$$\begin{array}{ccc} L & \xrightarrow{\psi} & F \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ K & \xrightarrow[\varphi]{} & E \end{array}$$

be a commutative diagram in the category of ordered simplicial complexes and increasing mappings. Assume that the canonical map $L \to K \times_E F$ is surjective. Let $p : C(|\psi|) \longrightarrow C(|\varphi|)$ denote the canonical map induced by $|\mu|$ and $|\nu|$. Does there exist a triangulation of p extending the given triangulations of $|\mu|$ and $|\nu|$? The answer is positive in two particular cases : (1) $|\nu|$ is finite to one, and (2) $L = K \times_E F$, ψ and μ being the canonical propjections. These cases appear in the proof of Theorem 8.9.

9. APPENDIX

9.1.1. We consider only countable and locally finite simplicial complexes (s.c.). Given a s.c. K we denote by $|K|$ its geometric realization and assume that $|K|$ is contained in some Euclidean space R^n , the inclusion $|K| \subset R^n$ being linear on the simplexes of K . If K' is a subdivision of K , we identify $|K'|$ and $|K|$ in the usual way.

By subcomplex we always mean full subcomplex and use the notation $L \triangleleft K$ to express the fact that L is a (full) subcomplex of K (thus if the vertices of a simplex σ of K are in L , then σ itself is in L). A simplex of K is also viewed, as usually, as a subcomplex of K . If $L \triangleleft K$, then $|L|$ is considered as a subspace of $|K|$, $|L| \subset |K|$, in the usual way.

A <u>triangulation</u> of a topological space A consists of a pair (K,φ), K being a s.c. and $\varphi : |K| \to A$ a homeomorphism.

If (K,φ) is a triangulation of A and K' is a subdivision of K, then, since $|K'| = |K|$, the pair (K',φ) is also a triangulation of A ; (K',φ) is called a <u>subdivision</u> of (K,φ) . If $A \subset R^n$ is a polyhedron, a triangulation (K,φ) of A is called <u>linear</u> if for any simplex σ of K

the restriction of φ to $|\sigma|$ is linear. For example $(K, 1_{|K|})$ is a linear triangulation of $|K|$.

Let (K,φ) be a triangulation of A and let $A_1 \subset A$ be a closed subset. We say that the restriction of (K,φ) to A_1 exists if $\varphi^{-1}(A_1) = |K_1|$ for some subcomplex K_1 of K ; if this is the case, then $(K_1, \varphi||K_1|)$ is a triangulation of A_1 , called the <u>restriction</u> of (K,φ) and denoted $(K,\varphi)|A_1$ (we shall also say that (K,φ) is an <u>extension</u> of $(K,\varphi||K_1|)$).

9.1.2. Given a simplicial map (s.m.) $s : K \to L$ we denote its geometric realization by $|s| : |K| \to |L|$. The s.m. is called <u>proper</u> if and only if the preimage of any vertex of L consists of finitely many vertices of K . Clearly s is proper if and only if $|s|$ is proper (i.e. the preimge through $|s|$ of any compact subset of L is compact).

A continuous map $f : B \to A$ is called <u>triangulable</u> if there exist triangulations (K,φ) and (L,ψ) of A and B respectively and a simplicial map $s_f : L \to K$ such that $f = \varphi \circ |s_f| \circ \psi^{-1}$ (we shall also say that f is <u>simplicial</u> with respect to (L,ψ) and (K,φ) , or that the pair $\{(L,\psi),(K,\varphi)\}$ is a <u>triangulation</u> of f); notice that s_f is completely determined by f , (K,φ) and (L,ψ) .

Let $\{(L,\psi),(K,\varphi)\}$ be a triangulation of $f : B \to A$ and let $B_1 \subset B$ and $A_1 \subset A$ be closed subsets such that $f(B_1) \subset A_1$. Set $f_1 = f|B_1 : B_1 \to A_1$. We say that the restriction of $\{(L,\psi),(K,\varphi)\}$ to f_1 exists if the restrictions of (L,ψ) to B_1 and (K,φ) to A_1 exist ; if this is so, then $\{(L,\psi)|B_1,(K,\varphi)|A_1\}$ is a triangulation of f_1 , called the <u>restriction</u> of $\{(L,\psi),(K,\varphi)\}$ and denoted $\{(L,\psi),(K,\varphi)\}|f_1$ (we shall also say that $\{(L,\psi),(K,\varphi)\}$ is an <u>extension</u> of $\{(L,\psi),(K,\varphi)\}|f_1$).

9.1.3. An <u>ordered simplicial complex</u> (o.s.c.) is a s.c. K such that any simplex σ of K is a totally ordered set, the order of any face τ of σ being induced by the order of σ .

Any s.c. can be endowed with a structure of an o.s.c. More generally, if $L \triangleleft K$ and L is endowed with a structure of an o.s.c., then this structure can be extended (the meaning is obvious) to a structure of an o.s.c. on K (the fact that L is a full subcomplex is essential here).

9.1.4. Let K be a s.c. and K' be a barycentric subdivision of K. (not necessarilly the standard one: the barycenter $\hat{\sigma}$ of a simplex σ of K may be any interior point of $|\sigma|$). On K' we shall always consider the following order: $\hat{\sigma} < \hat{\tau}$ if and only if σ is a face of τ (denote this by $\sigma \prec \tau$). Together with this order K' is an o.s.c.

9.2.1. Let K and L be o.s.c.'s. We shall construct a new o.s.c. $K \times L$, called the <u>product</u> of K and L, as follows: the set of vertices of $K \times L$ is the cartesian product of the set of vertices of K and the set of vertices of L; $((v_1,w_1),\dots,(v_n,w_n))$ is a simplex of $K \times L$ if and only if $v_1,\dots,v_n$ (resp. $w_1,\dots,w_n$) are vertices of a simplex of K (resp. L) and $v_1 \le v_2 \le \dots \le v_n$ (resp. $w_1 \le w_2 \le \dots \le w_n$); if (v_1,w_1) and (v_2,w_2) are vertices of a simplex of $K \times L$ then $(v_1,w_1) \le (v_2,w_2)$ if and only if $v_1 \le v_2$ and $w_1 \le w_2$.

Let $p_1 : K \times L \to K$ and $p_2 : K \times L \to L$ be the canonical projections (they are s.m.'s !). We can therefore consider $|p_1| \times |p_2| : |K \times L| \to |K| \times |L|$. The next lemma is well known (see for example [E-S], Chap. II, Lemma 8.9).

9.2.2. LEMMA. $(K \times L, |p_1| \times |p_2|)$ is a triangulation of $|K| \times |L|$.

We shall also need the following result.

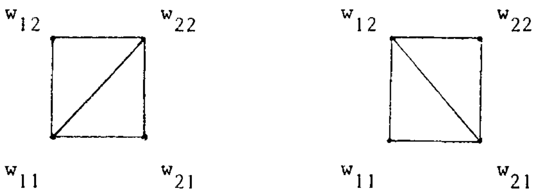
9.2.3. LEMMA. Let K and L be s.c.'s and let (N,ψ) be a triangulation of $|K| \times |L|$ such that $\{(N,\psi),(K,1_{|K|})\}$ (resp. $\{(N,\psi),(L,1_{|L|})\}$) is a triangulation of the projection $|K| \times |L| \longrightarrow |K|$ (resp. $|K| \times |L| \longrightarrow |L|$). Then K and L can be endowed with structures of o.s.c.'s such that

(1) there exists an isomorphism $\mu : K \times L \to N$;

(2) $\psi \circ |\mu| = |p_1| \times |p_2|$, p_1 and p_2 being as in 9.2.2.

(3) if $\dim(K) \geq 1$ and $\dim(L) \geq 1$, then the orders on K and L with the above properties are unique up to a simultaneous change to the oposite orders.

Proof. Assume that $\dim(K) \geq 1$ and $\dim(L) \geq 1$, the other cases being trivial. Let $\sigma = (u_1,u_2)$ be a 1-dimensional simplex of K. Set $u_1 \leq u_2$. Let $\tau = (v_1,v_2)$ be a 1-dimensional simplex of L. From the hypotheses it follows easily that $\psi^{-1}(|\sigma| \times |\tau|)$ is one of the following polyhedra



and $\psi(w_{ij}) = (u_i,v_j)$. In the first case set $v_1 < v_2$, while in the second case set $v_2 < v_1$. One can verify that this procedure determines a structure of o.s.c. on L. Next, starting with $\tau = (v_1,v_2)$ and with the order already determined on $\{v_1,v_2\}$, we endow K with a structure of o.s.c. The lemma follows without difficulty. Q.E.D.

The next lemma is now obvious.

9.2.4. LEMMA. Let (K,φ) and (L,ψ) be triangulations of A and B respectively. If K and L are endowed with structures of o.s.c.'s, then $(K \times L,\theta)$, with $\theta : |K \times L| \to A \times B$ given by $\theta(z) = (\varphi(|p_1|(z)), \psi(|p_2|(z)))$, is a triangulation of $A \times B$, called the product of (K,φ) and (L,ψ), and denoted $(K,\varphi) \times (L,\psi)$; the projections $A \times B \longrightarrow A$ and $A \times B \longrightarrow B$ are simplicial with respect to this triangulation of $A \times B$ and the given triangulations of A and B. Any triangulation of $A \times B$ with the last property is (up to an isomorphism) of the form $(K,\varphi) \times (L,\psi)$, for some orders on K and L.

9.3.1. Let K, L and M be o.s.c.'s and let $s : K \to M$ and $t : L \to M$ be increasing simplicial mappings. Let $K \times_M L$ be the full subcomplex of $K \times L$ whose vertices are all the vertices of $K \times L$ of the form (u,v) with $s(u) = t(v)$. $K \times_M L$ is called the <u>fibre product</u> of K and L over M (with respect to s and t). Let $\bar{p}_1 : K \times_M L \longrightarrow K$ and $\bar{p}_2 : K \times_M L \longrightarrow L$ denote the restrictions of the projections $p_1 : K \times L \longrightarrow K$ and $p_2 : K \times L \longrightarrow L$ respectively; define $p : K \times_M L \longrightarrow M$ by setting $p = s \circ \bar{p}_1 = t \circ \bar{p}_2$.

Consider now the (topological) fibre product $|K| \times_{|M|} |L| = \{(x,y) \in |K| \times |L| \; ; \; |s|(x) = |t|(y)\}$. One checks easily that $|\bar{p}_1| \times_{|M|} |\bar{p}_2| : |K \times_M L| \longrightarrow |K| \times_{|M|} |L|$ given by $(|\bar{p}_1| \times_{|M|} |\bar{p}_2|)(z) = (|\bar{p}_1|(z), |\bar{p}_2|(z)) = (|p_1|(z), |p_2|(z))$ is well defined and that $(|p_1| \times |p_2|)^{-1}(|K| \times_{|M|} |L|) = |K \times_M L|$. From Lemma 9.2.2 we get

9.3.2. LEMMA. $(K \times_M L, |\bar{p}_1| \times_{|M|} |\bar{p}_2|)$ is a triangulation of $|K| \times_{|M|} |L|$.

We shall also need the following generalization of Lemma 9.2.3.

9.3.3. LEMMA. Let K, L and M be s.c.'s, let $s : K \to M$ and $t : L \to M$ be s.m.'s and let (N,ψ) be a triangulation of $|K| \times_{|M|} |L|$ such that $\{(N,\psi),(K,1_{|K|})\}$ (resp. $\{(N,\psi),(L,1_{|L|})\}$) is a triangulation of the canonical projection $|K| \times_{|M|} |L| \to |K|$ (resp. $|K| \times_{|M|} |L| \longrightarrow |L|$). Then we can endow K, L and M with structures of o.s.c.'s such that

(1) s and t are increasing with respect to these orders;

(2) there exists an isomorphism $\mu : K \times_M L \to N$;

(3) $\psi \circ |\mu| = |\bar{p}_1| \times_{|M|} |\bar{p}_2|$.

<u>Proof</u>. Choose any structure of o.s.c. on M. Given a vertex w of M let K_w be the (full) subcomplex of K whose vertices are all the vertices u of K with $s(u) = w$. Define similarly $L_w \triangleleft L$ (with respect to t). Notice that $\{(N,\psi),(M,1_{|M|})\}$ is a triangulation of the map $|p| \circ \psi : |N| \to |M|$; let $r : N \to M$ be the corresponding simplicial map. As above

we can define $N_w \triangleleft N$ (with respect to r). From the hypotheses it follows that $\psi(|N_w|) = |K_w| \times |L_w| \subset |K| \times_{|M|} |L|$ and therefore $(N_w, \psi \big| |N_w|)$ is a triangulation of $|K_w| \times |L_w|$. We can apply Lemma 9.2.3 and endow K_w and L_w with structures of o.s.c.'s with the properties stated there. Together with assertion (1) in the present lemma, these structures determine the required structures of o.s.c.'s on K and L . The remaining verifications are left to the reader. Q.E.D.

The next lemma is now obvious.

9.3.4. LEMMA. Let (K,φ) , (L,ψ) and (M,ξ) be triangulations of A , B and C . Let $f : A \to C$ and $g : B \to C$ be simplicial with respect to these triangulations and let $s_f : K \to M$ and $s_g : L \to M$ be the corresponding s.m.'s. If K , L and M are endowed with structures of o.s.c.'s such that s_f ans s_g are increasing, then $(K \times_M L, \theta)$, with $\theta : |K \times_M L| \longrightarrow A \times_C B$ given by $\theta(z) = (\varphi(|\bar{p}_1|(z)), \psi(|\bar{p}_2|(z)))$, is a triangulation of $A \times_C B$, called the <u>fibre product</u> of (K,φ) and (L,ψ) over C (with respect to f and g) and denoted $(K,\varphi) \times_C (L,\psi)$; the projections $A \times_C B \longrightarrow A$ and $A \times_C B \longrightarrow B$ are simplicial with respect to this triangulation of $A \times_C B$ and the given triangulations of A and B . Any triangulation of $A \times_C B$ with the last property is, up to an isomorphism, of the form $(K,\varphi) \times_C (L,\psi)$, for some orders on K , L and M.

9.4.1. Let $s : K \to L$ be a proper s.m. Given a barycentric subdivision L' of L we can always find a barycentric subdivision K' of K such that s extends to a s.m. $s' : K' \to L'$ and $|s'| = |s|$. For any simplex σ we denote by $\hat{\sigma}$ the corresponding vertex of the barycentric subdivision.

We recall first the definition of the <u>simplicial mapping cylinder</u> M_s of s . The set of vertices of M_s is the disjoint union of the sets of vertices of L and K' . A finite set $(v_1,\dots,v_m,\hat{\sigma}_1,\dots,\hat{\sigma}_n)$ is a simplex of M_s if $\tau = (v_1,\dots,v_m)$ is a simplex of L and $\hat{\tau} \le s'(\hat{\sigma}_1)$, $\hat{\sigma}_1 < \hat{\sigma}_2 < \dots < \hat{\sigma}_n$ (recall that $\hat{\sigma}' < \hat{\sigma}''$ means that σ' is a face of

σ''). In the above definition the v_i's (or the $\hat{\sigma}_i$'s) may be omitted, with the obvious changes in the defining conditions. Thus a simplex of M_s is of the form $\tau * A$ (the join of τ and A), τ being a simplex of L , A being a simplex of K' , $A = (\hat{\sigma}_1, \dots, \hat{\sigma}_n)$ and $\hat{\tau} < s'(\hat{\sigma}_1)$. It is therefore clear that M_s is a subcomplex of the s.c. $L * K'$, the join of L and K' . There exists a canonical subdivision M'_s of M_s defined as follows: the set of vertices of M'_s is the disjoint union of the sets of vertices of L' and K' , and a set $(\hat{\tau}_1, \dots, \hat{\tau}_m, \hat{\sigma}_1, \dots, \hat{\sigma}_n)$ of vertices of M'_s is a simplex of M'_s if $\hat{\tau}_1 < \dots < \hat{\tau}_m \leq s'(\hat{\sigma}_1)$ and $\hat{\sigma}_1 < \dots < \hat{\sigma}_n$. Thus M'_s is a subcomplex of $L' * K'$.

There exist canonical s.m.'s $r_s : M'_s \to L'$, $i_s : L \to M_s$, $i'_s : L' \to M'_s$, $j_s : K' \to M_s$ and $j'_s : K' \to M'_s$ defined by $r_s(\hat{\tau}) = \hat{\tau} = i'_s(\hat{\tau})$ if τ is a simplex of L, $r_s(\hat{\sigma}) = s(\hat{\sigma}) = s'(\hat{\sigma})$, $j_s(\hat{\sigma}) = j'_s(\hat{\sigma}) = \hat{\sigma}$ if σ is a simplex of K and $i_s(v) = v$ if v is a vertex of L . We shall use i_s (resp. j_s) to identify L (resp. K') with a simplicial subcomplex of M_s .

Consider now the <u>topological mapping cylinder</u> $C_{|s|}$ of $|s| : |K| \to |L|$. Recall that $C_{|s|}$ is obtained from $(|K| \times [0,1]) \sqcup (|L| \times \{0\})$ (disjoint union) by identifying $(x,0) \in |K| \times [0,1]$ with $(|s|(x),0) \in |L| \times \{0\}$. The image of $(z,t) \in (|K| \times [0,1]) \sqcup (|L| \times \{0\})$ in $C_{|s|}$ is denoted $[z,t]$. Define $r_{|s|} : C_{|s|} \to |L|$, $i_{|s|} : |L| \to C_{|s|}$ and $j_{|s|} : |K| \to C_{|s|}$ by setting $r_{|s|}([x,t]) = |s|(x)$, $j_{|s|}(x) = [x,1]$ if $x \in |K|$ and $t \in [0,1]$ and $r_{|s|}([y,0]) = y$, $i_{|s|}(y) = [y,0]$ if $y \in |L|$. We shall use $i_{|s|}$ (resp. $j_{|s|}$) to identify $|L|$ (resp. $|K|$) with a closed subspace of $C_{|s|}$.

We shall prove now that there exists a triangulation of $C_{|s|}$ of the form (M'_s , φ) . We shall follow Goresky's approach (see Section 4 of $[G_1]$) since the map φ he constructs has a smoothness property (see 9.4.2 (4) below) which is necessary in obtaining smooth triangulations of a.s.'s.

9.4.2. PROPOSITION. Let $s: K \to L$ be a proper s.m. Then there exists a homeomorphism $\varphi_s : |M_s| \to C_{|s|}$ with the following properties:

(1) If $K_1 \lhd K$ and $L_1 \lhd L$ are subcomplexes, $s(K_1) \lhd L_1$ and $s_1 : K_1 \to L_1$ is the restriction of s, then $\varphi_s(|M_{s_1}|) = C_{|s_1|}$ and $\varphi_s \big| |M_{s_1}| = \varphi_{s_1}$ (of course, the barycentric subdivisions of K_1 and L_1 are those induced by K' and L').

(2) $j_{|s|} = \varphi_s \circ |j_s|$, $i_{|s|} = \varphi_s \circ |i_s|$, $|r_s| = r_{|s|} \circ \varphi_s$.

(3) Let J be a s.c. and let $g: K' \to J$ and $h: L' \to J$ be s.m.'s such that $g = h \circ s'$. Define $\psi: C_{|s|} \to |J|$ and $s_\psi : M_s' \to J$ by setting $\psi([x,t]) = |g|(x)$ for $x \in |K'| = |K|$, $\psi([y,0]) = |h|(y)$ for $y \in |L'| = |L|$, $s_\psi(\hat{\tau}) = h(\hat{\tau})$ for $\hat{\tau} \in L'$ and $s_\psi(\hat{\sigma}) = g(\hat{\sigma})$ for $\hat{\sigma} \in K'$. Then $\{(M_s', \varphi_s), (J, 1_{|J|})\}$ is a triangulation of ψ, the corresponding s.m. being s_ψ .

(4) The restiction $\varphi_s \big| |M_s| \smallsetminus |L| : |M_s| \smallsetminus |L| \to C_{|s|} \smallsetminus |L| = |K| \times (0,1]$ of φ_s is smooth (see [Mu], Section 8.1).

<u>Proof</u>. Let $\sigma = (u_1,\dots,u_m)$ be a simplex of K, $\tau = (v_1,\dots,v_n)$ be a simplex of L and assume that $s(\sigma) \preccurlyeq \tau$. Given $x = \sum \alpha_i u_i \in |\sigma|$ and $1 \le j \le n$ set $\alpha^j = \sum_{s(u_i)=v_j} \alpha_i$ and, if $\alpha^j > 0$, $x^j = \sum_{s(u_i)=v_j} (\alpha_i/\alpha_j)u_i$.

Let now $\theta * A$ be a simplex of M_s with $A = (\hat{\sigma}_1, \dots, \hat{\sigma}_r)$, $\theta \preccurlyeq s(\sigma_1)$ and $\hat{\sigma}_1 < \hat{\sigma}_2 < \dots < \hat{\sigma}_r$ (thus $\theta \preccurlyeq \tau$); let $z \in |\theta * A| \subset |M_s|$. Then there exist $t \in [0,1]$, $x \in |A| \subset |\sigma|$ and $y = \sum \beta_j v_j \in |\theta|$ such that $z = tx + (1-t)y$ (if $t \neq 0$ and $t \neq 1$, then x , y and t are unique). Since $\theta \preccurlyeq s(\sigma_1)$, it follows that for any vertex $v_j \in \theta$, $\alpha^j \neq 0$ and we can define $x_y \in |\sigma|$ by setting $x_y = \sum_{v_j \in \theta} \beta_j x^j$. Define next $\varphi_{s,\sigma,\tau}(z) \in C_{|s|}$ by

$$\varphi_{s,\sigma,\tau}(z) = [tx + (1-t)x_y,\ t]$$

(notice that $|s|(x_y) = y$; thus when $t = 0$, $\varphi_{s,\sigma,\tau}(z) = [x_y,0] = [y,0]$ is still well defined ; when $t = 1$, $\varphi_{s,\sigma,\tau}(z) = [x,1]$ is also well defined).

If $\tilde{\sigma}$, $\tilde{\tau}$, $\tilde{\theta}$ and $\tilde{A}$ are other simplexes satisfying the same conditions as σ , τ , θ and A and if $z \in |\tilde{\theta} * \tilde{A}|$, then one can check that $\varphi_{s,\sigma,\tau}(z) = \varphi_{s,\tilde{\sigma},\tilde{\tau}}(z)$. We can therefore define $\varphi_s : |M_s| \to C_{|s|}$ by $\varphi_s(z) = \varphi_{s,\sigma,\tau}(z)$, z , σ , τ and A being as above.

Assertions (1), (2), (3) and the continuity of φ_s are easy to verify. In order to prove that φ_s is a homeomorphism it is sufficient (by (1)) to consider the case $K = \sigma$ and $L = \tau = s(\sigma)$. If $\sigma = \tau$ and $s = 1_\sigma$, denote $M_s = M_\sigma$, $C_{|s|} = C_{|\sigma|}$ and $\varphi_s = \varphi_\sigma$. In this case $C_{|\sigma|} = |\sigma| \times [0,1] = |\sigma| \times |\Delta^1|$ (Δ^1 is the standard 1-dimensional simplex with vertices 0 and 1), $M_\sigma = \sigma' \times \Delta^1$ (σ' is considered with the order defined in 9.1.4 and Δ^1 with the order $0 < 1$) and $\varphi_\sigma : |\sigma' \times \Delta^1| \to |\sigma'| \times |\Delta^1|$ is the map determined by the projections of $\sigma' \times \Delta^1$ on σ' and Δ^1 . By Lemma 9.2.2 it is a homeomorphism.

Return now to the general case $s : \sigma \to \tau = s(\sigma)$. Let $\sigma = (u_1,\ldots,u_m)$ and $\tau = (v_1,\ldots,v_n)$. For $x \in |s|^{-1}(\hat{\tau})$ define $s_x : |\tau| \to |\sigma|$ by

$$s_x(y) = \Sigma\, y_j x^j \ , \quad y = \Sigma\, y_j v_j \in |\tau| \ .$$

It is clear that

$$(9.4.2.1) \qquad |s| \circ s_x = 1_{|\sigma|} \ .$$

Define also $F_x : |M_\tau| \to |M_s|$ and $G_x : C_{|\tau|} \to C_{|s|}$ as follows. Let $z \in |\theta * (\hat{\tau}_1,\ldots,\hat{\tau}_r)| \subset |M_\tau| = |M_{1_\tau}|$ (where $\theta \prec 1_\tau(\tau_1) = \tau_1 \prec \ldots \prec \tau_r \prec \tau$), $z = ty + \Sigma t_k \hat{\tau}_k$ with $y \in |\theta|$ and $t + \Sigma t_k = 1$. Let $\sigma_i = s^{-1}(\tau_i)$ and let A be the simplex of σ' generated by $\hat{\sigma}_1 \prec \ldots \prec \hat{\sigma}_r$. Then $\theta * A$ is a simplex of M_s and we can define $F_x(z) \in |\theta * A| \subset |M_s|$ by setting

$$F_x(z) = ty + \Sigma\, t_k s_x(\hat{\tau}_k) \ .$$

In view of (9.4.2.1) the definition is correct. G_x is defined by

$$G_x([y,t]) = [s_x(y),t] \ , \quad [y,t] \in C_{|\tau|} \ .$$

There are also canonical projections $\pi_s : M_s \to M_\tau$ and $p_s : C_{|s|} \to C_{|\tau|}$ given by

$$\pi_s(v_j) = v_j \quad \text{and} \quad \pi_s(\hat{\sigma}_0) = s(\hat{\sigma}_0) \quad \text{for} \quad \hat{\sigma}_0 < \sigma$$
$$p_s([x,t]) = [|s|(x),t] \; , \quad [x,t] \in C_{|s|} \; .$$

A direct verification shows that the diagram

(9.4.2.2)
$$\begin{array}{ccccc} |M_\tau| & \xrightarrow{F_x} & |M_s| & \xrightarrow{|\pi_s|} & |M_\tau| \\ \varphi_\tau \downarrow & & \downarrow \varphi_s & & \downarrow \varphi_\tau \\ C_{|\tau|} & \xrightarrow[G_x]{} & C_{|s|} & \xrightarrow[|p_s|]{} & C_{|\tau|} \end{array}$$

is commutative. From (9.4.2.1) it follows that

(9.4.2.3) $$|\pi_s| \circ F_x = 1_{|M_\tau|}$$

and

(9.4.2.4) $$p_s \circ G_x = 1_{C_{|\tau|}} \; .$$

Since $|\sigma| = \bigcup s_x(|\tau|)$, x running in $|s|^{-1}(\hat{\tau})$, and since $s_x(|\tau|) \cap s_{x'}(|\tau|) \cap |\sigma|^\circ = \emptyset$ if $x \neq x'$ ($|\sigma|^\circ$ is the interior of σ), the commutativity of the diagram (9.4.2.2), the relations (9.4.2.3) and (9.4.2.4) and the fact that φ_τ is a homeomorphism prove that φ_s is a homeomorphism too. Q.E.D.

9.4.3. LEMMA. Consider the following commutative diagram of s.m.'s

$$\begin{array}{ccc} K & \xrightarrow{s} & L \\ \alpha \downarrow & & \downarrow \beta \\ M & \xrightarrow[t]{} & N \end{array}$$

and assume that s and t are proper and $|\beta| : |L| \to |N|$ is finite to one. Let N' be a barycentric subdivision of N and chose barycentric subdivisions M' , K' and L' of M , K and L respectively such that

t , β , α and s extend to s.m.'s $t' : M' \to N'$, $\beta' : L' \to N'$, $\alpha' : K' \to M'$ and $s' : K' \to L'$ and $|t| = |t'|$, $|\beta| = |\beta'|$, $|\alpha| = |\alpha'|$ and $|s| = |s'|$ (this is always possible: one choses first M' and then K' such that $|t'| = |t|$ and $|\alpha| = |\alpha'|$; L' is uniquely determined by the condition $|\beta'| = |\beta|$). Define $s_\psi : M_s \to M_t$ and $\psi : C_{|s|} \to C_{|t|}$ by setting

$s_\psi(v) = \beta(v)$ for any vertex v of L ;

$s_\psi(\hat{\sigma}) = \alpha'(\hat{\sigma})$ for any simplex σ of K ;

$\psi([x,t]) = [|\alpha|(x),t]$ for $[x,t] \in C_{|s|}$ with $x \in |K|$;

$\psi([y,0]) = [|\beta|(y),0]$ for $[y,0] \in C_{|s|}$ with $y \in |L|$.

Then $\{(M_s,\varphi_s),(M_t,\varphi_t)\}$ is a triangulation of ψ , the corresponding s.m. being s_ψ .

Proof. Using (1) of Lemma 9.4.2 and the fact that $|\beta|$ is finite to one, it is sufficient to consider the case $K = \sigma$, $L = N = \tau$, $M = \theta$ and $\beta = 1_\tau$. This case can be settled by a direct verification. Q.E.D.

9.5.1. LEMMA. Let $f : A \to B$ be a proper continuous map and let $\beta : B_1 \to B$ be continuous. Let $A_1 = A \times_B B_1$ (with respect to f and β) and let $f_1 : A_1 \to B_1$ and $\alpha : A_1 \to A$ be the canonical projections. Let $\gamma : C_{f_1} \to C_f$ be the continuous map induced by α and β and let $r_f : C_f \to B$ and $r_{f_1} : C_{f_1} \to B_1$ be the canonical retractions. Define $h : C_{f_1} \to C_f \times_B B_1$ by $h(z) = (\gamma(z),r_{f_1}(z))$ (the fibre product is taken with respect to r_f and β). Then h is a homeomorphism and $h^{-1}\circ\gamma$ and $h^{-1}\circ r_{f_1}$ are the canonical projections of $C_f \times_B B_1$ on C_f and B_1 respectively.

The proof of this lemma is simple and left to the reader.

9.5.2. LEMMA. The notation is as above. Assume that f and β are triangulable with respect to some triangulations of A and B_1 and the same triangulation of B . Then γ is triangulable.

Proof. There is no loss of generality in assuming that $A = |K|$, $B_1 = |L_1|$, $B = |L|$, $f = |s|$ and $\beta = |t|$. By subdividing first L and then K and L_1, we may further assume that $A = |K'|$, $B = |L'|$, $B_1 = |L_1'|$, $f = |s'|$ and $\beta = |t'|$. We can therefore consider $K_1 = K' \times_{L'} L_1'$ and, by 9.3.2, we can identify A_1 with $|K_1|$; under this identification $\alpha = |\bar{p}_1|$ and $f_1 = |\bar{p}_2|$, $\bar{p}_1$ and $\bar{p}_2$ being the projections of K_1 on K' and L_1'. Let $r_s : M_s' \to L'$ be the retraction. Since M_s' has an obvious structure of o.s.c., we can consider $M = M_s' \times_{L'} L_1'$, with respect to r_s and t'. Using the homeomorphisms $|\tilde{M}| \to |M_s'| \times_{|L|} |L_1'|$ of 9.3.2 and $|M_s'| \to C_{|s|} = C_f$ of 9.4.2 and Lemma 9.5.1 we get a homeomorphism $\psi : |M| \to C_{f_1}$. One checks that $\{(M,\psi),(M_s',\varphi_s)\}$ is a triangulation of γ. Q.E.D.

9.6. Let $A \subset R^m$ and $B \subset R^n$ be polyhedra. A continuous map $f : A \to B$ is called piecewise linear if its graph is a polyhedron in $R^m \times R^n = R^{m+n}$.

If $s : K \to L$ is a s.m., then $|s| : |K| \to |L|$ is piecewise linear. The following partial converse is also true: given a proper piecewise linear map $f : A \to B$, there exist linear triangulations of A and B such that f is simplicial with respect to them (see [Hu], Theorem 3.6).

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