A

Applications of the Representation Theory of Finite Groups

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This appendix consists of two sections. In the first we give a self-contained and fairly complete introduction to the representation and character theory of finite groups, including Frobenius's formula and a higher genus generalization. In the second we give several applications related to topics treated in this book.

A.1 Representation Theory of Finite Groups

A.1.1 Irreducible Representations and Characters

Let G be a finite group. A (finite-dimensional; we will always assume this) *representation* (V, π) of G is a finite-dimensional complex vector space V and a homomorphism $\pi : G \to \mathrm{GL}(V)$. Thus each $g \in G$ defines a linear map $v \mapsto \pi(g)v$ from V to V, with $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$. One often drops the "$\pi$" and writes the action simply as $v \mapsto gv$. Alternatively, one often drops the "V" and simply denotes the representation itself by π. The definition given here corresponds to left representations; one also has right representations (where $v \mapsto vg$ with $v(g_1 g_2) = (vg_1)g_2$); this leads to an isomorphic theory, by replacing g by g^{-1}.

We call two representations V and V' *isomorphic*, denoted $V \simeq V'$, if there is a G-equivariant isomorphism from V to V', and write $V \cong V'$ if such an isomorphism has been fixed. If V is a (left) representation of G and A a complex vector space, then both $V \otimes_{\mathbb{C}} A$ and $\mathrm{Hom}_{\mathbb{C}}(A, V)$ are (left) representations in the obvious way $(g(v \otimes a) = (gv) \otimes a, (g\phi)(a) = g\phi(a))$. If $\dim_{\mathbb{C}} A = k$, then both of these representations are isomorphic to $V \oplus \cdots \oplus V$ (k copies), the isomorphisms being canonical if one has chosen a basis of A. Similarly, the dual space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is in a natural way a right representation of G via $(\phi g)(v) = \phi(gv)$ and the space $\mathrm{Hom}_{\mathbb{C}}(V, A)$ is also a right representation, isomorphic to k copies of V^*. Finally, if V and V' are two representations of G, then we write $V \otimes_G V'$ for the quotient of $V \otimes_{\mathbb{C}} V'$ by the relation $gv \otimes v' = v \otimes gv'$ and $\mathrm{Hom}_G(V, V')$ for the set of G-equivariant

linear maps from V to V'; these are simply vector spaces, without any natural G-action.

A representation V of G is called *irreducible* if it contains no proper subspace which is invariant under the action of G. As a simple example, let $G = S_n$ and $V = \mathbb{C}^n$ with the obvious action of G by permutation of the coordinates. Then V is not irreducible, since it contains the two subspaces $W_1 = \{(x, \ldots, x) \mid x \in \mathbb{C}\}$ and $W_2 = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_1 + \cdots + x_n = 0\}$, of dimensions 1 and $n - 1$ respectively, which are obviously invariant under the action of G. On the other hand, these two representations are irreducible and V is their direct sum. More generally, one has:

Lemma A.1.1. *Any representation of G is a direct sum of irreducible ones.*

Proof. Pick a G-invariant non-degenerate scalar product on V. (To obtain one, start with any positive-definite scalar product and replace it by the obvious average over G.) If V is not already irreducible, it contains a proper G-invariant subspace W. But then the orthogonal complement $W^\perp$ of W is also G-invariant, and $V = W \oplus W^\perp$. The result now follows by induction on the dimension. $\square$

This will be used in conjunction with the following property of irreducible representations:

Lemma A.1.2 (Schur's Lemma). *Let V and V' be two irreducible representations of G. Then the complex vector space $\mathrm{Hom}_G(V, V')$ is 0-dimensional if $V \not\simeq V'$ and 1-dimensional if $V \simeq V'$. The space $\mathrm{Hom}_G(V, V)$ is canonically isomorphic to $\mathbb{C}$.*

Proof. Since neither V nor V' has a non-trivial G-invariant subspace, any non-zero G-equivariant map $\phi : V \to V'$ has trivial kernel and cokernel. Hence $\mathrm{Hom}_G(V, V') = \{0\}$ if V and V' are not isomorphic. If they are, then we may assume that $V' = V$. Then for any eigenvalue λ of ϕ, the map $\phi - \lambda$ has a kernel and therefore is zero. Hence $\mathrm{Hom}_G(V, V) \cong \mathbb{C}$ canonically. $\square$

These two lemmas already suffice to prove one of the first basic facts of the theory, the "first orthogonality relation for characters." If (V, π) is an irreducible representation, we define its *character* as the function $\chi_\pi(g) = \mathrm{tr}\,(\pi(g), V)$ from G to $\mathbb{C}$. Then we have:

Corollary A.1.3 (First orthogonality relation). *Let (V, π) and (V', π') be two irreducible representations of G. Then*

$$\frac{1}{|G|} \sum_{g \in G} \chi_\pi(g) \overline{\chi_{\pi'}(g)} = \begin{cases} 1 \ \textit{if } \pi \simeq \pi', \\ 0 \ \textit{otherwise.} \end{cases} \tag{A.1}$$

Proof. The dimension of the space V^G of G-invariant vectors in any representation V of G is given by $\dim(V^G) = |G|^{-1} \sum_{g \in G} \operatorname{tr}(g, V)$. (*Proof.* The linear map $v \mapsto |G|^{-1} \sum_{g \in G} gv$ is a projection from V to V^G, so its trace equals $\dim V^G$.) Apply this to the G-representation $\operatorname{Hom}_{\mathbb{C}}(V, V') \cong V \otimes_{\mathbb{C}} V'^*$, with G acting by $g(v, \phi) = (gv, \phi \circ g^{-1})$. (Recall that V'^* is naturally a right representation of G, so that we have to replace g by g^{-1} when we act from the left.) Then the trace of $g \in G$ equals $\chi_\pi(g)\overline{\chi_{\pi'}(g)}$, and the dimension of $\big(\operatorname{Hom}(V, V')\big)^G = \operatorname{Hom}_G(V, V')$ is 1 or 0 by Schur's Lemma. $\square$

Now let $\{(V_i, \pi_i)\}_{i \in I}$ be a full set of non-isomorphic irreducible representations of G. Lemma A.1.1 tells us that any representation V of G is isomorphic to a direct sum $\bigoplus_{i \in I} \underbrace{V_i \oplus \cdots \oplus V_i}_{k_i}$ of the representations V_i or equivalently, by what was said above, that

$$V \cong \bigoplus_i V_i \otimes_{\mathbb{C}} A_i \cong \bigoplus_i \operatorname{Hom}_{\mathbb{C}}(B_i, V_i) \tag{A.2}$$

for some k_i-dimensional vector spaces A_i and B_i over $\mathbb{C}$, but we do not yet know that these spaces, or even the multiplicities k_i, are independent of the decomposition chosen. The following lemma shows that this is true and gives a canonical description of the spaces A_i and B_i.

Lemma A.1.4. *Let V be an arbitrary representation of G. Then we have canonical G-equivariant isomorphisms*

$$\bigoplus_{i \in I} V_i \otimes_{\mathbb{C}} \operatorname{Hom}_G(V_i, V) \overset{\sim}{\longrightarrow} V, \quad V \overset{\sim}{\longrightarrow} \bigoplus_{i \in I} \operatorname{Hom}_{\mathbb{C}}(\operatorname{Hom}_G(V, V_i), V_i) \tag{A.3}$$

given by sending $x \otimes \phi \in V_i \otimes \operatorname{Hom}_G(V_i, V)$ to $\phi(x)$ and $v \in V$ to the homomorphism $\phi \mapsto \phi(v)$ from $\operatorname{Hom}_G(V, V_i)$ to V_i. Conversely, given any decompositions of V of the form (A.2), *there are canonical isomorphisms $A_i \cong \operatorname{Hom}_G(V_i, V)$ and $B_i \cong \operatorname{Hom}_G(V, V_i)$ as complex vector spaces.*

Proof. Since both isomorphisms in (A.2) are additive under direct sums, we may assume by Lemma A.1.1 that V is irreducible, say $V = V_j$ for some $j \in I$. Then both statements in (A.3) follow immediately from Lemma A.1.2, since $V_j \otimes_{\mathbb{C}} \mathbb{C} \cong \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}, V_j) \cong V_j$. The proof of the last statement, which will not be used in what follows, is similar and will be left to the reader. $\square$

We next introduce the *group algebra* $\mathbb{C}[G]$. This is the set of linear combinations $\sum_{g \in G} \alpha_g [g]$ ($\alpha_g \in \mathbb{C}$) of formal symbols $[g]$ ($g \in G$), with the obvious addition and multiplication. It can be identified with $\operatorname{Maps}(G, \mathbb{C})$ via $\alpha(g) = \alpha_g$. The group algebra is a left *and* right representation of G via $g_1[g]g_2 = [g_1 g g_2]$, or equivalently $(g_1 \alpha g_2)(g) = \alpha(g_1^{-1} g g_2^{-1})$ if α is a map from G to $\mathbb{C}$. The central result of the representation theory of finite groups is the following assertion.

Theorem A.1.5. *Let G be a finite group. Then there is a canonical $(G \times G)$-equivariant algebra isomorphism*

$$\mathbb{C}[G] \cong \bigoplus_{i \in I} \operatorname{End}_{\mathbb{C}}(V_i) \tag{A.4}$$

sending $[g]$ to the collection of linear maps $\pi_i(g) : V_i \to V_i$.

Proof. For any representation V of G, $\operatorname{Hom}_G(\mathbb{C}[G], V) \cong V$ as G-representations, since an equivariant map $\phi : \mathbb{C}[G] \to V$ is uniquely determined by $\phi([1]) \in V$, which is arbitrary. Applying this to $V = V_i$ ($i \in I$) and then applying the second isomorphism in (A.3) to $V = \mathbb{C}[G]$, we obtain the assertion of the theorem. We can also obtain the isomorphism (A.4), in the reverse direction, by applying the first isomorphism in (A.3) to $V = \mathbb{C}[G]$ and using the canonical isomorphisms $\operatorname{Hom}_G(V_i, \mathbb{C}[G]) \cong V_i^*$ and $V_i \otimes_{\mathbb{C}} V_i^* \cong \operatorname{Hom}_{\mathbb{C}}(V_i, V_i) \cong \operatorname{End}_{\mathbb{C}}(V_i)$. $\square$

Essentially all important general facts about representations of finite groups are corollaries of this theorem. To state them, let us introduce the notation $\mathcal{C}$ for the set of conjugacy classes in G and $\mathcal{R}$ for the set of isomorphism classes of irreducible representations. (Of course $\mathcal{R}$ and the index set I used above are in canonical bijection, but we will no longer need to have picked representatives for the elements of $\mathcal{R}$.) Since the value of the character $\chi_\pi(g)$ depends only on the isomorphism class of π and the conjugacy class of g, we can write, with some abuse of notation, $\chi_\pi(C)$ for any $\pi \in \mathcal{R}$ and $C \in \mathcal{C}$. For instance, with these notations the first orthogonality relation (A.1) becomes

$$\sum_{C \in \mathcal{C}} |C| \chi_\pi(C) \overline{\chi_{\pi'}(C)} = |G| \delta_{\pi, \pi'} \qquad (\pi, \pi' \in \mathcal{R}). \tag{A.5}$$

Then Theorem A.1.5 has the following consequences.

Corollary A.1.6. *The cardinality of $\mathcal{R}$ is finite and*

$$\sum_{\pi \in \mathcal{R}} (\dim \pi)^2 = |G|. \tag{A.6}$$

Proof. Compare the dimensions on both sides of (A.4). $\square$

Corollary A.1.7. *The sets $\mathcal{C}$ and $\mathcal{R}$ have the same cardinality: there are as many irreducible representations of G as there are conjugacy classes in G.*

Proof. A basis for the center $Z(\mathbb{C}[G])$ of $\mathbb{C}[G]$ is clearly given by the elements $e_C = \sum_{g \in C} [g]$ ($C \in \mathcal{C}$). On the other hand, $\operatorname{End}_{\mathbb{C}}(V_i)$ is the matrix algebra $M_{\dim V_i}(\mathbb{C})$, with 1-dimensional center. Hence the algebra isomorphism (A.4) tells us that $|\mathcal{C}| = \dim_{\mathbb{C}} Z(\mathbb{C}[G]) = |\mathcal{R}|$. $\square$

Corollary A.1.8 (Second orthogonality relation). *Let C_1, $C_2 \in \mathcal{C}$. Then*

$$\sum_{\pi \in \mathcal{R}} \chi_\pi(C_1)\overline{\chi_\pi(C_2)} = \begin{cases} |G|/|C_1| & \text{if } C_1 = C_2, \\ 0 & \text{otherwise.} \end{cases} \tag{A.7}$$

Notice that this formula agrees with (A.6) when $C_1 = C_2 = \{1\}$, since $\chi_\pi(1) = \dim \pi$.

Proof. This follows from (A.5) and Corollary A.1.7, since these imply that the matrix $\left(|C|^{1/2}|G|^{-1/2}\chi_\pi(C)\right)_{\pi \in \mathcal{R}, C \in \mathcal{C}}$ is square and unitary, and the inverse of a unitary matrix is also unitary. But we can also obtain (A.7) directly (and then, if we wish, deduce (A.5) from it) by computing the trace of the action of $(g_1, g_2) \in C_1 \times C_2$ on both sides of (A.4). The action of (g_1, g_2) on the basis $\{[g]\}_{g \in G}$ of $\mathbb{C}[G]$ is given by the permutation $[g] \mapsto [g_1 g g_2^{-1}]$ (as before, we have to invert g_2 to turn the right action into a left one), so its trace is the number of fixed points of this permutation, which is clearly $|G|/|C_1|$ if g_1 and g_2 are conjugate and 0 otherwise. On the other hand, the trace of (g_1, g_2) on $\mathrm{End}_{\mathbb{C}}(\pi) = \pi \otimes_{\mathbb{C}} \pi^*$ equals $\chi_\pi(g_1)\overline{\chi_\pi(g_2)}$. $\square$

A.1.2 Examples

We illustrate the theory explained in Sec. A.1.1 in a few important special cases.

Abelian groups. If G is a cyclic group of order n, with generator γ, then there are n obvious 1-dimensional (and hence irreducible!) representations of G given by $V = \mathbb{C}$ and $\gamma v = \zeta v$ with $\zeta \in \mathbb{C}$ a (not necessarily primitive) nth root of unity. By the dimension formula (A.6), these are the only irreducible representations. More generally, for any finite abelian group one sees easily that all irreducible representations of G are 1-dimensional (because the commuting operators $\pi(g)$ on any representation V have a common eigenvector, or alternatively by reducing to the cyclic case) and that the corresponding characters are simply the homomorphisms from G to $\mathbb{C}^*$.

Symmetric groups of small order. The symmetric group S_n has two 1-dimensional representations $\mathbf{1}$ (the trivial representation, $V = \mathbb{C}$ with all elements of G acting as $+1$) and $\boldsymbol{\varepsilon}_n$ (the *sign* representation, $V = \mathbb{C}$ with odd permutations acting as -1) and an $(n-1)$-dimensional irreducible representation $\mathbf{St}_n$ which is the space $W_2 \subset \mathbb{C}^n$ mentioned at the beginning of this section. For $n = 2$ and $n = 3$, the dimension formula (A.6) shows that these are the only irreducible representations (and $\mathbf{St}_n \simeq \varepsilon_n$ for $n = 2$), with character tables given by

C	Id	$(1,2)$
$\lvert C\rvert$	1	1
$\mathbf{1}$	1	1
ε_2	1	-1

C	Id	$(1,2)$	$(1,2,3)$
$\lvert C\rvert$	1	3	2
$\mathbf{1}$	1	1	1
ε_3	1	-1	1
$\mathbf{St}_3$	2	0	-1

(Here the numbers in the second row show the size of the conjugacy classes, needed as weights to make the rows of the table orthogonal; the columns are orthogonal as they stand.) For $n = 4$, the orthogonality relations again have a unique solution and the character table must take the form

C	Id	$(1,2)$	$(1,2,3)$	$(1,2)(3,4)$	$(1,2,3,4)$
$\lvert C\rvert$	1	6	8	3	6
$\mathbf{1}$	1	1	1	1	1
ε_4	1	-1	1	1	-1
A	2	0	-1	2	0
$\mathbf{St}_4$	3	1	0	-1	-1
$\mathbf{St}_4 \otimes \varepsilon_4$	3	-1	0	-1	1

for some 2-dimensional irreducible representation A of S_4. We can construct A explicitly as $\{(x_s)_{s\in S} \mid \sum x_s = 0\}$, where S is the 3-element set of decompositions of $\{1,2,3,4\}$ into two (unordered) subsets of cardinality 2. The reader may wish to attempt constructing "by hand" the 7×7 character table for the group S_5, where the dimensions of the irreducible representations are 1, 1, 4, 4, 5, 5 and 6.

Observe that in the above tables for S_n the character values $\chi_\pi(g)$ are all integers. This is a general fact. For arbitrary finite group representations, $\chi_\pi(g)$ is a sum of roots of unity and hence an algebraic integer, and its Galois conjugates are simply the values of $\chi_\pi(g^\ell)$ with $\ell \in \mathbb{Z}$ prime to the order of g. (This is because the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on roots of unity is given by $\zeta \mapsto \zeta^\ell$ with ℓ prime to the order of ζ.) For $G = S_n$, however, g^ℓ and g are conjugate since they have the same cycle structure, so $\chi_\pi(g)$ is Galois invariant and hence belongs to $\mathbb{Z}$.

Symmetric groups of arbitrary order. We will not give a complete account of the general representation theory of S_n here, since it is a little complicated and there are many good accounts, but will only mention some highlights, following the approach given in the beautiful paper [OV], which we highly recommend to the reader. The statements given here will be used only in Sec. A.2.3. We denote by $\mathcal{R}_n$ the sets of isomorphism classes of irreducible representations of S_n. We will also consider S_{n-1} as a subgroup of S_n (namely, the set of elements fixing n), and similarly for all S_i, $i < n$.

A. The first basic fact is that each irreducible representation of S_n, when restricted to the subgroup S_{n-1}, splits into the direct sum of distinct irreducible representations of S_{n-1}. For $\pi \in \mathcal{R}_n$ and $\pi' \in \mathcal{R}_{n-1}$ we write $\pi' \prec \pi$ if π' occurs in $\pi|_{S_{n-1}}$, so $\pi|_{S_{n-1}} = \oplus_{\pi' \prec \pi} \pi'$. Similarly, each $\pi' \prec \pi$ when restricted to S_{n-2} splits into a sum of irreducible rerpresentations π'' of S_{n-2}, and continuing this process, we see that π splits canonically into a direct sum of 1-dimensional spaces V_ξ indexed by all possible chains $\xi : \pi_1 \prec \cdots \prec \pi_n = \pi$ with $\pi_i \in \mathcal{R}_i$. (Specifically, we have $V_\xi = V_1 \subset \cdots \subset V_n = \pi$ where each V_i is S_i-invariant and isomorphic to π_i.) Notice how unusual this behavior is: for general finite groups G it is more modern, and of course better, to think of representations as actions of G on abstract vector spaces, without any choice of basis, rather than as collections of matrices satisfying the same relations as the elements of G, but for the symmetric groups the irreducible representations come equipped with their own nearly (i.e., up to scalar multiples) canonical bases, and we have matrices after all!

B. The next key idea is to introduce the so-called *Jucys-Murphy element*

$$X_n = (1, n) + (2, n) + \cdots + (n-1, n)$$

of the group algebra $\mathbb{Z}[S_n]$. We can write this element as $e_{[T]_n} - e_{[T]_{n-1}}$, where $e_{[T]_n} \in Z(\mathbb{Z}[S_n])$ as in the proof of Corollary A.1.7 above is the sum of all elements in the conjugacy class $[T]_n$ of $T = (1, 2) \in S_n$ and $e_{[T]_{n-1}}$ is the corresponding element for $n-1$. Since $e_{[T]_n}$ is central, by Schur's lemma it acts on each $\pi \in \mathcal{R}_n$, and hence on each subrepresentation $\pi' \prec \pi$, as multiplication by a scalar $\nu_\pi(T)$ (which belongs to $\mathbb{Z}$ by the remark on integrality made above), and similarly $e_{[T]_{n-1}}$ acts on π' as a scalar $\nu_{\pi'}(T)$, so X_n acts on π' as multiplication by the number $a_n = \nu_\pi(T) - \nu_{\pi'}(T)$. By induction on i it follows that for each chain $\xi = (\pi_1, \ldots, \pi_n)$ each element $X_i \in \mathbb{Z}[S_i] \subset \mathbb{Z}[S_n]$ acts on π_{i-1} as multiplication by some integer $a_i(\xi)$, so we can associate to the chain ξ a *weight vector* $a(\xi) = (a_1(\xi), \ldots, a_n(\xi)) \in \mathbb{Z}^n$.

C. Conversely, the weight vector $a(\xi)$ determines ξ (and hence also π) completely, and there is a complete description of which vectors $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ occur as weight vectors. (The conditions are (i) $a_1 = 0$, (ii) for each $j > 1$, we have $|a_j - a_i| = 1$ for some $i < j$, and (iii) if $a_i = a_j$ for some $i < j$ then both $a_i - 1$ and $a_i + 1$ occur among the a_k with $i < k < j$.) Furthermore, two weight vectors $a, a' \in \mathbb{Z}^n$ correspond to the same representation π if and only if they are permutations of one another, so that π is uniquely characterized by the function $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ defined by $f(r) = \#\{i \mid a_i = r\}$, and this sets up a bijection between $\mathcal{R}_n$ and the set of finitely supported functions $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ satisfying $f(r+1) - f(r) \in \{0, |r| - |r+1|\}$ for all r and $\sum_{r \in \mathbb{Z}} f(r) = n$. These functions in turn correspond bijectively to the elements of the set $\mathcal{Y}_n$ of *Young diagrams* ($=$ subsets $Y \subset \mathbb{N}^2$ such that $(x, y) \in Y \Rightarrow (x', y') \in Y$ whenever $1 \leq x' \leq x$, $1 \leq y' \leq y$) of cardinality n, the correspondence $f \leftrightarrow Y$ being given by $f(r) = \#\{(x, y) \in Y \mid x - y = r\}$ and $Y = \{(x, y) \in \mathbb{N}^2 \mid \min(x, y) \leq f(x - y)\}$. In the bijection between $\mathcal{R}_n$

and $\mathcal{Y}_n$ obtained in this way one has $\pi' \prec \pi$ if and only if the corresponding Young diagrams $Y' \in \mathcal{Y}_{n-1}$ and $Y \in \mathcal{Y}_n$ satisfy $Y' \subset Y$.

D. Finally, there is an explicit inductive procedure, given by the so-called *Murnaghan-Nakayama rule*, to compute the value of the character $\chi_\pi(C)$ for any $\pi \in \mathcal{R}_n$ and conjugacy class $C \subset \mathrm{S}_n$ in terms of the Young diagram associated to π and the partition of n associated to C. However, we will not use this and do not give the details here, noting only that the case $C = [T]_n$, which will be needed in Sec. A.2.3, follows from **B.** and **C.** above.

A.1.3 Frobenius's Formula

We end Sec. A.1 by describing a formula which has applications to combinatorial problems in many parts of mathematics and in particular to several of the topics treated in this book; some of these applications will be discussed in Sec. A.2. This formula computes the number

$$\mathcal{N}(G; C_1, \ldots, C_k) := \# \left\{ (c_1, \ldots, c_k) \in C_1 \times \cdots \times C_k \mid c_1 \cdots c_k = 1 \right\}$$

for arbitrary conjugacy classes $C_1, \ldots, C_k \in \mathcal{C}$ in terms of the characters of the irreducible representations of G. Note that $\mathcal{N}(G; C_1, \ldots, C_k)$ is independent of the order of the arguments, because the identity $c_i c_{i+1} = c_{i+1}(c_{i+1}^{-1} c_i c_{i+1})$ lets us interchange C_i and C_{i+1}.

Theorem A.1.9 (Frobenius's formula). *Let G be a finite group and C_1, ..., C_k conjugacy classes in G. Then*

$$\mathcal{N}(G; C_1, \ldots, C_k) = \frac{|C_1| \cdots |C_k|}{|G|} \sum_\chi \frac{\chi(C_1) \cdots \chi(C_k)}{\chi(1)^{k-2}}, \qquad (A.8)$$

where the sum is over all characters of irreducible representations of G.

Before giving the proof, we note three special cases. If $k = 1$ or $k = 2$, then (A.8) reduces to the orthogonality relation (A.7), applied to $(C_1, 1)$ or (C_1, C_2^{-1}), respectively. For $k = 3$, we write $(C_1, C_2, C_3) = (A, B, C^{-1})$ with $A, B, C \in \mathcal{C}$. Then $\mathcal{N}(G; C_1, C_2, C_3) = n_{AB}^C$, where

$$n_{AB}^C = \#\{(a, b) \in A \times B \mid ab \in C\}.$$

The integers n_{AB}^C are nothing but the structure constants (i.e., the numbers defined by $e_A e_B = \sum_C n_{AB}^C e_C$) of the center of the group ring $\mathbb{Z}[G]$ with respect to the basis $\{e_C\}$ defined in the proof of Corollary A.1.7 above. Formula (A.8) therefore describes the ring structure of this commutative ring in terms of the character theory of G. This formula plays a role in mathematical physics in connection with the so-called "fusion algebras."

Proof. If C is any conjugacy class of G, then the element $e_C = \sum_{g \in C}[g]$ is central and hence, by Schur's lemma, acts on any irreducible representation

π of G as multiplication by a scalar $\nu_\pi(C)$. Since each element $g \in C$ has the same trace $\chi_\pi(g) = \chi_\pi(C)$, we find

$$|C| \chi_\pi(C) = \sum_{g \in C} \chi_\pi(g) = \operatorname{tr}(\pi(e_C), V) = \operatorname{tr}(\nu_\pi(C) \cdot \operatorname{Id}, V) = \nu_\pi(C) \dim \pi$$

and hence

$$\nu_\pi(C) = \frac{|C|}{\dim \pi} \chi_\pi(C) = \frac{\chi_\pi(C)}{\chi_\pi(1)} |C|. \tag{A.9}$$

Now we compute the trace of the action by left multiplication of the product of the elements $e_{C_1}, \ldots, e_{C_k}$ on both sides of (A.4). On the one hand, this product is the sum of the elements $[c_1 \cdots c_k]$ with $c_i \in C_i$ for all i, and since the trace of left multiplication by $[g]$ on $\mathbb{C}[G]$ is clearly $|G|$ for $g = 1$ and 0 otherwise, the trace equals $|G| \mathcal{N}(G; C_1, \ldots, C_k)$. On the other hand, the product of the e_{C_i} acts as scalar multiplication by $\prod \nu_\pi(C_i)$ on π and hence also on the $(\dim \pi)^2$-dimensional space $\operatorname{End}_{\mathbb{C}}(\pi)$. Formula (A.8) follows immediately. □

Theorem A.1.9 has a clear topological interpretation. Let X be the 2-sphere with k (numbered) points $P_1, \ldots, P_k$ removed. Then $\pi_1(X)$ is a (free) group on k generators $x_1, \ldots, x_k$ with $x_1 \cdots x_k = 1$, and $\mathcal{N}(G; C_1, \ldots, C_k)$ simply counts the number of homomorphisms ρ from $\pi_1(X)$ to G with $\rho(x_i) \in C_i$ for each i. As explained in the text of the book, if G acts faithfully on some finite set F, then each such homomorphism corresponds to a (not necessarily connected) Galois covering of X with fibre F, Galois group G, and ramification points P_i, such that for each i the permutation of the elements of a fixed fibre induced by the local monodromy at P_i belongs to the conjugacy class C_i. Hence $\mathcal{N}(G; C_1, \ldots, C_k)$ counts the coverings with these properties. Observe that the above-mentioned invariance of $\mathcal{N}(G; C_1, \ldots, C_k)$ under permutation of its arguments is clear from this topological point of view.

A natural idea is now to generalize this to ramified coverings of Riemann surfaces of arbitrary genus $g \geq 0$. In view of the structure of the fundamental group of a k-fold punctured surface of genus g, this boils down to computing the number

$$\mathcal{N}_g(G; C_1, \ldots, C_k) = \#\{(a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_k) \in$$
$$G^{2g} \times C_1 \times \cdots \times C_k \mid [a_1, b_1] \cdots [a_g, b_g] \, c_1 \cdots c_k = 1\}.$$

The result, given in the theorem below, turns out to be surprisingly simple, but was apparently not given in the classical group-theoretical literature and was discovered first in the context of mathematical physics ([DW], [FQ]). The proof is an almost immediate consequence of the special case $g = 0$.

Theorem A.1.10. *With the same notations as above, we have for all $g \geq 0$*

$$\mathcal{N}_g(G; C_1, \ldots, C_k) = |G|^{2g-1} |C_1| \cdots |C_k| \sum_\chi \frac{\chi(C_1) \cdots \chi(C_k)}{\chi(1)^{k+2g-2}}. \tag{A.10}$$

Proof. Note that $[a_1, b_1] \cdots [a_g, b_g] = a_1(b_1 a_1 b_1^{-1})^{-1} \cdots a_g (b_g a_g b_g^{-1})^{-1}$ and that, for a and a' in the same conjugacy class A, there are $|G|/|A|$ elements $b \in G$ with $bab^{-1} = a'$. Hence

$$\mathcal{N}_g(G; C_1, \ldots, C_k) = \sum_{A_1, \ldots, A_g \in \mathcal{C}} \frac{|G|}{|A_1|} \cdots \frac{|G|}{|A_g|} \times$$
$$\mathcal{N}(G; A_1, A_1^{-1}, \ldots, A_g, A_g^{-1}, C_1, \ldots, C_k).$$

Now applying Theorem A.1.9 we find

$$\mathcal{N}_g(G; C_1, \ldots, C_k) = |G|^{g-1}|C_1| \cdots |C_k| \sum_{A_1, \ldots, A_g \in \mathcal{C}} |A_1| \cdots |A_g| \times$$
$$\left(\sum_\chi \frac{\chi(A_1)\overline{\chi(A_1)} \cdots \chi(A_g)\overline{\chi(A_g)}\chi(C_1) \cdots \chi(C_k)}{\chi(1)^{k+2g-2}} \right)$$
$$= |G|^{g-1}|C_1| \cdots |C_k| \sum_\chi \frac{\chi(C_1) \cdots \chi(C_k)}{\chi(1)^{k+2g-2}} \left(\sum_{A \in \mathcal{C}} |A| \chi(A) \overline{\chi(A)} \right)^g.$$

The theorem now follows from the case $\pi = \pi'$ of the orthogonality relation (A.5). $\square$

We observe that, by formula (A.9), both Frobenius's formula and its higher genus generalization can be written more naturally in terms of the eigenvalues $\nu_\pi(C)$ than the character values $\chi_\pi(C)$. In particular, equation (A.10) when expressed in terms of the $\nu_\pi(C)$ takes the simple form

$$\mathcal{N}_g(G; C_1, \ldots, C_k) = |G|^{2g-1} \sum_{\pi \in \mathcal{R}} \frac{\nu_\pi(C_1) \cdots \nu_\pi(C_k)}{(\dim \pi)^{2g-2}} . \tag{A.11}$$

A.2 Applications

In this section we give several applications of the character theory of finite groups in the case when $G = \mathrm{S}_n$, the symmetric group on n letters. In this case the conjugacy classes are described simply by the partitions of n, the conjugacy class of an element $g \in \mathrm{S}_n$ with d_1 one-cycles, d_2 two-cycles, etc., corresponding to the partition $\lambda = 1^{d_1} 2^{d_2} \cdots$ of n. It turns out that much of the part of the information which we are interested in can be captured by any of three polynomials $\Phi_g(X)$, $P_g(r)$ or $Q_g(k)$ which mutually determine each other but which encode the interesting combinatorial information about g in different ways.

A.2.1 Representations of S_n and Canonical Polynomials Associated to Partitions

In this subsection we define the three polynomials $\Phi_g(X)$, $P_g(r)$ or $Q_g(k)$ mentioned above and prove their main properties. Applications to several of the topics treated in the main text of the book will then be given in Sec. A.2.2– A.2.4. Since each of these three polynomials depends only on the conjugacy class C of $g \in S_n$ or equivalently only on the partition $\lambda \vdash n$ corresponding to C, we will also use the notations $\Phi_C(X)$, $P_C(r)$ or $Q_C(k)$ and $\Phi_\lambda(X)$, $P_\lambda(r)$ or $Q_\lambda(k)$.

We recall from Sec. A.1.2 the standard irreducible representation $\mathbf{St}_n = \mathbb{C}^n/\mathbb{C}$ of S_n of dimension $n-1$. We define $\Phi_g(X)$ as the characteristic polynomial of g on $\mathbf{St}_n$:

$$\Phi_g(X) = \det\left(1 - g\,X, \mathbf{St}_n\right). \tag{A.12}$$

It is easy to see that, if g has the cycle structure $1^{d_1}2^{d_2}\cdots$, then the characteristic polynomial of g on $\mathbb{C}^n = \mathbf{St}_n \oplus \mathbf{1}$ is simply $\prod(1-X^i)^{d_i}$, so in terms of λ we have

$$\Phi_\lambda(X) = \frac{(1-X)^{d_1}(1-X^2)^{d_2}\cdots}{1-X} \qquad (\lambda = 1^{d_1}2^{d_2}\cdots). \tag{A.13}$$

This is the first of our three polynomials.

From linear algebra, we know that the coefficient of X^r in the characteristic polynomial of any endomorphism ϕ of a finite-dimensional vector space V equals $(-1)^r$ times the trace of the endomorphism induced by ϕ on the rth exterior power $\bigwedge^r(V)$. We therefore have

$$\Phi_g(X) = \sum_{r=0}^{n-1}(-1)^r \chi_r(g)\, X^r, \tag{A.14}$$

where we have abbreviated

$$\chi_r(g) := \mathrm{tr}(g, \pi_r), \qquad \pi_r := \bigwedge^r(\mathbf{St}_n) \qquad (0 \le r \le n-1).$$

We shall show below that the representations π_r are irreducible and distinct, so the χ_r are distinct characters of S_n. We have $\chi_r(1) = \dim \pi_r = \binom{n-1}{r}$. Now, since there is a unique polynomial of degree $n-1$ having specified values at any n specified points, we can define a polynomial $P_g(r)$ by the requirements that $\deg P_g \le n-1$ and that

$$P_g(r) = \frac{\chi_r(g)}{\chi_r(1)} = \binom{n-1}{r}^{-1}\chi_r(g) \qquad (0 \le r \le n-1). \tag{A.15}$$

This is our second polynomial attached to (the conjugacy class of) g.

Finally, we recall the notations $\ell(g) = \sum d_i$ and $v(g) = \sum(i-1)d_i$ for the number of cycles of an element g of S_n and for its complement $v(g) = n-\ell(g)$.

(Again, since both depend only on the conjugacy class of g, we will use the alternative notations $\ell(C)$, $v(C)$ or $\ell(\lambda)$, $v(\lambda)$ as the situation requires.) Let σ denote the cyclic element $(1\,2\,\ldots\,n)$ of S_n. We then define

$$Q_C(k) \;=\; \frac{1}{|C|} \sum_{g\in C} k^{\ell(g\sigma)} \tag{A.16}$$

for any conjugacy class C of S_n. This is our third polynomial.

Before stating the main result, which describes how each of the three polynomials just defined determines the other two, we need two simple (and well-known) lemmas concerning the representations π_r.

Lemma A.2.1. *The representations $\pi_r = \bigwedge^r(\mathbf{St}_n)$ $(0 \le r \le n-1)$ of S_n are irreducible and distinct.*

Proof. Set $V = \pi_0 \oplus \cdots \oplus \pi_{n-1}$, a representation of $G = \mathrm{S}_n$ of dimension 2^{n-1}. By the results of Sec. A.1, we know that V can be decomposed uniquely as $\bigoplus_{\pi\in\mathcal{R}} m(\pi)\,\pi$, where $\mathcal{R}$ is a set of representatives of the isomorphism classes of irreducible representations of G and the multiplicities $m(\pi)$ are non-negative integers. Since V is by construction a sum of n non-trivial representations, we have $\sum m(\pi) \ge n$. Define $\chi_V : \mathrm{S}_n \to \mathbb{C}$ by $\chi_V(g) = \mathrm{tr}(g, V)$. Then χ_V decomposes as $\chi_V = \sum m(\pi)\chi_\pi$. We define a scalar product on the vector space of conjugacy-invariant functions $f : \mathrm{S}_n \to \mathbb{C}$ by $(f_1, f_2) = |G|^{-1} \sum_{g\in G} f_1(g)\overline{f_2(g)}$. By equation (A.1) (first orthogonality relation) we know that the characters χ_π of the irreducible representations of G form an orthogonal basis for this vector space with respect to this scalar product, so $(\chi_V, \chi_V) = \sum m(\pi)^2$. If we can show that $(\chi_V, \chi_V) = n$, it will follow that $\sum(m(\pi)^2 - m(\pi)) \le 0$ and hence that each multiplicity $m(\pi)$ equals 0 or 1, with $m(\pi) = 1$ for exactly n distinct irreducible representations of G, proving the lemma.

To calculate (χ_V, χ_V), we first use equations (A.14) and (A.13) to obtain that

$$\chi_V(g) \;=\; \sum_{r=0}^{n-1} \chi_r(g) \;=\; \Phi_g(-1) \;=\; \begin{cases} 2^{(d_1+d_3+\cdots)-1} & \text{if } d_2 = d_4 = \cdots = 0, \\ 0 & \text{otherwise} \end{cases}$$

for g with the cycle structure $\prod i^{d_i}$. Since the number of elements with this cycle structure is $n!/\prod\big(d_i!\,i^{d_i}\big)$, we obtain

$$(\chi_V,\,\chi_V) = \frac{1}{n!} \sum_{g\in\mathrm{S}_n} |\chi_V(g)|^2$$

$$= \frac{1}{4} \sum_{\substack{d_1,\,d_3,\,d_5,\cdots\ge 0 \\ d_1+3d_3+5d_5+\cdots=n}} \frac{4^{d_1+d_3+d_5+\cdots}}{d_1!\,d_3!\,d_5!\,\cdots\,1^{d_1}\,3^{d_3}\,5^{d_5}\,\cdots}$$

$$= \frac{1}{4}\cdot\text{coefficient of } x^n \text{ in } \prod_{\substack{i\ge 1 \\ i \text{ odd}}} e^{4x^i/i} \;=\; n\,,$$

since $4 \sum_{i \text{ odd}} \dfrac{x^i}{i} = 2 \log \dfrac{1+x}{1-x}$ and $\left(\dfrac{1+x}{1-x}\right)^2 = 1 + 4 \sum_{n \geq 1} nx^n.$ $\square$

Lemma A.2.2. *The value of the character of an irreducible representation π of S_n on the conjugacy class of the cyclic element $\sigma \in S_n$ is given by*

$$\chi_\pi(\sigma) = \begin{cases} (-1)^r & \text{if } \pi \simeq \pi_r \text{ for some } r,\ 0 \leq r \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first statement follows immediately from formulas (A.14) and (A.13), since

$$\sum_{r=0}^{n-1} (-1)^r \chi_r(\sigma)\, X^r = \Phi_\sigma(X) = \dfrac{1 - X^n}{1 - X} = \sum_{r=0}^{n-1} X^r.$$

The second then follows immediately from the second orthogonality relation for characters, since $\sum_{\pi \in \mathcal{R}} |\chi_\pi(\sigma)|^2 = n!/(n-1)! = n = \sum_{r=0}^{n-1} |\chi_r(\sigma)|^2$. $\square$

Theorem A.2.3. *For any integer $n \geq 1$ there are canonical linear isomorphisms between the three n-dimensional vector spaces*

$$\langle 1, X, \ldots, X^{n-1} \rangle, \qquad \langle 1, r, \ldots, r^{n-1} \rangle, \qquad \langle k, k^2, \ldots, k^n \rangle,$$

such that the three polynomials $\Phi_C(X)$, $P_C(r)$ and $Q_C(k)$ correspond to one another for every conjugacy class C in S_n. These isomorphisms are given by $\Phi \leftrightarrow P \leftrightarrow Q$, where $\Phi(X)$, $P(r)$ and $Q(k)$ are related by the generating function identities

$$\Phi(X) = \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} P(r) X^r = (1-X)^{n+1} \sum_{k=1}^{\infty} Q(k)\, X^{k-1}, \tag{A.17}$$

or alternatively in terms of bases of the three vector spaces by

$$(1-X)^{n-1-s} \quad \leftrightarrow \quad \binom{n-1-r}{s} \Big/ \binom{n-1}{s} \quad \leftrightarrow \quad \binom{s+k}{s+1} \tag{A.18}$$

or

$$X^s (1-X)^{n-1-s} \quad \leftrightarrow \quad (-1)^s \binom{r}{s} \Big/ \binom{n-1}{s} \quad \leftrightarrow \quad \binom{k}{s+1} \tag{A.19}$$

for $0 \leq s \leq n-1$. These isomorphisms are equivariant with respect to the three involutions $*$ defined by

$$\Phi^*(X) = (-X)^{n-1}\Phi(1/X),\ \ P^*(r) = P(n-1-r),$$
$$Q^*(k) = -Q(-k), \tag{A.20}$$

under which the polynomials $\Phi_C(X)$, $P_C(r)$ and $Q_C(k)$ are invariant or anti-invariant according to whether C is a conjugacy class of even or odd permutations.

Proof. The fact that the formulas (A.17) give isomorphisms between the spaces in question can be checked either directly or by using the binomial theorem to check that the collections of functions listed in (A.18) or in (A.19), each of which forms a basis for the relevant vector space, satisfy the identities in (A.17). The equivariance of the isomorphisms with respect to the involutions (A.20) follows most easily by noting that these involutions exchange the bases given in (A.18) and (A.19) up to a factor $(-1)^s$. The fact that $\Phi_g(X)$ and $P_g(r)$ are related by the first formula in (A.17) follows immediately from equations (A.13) and (A.14) above. The statement about the invariance or anti-invariance of Φ_g, P_g and Q_g under the involutions $*$ is easy in each case (although of course it would suffice to verify only one of them and then use the rest of the theorem): for Φ_g it follows immediately from formula (A.13) and the fact that $\operatorname{sgn}(g) = (-1)^{v(g)}$ with $v(g) = \sum(i-1)d_i$; for P_g it follows from (A.15) and the fact that $\bigwedge^{n-1-r}(\mathbf{St}_n)$ is dual to the tensor product of $\bigwedge^r(\mathbf{St}_n)$ with the sign representation ε_n (because $\bigwedge^{d-r} V \otimes \bigwedge^r(V) \xrightarrow{\wedge} \bigwedge^d(V)$ is a perfect pairing for any d-dimensional vector space V and $\bigwedge^{n-1}(\mathbf{St}_n) = \varepsilon_n$); and for Q_g it follows from the fact that $\ell(g\sigma) = n - v(g\sigma) \equiv v(g) + 1 \pmod 2$ for any $g \in \mathrm{S}_n$. The only thing we have to show is therefore that the polynomials $\Phi_C(X)$ and $Q_C(k)$ are related by the generating series identity

$$\sum_{k=1}^{\infty} Q_C(k)\, X^{k-1} \;=\; \frac{\Phi_C(X)}{(1-X)^{n+1}}\,, \tag{A.21}$$

for any conjugacy class C in S_n. For this we use the following lemma.

Lemma A.2.4. *For any integers n, r and k with $0 \le r \le n-1$ we have*

$$\frac{1}{n!} \sum_{g \in \mathrm{S}_n} P_g(r)\, k^{\ell(g)} \;=\; \text{coefficient of } X^{k-1} \text{ in } \frac{X^r}{(1-X)^{n+1}}\,.$$

Proof. By a calculation similar to the one in the proof of Lemma A.2.1, we have

$$\frac{1}{n!} \sum_{g \in \mathrm{S}_n} \Phi_g(Y)\, k^{\ell(g)} = \sum_{\substack{d_1,\, d_2,\, \cdots \\ d_1 + 2d_2 + \cdots = n}} \frac{k^{d_1 + d_2 + \cdots}}{d_1!\, d_2!\, \cdots\, 1^{d_1}\, 2^{d_2}\, \cdots} \times$$

$$\frac{(1-Y)^{d_1}(1-Y^2)^{d_2}\cdots}{1-Y}$$

$$= \frac{1}{1-Y} \cdot \text{coefficient of } u^n \text{ in } \prod_{i=1}^{\infty} \exp\!\left(\frac{k(1-Y^i)u^i}{i}\right)$$

$$= \frac{1}{1-Y} \cdot \text{coefficient of } u^n \text{ in } \left(\frac{1-uY}{1-u}\right)^k$$

$$= \text{coefficient of } X^{k-1} \text{ in } \frac{(1-XY)^{n-1}}{(1-X)^{n+1}}\,, \tag{A.22}$$

where the last equality follows either by using residue calculus to get

$$\frac{1}{1-Y} \operatorname{Res}_{u=0} \left(\left(\frac{1-uY}{1-u} \right)^k \frac{du}{u^{n+1}} \right) = -\operatorname{Res}_{X=1} \left(\frac{(1-XY)^{n-1}}{(1-X)^{n+1}} \frac{dX}{X^k} \right)$$

$$= +\operatorname{Res}_{X=0} \left(\frac{(1-XY)^{n-1}}{(1-X)^{n+1}} \frac{dX}{X^k} \right)$$

(here $u = \frac{1-X}{1-XY}$) or else from the geometric series identity

$$\sum_{k=1}^{\infty} X^{k-1} \left(\frac{1-uY}{1-u} \right)^k = \frac{1-uY}{1-u-X+uXY}$$

$$= \frac{1}{1-X} + (1-Y) \sum_{n=1}^{\infty} \frac{(1-XY)^{n-1}}{(1-X)^{n+1}} u^n.$$

The lemma follows by comparing the coefficients of $(-1)^r \binom{n-1}{r} Y^r$ on both sides of (A.22). $\square$

Remark A.2.5. Lemma A.2.4 implies that

$$\frac{1}{n!} \sum_{g \in S_n} P_g(r) \, k^{\ell(g)} = \binom{k+n-r-1}{n} - \binom{n-r-1}{n}$$

as polynomials in r and n, since both sides of this identity are polynomials in k and r, both have degree $\leq n-1$ in r, and they agree for $k \in \mathbb{N}$ and for $r = 0, 1, \ldots, n-1$.

We can now complete the proof of equation (A.21) and hence of Theorem A.2.3. Frobenius's formula (Theorem A.1.9) in the case $k = 3$ can be rewritten in the form

$$\frac{1}{|C|} \sum_{c \in C} F(bc) = \sum_{\pi} \chi_\pi(b) \chi_\pi(C) \left(\frac{1}{|G|} \sum_{A \in \mathcal{C}} |A| \frac{\chi_\pi(A)}{\chi_\pi(1)} F(A^{-1}) \right)$$

for any finite group G, conjugacy classes A and C of G, and class function (= conjugacy-invariant function) $F : G \to \mathbb{C}$, where the sum is over all irreducible representations of G. Specializing to $G = S_n$ and $b = \sigma$ and using Lemma A.2.2, we find

$$\frac{1}{|C|} \sum_{c \in C} F(\sigma c) = \sum_{r=0}^{n-1} (-1)^r \chi_r(C) \left(\frac{1}{n!} \sum_{g \in G} P_g(r) F(g^{-1}) \right).$$

Now specializing further to $F(g) = k^{\ell(g)}$ and using Lemma A.2.4 and equation (A.14) gives

$$Q_C(k) = \text{coefficient of } X^{k-1} \text{ in } \frac{\Phi_C(X)}{(1-X)^{n+1}}.   \square$$

We mention two simple consequences of Theorem A.2.3 which will be used later.

Corollary A.2.6. *Let* $\lambda = 1^{d_1} 2^{d_2} \cdots$ *be a partition of* n. *Then the polynomial* $P_\lambda(r)$ *has degree* $v(\lambda) = \sum (i - 1) d_i$ *and leading coefficient* $(-1)^{v(\lambda)} K(\lambda)$, *where*

$$K(\lambda) = \frac{(n - 1 - v(\lambda))!}{(n - 1)!} 1^{d_1} 2^{d_2} \cdots n^{d_n} . \tag{A.23}$$

The polynomial $Q_\lambda(k)$ *has degree* $v(\lambda) + 1$ *and leading coefficient*

$$\frac{\prod i^{d_i}}{(v(\lambda) + 1)!} .$$

Proof. From equation (A.13) we see that $\Phi_\lambda(X) \sim \frac{(n-1)!}{(n-1-v(\lambda))!} K(\lambda) \times (1 - X)^{n-1-v(\lambda)}$ as $X \to 1$. But this means that $\Phi_\lambda(X)$ is a linear combination of the basis elements (A.18) with $s \le v(\lambda)$, with the coefficient for $s = v(\lambda)$ being $\frac{(n-1)!}{(n-1-v(\lambda))!} K(\lambda)$. The correspondence (A.18) then tells us that P_λ and Q_λ have the degrees and leading coefficients stated.   □

Corollary A.2.7. *The polynomial* $Q_\lambda(k)$ *takes integer values for* $k \in \mathbb{Z}$. *Moreover, the value of* $Q_\lambda(k)$ *for* $k \in \mathbb{N}$ *depends only on the numbers* $n - d_1$, d_2, ..., d_{k-1}, *the first few being*

$$Q_\lambda(0) = 0, \quad Q_\lambda(1) = 1, \quad Q_\lambda(2) = n + 2 - d_1, \quad Q_\lambda(3) = \binom{n + 3 - d_1}{2} - d_2 .$$

Proof. Equations (A.21) and (A.13) give

$$Q_C(k) = \text{coefficient of } X^{k-1} \text{ in}$$
$$(1 - X)^{d_1 - n - 2} (1 - X^2)^{d_2} (1 - X^3)^{d_3} \cdots (1 - X^{k-1})^{d_{k-1}} .   □$$

Remark A.2.8. The fact that $\deg Q_\lambda \le v(\lambda)$ says that $\ell(g\sigma) \le v(g)$ for all $g \in S_n$. Equivalently, $v(g) + v(g\sigma) \ge n - 1 = v(\sigma)$. This is a special case of the general fact that $v(g_1) + v(g_2) \ge v(g_3)$ for any three elements g_1, g_2, $g_3 \in S_n$ with product 1, which can be seen most easily by noticing that $v(g_i)$ is the codimension of the fixed point set of g_i acting on $\mathbb{C}^n$ (or on its irreducible subspace $\mathbf{St}_n$) and that codimensions of subspaces behave subadditively. The statement that the polynomial $Q_\lambda(k)$ depends only on $n - d_1$ and the d_i with $i \ge 2$ says that it is stable under the inclusions $S_n \subset S_{n+1}$ and hence depends only on the class of g in S_∞. This is not difficult to see directly from the definition, but there does not seem to be any obvious reason why the value of $Q_\lambda(k)$ for $k \in \mathbb{N}$ depends only on the d_i with $i < k$ or why it is an integer.

A.2.2 Examples

We give a number of examples of the polynomials introduced in Sec. A.2.1 for special conjugacy classes and for small values of n.

Trivial element: $g = 1$. Here the cycle structure is simply 1^n, so equation (A.13) gives $\Phi_1(X) = (1 - X)^{n-1}$, while from equations (A.4) and (A.5) we immediately get $P_1(r) = 1$ and $Q_1(k) = k$.

Transposition: $T = (1, 2)$. Now the cycle structure is $1^{n-2}2$, so equation (A.13) gives

$$\Phi_T(X) = (1 - X)^{n-3}(1 - X^2) = (1 - X)^{n-2}(1 + X). \qquad (A.24)$$

From equation (A.14) we therefore obtain

$$\chi_r(T) = \binom{n-2}{r} - \binom{n-2}{r-1} = \binom{n-3}{r} - \binom{n-3}{r-2}$$

(with the obvious conventions when $r < 2$), so from (A.15) and the formula $\chi_r(1) = \binom{n-1}{r}$ we obtain

$$P_T(r) = 1 - \frac{2r}{n-1} \qquad (A.25)$$

while equation (A.17) or (A.18) or (A.19) gives

$$Q_T(k) = k^2.$$

Note that this last formula, unlike the formulas for $\Phi_T(X)$ and $P_T(r)$, is independent of n. This is a special case of the above-mentioned fact that $Q_g(k)$ is stable under $S_n \subset S_{n+1}$.

Cyclic permutation: $\sigma = (1, 2, \cdots, n)$. Here the cycle structure is simply n^1, so, as we have already seen, $\Phi_\sigma(X) = 1 + X + \cdots + X^{n-1}$ and $\chi_r(\sigma) = (-1)^r$. However, the explicit forms of the polynomials P_σ and Q_σ are quite complicated, as can be seen from the examples for $n = 5$ and 6 below.

Free involution: $\tau = (1, 2)(3, 4) \cdots (n - 1, n)$. Here n must be even, say $n = 2m$. The cycle structure of the conjugacy class of τ is 2^m, so (A.13) gives

$$\Phi_\tau(X) = \frac{(1 - X^2)^m}{1 - X} = (1 + X)(1 - X^2)^{m-1} \qquad (A.26)$$

and hence

$$\chi_r(\tau) = (-1)^{[(r+1)/2]} \binom{m-1}{[r/2]},$$

but here again the polynomials P_τ and Q_τ have no simple closed form.

Small n. Finally, to give the reader a better feel for the $\Phi_\lambda \leftrightarrow P_\lambda \leftrightarrow Q_\lambda$ correspondence, we give a complete table of these three polynomials for partitions of n with $1 \leq n \leq 6$ (see the next page). For convenience we tabulate the polynomial $\widetilde{P}_\lambda(t) := P\left(\frac{n-1}{2} - t\right)$ instead of $P_\lambda(r)$, since by the last statement of Theorem A.2.3 this is an even or an odd polynomial in t for all λ. The stability of Q_λ mentioned in the remark at the end of the last section is visible in this table: for example, the first seven entries in the column giving $Q_\lambda(k)$ for $n = 6$ agree with the values for $n = 5$.

A.2.3 First Application: Enumeration of Polygon Gluings

This is the combinatorial question which was discussed in Chapter 3 in connection with the evaluation of the (orbifold) Euler characteristic of the moduli space of curves of genus g. The problem was to count the number $\varepsilon_g(m)$ of ways to identify in pairs (with reverse orientation) the sides of a $(2m)$-gon such that the closed oriented surface obtained has a given genus g. Clearly this is the same as the number of free involutions in S_{2m} whose product with the standard cyclic permutation σ has $m + 1 - 2g$ cycles, so formula (A.16) gives

$$Q_\tau(k) \;=\; \frac{1}{(2m-1)!!} \sum_{0 \leq g \leq m/2} \varepsilon_g(m)\, k^{m+1-2g}\,, \qquad (\text{A.27})$$

where τ denotes any free involution (and where we have used that the conjugacy class of τ has cardinality $(2m-1)!!$). On the other hand, from equations (A.21) and (A.26) we get

$$\sum_{k=1}^{\infty} Q_\tau(k)\, X^{k-1} \;=\; \frac{\Phi_\tau(X)}{(1-X)^{2m+1}} \;=\; \frac{(1+X)^m}{(1-X)^{m+2}}\,,$$

which we can rewrite equivalently by using the same calculation as in the last line of (A.22) (with n replaced by $m + 1$ and Y by -1) as

$$Q_\tau(k) \;=\; \frac{1}{2} \cdot \text{coefficient of } u^{m+1} \text{ in } \left(\frac{1+u}{1-u}\right)^k.$$

Since the sum on the right-hand side of (A.27) is the polynomial denoted $T_m(k)$ in Sec. 3.1, this reproduces the evaluation of $\varepsilon_g(m)$ in terms of generating functions given in Theorem 3.1.5.

n	λ	$\widetilde{P}_\lambda(t)$	$Q_\lambda(k)$	$\Phi_\lambda(X)$
1	1^1	1	k	1
2	1^2	1	k	$1 - X$
	2^1	$2t$	k^2	$1 + X$
3	1^3	1	k	$1 - 2X + X^2$
	$1^1 2^1$	t	k^2	$1 - X^2$
	3^1	$\frac{1}{2}(3t^2 - 1)$	$\frac{1}{2}(k^3 + k)$	$1 + X + X^2$
4	1^4	1	k	$1 - 3X + 3X^2 - X^3$
	$1^2 2^1$	$\frac{2}{3}t$	k^2	$1 - X - X^2 + X^3$
	$1^1 3^1$	$\frac{1}{8}(4t^2 - 1)$	$\frac{1}{2}(k^3 + k)$	$1 - X^3$
	2^2	$\frac{1}{6}(4t^2 - 3)$	$\frac{1}{3}(2k^3 + k)$	$1 + X - X^2 - X^3$
	4^1	$\frac{1}{6}(4t^3 - 5t)$	$\frac{1}{6}(k^4 + 5k^2)$	$1 + X + X^2 + X^3$
5	1^5	1	k	$1 - 4X + 6X^2 - 4X^3 + X^4$
	$1^3 2^1$	$\frac{1}{2}t$	k^2	$1 - 2X + 2X^3 - X^4$
	$1^2 3^1$	$\frac{1}{4}t^2$	$\frac{1}{2}(k^3 + k)$	$1 - X - X^3 + X^4$
	$1^1 2^2$	$\frac{1}{3}(t^2 - 1)$	$\frac{1}{3}(2k^3 + k)$	$1 - 2X^2 + X^4$
	$1^1 4^1$	$\frac{1}{6}(t^3 - t)$	$\frac{1}{6}(k^4 + 5k^2)$	$1 - X^4$
	$2^1 3^1$	$\frac{1}{4}(t^3 - 2t)$	$\frac{1}{4}(k^4 + 3k^2)$	$1 + X - X^3 - X^4$
	5^1	$\frac{1}{24}(5t^4 - 15t^2 + 4)$	$\frac{1}{24}(k^5 + 15k^3 + 8k)$	$1 + X + X^2 + X^3 + X^4$
6	1^6	1	k	$1 - 5X + 10X^2 - 10X^3 + 5X^4 - X^5$
	$1^4 2^1$	$\frac{2}{5}t$	k^2	$1 - 3X + 2X^2 + 2X^3 - 3X^4 + X^5$
	$1^3 3^1$	$\frac{1}{80}(12t^2 + 5)$	$\frac{1}{2}(k^3 + k)$	$1 - 2X + X^2 - X^3 + 2X^4 - X^5$
	$1^2 2^2$	$\frac{1}{20}(4t^2 - 5)$	$\frac{1}{3}(2k^3 + k)$	$1 - X - 2X^2 + 2X^3 + X^4 - X^5$
	$1^2 4^1$	$\frac{1}{60}(4t^3 - t)$	$\frac{1}{6}(k^4 + 5k^2)$	$1 - X - X^4 + X^5$
	$1^1 2^1 3^1$	$\frac{1}{90}(4t^3 - 9t)$	$\frac{1}{4}(k^4 + 3k^2)$	$1 - X^2 - X^3 + X^5$
	$1^1 5^1$	$\frac{1}{384}(16t^4 - 40t^2 + 9)$	$\frac{1}{24}(k^5 + 15k^3 + 8k)$	$1 - X^5$
	2^3	$\frac{1}{30}(4t^3 - 13t)$	$\frac{1}{3}(k^4 + 2k^2)$	$1 + X - 2X^2 - 2X^3 + X^4 + X^5$
	$2^1 4^1$	$\frac{1}{240}(16t^4 - 64t^2 + 15)$	$\frac{1}{15}(k^5 + 10k^3 + 4k)$	$1 + X - X^4 - X^5$
	3^2	$\frac{1}{640}(48t^4 - 216t^2 + 115)$	$\frac{1}{40}(3k^5 + 25k^3 + 12k)$	$1 + X + X^2 - X^3 - X^4 - X^5$
	6^1	$\frac{1}{960}(48t^5 - 280t^3 + 259t)$	$\frac{1}{120}(k^6 + 35k^4 + 84k^2)$	$1 + X + X^2 + X^3 + X^4 + X^5$

The proof just given follows the exposition in [Z], where the identity (A.21) was proved and a few other applications were given. We mention two briefly:

- the probability that the product of two random cyclic permutations in S_n has exactly ℓ cycles is $(1 + (-1)^{n-\ell})$ times the coefficient of x^ℓ in $\binom{x+n}{n+1}$. In particular, for n odd the probability that such a product is cyclic equals $\frac{2}{n+1}$, as opposed to the probability that a random even element of S_n is cyclic, which equals $\frac{2}{n}$.

- the number of representations of an arbitrary even element of S_n as a product of two cyclic permutations is $\geq \frac{2(n-1)!}{n+2}$, as opposed to the average number of such representations for even permutations, which equals $\frac{2(n-1)!}{n}$.

A.2.4 Second Application: the Goulden–Jackson Formula

Our second application, again reproducing a result proved by a different method within the main text of the book, is the formula of Goulden-Jackson given in various forms in Theorems 1.5.12, 1.5.15, 1.6.6 and 5.2.2. The problem, in our present language, is to count the "Frobenius number" $\mathcal{N}(S_n; C_0, C_1, \cdots, C_k)$ of $(k+1)$-tuples $(c_0, \ldots, c_k) \in C_0 \times \cdots \times C_k$ when C_0 is the class of the cyclic element $\sigma \in S_n$ and $C_1, \ldots, C_k$ are arbitrary conjugacy classes in S_n. The Goulden-Jackson formula says that this number is 0 if $v = v(C_1) + \cdots + v(C_k)$ is less than $n-1$ and gives an explicit formula for it when $v(C_1) + \cdots + v(C_k) = n-1$. We give a somewhat more general result.

Theorem A.2.9. *Let $C_1, \ldots, C_k$ be arbitrary conjugacy classes in S_n and C_0 the class of the cyclic element σ. Then*

$$\mathcal{N}(S_n; C_0, C_1, \ldots, C_k) = \frac{(-1)^{n-1}}{n} |C_1| \cdots |C_k| \cdot \Delta^{n-1}\big(P_{C_1} \cdots P_{C_k}\big)(0),$$
(A.28)

where $P_{C_i}(r)$ is the polynomial of degree $v(C_i)$ associated to the conjugacy class C_i as in Sec. A.2.1 and Δ denotes the forward differencing operator $\Delta P(r) = P(r+1) - P(r)$.

For the conjugacy class C of S_n corresponding to a partition $\lambda = 1^{d_1} \cdots n^{d_n}$ of n, let us write $N(C)$ for the number $N(\lambda) = \frac{(d_1 + \cdots + d_n - 1)!}{d_1! \cdots d_n!}$ defined in 1.5.11. Then we have:

Corollary A.2.10 (Goulden-Jackson formula). *Let the C_i be as in the theorem. Then the number $\mathcal{N}(S_n; C_0, C_1, \ldots, C_k)$ vanishes if the number $v := v(C_1) + \cdots + v(C_k)$ is less than $n-1$ and is given by*

$$\frac{1}{(n-1)!} \mathcal{N}(S_n; C_0, C_1, \ldots, C_k) = n^{k-1} N(C_1) \cdots N(C_k)$$
(A.29)

if $v = n-1$.

Proof. The theorem is an immediate consequence of Frobenius's formula (Theorem A.1.9) and Lemma A.2.2, which says that $\chi(C_0)$ equals $(-1)^r$ if $\chi = \chi_r$ and vanishes for all other irreducible characters, together with definition (A.15), which together give

$$\mathcal{N}(S_n; C_0, C_1, \ldots, C_k) = \frac{1}{n} |C_1| \cdots |C_k| \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} P_{C_1}(r) \cdots P_{C_k}(r).$$

The corollary follows immediately from the theorem together with Corollary A.2.6 above and the formula $|C_i|K(\lambda_i) = nN(\lambda_i)$ (compare equation (A.23) and the definition of $N(\lambda)$), since the polynomial $P_{C_1}(r) \cdots P_{C_k}(r)$ has degree v and the mth difference of a polynomial of degree v is 0 for $m > v$ and equal to $m!$ times the leading coefficient of the polynomial if $m = v$.   □

Remark A.2.11. The fact that $\mathcal{N}(\mathrm{S}_n; C_0, C_1, \ldots, C_k) = 0$ unless $v \geq n - 1$ is also obvious from the remark at the end of Sec. A.2.1, which says that the function v satisfies the triangle inequality and hence necessarily $v(c_1) + \cdots + v(c_k) \geq v(c_0)$ if $c_0 c_1 \cdots c_k = 1$. It also follows from the topological interpretation of $\mathcal{N}(\mathrm{S}_n; C_0, C_1, \ldots, C_k)$ as the counting function for ramified coverings of S^2 with ramification types C_i, since $v - n - 1$ equals the Euler characteristic of the total space of the covering, which has the form $2 - 2g \leq 2$ because a covering one of whose ramification types is cyclic is necessarily connected. This shows also that the number $\mathcal{N}(\mathrm{S}_n; C_0, C_1, \ldots, C_k)$ vanishes unless $v \equiv n - 1 \pmod 2$. To see this from Theorem A.2.9, it is convenient to rewrite formula (A.28) in the form

$$\frac{n\,\mathcal{N}(\mathrm{S}_n; C_0, C_1, \ldots, C_k)}{|C_1| \cdots |C_k|} \;=\; \Delta_+^{n-1}\big(\widetilde{P}_{C_1} \cdots \widetilde{P}_{C_k}\big)(0)\,, \qquad (\mathrm{A}.30)$$

where $\widetilde{P}_C(t) = P_C\big(\frac{n-1}{2} - t\big)$ is the shifted version of $P_C(r)$ mentioned in Sec. A.2.2 and Δ_+ is the symmetric difference operator $\Delta_+ f(t) = f(t + \frac{1}{2}) - f(t - \frac{1}{2})$. This expression obviously vanishes if $v \not\equiv n - 1 \pmod 2$ because of the symmetry property $\widetilde{P}_C(-t) = (-1)^{v(C)}\widetilde{P}_C(t)$ and the fact that Δ_+ reverses the parity of an even or odd function.

Remark A.2.12. As a special case, we can take $C_1 = \cdots = C_k = [T]$, the class of transpositions in S_n. Then $\widetilde{P}_{C_i}(t) = \frac{2t}{n-1}$ by formula (A.25), so equation (A.30) gives

$$\mathcal{N}\big(\mathrm{S}_n;\, \sigma,\, \underbrace{T, \ldots, T}_{k}\big) = n^{k-1}\, \Delta_+^{n-1}\big(t^k\big)\big|_{t=0}\,.$$

Using the identity

$$\Delta_+^{n-1}\big(t^k\big)\big|_{t=0} = \sum_{r=0}^{n-1}(-1)^r \binom{n-1}{r}\left(\frac{n-1}{2} - r\right)^k = \frac{d^k}{du^k}\big[(e^{u/2} - e^{-u/2})^{n-1}\big]\big|_{u=0}$$

and the residue theorem, we can write this equivalently as

$$\mathcal{N}\big(G;\, \sigma,\, \underbrace{T, \ldots, T}_{k}\big) = k! \cdot n^{k-1} \cdot S_g(n) \quad \text{for } k = n - 1 + 2g \quad (g \geq 0), \ (\mathrm{A}.31)$$

where $S_g(n)$ is the polynomial of degree g in n given by

$$S_g(n) = \text{coefficient of } u^{2g} \text{ in } \left(\frac{\sinh u/2}{u/2} \right)^{n-1}$$

$$= \frac{n-1}{n-1+2g} \cdot \text{coefficient of } s^{2g} \text{ in } \left(\frac{s/2}{\text{asinh } s/2} \right)^{n+2g-1}, \quad \text{(A.32)}$$

the first values of which are given by

$$S_0(n) \; = \; 1, \quad S_1(n) \; = \; \frac{n-1}{24}, \quad S_2(n) \; = \; \frac{(n-1)(5n-7)}{5760}. \quad \text{(A.33)}$$

Formula (A.31) for $\mathcal{N}(S_n; \sigma, T, \ldots, T)$ was given in [SSV].

Remark A.2.13. Another observation is that, from formula (A.10) together with the fact that $\chi(\sigma) = \chi(\sigma)^3$ for all irreducible characters χ of S_n (Lemma A.2.2), we have the formula

$$\mathcal{N}_g(S_n; \sigma, C_1, \ldots, C_k) \; = \; n^{2g} \mathcal{N}(S_n; \underbrace{\sigma, \ldots, \sigma}_{2g+1}, C_1, \ldots, C_k) \quad \text{(A.34)}$$

for the "generalized Frobenius number" of Theorem A.1.10, for any integer $g \geq 0$ and any conjugacy classes $C_1, \ldots, C_k$ in S_n. This allows one to generalize Theorem A.2.9 to the case of ramified coverings of a Riemann surface of arbitrary genus with cyclic ramification at at least one point.

In view of the very simple form of equation (A.34), it might be of interest to give a direct combinatorial or topological proof, without using character theory.

In the rest of this subsection, we show how one can use Theorem A.2.9 to give explicit formulas for $\mathcal{N}(S_n; C_0, C_1, \ldots, C_k)$ for arbitrary values of v. The calculation and the results are considerably simplified if we use the symmetric formula (A.30) instead of (A.28), but still rapidly become complicated as the number $2g = v - n + 1$ grows.

The main step in the calculation is a refinement of Corollary A.2.6 of Theorem A.2.3 giving more leading terms of $P_\lambda(r)$ (or equivalently, of $\widetilde{P}_\lambda(t)$) for an arbitrary conjugacy class λ of S_n. Write $\lambda = n_1 + \cdots + n_s = 1^{d_1} 2^{d_2} \cdots$ and define invariants $\ell_h(\lambda)$ (higher moments) of λ by

$$\ell_h(\lambda) \; = \; n_1^h + \cdots + n_s^h \; = \; \sum_{i \geq 1} i^h d_i \qquad (h = 0, 1, \ldots), \quad \text{(A.35)}$$

so that $\ell_0(\lambda) = \ell(\lambda)$, $\ell_1(\lambda) = n$. Assign to $\ell_h(\lambda)$ the weight h. Then we have:

Lemma A.2.14. *The polynomial* $\widetilde{P}_\lambda(t) = P_\lambda\left(\frac{n-1}{2} - t\right)$ *has the form*

$$\widetilde{P}_\lambda(t) \; = \; K(\lambda) \left(1 + a_1(\lambda) \, t^{v(\lambda)-2} + a_2(\lambda) \, t^{v(\lambda)-4} + \cdots \right),$$

where $K(\lambda)$ is given by (A.23) and each $a_j(\lambda)$ is a universal polynomial of weighted degree $\leq 2j$ in n and the invariants $\ell_h = \ell_h(\lambda)$ with h even, the first two values being

$$a_1(\lambda) = (\ell_0 + 1)\ell_0 \frac{\ell_2 - 1}{24} - \frac{(n-1)n(n+1)}{24},$$

$$a_2(\lambda) = (\ell_0 + 3)(\ell_0 + 2)(\ell_0 + 1)\ell_0 \frac{5(\ell_2 - 1)^2 - 2(\ell_4 - 1)}{5760}$$

$$- \frac{(n-1)n(n+1)}{24}(\ell_0 + 1)\ell_0 \frac{\ell_2 - 1}{24}$$

$$+ \frac{(n-3)(n-2)(n-1)n(n+1)(5n+7)}{5760}.$$

Proof. We already know that $\widetilde{P}_\lambda(t)$ is a polynomial of degree and parity $v(\lambda)$ and hence has the form $K(\lambda) \sum_j a_j(\lambda) t^{v(\lambda)-2j}$ for some numbers $a_j(\lambda)$. Define a power series $\widetilde{\Phi}_\lambda(u) \in \mathbb{Q}[[u]]$ by $\widetilde{\Phi}_\lambda(u) = e^{(n-1)u/2} \Phi_\lambda(e^{-u})$. Then equations (A.14) and (A.15) give

$$\widetilde{\Phi}_\lambda(u) = \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} \widetilde{P}_\lambda\Big(\frac{n-1}{2} - r\Big) e^{(\frac{n-1}{2}-r)u}$$

$$= K(\lambda) \sum_{j \geq 0} a_j(\lambda) \frac{d^{v(\lambda)-2j}}{du^{v(\lambda)-2j}} \Big[\sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} e^{(\frac{n-1}{2}-r)u}\Big]$$

$$= K(\lambda) \sum_{j \geq 0} a_j(\lambda) \frac{d^{v(\lambda)-2j}}{du^{v(\lambda)-2j}} \Big[\big(e^{u/2} - e^{-u/2}\big)^{n-1}\Big]$$

$$= K(\lambda) \sum_{j \geq 0} a_j(\lambda) \frac{d^{v(\lambda)-2j}}{du^{v(\lambda)-2j}} \Big[\sum_{g \geq 0} S_g(n) u^{n-1+2g}\Big]$$

$$= (n-1)! \, K(\lambda) \sum_{j,\,g \geq 0} a_j(\lambda) \widehat{S}_g(n) \frac{u^{n-v(\lambda)-1+2j+2g}}{(n - v(\lambda) - 1 + 2j + 2g)!},$$

where $S_g(n)$ is as in (A.32) and

$$\widehat{S}_g(n) = \frac{(n-1+2g)!}{(n-1)!} S_g(n) = \frac{1}{(n-1)!} \Delta_+^{n-1}\big(t^{n-1+2g}\big)\big|_{t=0}, \quad (A.36)$$

a polynomial of degree $3g$ in n. On the other hand, from formula (A.13) and the expansion

$$e^{x/2} - e^{-x/2} = x \exp\Big(\sum_{h=2}^\infty \frac{B_h}{h} \frac{x^h}{h!}\Big) = x \exp\Big(\frac{x^2}{24} - \frac{x^4}{2880} + \cdots\Big),$$

where B_h denotes the hth Bernoulli number, we have

$$\widetilde{\Phi}_\lambda(u) = \prod_{i \geq 1} \left(e^{iu/2} - e^{-iu/2} \right)^{d_i^*}$$

$$= \left(\prod_i i^{d_i} \right) \cdot u^{\ell(\lambda)-1} \cdot \exp\left(\sum_{h=2}^{\infty} \frac{B_h}{h} \ell_h^*(\lambda) \frac{x^h}{h!} \right)$$

$$= \frac{(n-1)! \, K(\lambda)}{(n-1-v(\lambda))!} u^{n-v(\lambda)-1}$$

$$\times \left(1 + \frac{\ell_2^*(\lambda)}{24} u^2 + \frac{5\ell_2^*(\lambda)^2 - 2\ell_4^*(\lambda)}{5760} u^4 + \cdots \right),$$

where $d_i^* = d_i - \delta_{i,1}$ and $\ell_h^*(\lambda) = \sum_{i \geq 1} i^h d_i^* = \ell_h(\lambda) - 1$. Comparing with the previous formula for $\widetilde{\Phi}_\lambda(u)$, we find

$$a_0(\lambda) = 1,$$

$$a_1(\lambda) + \widehat{S}_1(n) = \frac{(n+1-v(\lambda))!}{(n-1-v(\lambda))!} \frac{\ell_2^*(\lambda)}{24},$$

$$a_2(\lambda) + \widehat{S}_1(n)\, a_2(\lambda) + \widehat{S}_2(n) = \frac{(n+3-v(\lambda))!}{(n-1-v(\lambda))!} \frac{5\ell_2^*(\lambda)^2 - 2\ell_4^*(\lambda)}{5760},$$

etc. The lemma now follows by induction on j. $\square$

Returning to the situation of Theorem A.2.9, we consider arbitrary conjugacy classes $C_i \leftrightarrow \lambda_i$ $(i = 1, \ldots, k)$ of S_n. From Lemma A.2.14 we get

$$\prod_{i=1}^k \widetilde{P}_{\lambda_i}(t) = \left(\prod_{i=1}^k K(\lambda_i) \right) \left(t^v + a_1(\lambda_1, \ldots, \lambda_k) t^{v-2} + a_2(\lambda_1, \ldots, \lambda_k)\, t^{v-4} + \cdots \right)$$

where $v = \sum v(\lambda_i)$ as before and

$$a_1(\lambda_1, \ldots, \lambda_k) = \sum_{1 \leq i \leq k} a_1(\lambda_i),$$

$$a_2(\lambda_1, \ldots, \lambda_k) = \sum_{1 \leq i < j \leq k} a_1(\lambda_i) a_2(\lambda_j) + \sum_{1 \leq i \leq k} a_2(\lambda_i),$$

etc. Substituting this into equation (A.30) and recalling that $|C_i| K(C_i) = nN(C_i)$, we find

$$\frac{1}{(n-1)!} \frac{\mathcal{N}(S_n; C_0, C_1, \ldots, C_k)}{n^{k-1} N(\lambda_1) \cdots N(\lambda_k)}$$

$$= \frac{1}{(n-1)!} \Delta_+^{n-1} \left(\sum_{j \geq 0} a_j(\lambda_1, \ldots, \lambda_k)\, t^{v-2j} \right) \bigg|_{t=0}$$

$$= \sum_{j=0}^g \widehat{S}_{g-j}(n)\, a_j(\lambda_1, \ldots, \lambda_k) \qquad (v = n - 1 + 2g),$$

where the polynomials $\widehat{S}_{g-j}(n)$ are given by (A.36) and (A.32). This is the desired generalization of the Goulden-Jackson formula (A.29) to the case of ramified coverings of S^2 by surfaces of arbitrary genus g with at least one cyclic ramification point. We note the case $g = 1$ separately since it is not too complicated:

Corollary A.2.15. *With the notations of* Theorem A.2.9 *and* Corollary A.2.10, *we have*

$$\frac{1}{(n-1)!} \frac{\mathcal{N}(\mathrm{S}_n; C_0, C_1, \ldots, C_k)}{n^{k-1} N(C_1) \cdots N(C_k)}$$
$$= \sum_{i=1}^{k} \ell(C_i)(\ell(C_i) + 1) \frac{\ell_2(C_i) - 1}{24} - (k-1) \frac{n^3 - n}{24}$$

if $v(C_1) + \cdots + v(C_k) = n + 1$, where $\ell_2(C_i)$ denotes the sum of the squares of the lengths of the cycles in the conjugacy class C_i.

A.2.5 Third Application: "Mirror Symmetry" in Dimension One

"Mirror symmetry", originally discovered in the context of mathematical physics (string theory) and intensively studied during recent years, is a predicted duality between certain families of Calabi–Yau manifolds. It manifests itself on several levels, one of which has to do with the counting functions (Gromov–Witten invariants) that enumerate the holomorphic mappings $C \to X$ of complex curves C into a Calabi–Yau manifold X representing a given class in the second homology group of X.

A Calabi–Yau manifold of dimension n is a complex (projective) n-manifold X such that the space of holomorphic i-forms on X is 0-dimensional for
$0 < i < n$ and 1-dimensional for $i = n$. The original mirror symmetry phenomenon concerned the case $n = 3$, but it was observed by Dijkgraaf [D] that it also occurs in the far simpler case $n = 1$. Here "Calabi–Yau" is just a synonym of "genus one", and the coefficients of the counting functions merely enumerate the generic mappings $Y \to X$, appropriately weighted, from curves of genus g to a given complex curve X of genus 1, where both g and the degree n of the mapping are fixed. Here "generic" means that exactly two sheets of the covering come together over every ramification point, so that each point of X has either n or $n-1$ preimages, and "appropriately weighted" means, as usual, that each covering is counted with a weight equal to the reciprocal of the number of its automorphisms over X. Note that, by the Riemann–Hurwitz formula, a covering of the above type will be ramified over precisely $2g - 2$ points of X.

The problem is therefore to compute $h^0(2g - 2, n)$, where $h^0(k, n)$ denotes the number of (weighted) isomorphism classes of connected n-sheeted coverings of a given Riemann surface X of genus 1 with generic ramification over k

given points of X. If we denote by $h^0(k, n)$ the corresponding number without the connectedness condition, then a standard argument gives the relation

$$\sum_{n>0}\sum_{k\geq 0} h^0(k,n) \frac{X^k}{k!} q^n = \log\left(1 + \sum_{n>0}\sum_{k\geq 0} h(k,n) \frac{X^k}{k!} q^n\right), \quad \text{(A.37)}$$

while another standard argument gives the formula

$$h(k,n) = \frac{1}{n!} \mathcal{N}_1\big(\mathrm{S}_n; \underbrace{T,\ldots,T}_{k}\big) \quad \text{(A.38)}$$

for $h(k, n)$ in terms of the generalized Frobenius number $\mathcal{N}_1(\mathrm{S}_n; T, \ldots, T)$ of $(k + 2)$-tuples $(a, b, c_1, \ldots, c_k) \in (\mathrm{S}_n)^2 \times [T]^k$ with $[a, b]c_1 \cdots c_k = 1$ as introduced in Sec. A.1.3; here $[T]$ denotes the conjugacy class of transpositions, see Sec. A.2.2. (The number $h^0(n, k)$ counts the tuples which generate a subgroup of S_n acting transitively on $\{1, \ldots, n\}$, but since we have no closed formula for this we must use (A.37) and (A.38) instead.)

From formulas (A.38) and (A.11) with $g = 1$ we find

$$h(k,n) = \sum_{\pi \in \mathcal{R}_n} \nu_\pi(T)^k$$

where $\mathcal{R}_n$ as in Sec. A.1.2 denotes the set of irreducible representations of S_n and $\nu_\pi(T)$ $(\pi \in \mathcal{R}_n)$ as in Sec. A.1.2 and A.1.3 is the eigenvalue of $\sum_{g \in [T]_n} g$ on the representation π. From this we obtain

$$\sum_{k\geq 0} h(k,n) \frac{X^k}{k!} = \sum_{\pi \in \mathcal{R}_n} e^{\nu_\pi(T)X} = H_n(e^X), \quad \text{(A.39)}$$

where

$$H_n(u) = \sum_{\pi \in \mathcal{R}_n} u^{\nu_\pi(T)} \quad \in \mathbb{Z}[u, u^{-1}],$$

a symmetric Laurent polynomial in u which for small values of n can be computed directly from the information given in Sec. A.1.2:

$$H_2(u) = u + u^{-1},$$
$$H_3(u) = u^3 + 1 + u^{-3},$$
$$H_4(u) = u^6 + u^2 + 1 + u^{-2} + u^{-6},$$
$$H_5(u) = u^{10} + u^5 + u^2 + 1 + u^{-2} + u^{-5} + u^{-10},$$
$$H_6(u) = u^{15} + u^9 + u^5 + 2u^3 + 1 + 2u^{-3} + u^{-5} + u^{-9} + u^{-15}.$$

In general, we see from **B.** and **C.** of Sec. A.1.2 that, if $\pi \in \mathcal{R}_n$ corresponds to the function $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ and to the Young diagram $Y \in \mathcal{Y}_n$, then

$$\nu_\pi(T) = \sum_{r \in \mathbb{Z}} rf(r) = \sum_{(x,y) \in Y} (x - y).$$

Either of these descriptions leads to a formula for $H_n(u)$, but a more convenient formula is obtained by using a third parametrization of the irreducible representations of S_n: if π corresponds to the function f and the Young diagram Y, denote by $s = f(0) = \max\{i \mid (i,i) \in Y\}$ the number of diagonal elements in Y and let

$$a_i = \max\{j \mid (i+j-1, i) \in Y\}, \qquad b_i = \max\{j \mid (i, i+j-1) \in Y\}$$

be the number of elements of Y to the right of or above the point $(i,i) \in Y$, respectively, so that Y is described as $\bigcup_{i=1}^{s} \left([i, i+a_i-1] \times \{i\} \cup \{i\} \times [i, i+b_i-1]\right)$. Then we have

$$n = \sum_{i=1}^{s}(a_i + b_i - 1), \qquad \nu_\pi(T) = \sum_{i=1}^{s}\left(\binom{a_i}{2} - \binom{b_i}{2}\right), \qquad (A.40)$$

and $\mathcal{R}_n$ is parametrized by the set of tuples $(s, (a_1, \ldots, a_s), (b_1, \ldots, b_s))$ satisfying the inequalities $a_1 > \cdots > a_s > 0$, $b_1 > \cdots > b_s > 0$ and the first of equations (A.40). This gives the generating function identity

$$\sum_{n\geq 0} H_n(u)\, q^n = \text{coefficient of } \zeta^0$$

$$\text{in } \prod_{a\geq 1}\left(1 + u^{\binom{a}{2}} q^{a-1}\, \zeta\right) \cdot \prod_{b\geq 1}\left(1 + u^{-\binom{b}{2}} q^{b-1}\, \zeta^{-1}\right).$$

This can be written more symmetrically by shifting a and b by $\frac{1}{2}$ to get

$$\sum_{n\geq 0} H_n(u)\, q^n = \text{coefficient of } \zeta^0 \text{ in}$$

$$\prod_{m\in\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}} \left(1 - u^{m^2/2}\, q^m\, \zeta\right)\left(1 - u^{-m^2/2}\, q^m\, \zeta^{-1}\right).$$

(Here we have multiplied ζ by $-q^{1/2}u^{1/8}$, which does not affect the coefficient of ζ^0.) Combining this with the previous formulas, we obtain the following theorem, due to Douglas [Dou] and Dijkgraaf [D].

Theorem A.2.16. *For $g \geq 1$, let $F_g(q) = \sum_{n\geq 1} h^0(2g - 2, n)\, q^n \in \mathbb{Q}[[q]]$ denote the counting function of generically ramified coverings of a genus 1 Riemann surface by Riemann surfaces of genus g. Then*

$$\sum_{g=1}^{\infty} F_g(q)\, \frac{X^{2g-2}}{(2g-2)!} = \log\left(\sum_{n=0}^{\infty} H_n(e^X)\, q^n\right)$$

$$= \log\left(\text{coefficient of } \zeta^0 \text{ in} \prod_{m\in\mathbb{Z}_{\geq 0}+\frac{1}{2}} \left(1 - u^{m^2/2}\, q^m\, \zeta\right)\left(1 - u^{-m^2/2}\, q^m\, \zeta^{-1}\right)\right).$$

This theorem has an interesting corollary, which was discovered and proved in the language of mathematical physics by Dijkgraaf [Dij] and Rudd [Ru] and proved from a purely mathematical point of view in [KZ]. Recall that a *modular form* of weight k on the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ is a function $F(z)$, defined for complex numbers z with $\Im(z) > 0$, which has a Fourier expansion of the form $F(z) = \sum_{n=0}^{\infty} a(n)e^{2\pi i n z}$ with coefficients $a(n)$ of polynomial growth and which satisfies the functional equation $F\left(\frac{az+b}{cz+d}\right) = (cz+d)^k F(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. A *quasimodular form* of weight k on Γ is a function having a Fourier expansion with the same growth condition and such that for each value of z the function $(cz+d)^{-k}F\left(\frac{az+b}{cz+d}\right)$ is a polynomial in $(cz+d)^{-1}$ as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ varies. The ring of modular forms is generated by the two functions

$$E_4(z) \;=\; 1+240 \sum_{n=1}^{\infty} \frac{n^3\, q^n}{1-q^n} \;=\; 1 + 240q + 2160q^2 + 6720q^3 + \cdots$$

and

$$E_6(z) \;=\; 1-504 \sum_{n=1}^{\infty} \frac{n^5\, q^n}{1-q^n} \;=\; 1 - 504q - 16632q^2 - 122976q^3 - \cdots$$

of weight 4 and 6, respectively, where $q = e^{2\pi i z}$, while the ring of quasimodular forms is generated by these two functions together with the quasimodular form

$$E_2(z) \;=\; 1-24 \sum_{n=1}^{\infty} \frac{n\, q^n}{1-q^n} \;=\; 1 - 24q - 72q^2 - 96q^3 - \cdots$$

of weight 2. Theorem A.2.16 then implies:

Corollary A.2.17. *For all $g \geq 2$, $F_g(q)$ is the q-expansion of a quasimodular form of weight $6g - 6$.*

We omit the proof, referring the reader to the papers cited above. The first example is

$$F_2(q) \;=\; \frac{1}{2^7 3^4 5}\left(5E_2^3 - 3E_2 E_4 - 2E_6\right) \;=\; q^2 + 8q^3 + 30q^4 + 80q^5 + 180q^6 + \cdots ,$$

which leads easily to the amusing closed formula $h^0(2,n) = \dfrac{n}{6} \sum_{d|n}(d^3 - nd)$ for the number of coverings (generically ramified, with fixed ramification points) of an elliptic curve by Riemann surfaces of genus 2. In general, however, the generating functions $F_g(q)$ do not have integral coefficients, e.g., $F_3(q)$ begins

$$\frac{1}{12}q^2 + \frac{20}{3}q^3 + 102q^4 + \frac{2288}{3}q^5 + \cdots .$$

References

[D]   R. Dijkgraaf, Mirror symmetry and elliptic curves, in: The Moduli Spaces of Curves, R. Dijkgraaf, C. Faber, G. van der Geer eds., Progress in Mathematics, **vol. 129**, Birkhäuser, Boston, 149–164 (1995)

[Dij]   R. Dijkgraaf, Chiral deformations of conformal field theories, Nuclear Phys. B, **493**, 588–612 (1997)

[Dou] M. R. Douglas, Conformal string theory techniques in large N Yang–Mills theory, in: Quantum field theory and string theory (Cargese 1993), 119–135, NATO Adv. Sci. Inst. Ser. B Phys., **328**, Plenum, New York (1995)

[DW]   R. Dijkgraaf and E. Witten, Topological gauge theories and group cohomology, Commun. Math. Phys., **129**, 393–429 (1990)

[FQ]   D. Freed and F. Quinn, Chern–Simons gauge theory with a finite gauge group, Commun. Math. Phys., **156**, 435–472 (1993)

[KZ]   M. Kaneko and D. Zagier, A generalized Jacobi theta function and quasimodular forms, in: The Moduli Spaces of Curves, R. Dijkgraaf, C. Faber, G. van der Geer eds., Progress in Mathematics, **vol. 129**, Birkhäuser, Boston, 165–172 (1995)

[OV]   A. Okounkov and A. Vershik, A new approach to representation theory of symmetric groups, Selecta Math. (N.S.), **2**, 581–605 (1996)

[Ru]   R. Rudd, The string partition function for QCD on the torus, hep-th/9407176

[SSV] B. Shapiro, M. Shapiro and A. Vainshtein, Ramified coverings of S^2 with one degenerate branch point and enumeration of edge-ordered graphs, in: Topics in singularity theory, Amer. Math. Soc. Transl. Ser. 2, 180, AMS. Providence RI, 219–227 (1997)

[Z]   D. Zagier, On the distribution of the number of cycles of elements in symmetric groups, Nieuw Arch. Wiskd., IV Ser., **13**, 489–495 (1995)

References

1. **Abel N. H.** Über die Integration der Differetial-Formen $\frac{\rho dx}{\sqrt{R}}$ wenn ρ und R ganze Funktionen sind. – *J. Reine Angew. Math.*, 1826, vol. **1**, 185–221. See also: Sur l'intégration de la formule différentielle $\frac{\rho dx}{\sqrt{R}}$, R et ρ étant des fonctions entières. – In "Oeuvres Complètes de Niels Henrik Abel" (L. Sylow, S. Lie eds.), Christiania, 1881, t. **1**, 104–144.

2. **Abramovitz M., Stegun I. A.** Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables. – Dover, 1965.

3. **Adrianov N. M.** Classification of primitive edge rotation groups of plane trees. – *Fundamentalnaya i Prikladnaya Matematika*, 1997, vol. **3**, no. 4, 1069–1083 (in Russian).

4. **Adrianov N. M.** Arithmetic theory of graphs on the surfaces. – Ph.D. thesis, Moscow State University, 1997, 116 pp. (in Russian).

5. **Adrianov N. M., Kochetkov Yu. Yu., Shabat G. B., Suvorov A. D.** Mathieu groups and plane trees. – *Fundamentalnaya i Prikladnaya Matematika*, 1995, vol. **1**, no. 2, 377–384 (in Russian).

6. **Adrianov N. M., Kochetkov Yu. Yu., Suvorov A. D.** Plane trees with special primitive edge rotation groups. – *Fundamentalnaya i Prikladnaya Matematika*, 1997, vol. **3**, no. 4, 1085–1092 (in Russian).

7. **Adrianov N. M., Shabat G. B.** Unicellular cartography and Galois orbits of plane trees. – In: "Geometric Galois Action" (L. Schneps, P. Lochak eds.), Vol. 2: "The Inverse Galois Problem, Moduli Spaces and Mapping Class Groups", London Math. Soc. Lecture Notes Series, vol. **243**, 1997, 13–24.

8. **Adrianov N. M., Zvonkin A. K.** Composition of plane trees. – *Acta Applicandae Mathematicae*, 1998, vo. **52**, no. 1–3, 239–245.

9. **Appel K., Haken W.** Every Planar Map Is Four-Colorable. – AMS, "Contemporary Mathematics", 1989, vol. **98**.

10. **Arnold V. I.** Braids of algebraic functions and cohomology of swallowtails. – *Russian Mathematical Surveys*, 1968, vol. **23**, 247–248.

11. **Arnold V. I.** Critical points of functions and the classification of caustics. – *Russ. Math. Surveys*, 1974, vol. **29**, 243–244

12. **Arnold V. I.** Plane curves, their invariants, perestroikas and classification. – In: Adv. Soviet Math., AMS, Providence RI, 1994, vol. **21**, 33-91.

13. **Arnold V. I.** Topological classification of trigonometric polynomials and combinatorics of graphs with an equal number of vertices and edges. – *Funct. Anal. and its Appl.*, 1996, vol. **30**, no. 1, 1–14.

14. **Artin E.** Theorie der Zöpfe. – *Hamburger Abhandlungen*, 1925, vol. **4**, 47–72.

15. **Ashkinuze V. G.** On the number of semi-regular polyhedra. – *Matematicheskoe Prosveshchenie*, 1957, vol. **1**, 107–118 (in Russian).

16. **Atkinson M. D.** An algorithm for finding the blocks of a permutation group. – *Math. Comp.*, 1975, vol. **29**, 911–913.

17. **Babai L.** The probability of generating the symmetric group. – *J. of Combinat. Theory A*, 1989, vol. **52**, 148–153.

18. **Bar-Natan D.** On Vassiliev knot invariants. – *Topology*, 1995, vol. **34**, no. 2, 423–472.

19. **Bar-Natan D., Garoufalidis S.** On the Melvin–Morton–Rozansky conjecture. – *Invent. Math.*, 1996, vol. **125**, 103–133.

20. **Bauer M., Itzykson C.** Triangulations. – In: "Formal Power Series and Algebraic Combinatorics 93" (A. Barlotti, M. Delest, R. Pinzani eds.), Florence, June 1993, 1–45. Also in: "The Grothendieck Theory of Dessins d'Enfant" (L. Schneps ed.), London Math. Soc. Lecture Notes series, vol. **200**, Cambridge Univ. Press, 1994, 179–236.

21. **Belyĭ G. V.** On Galois extensions of a maximal cyclotomic field. – *Math. USSR Izvestija*, 1980, vol. **14**, no. 2, 247–256. (Original Russian paper: *Izv. Acad. Nauk SSSR, cer. mat.*, 1979, vol. **43**, no. 2, 269–276.)

22. **Belyĭ G. V.** A new proof of the three-point theorem. – *Matematicheskii Sbornik*, 2002, vol. **193**, no. 3, 21–24.

23. **Bender E. A., Richmond L. B.** A survey of the asymptotic behaviour of maps. – *J. of Combinat. Theory B*, 1986, vol. **40**, 297–329.

24. **Bergeron F., Labelle G., Leroux P.** Combinatorial Species and Tree-like Structures. – Cambridge University Press, Cambridge, 1998.

25. **Bessis D., Itzykson C., Zuber J.-B.** Quantum field theory techniques in graphical enumeration. – *Adv. Appl. Math.*, 1980, vol. **1**, no. 2, 109–157.

26. **Bétréma J., Péré D., Zvonkin A. K.** Plane trees and their Shabat polynomials. Catalog. – Rapport interne du LaBRI, Bordeaux, 1992, 119 pp.

27. **Bétréma J., Zvonkin A. K.** La vraie forme d'un arbre. – In: "TAPSOFT '93: Theory and Practice of Software Development. 4-th International Joint Conference CAAP/FASE, Orsay, April 1993. Proceedings" (M.-C. Gaudel, J.-P. Jouannaud eds.), Lecture Notes in Computer Science, Springer, 1993, vol. **668**, 599–612.

28. **Bétréma J., Zvonkin A. K.** Plane trees and Shabat polynomials. – *Discrete Mathematics*, 1996, vol. **153**, 47–58.

29. **Biane P., Capitaine M., Guionnet A.** Large deviation bounds for matrix Brownian motion. – *Invent. Math.*, 2003, vol. **152**, no. 2, 433–459.

30. **Birch B. J.** Noncongruence subgroups, covers and drawings. – In: "The Grothendieck Theory of Dessins d'Enfants" (L. Schneps ed.), London Math. Soc. Lecture Notes Series, Cambridge Univ. Press, 1994, vol. **200**, 25–46.

31. **Birch B. J.** Shabat trees of diameter 4: Appendix to a paper of Zvonkin. – 1995, unpublished.

32. **Birch B. J., Chowla S., Hall M., Jr., Schinzel A.** On the difference $x^3 - y^2$. – *Norske Vid. Selsk. Forh. (Trondheim)*, 1965, vol. **38**, 65–69.

33. **Birman J. S.** Braids, Links and Mapping Class Groups. – Princeton University Press, 1974 (Annals of Math. Studies, vol. **82**).

34. **Birman J., Lin X.-S.** Knot polynomials and Vassiliev invariants. – *Invent. Math.*, 1993, vol. **111**, 225–270.

35. **Bóna M., Bousquet M., Labelle G., Leroux P.** Enumeration of m-ary cacti. – *Advances in Applied Math.*, 2000, vol. **24**, 22–56.

36. **Bonnington C. P., Little C. H.** The Foundation of Topological Graph Theory. – Springer, 1995, 180 pp.

37. **Bouchet A.** Circle graph obstructions. – *J. of Combinat. Theory B*, 1994, vol. **60**, 107–144.

38. **Boulatov D. V., Kazakov V. A., Kostov I. K., Migdal A. A.** Analytical and numerical study of a model of dynamically triangulated random surfaces. – *Nuclear Physics B*, 1986, vol. **275** [FS17], 641–686.

39. **Boulatov D. V., Kazakov V. A.** The Ising model on a random planar lattice: the structure of the phase transition and the exact critical exponents. – *Phys. Lett. B*, 1987, vol. **186**, no. 3-4, 379–384.

40. **Bouttier J., Di Francesco P., Guitter E.** Critical and tricritical hard objects on bicolourable random lattices: exact solutions. – *J. Phys. A*, 2002, vol **35**, no. 17, 3821–3854.

41. **Bouya D.** Mégacartes. – Mémoire de DEA, Université Bordeaux I, 1997.

42. **Bouya D., Zvonkin A. K.** Topological classification of complex polynomials: New experimental results. – Rapport interne du LaBRI no. 1219-99, May 1999, 25 pp. (`http://dept-info.labri.u-bordeaux.fr/~zvonkin`)

43. **Brezin E., Byrne R., Levy J., Pilgrim K., Plummer K.** A census of rational maps. – *Conformal Geometry and Dynamics. An Electronic Journal of the AMS*, 2000, vol. **4**, 35–74.
    (`http://www.ams.org/ecgd/home-2000.html`)

44. **Brézin E., Itzykson C., Parisi G., Zuber J.-B.** Planar diagrams. – *Commun. Math. Phys.*, 1978, vol. **59**, 35–51.

45. **Brézin E., Kazakov V. A.** Exactly solvable field theories of closed strings. – *Physics Letters B*, 1990, vol. **236**, no. 2, 144–150.

46. **Browkin J., Brzezinski J.** Some remarks on the abc-conjecture. – *Math. Comp.*, 1994, vol. **62**, 931–939.

47. **Butler G., McKay J.** The transitive groups of degree up to eleven. – *Commun. in Algebra*, 1983, vol. **11**, no. 8, 863–911.

48. **Cayley A.** A theorem on trees. – *Quarterly J. of Pure and Appl. Math.*, 1889, vol. **23**, 376–378. (Reprinted in: "The Collected Mathematical Papers of A. Cayley", in 13 volumes, Cambridge Univ. Press, 1889–1897, vol. **13**, 26–28.)

49. **Chmutov S. V., Duzhin S. V.** An upper bound for the number of Vassiliev knot invariants. – *J. of Knot Theory and its Ramifications*, 1994, vol. **3**, no. 2, 141–151.

50. **Chmutov S. V., Duzhin S. V.** The Kontsevich integral. – *Acta Appl. Math.*, 2001, vol. **66**, no. 2, 155–190.

51. **Chmutov S. V., Duzhin S. V., Kaishev A. I.** The algebra of 3-graphs. – In: *Proc. Steklov. Inst. Math.*, 1998, **vol. 221**, 157–186.

52. **Chmutov S. V., Duzhin S. V., Lando S. K.** Vassiliev knot invariants I. Introduction. – In: Advances in Sov. Math., 1994, vol. **21**, 117–126.

53. **Chmutov S. V., Duzhin S. V., Lando S. K.** Vassiliev knot invariants II. Intersection graph conjecture for trees. – In: Advances in Sov. Math., 1994, vol. **21**, 127–134.

54. **Chmutov S. V., Duzhin S. V., Lando S. K.** Vassiliev knot invariants III. Forest algebra and weighted graphs. – In: Advances in Sov. Math., 1994, vol. **21**, 135–146.

55. **Chmutov S.V., Varchenko A.N.** Remarks on the Vassiliev knot invariants coming from sl$_2$. – *Topology*, 1997, vol. **36**, 153–178.

56. **Clebsch A.** Zur Theorie der Riemann'schen Fläche. – *Mathematische Annalen*, 1873, vol. **6**, 216–230.

57. **Cohen H.** A Course in Computational Algebraic Number Theory. – Springer (Graduate Texts in Mathematics, vol. 138), 1993.

58. **Cohen P. B., Itzykson C., Wolfart J.** Fuchsian triangle groups and Grothendieck dessins: Variations on a theme of Belyi. – *Comm. Math. Phys.*, 1994, vol. **163**, no. 3, 605–627.

59. **Cohen P. B., Wolfart J.** Dessins de Grothendieck et variétés de Shimura. – *C. R. Acad. Sci. Paris, Sér. I*, 1992, t. **315**, 1025–1028.

60. **Colin de Verdière Y.** Spectres de Graphes. – Société Mathématique de France, 1998 (Cours Spécialisés, vol. **4**).

61. **Connes A., Kreimer D.** Renormalization in quantum field theory and the Riemann–Hilbert problem I. The Hopf algebra of graphs and the main theorem. – *Comm. Math. Phys.*, 2000, vol. **210**, no. 1, 249–273.

62. **Conway J. H., Curtis R. T., Norton S. P., Parker R. A., Wilson R. A.** (with computational assistance from **Thackray J. G.**) ATLAS of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups. – Clarendon Press, Oxford, 1985.

63. **Cori R.** Un Code pour les Graphes Planaires et ses Applications. – Astérisque, 1975, vol. **27**.

64. **Cori R., Machì A.** Construction of maps with prescribed automorphism group. – *Theoret. Comput. Sci.*, 1982, vol. **21**, 91–98.

65. **Cori R., Machi A.** Maps, hypermaps and their automorphisms: a survey, I, II, III. — *Expositiones Mathematicae*, 1992, vol. **10**, 403–427, 429–447, 449-467.

66. **Cori R., Penaud J.-G.** The complexity of a planar hypermap and that of its dual. – *Ann. of Discr. Math.*, 1980, vol. **9**, 53–62.

67. **Cormen T. H., Leiserson C. E., Rivest R. L.** Introduction to Algorithms. – The MIT Press, 1990.

68. **Couveignes J.-M.** Calcul et rationalité de fonctions de Belyi en genre 0. – *Ann. de l'Inst. Fourier*, 1994, vol. **44**, no. 1, 1–38.

69. **Couveignes J.-M.** Quelques revêtements définis sur $\mathbb{Q}$. – *Manuscripta Math.*, 1997, vol. **92**, no. 4, 409–445.

70. **Couveignes J.-M., Granboulan L.** Dessins from a geometric point of view. – In: "The Grothendieck Theory of Dessins d'Enfant" (L. Schneps ed.), Cambridge Univ. Press, 1994, vol. **200**, 79–113.

71. **Cox D. A., Katz S.** Mirror Symmetry and Algebraic Geometry. – AMS, 1999.

72. **Crescimanno M., Taylor W.** Large N phases of chiral QCD$_2$. – *Nuclear Phys. B*, vol. **437**, 1995, 3–24.

73. **Cromwell P. R.** Kepler's work on polyhedra. – *The Math. Intelligencer*, 1995, vol. **17**, no. 3, 23–33.

74. **Crowell R. H., Fox R. H.** Introduction to Knot Theory. – Ginn, 1963.

75. **Davenport H.** On $f^3(t) - g^2(t)$. – *Norske Vid. Selsk. Forh. (Trondheim)*, 1965, vol. **38**, 86–87.

76. **Davenport J. H.** On the Integration of Algebraic Functions. – Springer, 1981.

77. **Davis C.** Extrema of a polynomial. – *Amer. Math. Monthly*, 1957, vol. **64**, 679–680.

78. **Deligne P.** Letter to Looijenga of March 9, 1974. – Unpublished (cited in [171]).

79. **Deligne P., Mumford D.** The irreducibility of the space of curves of given genus. – *Inst. Hautes Études Sci., Publ. Math.*, 1969, vol. **36**, 75–109.

80. **Di Francesco P.** 2-D quantum gravity and topological gravities, matrix models, and integrable differential systems. – In: "The Painlevé Property", CRM Ser. Math. Phys., Springer, New York, 1999, 229–285.

81. **Di Francesco P., Ginsparg P., Zinn-Justin J.** 2*D* gravity and random matrices. – *Phys. Rep.*, 1995, vol. **254**, 1–133.

82. **Di Francesco P., Golinelli O., Guitter E.** Meanders and the Temperley–Lieb algebra. – *Comm. in Math. Phys.*, 1997, vol. **186**, no. 1, 1–59.

83. **Di Francesco P., Golinelli O., Guitter E.** Meanders: exact asymptotics. – *Nuclear Physics B*, 2000, vol. **570 [FS]**, 699–712.

84. **Di Francesco P., Itzykson C., Zuber J.-B.** Polynomial averages in the Kontsevich model. – *Commun. Math. Phys.*, 1993, vol. **151**, 193–219.

85. **Dixon J. D.** The probability of generating the symmetric group. – *Mathematische Zeitschrift*, 1969, vol. **110**, 199–205.

86. **Douady A., Hubbard J.** A proof of Thurston's characterization of rational functions. – *Acta Mathematica*, 1993, vol. **171**, 263–297.

87. **Douglas M. R., Shenker S. H.** Strings in less than one dimension. – *Nuclear Physics B*, 1990, vol. **335**, 635–654.

88. **Dyck W.** Über das Problem der Nachbargebiete. – *Math. Ann.*, 1888, vol. **32**, 457–512.

89. **Edmonds A. L., Kulkarni R. S., Stong R. E.** Realizability of branched coverings of surfaces. – *Trans. Amer. Math. Soc.*, 1984, vol. **282**, no. 2, 773–790.

90. **Edmonds J. R.** A combinatorial representation for polyhedral surfaces. – *Notices Amer. Math. Soc.*, 1960, vol. **7**, p. 646.

91. **Ekedahl T., Lando S.K., Shapiro M., Vainshtein A.** On Hurwitz numbers and Hodge integrals. – *C. R. Acad. Sci. Paris Sér. I Math.*, 1999, vol. **328**, 1175–1180.

92. **Ekedahl T., Lando S.K., Shapiro M., Vainshtein A.** Hurwitz numbers and intersections on moduli spaces of curves. – *Invent. Math.*, 2001, vol. **146**, no. 2, 297–327.

93. **Elkies N.** *ABC* implies Mordell. – *Intern. Math. Res. Notes*, 1991, no. 7, 99–109.

94. **El Marraki M., Hanusse N., Zipperer J., Zvonkin A.** Cacti, braids and complex polynomials. – *Séminaire Lotharingien de Combinatoire*, 1997, vol. **37**, 36 pp. (`http://www.mat.univie.ac.at/~slc`)

95. **El Marraki M., Zvonkin A.** Composition des cartes et hypercartes. – *Séminaire Lotharingien de Combinatoire*, 1995, vol. **34**, 18 pp. (`http://www.mat.univie.ac.at/~slc`)

96. **Euler L.** Elementa doctrinae solidorum. – *Novi Comment. Acad. Sci. Petrop.*, 1752, vol. **4**, 109–140.

97. **Faber C., Pandharipande R.** Hodge integrals and Gromov-Witten theory. – *Invent. Math.*, 2000, vol. **139**, no. 1, 173–199.

98. **Feit W.** Some consequences of the classification of finite simple groups. – In: B. Cooperstein, G. Mason eds. "The Santa Cruz Conference on Finite Groups" (Proceedings of Symposia in Pure Mathematics, vol. **37**), AMS, 1980, 175–181.

99. **Feller W.** An Introduction to Probability Theory and its Application. – John Wiley & Sons, 1971, vol. **2**.

100. **Figueroa-O'Farrill J. M., Kimura T., Vaintrob A.** The universal Vassiliev invariant for the Lie superalgebra gl(1|1). – *Commun. Math. Phys.*, 1997, vol. **185**, 93–127.

101. **Filimonenkov V. O., Shabat G. B.** Fields of definition of Belyi function and Galois cohomology. – *Fundamentalnaya i Prikladnaya Matematika*, 1995, vol. **1**, no. 3, 781–799 (in Russian).

102. **Flajolet Ph.** Elements of a general theory of combinatorial structures. – In: Lect. Notes in Comp. Sci., vol. **199**, Springer, Berlin, 1985, 112–127.

103. **Foissy L.** Les algèbres de Hopf des arbres enracinés décorés. – Thèse, Université de Reims, 2002.

104. **Forster O.** Riemannsche Flächen. – Springer, 1977.

105. **Fried M. D.** Exposition of an arithmetic–group theoretic connection via Riemann's existence theorem. – In: "The Santa Cruz Conference on Finite Groups" (B. Cooperstein, G. Mason eds.), Proceedings of Symposia in Pure Mathematics, AMS, 1980, vol. **37**, 571–602.

106. **Fried M. D.** Arithmetic of 3 an 4 branch point covers. A bridge provided by noncongruence subgroups of $SL_2(\mathbb{Z})$. – In: "Séminaire de Théorie des Nombres, Paris, 1987–88", Birkhäuser, "Progress in Mathematics", 1990, vol. **81**, 77–117.

107. **Fried M. D., Jarden M.** Field Arithmetic. – Springer, 1986 ("Ergebnisse der Mathematik und ihrer Grenzgebiete, 3.Folge, Band 11).

108. **Fulton W.** Intersection Theory, 2nd ed. – Springer-Verlag, Berlin, 1998.

109. **Geissinger L.** Hopf algebras of symmetric functions and class functions. – In: "Combinatoire et Représentations du Groupe Symétrique", Springer, Lect. Notes in Math., 1977, vol. **579**, 168–181.

110. **Gelfand I. M., Dikii L. A.** Fractional powers of operators, and Hamiltonian systems. – *Funkcional. Anal. i Prilozhen*, 1976, vol. **10**, no. 4, 13–29.

111. **Getzler E.** Operads and moduli spaces on genus 0 Riemann surfaces. – In: "The Moduli Spaces of Curves", Birkhäuser, Boston, 1995, 199–230.

112. **Goryunov V., Lando S.K.** On enumeration of meromorphic functions on the line. – In: "The Arnoldfest", Fields Inst. Commun., AMS, Providence RI, 1999, vol. **24**, 209–223.

113. **Goulden I. P., Jackson D. M.** Combinatorial Enumeration, John Wiley & Sons, Inc., New York, 1983.

114. **Goulden I. P., Jackson D. M.** The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group. – *Europ. J. Comb.*, 1992, vol. **13**, 357–365.

115. **Goulden I. P., Jackson D. M.** Transitive factorization into transpositions, and holomorphic mappings on the sphere. – *Proc. Amer. Math. Soc.*, 1997, vol. **125**, 51–60.

116. **Goulden I. P., Jackson D. M.** A proof of a conjecture for the number of ramified coverings of the sphere by the torus. – *J. Combin. Theory, Ser. A*, 1999, vol. **88**, 246–258.

117. **Goulden I., Jackson D., Vakil R.** The Gromov–Witten potential of a point, Hurwitz numbers, and Hodge integrals. – *Proc. London Math. Soc.*, 2001, vol. **83**, no. 3, 563–581.

118. **Goulden I. P., Jackson D. M., Vainshtein A.** The number of ramified coverings of the sphere by the torus and surfaces of higher genera. – *Annals of Combin.*, 2000, vol. **4**, 27–46.

119. **Goupil A., Schaeffer G.** Factoring n-cycles and counting maps of given ginus. – *Europ. J. Combinat.*, 1998, vol. **19**, no. 7, 819–834.

120. **Graber T., Vakil R.** Hodge integrals and Hurwitz numbers via virtual localization. – *Composito Math.*, 2003, vol. **135**, no. 1, 25–36.

121. **Granboulan L.** Construction d'une extension régulière de $\mathbb{Q}(T)$ de groupe de Galois M_{24}. – *Experimental Math.*, 1996, vol. **5**, no. 1, 3–14.

122. **Granboulan L.** Calcul d'objets géométriques à l'aide de méthodes algébriques et numériques : Dessins d'Enfants. – Thèse, Université Paris-7, 1997.

123. **Griffiths P. A.** Introduction to Algebraic Curves. – AMS, 1989 (Translations of Mathematical Monographs, vol. **76**).

124. **Griffiths P. A., Harris J.** Principles of Algebraic Geometry, 2nd ed. – John Wiley & Sons, New York, 1994.

125. **Gross D. J., Migdal A. A.** A nonperturbative treatment of two-dimensional quantum gravity. – *Nuclear Physics B*, 1990, vol. **340**, 333–365.

126. **Gross D. J., Taylor W.** Two-dimensional QCD is a string theory. – *Nuclear Physics B*, 1993, **vol. 400**, 181–210.

127. **Gross D. J., Taylor W.** Twists and Wilson loops in the string theory of two dimensional QCD. – *Nuclear Physics B*, 1993, **vol. 403**, 395–452.

128. **Gross J. L., Tucker T. W.** Topological Graph Theory. – John Wiley & Sons, 1987.

129. **Grothendieck A.** Esquisse d'un programme (1984). – In: "Geometric Galois Action" (Schneps L., Lochak P. eds.), vol. 1: "Around Grothendieck's *Esquisse d'un Programme*", London Math. Soc. Lecture Notes Series, Cambridge Univ. Press, 1997, vol. **242**, 5–48. (English translation: "Scetch of a programme", the same volume, p. 243–284.)

130. **Guionnet A.** First order asymptotics of matrix integrals; a rigorous approach towards the understanding of matrix models. – Preprint, November 2002, `math.PR/0211131`

131. **Guralnick R. M., Thompson J. G.** Finite groups of genus zero. – *J. of Algebra*, 1990, vol. **131**, 303–341.

132. **Hamilton W. R.** Letter to John T. Graves "On the Icosian" (17th October 1856). – In: "W. R. Hamilton, Mathematical Papers, vol. III, Algebra" (H. Halberstam, R. E. Ingram eds.), Cambridge Univ. Press, 1967, 612–625.

133. **Hansen V. L.** Braids and Coverings: Selected Topics. – Cambridge University Press, 1989 (London Math. Soc. Student Texts, vol. **18**).

134. **Hanusse N.** Cartes, constellations et groupes : questions algorithmiques. – Thèse, Université Bordeaux I, 1997.

135. **Hanusse N., Zvonkin A. K.** Cartographic generation of Mathieu groups. – In: "Formal Power Series and Algebraic Combinatorics '99", Barcelona, June 1999, 241–253.

136. **Harary F., Palmer E. M.** Graphical Enumeration. – Academic Press, New York and London, 1973

137. **Harer J.** The cohomology of the moduli space of curves. – Lecture Notes in Math., Springer, 1988, vol. **1337**, 138–221.

138. **Harer J., Zagier D.** The Euler characteristic of the moduli space of curves. – *Invent. Math.*, 1986, vol. **85**, 457-485.

139. **Harris J., Morrison I.** Moduli of Curves. – Springer, 1998.

140. **Heffter L.** Über das Problem der Nachbargebiete. – *Math. Ann.*, 1891, vol. **38**, 477–508.

141. **Heffter L.** Über metacyklische Gruppen und Nachbarconfigurationen. – *Math. Ann.*, 1898, vol. **50**, 261–268.

142. **Herman M. R.** Exemples de fractions rationnelles ayant une orbite dense sur la sphère de Riemann. – *Bull. Soc. Math. France*, 1984, vol. **112**, 93–142.

143. **'t Hooft G.** A planar diagram theory for strong interactions. – *Nuclear Physics B*, 1974, vol. **72**, 461–473.

144. **Hurwitz A.** Über Riemann'sche Fläche mit gegebenen Verzweigungspunkten. – *Math. Ann.*, 1891, vol. **39**, 1–61.

145. **Hurwitz A.** Über die Anzahl Riemannschen Flächen mit gegebenen Verzweigungspunkten. – *Math. Ann.*, 1902, vol. **55**, 53–66.

146. **Itzykson C., Zuber J.-B.** The planar approximation. II. – *J. Math. Phys.*, 1980, vol. **21**, no. 3, 411–421.

147. **Itzykson C., Zuber J.-B.** Matrix integration and combinatorics of modular groups. – *Commun. Math. Phys.*, 1990, vol. **134**, 197–207.

148. **Itzykson C., Zuber J.-B.** Combinatorics of the modular group II. The Kontsevich integrals. – *Intern. J. of Modern Physics A*, 1992, vol. **7**, no. 23, 5661–5705.

149. **Jackson D. M.** Counting cycles in permutations by group characters, with an application to a topological problem. – *Trans. Amer. Math. Soc.*, 1987, vol. **299**, no. 2, 785–801.

150. **Jackson D. M., Visentin T. I.** An Atlas of the Smaller Maps in Orientable and Nonorientable Surfaces. – Chapman and Hall, 2001.

151. **Jacques A.** Constellations et graphes topologiques. – In: "Combinatorial Theory and its Applications" (P. Erdös, A. Renyi, V. Sos eds.), North-Holland, Amsterdam, 1970, vol. II, 657–673.

152. **James G., Kerber A.** The Representation Theory of Symmetric Groups. – Addison–Wesley, 1981.

153. **Jendroľ S.** A non-involutory self-duality. – *Discrete Math.*, 1989, vol. **74**, 325–326.

154. **Jensen I.** A transfer matrix approach to the enumeration of plane meanders. – *J. Phys. A*, 2000, vol. **33**, no. 34, 5953–5963.

155. **Jones G. A.** Maps on surfaces and Galois groups. – *Mathematica Slovaca*, 1997, vol. **47**, no. 1, 1–33.

156. **Jones G. A.** Characters and surfaces: a survey. – In: "The Atlas of Finite Groups: Ten Years on (Birmingham, 1995)", Cambridge Univ. Press, London Math. Soc. Lecture Notes Series, 1998, vol. **249**, 90–118.

157. **Jones G. A.** Cyclic regular subgroups of primitive permutation groups. – *J. Group Theory*, 2002, vol. **5**, no. 4, 403–407.

158. **Jones G. A., Singerman D.** Theory of maps on orientable surfaces. – *Proc. London Math. Soc.*, 1978, vol. **37**, 273–307.

159. **Jones G. A., Singerman D.** Maps, hypermaps and triangle groups. – In: "The Grothendieck Theory of Dessins d'Enfants" (L. Schneps ed.), London Math. Soc. Lecture Notes Series, Cambridge Univ. Press, 1994, vol. **200**, 115–145.

160. **Jones G. A., Singerman D.** Belyi functions, hypermaps and Galois groups. – *Bull. London Math. Soc.*, 1996, vol. **28**, no. 6, 561–590.

161. **Jones G. A., Streit M.** Galois groups, monodromy groups and cartographic groups. – In: "Geometric Galois Action" (L. Schneps, P. Lochak eds.), vol. 2: "The Inverse Galois Problem, Moduli Spaces and Mapping Class Groups", London Math. Soc. Lecture Notes Series, Cambridge Univ. Press, 1997, vol. **243**, 25–65.

162. **Jones G. A., Zvonkin A. K.** Orbits of braid groups on cacti. – *Moscow Mathematical Journal*, 2002, vol. **2**, no. 1, 129–162.

163. **Jost J.** Compact Riemann Surfaces. An Introduction to Contemporary Mathematics. – Springer, 1997.

164. **Kassel C., Turaev V.** Chord diagram invariants of tangles and graphs. – *Duke Math. J.*, 1998, vol. **92**, no. 3, 497–552.

165. **Kazakov V. A.** Ising model on a dynamical planar random lattice: exact solution. – *Phys. Lett. A*, 1986, vol. **119**, no. 3, 140–144.

166. **Kazakov V. A.** Bosonic strings and string field theories in one-dimensional target space. – In: "Random Surfaces and Quantum Gravity. Proceedings of the NATO Advanced Workshop, Cargèse, 1990" (O. Alvarez, E. Marinari, P. Windey eds.), Plenum Press, 1991, 269–306.

167. **Kepler J.** Harmonice Mundi. Linz, 1619. – In: M. Caspar "Johannes Kepler Gesammelte Werke", Münich, Beck, 1938, vol. 6.

168. **Kharchev S., Marshakov A., Mironov A., Morozov A., Zabrodin A.** Towards unified theory of 2d gravity. – *Nuclear Phys.*, 1992, vol. **B380**, 181-240.

169. **Khovanskii A. G., Zdravkovska S.** Branched covers of S^2 and braid groups. – *J. of Knot Theory and its Ramifications*, 1996, vol. **5**, no. 1, 55–75.

170. **Klein F.** Vorlesungen über das Ikosaeder und die Aflösung der Gleichungen vom fünften Grade. – Leipzig, 1884. (Reprinted: **Klein F.** The Icosahedron and the Solution of Equations of the Fifth Degree. – Dover Publ., 1956.)

171. **Kluitmann P.** Hurwitz action and finite quotients of braid groups. – In "Braids" (J. Birman, A. Libgober eds.), AMS, "Contemporary Mathematics", 1988, vol. **78**, 299–325.

172. **Kneissler J.** The number of primitive Vassiliev invariants up to degree 12, `q-alg/9706022`.

173. **Kneser H.** Die Deformationssätze der einfach zusammenhängenden Flächen. – *Mathematische Zeitschrift*, 1926, vol. **25**, 362–372.

174. **Knuth D.** The Art of Computer Programming. Vol. 3: Sorting and Searching. – Addison–Wesley, 1973.

175. **Ko K. H., Smolinsky L.** A combinatorial matrix in 3-manifold theory. – *Pacific J. Math.*, 1991, vol. **149**, no. 2, 319–336.

176. **Kochetkov Yu. Yu.** On non-trivially decomposable types. – *Uspekhi Matem. Nauk*, 1997, vol. **52**, no. 5, 205–206.

177. **Köck B.** Belyi's theorem revisited. – To appear in *Beiträge zur Algebra und Geometrie*.

178. **Kontsevich M.** Intersection theory on the moduli space of curves and matrix Airy function. – *Comm. Math. Phys.*, 1992, vol. **147**, no. 1, 1–23.

179. **Kontsevich M.** Vassiliev knot invariants. – *Adv. in Soviet Math.*, 1993, vol. **16**, part 2, 137–150.

180. **Kontsevich M., Manin Yu. I.** Gromov–Witten classes, quantum cohomology, and enumerative geometry. – *Comm. Math. Phys.*, 1994, vol. **164**, 525–562.

181. **Kostov I. K., Mehta M. L.** Random surfaces of arbitrary genus: exact results for $D = 2$ and -2 dimensions. – *Physics Letters B*, 1987, vol. **189**, no. 1-2, 118–124.

182. **Kwak J. H., Lee J.** Enumeration of graph coverings, surface branched coverings and related group theory. – In: "Combinatorial and Computational Mathematics: Present and Future", Papers from the workshop held at the Pohang University, Pohang, February 2000 (S. Hong, J. H. Kwak, K. H. Kim, F. W. Raush eds.), World Scientific, River Edge NJ, USA, 2001, 97–161.

183. **Labelle G.** Sur la symétrie et asymétrie des structures combinatoires. – *Theoret. Comput. Sci.*, 1993, vol. **117**, 3–22.

184. **Labelle G., Leroux P.** Enumeration of (uni- or bicolored) plane trees according to their degree distribution. – *Discrete Math.*, 1996, vol. **157**, no. 1-3, 227–240.

185. **Lando S. K.** On primitive elements in the bialgebra of chord diagrams. – *Amer. Math. Soc. Transl. Ser.* 2, AMS, Providence RI, 1997, vol. **180**, 167–174.

186. **Lando S. K.** On enumeration of unicursal curves. – In: *Amer. Math. Soc. Transl. Ser.* 2, AMS, Providence RI, 1999, vol. **190**, 77–81.

187. **Lando S. K.** On a Hopf algebra in graph theory. – *J. Comb. Theory, Ser. B*, 2000, vol. **80**, 104–121

188. **Lando S. K., Zvonkin A. K.** Meanders. – *Selecta Mathematica Sovietica*, 1992, vol. **11**, no. 2, 117–144.

189. **Lando S. K., Zvonkin A. K.** Plane and projective meanders. – *Theoret. Comput. Sci.*, 1993, vol. **117**, 227–241.

190. **Lando S. K., Zvonkine D.** On multiplicities of the Lyashko–Looijenga mapping on the discriminant strata. – *Functional Analysis and its Applications*, 1999, vol. **33**, no. 3, 21-34.

191. **Lang S.** Algebraic Number Theory. – Springer, 1994 (2nd edition).

192. **Lass B.** Démonstration combinatoire de la formule de Harer–Zagier. – *C. R. Acad. Sci. Paris*, 2001, vol. **333**, 155-160.

193. **Lass B.** Calcul combinatoire ensembliste. – Thèse, Strasbourg, 2001.

194. **Lattès S.** Sur l'itération des substitutions rationnelles et les fonctions de Poincaré. – *C. R. Acad. Sci. Paris*, 1918, vol. **166**, 26–28.

195. **Le T. Q. T., Murakami J.** The universal Vassiliev–Kontsevich invariant for framed oriented links. – *Compositio Math.*, 1996, vol. **102**, no. 1, 41–64.

196. **Lhuilier S. A. J.** Mémoire sur la polyèdrométrie; contenant une démonstration directe du Théorème d'Euler sur les polyèdres, et un examen des diverses exceptions auxquelles ce théorème est assujetti. **(Extrait) par M. Gergonne.** – *Ann. Math. Pures et Appliquées* (*Annales de Gergonnes*), 1812-13, t. **3**, 169–189.

197. **Lickorish W. B. R.** Invarinats for 3-manifolds from the combinatorics of the Jones polynomial. – *Pacific J. Math.*, 1991, vol. **149**, no. 2, 337–347.

198. **Lieberum J.** On Vassiliev invariants not coming from semisimple Lie algebras. – *J. Knot Theory Ramif.*, 1999, **vol. 8**, no. 5, 659–666.

199. **Lieberum J.** Chromatic weight system and the corresponding knot invariants. – *Math. Ann.*, 2000, **vol. 317**, no. 3, 459–482.

200. **Liu Y.** Enumerative Theory of Maps. – Kluwer, 1998.

201. **Looijenga E.** The complement of the bifurcation variety of a simple singularity. – *Inv. Math.*, 1974, vol. **23**, 105–116.

202. **Looijenga E.** Intersection theory on Deligne–Mumford compactifications (after Witten and Kontsevich). – In: Séminaire Bourbaki, 1993 (Astérisque, vol. **216**), exposé 768, 187–212.

203. **Looijenga E.** Cellular decompositions of compactified moduli spaces of pointed curves. – In: "The Moduli Space of Curves", Birkhäuser, Progress in Mathematics, 1995, vol. **129**, 369–400.

204. **Łuczak T., Pyber L.** On random generation of the symmetric group. – *Combin. Probab. Comput.*, 1993, vol. **2**, 505–512.

205. **Lüroth J.** Note über Verzweigungsschnitte und Querschnitte in einer Riemann'schen Fläche. – *Mathematische Annalen*, 1871, vol. **4**, 181–184.

206. **Magot N.** Cartes planaires et fonctions de Belyi : Aspects algorithmiques et expérimentaux. – Thèse, Université Bordeaux I, 1997.

207. **Magot N., Zvonkin A.** Belyi functions for Archimedean solids. – *Discrete Math.*, 2000, vol. **217**, 249–271.

208. **Malle G., Matzat B. H.** Inverse Galois Theory. – Springer, 1999.

209. **Malle G., Saxl J., Weigel Th.** Generation of classical groups. – *Geometriae Dedicata*, 1994, vol. **49**, 85–116.

210. **Malyshev V. A., Minlos R. A.** Gibbs Random Fields. – Kluwer, Dordrecht, 1991.

211. **Manin Yu.** Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces. – American Mathematical Society, Providence RI, 1999.

212. **Matiyasevich Yu. V.** Calculation of generalized Chebyshev polynomials on computer. – *Vestnik Mosk. Universiteta*, 1996, no. 6, 59–61 (in Russian).

213. **McCanna J.** Is self-duality always involutory? – *Congr. Numer.*, 1990, vol. **72**, 175–178.

214. **McMullen C. T.** Complex Dynamics and Renormalization. – Princeton University Press, NJ, 1994.

215. **Mednykh A. D.** Nonequivalent coverings of Riemann surfaces with a prescribed ramification type. – *Siber. Math. J.*, 1984, vol. **25**, no. 4, 606–625.

216. **Mehta M. L.** A method of integration over matrix variables. – *Comm. Math. Physics*, 1981, vol. **79**, 327–340.

217. **Mellor B.** The intersection graph conjecture for loop diagrams. – *J. Knot Theory Ramifications*, 2000, vol. **9**, 187–211.

218. **Milnor J., Moore J.** On the structure of Hopf algebras.– *Ann. Math.*, 1965, vol. **81**, 211–264.

219. **Mohar B., Thomassen C.** Graphs on Surfaces. – Johns Hopkins Univ. Press, 2001.

220. **Moran G.** Chords in a circle and linear algebra over GF(2). – *J. Combinatorial Theory, Ser. A*, 1984, vol. **37**, 239–247.

221. **Müller P.** Primitive monodromy groups for polynomials. – In "Recent Developments in the Inverse Galois Problem" (M. Fried ed.), AMS, Contemporary Mathematics, 1995, vol. **186**, 385–401.

222. **Natanzon S. M.** Topology of 2-dimensional coverings and meromorphic functions on real and complex algebraic curves. – *Selecta Mathematica Formerly Sovietica*, 1993, vol. **12**, no. 3, 1993, 251–291. (First published in Russian: *Trudy Seminara po Vektornomu i Tensornomu Analizu*, 1988, no. 23, 79–103.)

223. **Nedela R.** Regular maps – combinatorial objects relating different fields of mathematics. – *J. Korean Math. Soc.*, 2001, vol. **38**, no. 5, 1069–1105.

224. **Nitaj A.** An algorithm for finding good *abc*-examples. – *C. R. Acad. Sci. Paris*, 1993, vol. **317**, 811–815.

225. **Nitaj A.** Algorithms for finding good examples for the *abc* and Szpiro conjectures. – *Experimental Math.*, 1993, vol. **2**, no. 3, 223–230.

226. **Oesterlé J.** Nouvelles approches du "théorème" de Fermat. – Séminaire Bourbaki, exposé 694, Astérisque, 1988, vol. **161–162**, 165–186.

227. **Okounkov A., Pandharipande R.** Gromov–Witten theory, Hurwitz numbers, and matrix models, I. – Preprint, January 2001, 107 pp. (`math.AG/0101147`)

228. **Pak I.** What do we know about the product replacement algorithm? – In: "Groups and Computations III", Ohio State Univ. Math. Res. Inst. Publ., 2001, vol. **8**, 301–347.

229. **Pakovitch F. B.** Combinatoire des arbres planaires et arithmétique des courbes hyperelliptiques. – *Ann. Inst. Fourier*, 1998, vol. **48**, no. 2, 323–351.

230. **Patras F.** L'algèbre des descentes d'une bigèbre graduée. – *Journal of Alegbra*, 1994, vol. **170**, 547–566.

231. **Penner R. C.** The moduli space of a punctured surface and perturbative series. – *Bull. Amer. Math. Soc.*, 1986, vol. **15**, no. 1, 73–77.

232. **Penner R. C.** Perturbative series and the moduli space of Riemann surfaces. – *J. Diff. Geometry*, 1988, vol. **27**, 35–53.

233. **Penner R. C., Harer J. L.** Combinatorics of Train Tracks. – Princeton Univ. Press, 1992 (Annals of Math. Studies, vol. **125**).

234. **Peterson M. A.** The geometry of Piero della Francesca. – *The Math. Intelligencer*, 1997, vol. **19**, no. 3, 33–40.

235. **Pilgrim K. M.** Dessins d'enfants and Hubbard trees. – *Ann. Sci. de l'École Normale Sup.*, 2000, vol. **33**, no. 5, 671–693.

236. **Prasolov V. V., Sossinsky A.B.** Knots, Links, Braids and 3-Manifolds. An Introduction to the New Invariants in Low-Dimensional Topology. – Translations of Mathematical Monographs, 154, American Mathematical Society, Providence, RI, 1997.

237. **Protopopov A. N.** Topological classification of branched coverings of a two-dimensional sphere. – *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*), 1988, vol. **167**, "Issled. Topol.", no. 6, 135–156, 192.

238. **Ree R.** A theorem on permutations. – *J. Combinat. Theory*, 1971, vol. **10**, 174–175.

239. **Reshetikhin N. Yu., Turaev V. G.** Ribbon graphs and their invariants. – *Comm. Math. Physics*, 1990, vol. **127**, 1–26.

240. **Reyssat E.** Quelques Aspects des Surfaces de Riemann. – Birkhäuser, 1989.

241. **Ringel G.** Map Color Theorem. – Springer, 1974 (Grundlehren der Mathematischen Wissenschaften, vol. **209**).

242. **Ringel G., Youngs J. W. T.** Das Geschlecht des symmetrischen vollständingen dreifärbbaren Graphen. – *Comm. Math. Helv.*, 1970, vol. **45**, 152–158.

243. **Ritt J. F.** Prime and composite polynomials. – *Trans. Amer. Math. Soc.*, 1922, vol. **23**, 51–66.

244. **Robertson N., Seymour P. D.** Graph minors. XVII. Taming a vortex. – *J. Combinat. Theory, ser. B*, 1999, vol. **77**, no. 1, 162–210.

245. **Royle G. F.** Transitive groups of degree twelve. – *J. Symbolic Comput.*, 1987, vol. **4**, 255–268.

246. **Schmitt W. R.** Incidence Hopf algebras. – *Journal of Pure and Applied Algebra*, 1994, vol. **96**, 299–330.

247. **Schneps L. (ed.)** The Grothendieck Theory of Dessins d'Enfant. – London Math. Soc. Lecture Notes Series, vol. **200**, Cambridge Univ. Press, 1994.

248. **Schneps L.** Dessins d'enfants on the Riemann sphere. – In: "The Grothendieck Theory of Dessins d'Enfant" (L. Schneps ed.), London Math. Soc. Lecture Notes Series, Cambridge Univ. Press, 1994, vol. **200**, 47–77.

249. **Schneps L., Lochak P. (eds.)** Geometric Galois Action. Vol. 1: Around Grothendieck's *Esquisse d'un Programme*. – London Math. Soc. Lecture Notes Series, vol. **242**, Cambridge Univ. Press, 1997.

250. **Schneps L., Lochak P. (eds.)** Geometric Galois Action. Vol. 2: The Inverse Galois Problem, Moduli Spaces and Mapping Class Groups. – London Math. Soc. Lecture Notes Series, vol. **243**, Cambridge Univ. Press, 1997.

251. **Scott G. P.** Braid groups and the group of homeomorphisms of a surface. – *Proc. Camb. Phil. Soc.*, 1970, vol. **68**, 605–617.

252. **Segal G., Wilson G.** Loop groups and equations of KdV type. – *Inst. Hautes Études Sci., Publ. Math.*, 1985, vol. **61**, 5–65

253. **Serre J.-P.** Topics in Galois Theory. – Jones and Bartlett, Boston, 1992.

254. **Servatius B., Servatius H.** Self-dual maps on the sphere. – *Discr. Math.*, 1994, vol. **134**, 139–150.

255. **Servatius B., Servatius H.** The 24 symmetry pairings of self-dual maps on the sphere. – *Discr. Math.*, 1995, vol. **140**, 167–183.

256. **Shabat G. B.** Notes of the seminar on the "dessins d'enfants". – Moscow, 1990, unpublished (in Russian), 70 pp.

257. **Shabat G. B., Voevodsky V. A.** Drawing curves over number fields. – In "The Grothendieck Festschrift" (P. Cartier, L. Illusie, N. M. Katz, G. Laumon, Y. Manin, K. A. Ribet eds.), Birkhäuser, 1990, vol. **3**, 199–227.

258. **Shabat G. B., Zvonkin A. K.** Plane trees and algebraic numbers. – In "Jerusalem Combinatorics '93" (H. Barcelo, G. Kalai eds.), AMS, Contemporary Mathematics, 1994, vol. **178**, 233–275.

259. **Silverman J. H., Tate J.** Rational Points on Elliptic Curves. – Springer, 1992.

260. **Simon B.** The $P(\varphi)_2$ Euclidian (Quantum) Field Theory. – Princeton Univ. Press, 1974.

261. **Singerman D., Syddall R. I.** Belyi uniformization of elliptic curves. – *Bull. London Math. Soc.*, 1997, vol. **29**, 443–451.

262. **Singerman D., Watson P. D.** Weierstrass points on regular maps. – *Geom. Dedicata*, 1997, vol. **66**, no. 1, 69–88.

263. **Smit D.-J.** Summation over equilaterally triangulated surfaces and the critical string measure. – *Comm. Math. Phys.*, 1992, vol. **143**, no. 2, 253–285.

264. **Soboleva E.** Vassiliev knot invariants coming from Lie algebras and 4-invariants. – *J. Knot Theory and its Ramifications*, 2001, vol. **10**, 161–169.

265. **Stanley R. P.** Enumerative Combinatorics. Vol. I. – Wadsworth & Brooks, Monterey CA, 1986.

266. **Steinitz E.** Polyeder und Raumeinteilungen. – In: "Encyclopädie der Mathematischen Wissenschaften", Band 3 (Geometrie), Teil 3AB12, 1922, 1–139.

267. **Stillwell J.** Classical Topology and Combinatorial Group Theory. – Springer, 1993 (2nd edition), Graduate Texts in Mathematics, vol. **72**.

268. **Stothers W. W.** Polynomial identities and Hauptmoduln. – *Quart. J. Math. Oxford, Ser.* 2, 1981, vol. **32**, no. 127, 349–370.

269. **Strebel K.** Quadratic Differentials. – Springer, 1984.
270. **Streit M.** Homology, Belyi functions and canonical curves. – *Manuscripta Mathematica*, 1996, vol. **90**, 489–509.
271. **Streit M., Wolfart J.** Characters and Galois invariants of regular dessins. – *Rev. Mat. Complut.*, 2000, vol. **13**, no. 1, 49–81.
272. **Sullivan D.** Quasiconformal homeomorphisms and dynamics. I: Solution of the Fatou–Julia problem on wandering domains. – *Ann. Math.*, 1985, vol. **122**, 401–418.
273. **Sweedler M. E.** Hopf Algebras. – W. A. Benjamin, Inc., 1969.
274. **Szegö G.** Orthogonal Polynomials. – AMS, 1939 (American Mathematical Society Colloquium Publications, vol. **23**).
275. **Tchebicheff P. L.** Sur l'intégration de la différentielle

$$(x + A)/\sqrt{x^4 + \alpha x^3 + \beta x^2 + \gamma}\, dx\ .$$

– *Bull. Acad. Impériale de Saint-Petersbourg*, 1861, vol. **3**, 1–12. Reprinted in: *Jurnal des Math. Pures et Appl.*, 1864, vol. **9**, no. 2, 225–246.
276. **Temperley H., Lieb E.** Relations between the percolation and coloring problem and other graph-theoretical problems associated with regular planar lattices: some exact results the for percolation problem. – *Proc. Roy. Soc.*, 1971, vol. **A322**, 251-280.
277. **Thom R.** L'équivalence d'une fonction différentiable et d'un polynôme. – *Topology*, 1965, vol. **3**, 297–307.
278. **Thomassen C.** The graph genus problem is NP-complete. – *J. Algorithms*, 1989, vol. **10**, 568–576.
279. **Tutte W. T.** A ring in graph theory. – *Proc. Cambridge Philos. Soc.*, 1947, vol. **43**, 26–40. Reprinted in: "Selected Papers of W. T. Tutte," vol. **1**, Charles Babbage Research Center, Winnipeg, 1979.
280. **Tutte W. T.** Planted plane trees with a given partition. – *Amer. Math. Monthly*, 1964, vol. **71**, 272–277.
281. **Tutte W. T.** Graph Theory. – Addison-Wesley, 1984.
282. **Vaintrob A.** Vassiliev knot invariants and Lie S-algebras. – *Mathematical Research Letters*, 1994, vol. **1**, 579–595.
283. **Vassiliev V. A.** Cohomology of knot spaces. – In: "Theory of Singularities and Its Applications" (V. I. Arnold ed.), Advances in Soviet Math., vol. **1**, AMS, 1990.
284. **Vassiliev V. A.** Complements of Discriminants of Smooth Maps: Topology and Applications. – AMS, 1992.
285. **Vassiliev V. A.** Invariants of knots and complements of discriminants. – In: "Developments in Mathematics, The Moscow School" (V. I. Arnold, M. Monastyrsky eds.), Chapmann & Hall, 1993, 194–250.
286. **Vassiliev V. A.** Topology of Complements to Discriminants. – Fazis, Moscow, 1997 (in Russian).
287. **Vogel P.** Algebraic structure on modules of diagrams. – Preprint, 1995. (`http://www.math.jussieu.fr/~vogel/`)
288. **Völklein H.** Groups as Galois Groups. An Introduction. – Cambridge University Press, 1996 ("Cambridge Studies in Advanced Mathematics", vol. **53**).
289. **van der Waerden B. L.** Algebra. – Springer, 1966.
290. **Wajnryb B.** Orbits of Hurwitz action for coverings of a sphere with two special fibers. – *Indag. Math., New Ser.*, 1996, vol. **7**, no. 4, 549–558.

291. **Walsh T. R. S.** Hypermaps versus bipartite maps. – *J. Combinat. Theory B*, 1975, vol. **18**, 155–163.

292. **de Weger B. M. M.** Solving exponential diophantine equations using lattice basis reduction algorithms. – *J. Number Theory*, 1987, vol. **26**, 325–367.

293. **Weil A.** The field of definition of a variety. – *Amer. J. Math.*, 1956, vol. **78**, 509–524.

294. **Weil A.** Number Theory. An Approach through History, from Hammurapi to Legendre. – Birkhäuser, 1984.

295. **Weyl H.** Zur Theorie der Darstellung der einfachen kontinuierlichen Gruppen. (Aus einem Schreiben an Herrn I. Schur) – In: "Sitzungsberichte der Preußischen Akadämie der Wissenschaften zu Berlin", 1924, 338-345. – Reprinted in: H. Weyl, "Gesammelte Abhadlungen", Band II, Springer, 1968, 453–460.

296. **Weyl H.** The Classical Groups. – Princeton, 1946.

297. **White A. T.** The genus of the complete tripartite graph $K_{mn,n,n}$. – *J. Combinat. Theory*, 1969, vol. **7**, 283–285.

298. **White A. T.** Graphs of Groups on Surfaces. – Elsevier, Amsterdam, 2001.

299. **Witten E.** Two dimensional gravity and intersection theory on the moduli space. – *Surveys in Diff. Geom.*, 1991, vol. **1**, 243–310.

300. **Wolfart J.** The 'obvious' part of Belyi's theorem and Riemann surfaces with many automorphisms. – In: "Geometric Galois Action. Vol. **1**: Around Grothendieck's *Esquisse d'un Programme*" (Schneps L., Lochak P. eds.), London Math. Soc. Lecture Notes Series, Cambridge Univ. Press, 1997, vol. **242**, 97–112.

301. **Zagier D.** On the distribution of the number of cycles of elements in symmetric groups. – *Nieuw Arch. Wiskd., IV. Ser.*, 1995, vol. **13**, no. 3, 489–495.

302. **Zannier U.** On Davenport's bound for the degree of $f^3 - g^2$ and Riemann's Existence Theorem. – *Acta Arithmetica*, 1995, vol. **71**, no. 2, 107–137.

303. **Zapponi L.** Fleurs, arbres et cellules : un invariant galoisien pour une famille d'arbres. – *Composito Mathematica*, 2000, vol. **122**, 113–133.

304. **Zdravkovska S.** Topological classification of polynomial maps. – *Uspekhi Mat. Nauk*, 1970, vol. **25**, no. 4, 179–180.

305. **Zelevinsky A. V.** Representations of Finite Classical Groups. A Hopf Algebra Approach. – Springer, 1981 (Lecture Notes in Math., vol. **869**).

306. **Zinn-Justin P., Zuber J.-B.** On the counting of colored tangles. – *J. of Knot Theory and its Ramifications*, 2000, vol. **9**, no. 8, 1127–1141.

307. **Zvonkin A. K.** How to draw a group. – *Discrete Mathematics*, 1998, vol. **180**, 403–413.

308. **Zvonkin A. K.** Galois orbits of plane trees: a case study. – Unpublished note, 1996.

309. **Zvonkin A. K.** Matrix integrals and map enumeration: An accessible introduction. – *Computers and Mathematics with Applications: Mathematical and Computer Modelling*, special issue "Combinatorics and Physics" (M. Bousquet-Mélou, D. Loeb eds.), 1997, vol. **26**, no. 8-10, 281–304.

310. **Zvonkin A. K.** Towards topological classification of univariate complex polynomials. – In: "Formal Power Series and Algebraic Combinatorics, 12th International Conference, FPSAC'00, Moscow, Russia, June 2000, Proceedings" (D. Krob, A. A. Mikhalev, A. V. Mikhalev eds.), Springer, 2000, 76–87.

311. **Zvonkin A. K.** Megamaps: construction and examples. – *Discrete Mathematics and Theoretical Computer Science*, Conference edition: "Discrete Models: Combinatorics, Computation, and Geometry", July 2001, 329–339.

312. **Zvonkine D.** Multiplicities of the Lyashko–Looijenga map on its strata. –
     C. R. Acad. Sci. Paris, Sér. I, 1997, vol. **324**, no. 12, 1349–1353.
313. **Zvonkine D.** Énumération des revêtements ramifiés des surfaces de Riemann.
     – Thèse, Université Paris-Sud Orsay, 2003.

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