeros
no finite zeros

Designs of flight controllers for AIRC are given in Examples 7.3.2 and 7.3.3.

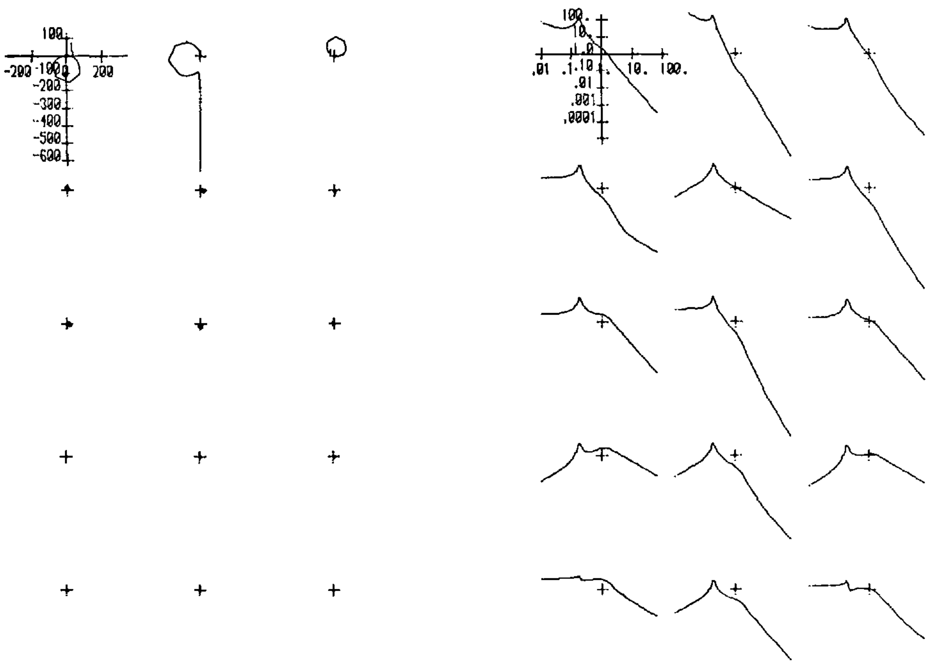

Fig.F.1 Nyquist and Bode magnitude arrays of the system AIRC.
(All elements are drawn to the same scale as the (1,1)-entry.)

$$A = \begin{bmatrix} 0.0000 & 0.0000 & 1.1320 & 0.0000 & -1.000 \\ 0.0000 & -.0538 & -.1712 & 0.0000 & 0.0705 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0485 & 0.0000 & -.8556 & -1.013 \\ 0.0000 & -.2909 & 0.0000 & 1.0532 & -.6859 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ -0.120 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 4.4190 & 0.0000 & -1.665 \\ 1.5750 & 0.0000 & -.0732 \end{bmatrix} \qquad C = I_5 \qquad D = O_{5\times 3}$$

Table F.2

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