Appendix A. Fourier Analysis

We give here some of the basic facts of Fourier analysis which can be found in Rudin [1]. On every locally compact abelian (LCA) group G there exists a positive regular Borel measure which is translation-invariant. It is called the Haar measure of G and is denoted m_G. The space M(G) is the collection of all finite regular Borel measures on G. It is a commutative Banach algebra with unit under the operation of convolution *: for $\mu,\nu \in M(G)$, $\mu*\nu \in M(G)$ is defined by

$$\int_G f d\mu*\nu = \int_G \int_G f(x+y)\,d\mu(x)\,d\nu(y), \quad f \in C_o(G).$$

Recall the dual space $C_o(G)^*$ of $C_o(G)$ is M(G) (Riesz representation theorem). The subspace $L^1(G)$ consists of all $\nu \in M(G)$ which are absolutely continuous with respect to m_G ($\nu << m_G$). It can be identified with the space of equivalence classes of Borel functions f on G with $\int_G |f|\,dm_G < \infty$.

The dual (or character) group $\hat{G}$ of G is the set of all continuous homomorphisms of G into the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$. The topology of $\hat{G}$ is the compact-open topology; and G is also an LCA group. The space G can be identified with the maximal ideal space of $L^1(G)$ with the Gelfand topology. The Pontryagin duality theorem asserts that $G^{\wedge\wedge} = G$.

For $\mu \in M(G)$, the Fourier-Stieltjes transform $\hat{\mu}$ is a continuous, bounded function on $\hat{G}$ and is defined by

$$\hat{\mu}(\gamma) = \int_G \gamma(x)\,d\mu(x), \; \gamma \in \hat{G}.$$

The map $\mu \mapsto \hat{\mu}: M(G) \to C^B(\hat{G})$ is a faithful, norm nonincreasing, algebra homomorphism. The Bochner theorem asserts that a function f on $\hat{G}$ is a continuous, positive-definite function if and only if $f = \hat{\mu}$, some $\mu \in M(G)$ with $\mu \geq 0$.

The space $M(G)^{\wedge}$ has been characterized by Eberlein as the space of functions $f \in C^B(\hat{G})$ with the property

$$|\Sigma_{i=1}^{n} c_i f(\gamma_i)| \leq K \sup\{|\Sigma_{i=1}^{n} c_i \gamma_i(x)| : x \in G\}$$

for all $c_1, \cdots, c_n \in \mathbb{C}$, $\gamma_1, \cdots, \gamma_n \in \hat{G}$ (K a constant depending only on f).

Appendix B. Spectral Theorem

This is a version of the spectral theorem adequate for our purposes, see Rudin [2] and Naimark [1].

For the σ-algebra M of all Borel sets on a locally compact Hausdorff space Ω, a resolution of the identity is a mapping E of M into the bounded operators $B(H)$ on some Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ such that:

(a) $E(\emptyset) = 0$, $E(\Omega) = I$

(b) $E(\omega)$ is a self-adjoint projection $(\omega \in M)$

(c) $E(\omega_1 \cap \omega_2) = E(\omega_1)E(\omega_2) \quad (\omega_1, \omega_2 \in M)$

(d) if $\omega_1 \cap \omega_2 = \emptyset$, then

$$E(\omega_1 \cup \omega_2) = E(\omega_1) + E(\omega_2) \quad (\omega_1, \omega_2 \in M)$$

(e) for each $x,y \in H$, the set function $E_{x,y}$ defined by $E_{x,y}(\omega) = \langle E(\omega)x,y\rangle$, $(\omega \in M)$ is a finite regular Borel measure on M.

For a resolution of the identity E on M, the Banach algebra $L^\infty(E)$ is the space of equivalence classes of all bounded complex M-measurable functions on Ω with the essential sup-norm (the equivalence relation is defined by $f \sim g$ if and only if $E(\{\omega \in \Omega : f(\omega) \neq g(\omega)\}) = 0$.

Given a resolution of the identity on M, there exists a isometric $*$-isomorphism ψ of $L^\infty(E)$ onto a closed normal subalgebra A of $B(H)$ given by

$$(*) \qquad \langle\psi(f)x,y\rangle = \int_\Omega f dE_{x,y} \quad (f \in L^\infty(E), x,y \in H).$$

We abbreviate $(*)$ to $\psi(f) = \int_\Omega f dE$.

Let A be a closed normal subalgebra of $\mathcal{B}(\mathcal{H})$ which contains the identity operator I. Let Δ_A denote the maximal ideal space (Δ is compact) of A. Then there exists a unique resolution of the identity E on the Borel subsets of Δ_A with

$$(**) \qquad \langle Sx,y\rangle = \int_{\Delta_A} \hat{S}dE_{x,y} \ ,$$

where $S \mapsto \hat{S}$ denotes the Gel'fand transform of A onto $C(\Delta_A)$. We abbreviate (**) by writing $S = \int_{\Delta_A} \hat{S}dE$.

Let A be a commutative C*-algebra with a unit, and let $f \mapsto T_f : A \to \mathcal{B}(\mathcal{H})$ be a continuous $*$-representation of A. The representation T is said to by <u>cyclic</u> if there exists a vector $x_o \in \mathcal{H}$ with $||x_o|| = 1$ and the set $\{T_f(x_o) : f \in A\}$ is dense in $\mathcal{H}$. Let p be the positive functional on A defined by x_o; that is $p(f) = \langle T_f(x_o), x_o\rangle$. By the Riesz representatinn theorem, there exists a measure $\mu \in M_p(\Delta_A)$ with

$$p(f) = \int_{\Delta_A} \hat{f}d\mu, \quad (f \in A).$$

For $f \in A$, let $j(T_f(x_o))$ be the function $\hat{f} \in C(\Delta_A) \subset L^2(\mu)$. The map j extends to a linear isometry of $\mathcal{H}$ onto $L^2(\mu)$; and for $\phi \in L^2(\mu)$, $jT_fj^{-1}(\phi) = \hat{f}\ \phi (f \in A)$. Thus T is equivalent to the representation $f \mapsto \hat{f} : A \to L^\infty(\mu) \subset \mathcal{B}(L^2(\mu))$.

Appendix C. The structure semigroup of the representation algebra

In this section the semigroup S is a commutative semitopological semigroup with identity 1. We will characterize the dual $R(S)^*$ $(=A(S))$ of $R(S)$.

1 Definition: For $f \in R(S)$ and $F \in R(S)^*$, define the function $E_F f$ on S by $E_F f(y) = F(f_y)$ $(y \in S)$ (recall 2.1.11).

The following is a crucial proposition.

2 Proposition: The operator $E_F: f \mapsto E_F f$ takes $R(S)$ into $R(S)$ with $||E_F|| \leq ||F||$ $(F \in R(S)^*)$.

Proof. Fix $f \in R(S)$. Recall from Chapter 2 (2.1.10), the spaces $W = \Sigma \oplus_{T \in S} L^\infty(\mu,\Omega)$ and $L = \Sigma \oplus_{T \in S} L^1(\mu,\Omega)$ $((T,\mu,\Omega) \in S)$ with $L^* = W$. For $w \in W$, $g \in L$, the canonical pairing between W and L is $\langle w,g\rangle = \Sigma_{T \in S} \int_\Omega w_T g \, d\mu$. For $x \in S$, $\rho x \in W$ is defined by $(\rho x)_T = Tx$ $(T \in S)$. For $x \in S$, $\rho^*: L \to R(S)$ is defined by $\rho^* g(x) = \langle \rho x, g\rangle$ $(g \in L)$. By Theorem 1.1.12, $L^*/\ker \rho^* = R(S)$. Thus for a fixed $f \in R(S)$, let $g \in L$ with $\rho^* g = f$ and $||f||_R = ||g||_L$, so that $f(z) = \rho^* g(z) = \langle \rho z, g\rangle$ $(z \in S)$. For $h \in L$ and $w \in W$, define $h \times w \in L$ by $h \times w = (w_T h_T)_{T \in S}$. Thus for $h \in L$ and $w_1, w_2 \in W$, $\langle w_1, h \times w_2\rangle = \langle w_2, h \times w_1\rangle = \langle w_1 w_2, h\rangle$; in particular, $\langle w(\rho y), g\rangle = \Sigma_{T \in S} \int_\Omega w_T(Ty) g_T d\mu = \langle w, g \times (\rho y)\rangle$ $(y \in S)$. Hence $f_x(y) = f(xy) = \langle \rho(xy), g\rangle = \langle (\rho x)(\rho y), g\rangle = \langle \rho y, g \times (\rho x)\rangle$ $= \rho^*(g \times (\rho x))(y)$, $(x,y \in S)$; that is, $f_x = \rho^*(g \times \rho(x))$.

Each $F \in R(S)^*$ defines a bounded linear functional $F^\# \in L^* = W$ by $\langle F^\#, g\rangle = F(\rho^* g)$, $(g \in L)$. So for $x \in S$, $(E_F f)(x) = F(f_x)$

$= F(\rho^*(g\times(\rho x))) = \langle F^{\#}, g\times(\rho x)\rangle = \langle \rho x, g\times F^{\#}\rangle = \rho^*(g\times F^{\#})(x)$; and so $E_F f \in R(S)$. Also $|||E_F f||_{R(S)} \leq ||\rho^*||\ ||g\times F^{\#}||_L$ $\leq ||g||_L ||F|| = ||f||_R ||F||$; so $||E_F|| \leq ||F||$. □

3 Definition: Let $B_S(R(S))$ denote the bounded operators on $R(S)$ which commute with translation; that is for $\Phi \in B_S(R(S))$, $\Phi f_x = (\Phi f)_x$ $(f \in R(S),\ x \in S)$.

4 Theorem: Let S be a commutative semitopological semigroup with identity 1. The map $F \mapsto E_F$ is a one-to-one linear isometry of $R(S)^*$ onto $B_S(R(S))$.

Proof. Let $F \in R(S)^*$. Then $||E_F|| \leq ||F||$, and $|F(f)| = |E_F(f_1)| \leq ||E_F||\ ||f_1||_R = ||E_F||\ ||f||_R$. Thus $||E_F|| = ||F||$; that is, $F \mapsto E_F$ is a linear isometry of $R(S)^*$ into $B(R(S))$. Also $E_F \in B_S(R(S))$: for $x,y \in S$, $(E_F f)_x(y) = E_F f(xy)$ $= F(f_{xy}) = F((f_x)_y) = E_F(f_x)(y)$, $(f \in R(S))$.

For $E \in B_S(R(S))$, define $F \in R(S)^*$ by $F(f) = E(f)(1)$, $(f \in R(S))$. Now $E_F = E$: for $f \in R(S)$, $(E_F f)(x) = F(f_x) = E(f_x)(1)$ $= E(f)(x)$, $(x \in S)$. □

5 Corollary: Let S be a commutative semitopological semigroup with identity 1. The maximal ideal space Δ of $R(S)$ is identified with the nonzero endomorphisms in $B_S(R(S))$.

Proof. The composition of two nonzero endomorphisms in nonzero since $1 \in S$. □

6 Corollary: Let S be a commutative semitopological semigroup with identity 1. The space $R(S)^*$ with the weak-* topology is homeomorphic to the space $B_S(R(S))$ with the weak operator topology. Thus Δ is a compact semitopological semigroup with an identity.

Proof. Recall from 2.1.13 that $R(S)^* = A(S) \subset W$ is isomorphic to the weak-* closed span of $\{\rho x : x \in S\}$ in W. For $\{F_\alpha\} \cup \{F\} \subset R(S)^*$, $E_{F_\alpha} \overset{\alpha}{\to} E_F$ WO if and only if $E_{F_\alpha} f \overset{\alpha}{\to} E_F f$ weakly in $R(S)$ $(f \in R(S))$ if and only if $\langle w, E_{F_\alpha} f\rangle \overset{\alpha}{\to} \langle w, E_F f\rangle$ $(w \in A(S),\ f \in R(S))$ if and only if $\langle w, g \times F_\alpha^\#\rangle \overset{\alpha}{\to} \langle w, g \times F^\#\rangle$ $(g \in L,\ w \in A(S))$ if and only if $\langle F_\alpha^\#, g \times w\rangle \overset{\alpha}{\to} \langle F^\#, g \times w\rangle$ $(g \in L,\ w \in A(S))$ if and only if $\langle F_\alpha^\#, g\rangle \overset{\alpha}{\to} \langle F^\#, g\rangle$ $(g \in L)$ (since $L \times A(S) = L$) if and only if $F_\alpha \overset{\alpha}{\to} F$ weak-* in $R(S)$. □

7 Remark: For S a locally compact abelian group, these results are due to Taylor [2].

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