APPENDIX: Chain Conditions

This brief appendix states facts referred to in the preceding text, with enough references that the interested reader may pursue the matter. One new result is presented here.

DEFINITIONS. (All rings will be Noetherian)

i) A chain of primes $P_0 \subset P_1 \subset \ldots \subset P_n$ is <u>saturated</u> if height $P_{i+1}/P_i = 1$ for $i = 0, 1, \ldots, n-1$.

ii) A saturated chain $P_0 \subset \ldots \subset P_n$ is a <u>maximal chain</u> if P_0 and P_n are respectively minimal and maximal primes.

iii) The ring R is <u>catenary</u> if any two saturated chains of primes with common end points have common length.

iv) The local ring (R,M) is <u>quasi-unmixed</u> if every minimal prime in the completion R^* has depth equal to dim R.

v) The domain R satisfies the <u>Altitude Formula</u> if for any finitely generated extension domain of R, T, and for any $Q \in \text{spec } T$ with $P = Q \cap R$, we have height $P + \text{TrD}(T/R) = \text{height } Q + \text{TrD}((T/Q)/(R/P))$. Here TrD(B/A) is the transcendence degree of the domain B over the subdomain A.

Remark: It is well known that the Altitude Inequality holds for any Noetherian domain. That is, with R, T, P and Q as above, height $P + \text{TrD}(T/R) \geq \text{height } Q + \text{TrD}((T/Q)/(R/P))$ [ZS].

It was long known that affine rings are catenary. As a consequence of his structure theorem for complete local rings, Cohen showed that complete local domains are catenary. Nagata's celebrated example [N , Example 2, pp. 203-205] gives a Noetherian domain R which is catenary but for which R[X], X an indeterminate, is not catenary. The following pair of results are fundamental (see [R1] or [MD]).

THEOREM A1. Let R be a Noetherian ring, and let X be an indeterminate. The following are equivalent.

i) Every finitely generated extension of R is catenary.

ii) for each maximal ideal M of R, $R[X]_{(M,X)}$ is catenary.

iii) R is locally quasi-unmixed.

iv) R_M is quasi-unmixed for each maximal ideal M of R.

THEOREM A2. Let R be a Noetherian domain. Then R satisfies the Altitude Formula if and only if R is locally quasi-unmixed.

We note that being quasi-unmixed is quite stable.

THEOREM A3. If the local ring (R,M) is quasi-unmixed, so is every finitely generated extension. If I is an ideal of R and if all minimal primes of I have the same depth, then R/I is quasi-unmixed.

Theorem A1 ((ii) $\Leftrightarrow$ (iv)) has been generalized (see [MD] or [RM]).

THEOREM A4. Let (R,M) be a local domain and let X be an indeterminate. Let $N \in \operatorname{spec} R[X]$ with $N \cap R = M$ but $N \neq MR[X]$. Then the following three sets of integers are all equal.

a) $\{n \mid$ there is a maximal chain of prime ideals of length n in $R[X]_N\}$

b) $\{n \mid$ there is a maximal chain of prime ideals of length n-1 in some integral extension domain of R$\}$.

c) $\{n \mid$ there is a minimal prime of depth n-1 in $R^*\}$.

If $R \subseteq T$ is an integral extension with (R,M) local, and if T contains a height 1 maximal then by going up to $\overline{T}$ and then going down to $\overline{R}$, we see that $\overline{R}$ contains a height 1 maximal. Thus Proposition 3.19 ((i) $\Leftrightarrow$ (iv)) is a special case of Theorem A4.

The next result is referred to in Chapter 5 and Chapter 10. As it does not appear in the literature, we prove it.

THEOREM A5. Let P be a prime in a local ring (R,M). Suppose that $(R/P)^*$ has a depth n minimal prime and $(R_P)^*$ has a depth m minimal prime. Then R^* has a depth $n+m$ minimal prime.

Proof: By Theorem A4, there is a maximal chain of length $n+1$ in $(R/P)[X]$ localized at $(M/P,X)$. Under the natural map $R[X] \to (R/P)[X]$, suppose the inverse image of that chain is $PR[X] \subset P_1 \subset \ldots \subset P_n \subset (M,X)$. Now this chain is saturated, and by [HM, Corollary 1.5] we may assume that $P_1 \cap R = P$ (but $P_1 \neq PR[X]$).

Considering $R \to R_P \to (R_P)^*$, we let q^* be a depth m minimal prime of $(R_P)^*$ and q be its inverse image in R. Thus $q^*/q(R_P)^*$ is a depth m minimal prime in $(R_P/qR_P)^* = ((R/q)_{P/q})^*$. By Theorem A4. applied to $(R/q)_{P/q}$ with $N = P_1R_P[X]/qR_P[X]$, we easily find a saturated chain of primes of length $m+1$ having the form $qR[X] \subset Q_1 \subset \ldots \subset Q_m \subset P_1$.

Clearly $qR[X] \subset Q_1 \subset \ldots \subset Q_m \subset P_1 \subset P_2 \subset \ldots \subset P_n \subset (M,X)$ is a maximal chain of length $m+n+1$. Applying Theorem A4 to $R[X]_{(M,X)}$, we see that R^* contains a depth $m+n$ minimal prime.

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LIST OF NOTATION

(Page numbers indicate where more information can be found.)

		Page
$A(I,n)$	$\mathrm{Ass}(R/I^n)$	3
$A^*(I)$	limit of $A(I,n)$, $n = 1,2,\ldots$,	
$\bar{A}(I,n)$	$\mathrm{Ass}(R/\overline{I^n})$	
$\bar{A}^*(I)$	limit of $\bar{A}(I,n)$, $n = 1,2,3,\ldots$,	12
$B(I,n)$	$\mathrm{Ass}(I^{n-1}/I)$	
$B^*(I)$	limit of $B(I,n)$, $n = 1,2,\ldots$,	3
$\bar{B}(I,n)$	$\mathrm{Ass}(\overline{I^{n-1}}/\overline{I^n})$	
$\bar{B}^*(I)$	limit of $\bar{B}(I,n)$, $n = 1,2,3,\ldots$,	100
$gr^*(I)$	the asymptotic grade of I	35
$\bar{I}$	the integral closure of the ideal I	3
$I \sim J$	projective equivalence	53
$\ell(I)$	the analytic spread of I	26
$Q(R)$	the total quotient ring of R	
$\bar{R}$	the integral closure of R in $Q(R)$	
R^*	the M-adic completion of the local ring (R,M)	
$R^{[1]}$	$\{y \in Q(R) \mid (R : y) \not\subseteq P$ whenever $z(P) = 1\}$	77
$R^{\langle 1\rangle}$	$\{y \in Q(R) \mid (R : y) \not\subseteq P$ whenever $\mathrm{Ass}((R_P)^*)$ contains a depth 1 prime$\}$	87
$T(I)$	the ideal transform of I	76
$V_I(x)$	n if $x \in I^n - I^{n+1}$; ∞ if $x \in \cap I^n$, $n = 1,2,3,\ldots$	
$\bar{V}_I(x)$	$\lim_{n\to\infty} V_I(x^n)/n$	39
$z(P)$	$\min\{\mathrm{depth}\ q^* \mid q^*$ is a minimal prime of $(R_P)^*\}$	
$\subseteq$, $\subset$	inclusion , proper inclusion	

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index (continued)