Appendix

The Appendix is to introduce basic notations and to recall certain definitions and fundamental theorems from functional analysis, measure theory and advanced calculus in an unsystematic manner. In particular we quote unconventional versions of standard theorems which we found useful in applications. There are no proofs, except when we cannot name a convenient reference.

1. Linear Functionals and the Hahn-Banach Theorem

Let E be a real vector space and E^* its dual space, defined to consist of all linear functionals $E \to \mathbb{R}$. A functional $\theta: E \to \mathbb{R}$ is defined to be sublinear iff $\theta(u+v) \leqq \theta(u)+\theta(v)$ and $\theta(tu)=t\theta(u)$ for all $u,v \in E$ and real $t \geqq 0$. We start with a powerful version of the Hahn-Banach theorem.

1.1 HAHN-BANACH THEOREM (CONVEX VERSION): Let $\theta: E \to \mathbb{R}$ be sublinear. Assume that $T \subset E$ is nonvoid and such that to $u,v \in T$ there exists $f \in T$ with $\theta(f-\frac{1}{2}(u+v)) \leqq 0$. Then there exists $\sigma \in E^*$ with $\sigma \leqq \theta$ such that

$$\inf_{f \in T} \sigma(f) = \inf_{f \in T} \theta(f).$$

If we specialize to $T=\{f\}$ and $=\{-f\}$ for fixed $f \in E$ then we obtain the subsequent conventional form of the Hahn-Banach theorem: Let $\theta: E \to \mathbb{R}$ be sublinear. Then there exist linear functionals $\sigma \in E^*$ with $\sigma \leqq \theta$. Furthermore

$$\{\sigma(f): \sigma \in E^* \text{ with } \sigma \leqq \theta\} = [-\theta(-f), \theta(f)] \quad \forall f \in E.$$

We quote another important consequence of the convex version 1.1.

1.2 HAHN-BANACH THEOREM (CONE VERSION): Let $\theta: E \to \mathbb{R}$ be sublinear. Assume that $T \subset E$ is nonvoid and such that to $u,v \in T$ there exists $f \in T$ with $\theta(f-(u+v)) \leqq 0$. If $\theta(f) \geqq 0$ $\forall f \in T$ then there exists $\sigma \in E^*$ with $\sigma \leqq \theta$ such that $\sigma(f) \geqq 0$ $\forall f \in T$.

The most familiar form of the Hahn-Banach theorem is the extension theorem: Let $\theta:E\to\mathbb{R}$ be sublinear. Then each $\varphi\in S^*$ on a linear subspace $S\subset E$ with $\varphi\leqq\theta|S$ can be extended to some $\sigma\in E^*$ with $\sigma\leqq\theta$. This is a consequence of the above conventional form applied to an appropriate modified sublinear functional.

There is an important special case which is valid in both the real and the complex situation: Let E be a (real or complex) vector space with E^* its (real or complex) dual space, and let $\|.\|:E\to[0,\infty[$ be a seminorm (relative to real or complex coefficients). Then each $\varphi\in S^*$ on a (real or complex) linear subspace $S\subset E$ with $|\varphi|\leqq\|.\|$ can be extended to some $\sigma\in E^*$ with $|\sigma|\leqq\|.\|$. It is of course natural to formulate this version in terms of the norm dual space E' of E relative to $\|.\|$. The vehicle which carries over to the complex situation is the subsequent simple but important remark.

<u>1.3</u> <u>REMARK</u>: Let E be a complex vector space. Then the complex-linear functionals $\phi:E\to\mathbb{C}$ and the real-linear functionals $\varphi:E\to\mathbb{R}$ are in one-to-one correspondence with each other via

$$\phi\mapsto\varphi:\varphi(x) = \mathrm{Re}\phi(x)\ \forall x\in E,$$

$$\phi(x) = \varphi(x)-i\varphi(ix)\ \forall x\in E:\phi\leftarrow\varphi.$$

An important aspect for linear functionals is positivity. We do not intend to consider vector spaces with order structures. Instead we shall restrict ourselves to B(X), defined to consist of the bounded complex-valued functions on the nonvoid set X, with its natural pointwise structure and with $\|.\|$ the supremum norm (= supnorm). The basic equivalence theorem will be formulated in both a real and a complex version. For $V\subset B(X)$ define $\mathrm{Re}V:=\{\mathrm{Re}\ f:f\in V\}$.

<u>1.4</u> <u>THEOREM</u>: Let $V\subset\mathrm{Re}B(X)$ be a real-linear subspace with $1\in V$ and $\varphi:V\to\mathbb{R}$ be a real-linear functional. Then the subsequent properties are equivalent.

i) If $f\in V$ and $f\geqq 0$ then $\varphi(f)\geqq 0$, and $\varphi(1)=1$.

ii) $\varphi(f)\leqq \mathrm{Sup}\ f$ for all $f\in V$.

iii) $|\varphi(f)|\leqq\|f\|$ for all $f\in V$, and $\varphi(1)=1$.

<u>1.5</u> <u>THEOREM</u>: Let $V\subset B(X)$ be a complex-linear subspace with $1\in V$ and $\phi:V\to\mathbb{C}$ be a complex-linear functional. Then the subsequent properties are equivalent.

i) If $f\in V$ and Re $f \geq 0$ then $\mathrm{Re}\,\phi(f)\geq 0$, and $\phi(1)=1$.

ii) $\mathrm{Re}\,\phi(f) \leq \mathrm{Sup}\,\mathrm{Re}\,f$ for all $f\in V$.

iii) $|\phi(f)|\leq\|f\|$ for all $f\in V$, and $\phi(1)=1$.

iv) $\phi(f)\in\overline{\mathrm{Konv}\, f(X)}$ (= the closed convex hull) for all $f\in V$.

If V is closed under complex conjugation then the equivalence extends to

i*) If $f\in V$ and $f\geq 0$ then $\phi(f)\geq 0$, and $\phi(1)=1$.

The steps i)⇒ii)⇒iii)⇒i) in the proof of 1.4 and i)⇒ii)⇒iii)⇒iv)⇒i) and i)⇔i*) in the proof of 1.5 are all conventional, except iii)⇒iv) in 1.5 which is an immediate consequence of the subsequent lemma.

1.6 LEMMA: Let $K\subset\mathbb{C}$ be convex bounded $\neq\emptyset$ and $a\in\mathbb{C}$. Then

$$|a-z| \leq \mathop{\mathrm{Sup}}_{u\in K}|u-z| \quad \forall z\in\mathbb{C} \quad \text{implies that } a\in\overline{K}.$$

Proof: Assume that $a\notin\overline{K}$. Then from 1.1 we obtain a complex number c of modulus $|c|=1$ such that

$$0<\varepsilon := \mathop{\mathrm{Inf}}_{u\in K}|u-a| = \mathop{\mathrm{Inf}}_{u\in K}\mathrm{Re}\,\overline{c}(u-a) \quad \text{and hence } \mathrm{Re}\,\overline{c}(u-a)\geq\varepsilon \;\; \forall u\in K.$$

Take an $R>0$ such that $|u-a|\leq R$ $\forall u\in K$. For $t>0$ then

$$|u-(a+tc)|^2 = |(u-a)-tc|^2 = |u-a|^2 - 2t\mathrm{Re}\,\overline{c}(u-a) + t^2 \leq R^2 - 2t\varepsilon + t^2,$$

$$t = |a-(a+tc)| \leq \mathop{\mathrm{Sup}}_{u\in K}|u-(a+tc)| \leq (R^2-2t\varepsilon+t^2)^{1/2}.$$

It follows that $2t\varepsilon\leq R^2$ $\forall t>0$ which is a contradiction. QED.

We conclude with an unconventional but useful version of the bipolar theorem.

1.7 BIPOLAR THEOREM: Let E be a real vector space and $F\subset E^*$ be a linear subspace. Then the closed convex hull in the weak topology $\sigma(E,F)$ of $M\subset E$ is

$$\overline{\mathrm{Konv}\, M} = \{u\in E: \varphi(u) \leq \mathop{\mathrm{Sup}}_{x\in M}\varphi(x) \text{ for all } \varphi\in F\}.$$

2. Measure Theory

Let (X,Σ) be a measurable space, that is a nonvoid set X with a σ-algebra Σ of subsets. Let $B(X,\Sigma)\subset B(X)$ consist of the measurable bounded complex-valued functions on X. Define $ca(X,\Sigma)$ to consist of the complex-valued measures on Σ, in particular $Pos(X,\Sigma)\subset ca(X,\Sigma)$ to consist of the measures with values $\geqq 0$, and $Prob(X,\Sigma)$ of the so-called probability measures $\sigma\in Pos(X,\Sigma)$ with $\sigma(X)=1$. $B(X,\Sigma)$ and $ca(X,\Sigma)$ form a dual system via the bilinear functional $(f,\theta)\mapsto \int fd\theta$. For $\theta\in ca(X,\Sigma)$ the variation $|\theta|\in Pos(X,\Sigma)$ is defined to be

$$|\theta|(A) = \operatorname{Sup}\{|\int fd\theta|: f\in B(X,\Sigma) \text{ with } |f|\leqq\chi_A\} \quad \forall A\in\Sigma,$$

with χ_A the characteristic function of $A\subset X$, whence in particular the norm

$$\|\theta\| = |\theta|(X) = \operatorname{Sup}\{|\int fd\theta|: f\in B(X,\Sigma) \text{ with } |f|\leqq 1\}.$$

In this norm $ca(X,\Sigma)$ is complete.

For $m\in Pos(X,\Sigma)$ define $L(m) = L(X,\Sigma,m)$ to consist of the equivalence classes modulo m of the measurable complex-valued functions on X. With the obvious precautions we can use the same notations and expressions for the members of $L(m)$ as for the functions themselves. For example, for a sequence of members $f_n\in L(m)$ the symbol $[f_n\to 0]:=\{x\in X: f_n(x)\to 0\}$ defines a measurable set $\in\Sigma$ modulo m-null sets. In $L(m)$ we have the usual Banach spaces $L^p(m)$ for $1\leqq p\leqq\infty$. Furthermore define $L^o(m)$ to consist of the $f\in L(m)$ with $(\log|f|)^+\in L^1(m)$, that is with $|f|\leqq e^F$ for some $F\in L^1(m)$.

For $\theta,m\in ca(X,\Sigma)$ the notation $\theta\ll m$ is to mean that θ is m-continuous (= absolutely continuous with respect to m), that is $|m|$-continuous in the usual sense. After the Radon-Nikodym theorem this means that $\theta=fm$ for some $f\in L^1(|m|)$. For $\theta,m\in ca(X,\Sigma)$ we have the Lebesgue decomposition $\theta=\theta_m+\theta_{\hat{m}}$ of θ into the m-continuous part $\theta_m = \frac{d\theta}{dm}\, m$ with $\frac{d\theta}{dm}\in L^1(|m|)$ and the m-singular part $\theta_{\hat{m}}$. The Lebesgue decomposition is a particular case of the notion of preband decomposition to be dealt with in Section II.2.

In the remainder of the section we consider the important particular measurable space (X,Σ) where X is a compact Hausdorff space and Σ is the σ-algebra of its Baire subsets. We write $B(X,\text{Baire})$ and observe that $C(X)\subset B(X,\text{Baire})$. Also we abbreviate $ca(X):=ca(X,\text{Baire})$ and likewise $Pos(X)$ and $Prob(X)$. In the present situation the fundamental fact is the F.Riesz representation theorem which states that the above bilinear functional $(f,\varphi)\mapsto\int fd\varphi$ in fact produces a norm isometric isomorphism between the supnorm dual space $(C(X))'$ and $ca(X)$.

2.1 F.RIESZ REPRESENTATION THEOREM: The linear functionals $\phi\in(C(X))'$ and the measures $\varphi\in ca(X)$ are in one-to-one correspondence with each other via $\phi(f) = \int fd\varphi\ \forall f\in C(X)$. Furthermore $\|\phi\|=\|\varphi\|$.

In particular $\varphi\in Pos(X)$ iff ϕ is a positive functional in the sense of 1.5. As usual the measures $\varphi\in ca(X)$ and their functionals $\phi\in(C(X))'$ will be identified, so that $\varphi(f) = \int fd\varphi$ for all $f\in C(X)$.

We do not intend to discuss the details of the standard extension procedure for Baire measures. Let us merely recall the extension of a positive measure $\sigma\in Pos(X)$ to the class $USC(X)$ of upper semicontinuous functions $X\to[-\infty,\infty[$ which is defined to be

$$\int fd\sigma := \text{Inf}\{\sigma(F):F\in ReC(X) \text{ with } F\geqq f\} \geqq -\infty \quad \forall f\in USC(X).$$

For a subset $K\subset X$ we have $\chi_K\in USC(X)$ iff K is closed. We then write $\int fd\sigma=:\sigma(K)$. At this point we can define the support $Supp(\sigma)$ of $\sigma\in ca(X)$ to be the smallest closed subset $K\subset X$ with full measure $|\sigma|(K)=|\sigma|(X)$, the existence of which is not hard to see.

Now we combine the F.Riesz representation theorem with the Hahn-Banach versions of Section 1 to obtain some efficient representation theorems. Note that $\text{Max}:ReC(X)\to\mathbb{R}$ is a sublinear functional. We see from 1.4 and 1.5 that the linear functionals $\sigma:ReC(X)\to\mathbb{R}$ with $\sigma\leqq\text{Max}$ are precisely the probability measures $\sigma\in Prob(X)$. Let us prove a certain refinement.

2.2 REMARK: Let $K\subset X$ be closed $\neq\emptyset$. Then the linear functionals $\sigma : ReC(X)\to\mathbb{R}$ with $\sigma\leqq\text{Max}(.|K)$ are precisely the probability measures $\sigma\in Prob(X)$ with $\sigma(K)=1$.

Proof: Let $\sigma:ReC(X)\to\mathbb{R}$ be a linear functional. i) If $\sigma\leqq\text{Max}(.|K)\leqq\text{Max}$ then $\sigma\in Prob(X)$. And for $\chi_K \leqq f\in ReC(X)$ we have $1-\sigma(f) = \sigma(1-f) \leqq \text{Max}(1-f|K) \leqq 0$, so that $\sigma(K)\geqq 1$ and hence $\sigma(K)=1$. ii) For the converse let $\sigma\in Prob(X)$ with $\sigma(K)=1$. For $f\in ReC(X)$ then $\text{Max}f - f\geqq(\text{Max}f-\text{Max}(f|K))\chi_K$ implies that $\sigma(f)\leqq\text{Max}(f|K)$. QED.

2.3 THEOREM: Let $K\subset X$ be closed $\neq\emptyset$. Assume that $T\subset USC(X)$ is nonvoid and such that to $u,v\in T$ there exists $f\in T$ with $f\leqq\frac{1}{2}(u+v)$ on K. Then there exists $\sigma\in Prob(X)$ with $\sigma(K)=1$ such that

$$\operatorname*{Inf}_{f\in T}\int f d\sigma = \operatorname*{Inf}_{f\in T}\operatorname{Max}(f|K).$$

Proof: Follows upon application of 1.1 to the subset $\{F\in ReC(X):F \geqq \text{some } f\in T\}\subset ReC(X)$. QED.

2.4 THEOREM: Let $K\subset X$ be closed $\neq\emptyset$. Assume that $T\subset USC(X)$ is nonvoid and such that to $u,v\in T$ there exists $f\in T$ with $f\leqq u+v$ on K. If $\operatorname{Max}(f|K)\geqq 0$ $\forall f\in T$ then there exists $\sigma\in Prob(X)$ with $\sigma(K)=1$ such that $\int f d\sigma\geqq 0$ $\forall f\in T$.

Proof: Follows upon application of 1.2 to the subset $\{F\in ReC(X):F \geqq \text{some } f\in T\}\subset ReC(X)$. QED.

In conclusion we use 1.4 and 1.5 and the Hahn-Banach extension theorem to obtain another important representation theorem, as before in both a real and a complex version.

2.5 THEOREM: Let $V\subset ReC(X)$ be a real-linear subspace with $1\in V$ and $\varphi:V\rightarrow\mathbb{R}$ a real-linear functional such that $\varphi(f)\geqq 0$ for all $0\leqq f\in V$. Then there exists $\sigma\in Pos(X)$ such that $\varphi(f) = \int f d\sigma$ $\forall f\in V$.

2.6 THEOREM: Let $V\subset C(X)$ be a complex-linear subspace with $1\in V$ and $\phi:V\rightarrow\mathbb{C}$ a complex-linear functional such that $Re\phi(f)\geqq 0$ for all $f\in V$ with $Re\ f \geqq 0$. Then there exists $\sigma\in Pos(X)$ such that $\phi(f)=\int f d\sigma$ $\forall f\in V$.

Proofs: We restrict ourselves to 2.5. We can assume that $\varphi(1)=1$. Then φ fulfills the equivalent condition iii) in 1.4. Hence it admits an extension $\sigma:ReC(X)\rightarrow\mathbb{R}$ which preserves the equivalent properties i)-iii) in 1.4. Then 2.1 can be applied. QED.

3. The Cauchy Formula via the Divergence Theorem

It is well-known that the Cauchy formula can be deduced from an appropriate version of the divergence theorem. It then appears in what can be considered to be its natural form. We take the opportunity to present the details. Let us fix a nonvoid bounded open subset $G\subset\mathbb{C}=\mathbb{R}^2$.

A boundary point $u \in \partial G$ is called regular iff there exists a neighbourhood $U \subset \mathbb{C}$ of u such that $U \cap \partial G$ is a smooth curve (= one-dimensional C^1-manifold). Otherwise $u \in \partial G$ is called a singular boundary point. We introduce

$$R(G) = \{u \in \partial G : u \text{ is regular}\},$$
$$S(G) = \{u \in \partial G : u \text{ is singular}\}.$$

Then R(G) is a smooth curve and S(G) is compact with $\partial G = R(G) \cup S(G)$. A regular boundary point $u \in R(G)$ is called outer iff one of the two unit normal vectors N to R(G) at u is such that $u - tN \in G$ and $u + tN \notin \overline{G}$ for small $t > 0$. Then this unique $N =: N(u)$ is called the outer normal of G at u. Otherwise $u \in R(G)$ is called inner, which means that u is an interior point of $\overline{G}$. We introduce

$$X(G) = \{u \in \partial G : u \text{ is outer}\},$$
$$Y(G) = \{u \in \partial G : u \text{ is inner}\}.$$

Then X(G) and Y(G) are smooth curves with $X(G) \cup Y(G) = R(G)$. The function $u \mapsto N(u)$ is continuous on X(G).

The divergence theorem holds true whenever S(G) is small. This statement is made precise in terms of the one-dimensional Minkowski content, defined for a compact set $S \subset \mathbb{C}$ to be

$$\tau(S) = \limsup_{\delta \downarrow 0} \frac{L(\{z \in \mathbb{C} : \operatorname{dist}(z,S) \leq \delta\})}{2\delta},$$

where L denotes two-dimensional Lebesgue masure on $\mathbb{C}$. Clearly $\tau(S) = 0$ if S consists of finitely many points.

<u>3.1</u> <u>THEOREM</u>: Assume that G satisfies $\tau(S(G)) = 0$. Let $A: G \cup R(G) \to \mathbb{R}^2$ be a continuous bounded vector function, differentiable on G (in the sense of real analysis) such that $\operatorname{div} A: G \to \mathbb{R}$ is continuous. Then we have

$$\int_{X(G)} \langle A(x), N(x) \rangle d\sigma(x) = \int_G \operatorname{div} A(x) dL(x),$$

whenever both integrals exists. Here σ denotes one-dimensional Lebesgue measure (= arc length) on X(G).

Observe that in the case $\tau(S(G)) = 0$ and $\sigma(X(G)) < \infty$ all assumptions of the above theorem are fulfilled if A is defined and C^1 on some open set

$U \subset \mathbb{C}$ with $G \subset \overline{G} \subset U$. From 3.1 we obtain the Green formula via a well known argument.

3.2 COROLLARY: Assume that G satisfies $\tau(S(G))=0$ and $\sigma(X(G))<\infty$. Let $f,g \in C^2(U)$ on some open set $U \subset \mathbb{C}$ with $G \subset \overline{G} \subset U$. Let $\frac{\partial f}{\partial n}(x)$ denote the directional derivative of f at the point $x \in X(G)$ in the direction $N(x)$. Then we have

$$\int_{X(G)} \left(f(x)\frac{\partial g}{\partial n}(x) - g(x)\frac{\partial f}{\partial n}(x) \right) d\sigma(x) = \int_G (f(x)\Delta g(x) - g(x)\Delta f(x))\, dL(x).$$

We derive from 3.1 the Cauchy theorem and the Cauchy formula. We introduce the differential operators

$$\frac{\partial}{\partial Z} = \frac{1}{2}\left(\frac{\partial}{\partial X} + \frac{1}{i}\frac{\partial}{\partial Y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{Z}} = \frac{1}{2}\left(\frac{\partial}{\partial X} - \frac{1}{i}\frac{\partial}{\partial Y}\right).$$

A function $f: G \to \mathbb{C}$ is holomorphic iff f is differentiable (in the sense of real analysis) with $\frac{\partial f}{\partial \overline{Z}} = 0$; in this case $f'(z) = \frac{\partial f}{\partial Z}(z) = \frac{\partial f}{\partial X}(z) = \frac{1}{i}\frac{\partial f}{\partial Y}(z)$ $\forall z \in G$.

3.3 THEOREM: Assume that G satisfies $\tau(S(G))=0$. Let $f: G \cup R(G) \to \mathbb{C}$ be a continuous bounded function such that f is differentiable on G with $\frac{\partial f}{\partial \overline{Z}}$ continuous. Then we have

$$\int_{X(G)} f(x)N(x)\,d\sigma(x) = 2\int_G \frac{\partial f}{\partial \overline{Z}}(x)\,dL(x),$$

whenever both integrals exist.

Proof: Apply 3.1 to the vector functions A=(Re f, -Im f) and B=(Im f, Re f) and sum the two equations. QED.

3.4 CONSEQUENCE: Assume that G satisfies $\tau(S(G))=0$ and $\sigma(X(G))<\infty$. Let $f: G \cup R(G) \to \mathbb{C}$ be a continuous bounded function such that f is differentiable on G with $\frac{\partial f}{\partial \overline{Z}}$ continuous. Suppose further that

$$\int_G \left|\frac{\partial f}{\partial \overline{Z}}(x)\right| dL(x) < \infty.$$

Then we have

$$f(u) = \frac{1}{2\pi}\int_{X(G)} \frac{f(x)}{x-u} N(x)\,d\sigma(x) - \frac{1}{\pi}\int_G \frac{1}{x-u}\frac{\partial f}{\partial \overline{Z}}(x)\,dL(x) \quad \forall u \in G.$$

The subsequent special case will be sufficient for our purposes.

3.5 SPECIAL CASE: Assume that G satisfies $\tau(S(G))=0$ and $\sigma(X(G))<\infty$. Let $f\in C^1(U)$ on some open set $U\subset\mathbb{C}$ with $G\subset\overline{G}\subset U$. Then we have

$$f(u) = \frac{1}{2\pi}\int_{X(G)} \frac{f(x)}{x-u} N(x)\,d\sigma(x) - \frac{1}{\pi}\int_G \frac{1}{x-u}\frac{\partial f}{\partial \overline{z}}(x)\,dL(x) \quad \forall u\in G.$$

Notes

There is an extensive literature on fortified Hahn-Banach type theorems. The ancestor of numerous versions is a well-known theorem due to MAZUR-ORLICZ [1953], see also PTÅK [1956]. The above presentation of 1.1-1.2 and 2.3-2.4 follows KÖNIG [1968][1970b]. The version 1.7 of the bipolar theorem is from KÖNIG [1972]. The presentation of the divergence theorem is adapted from KÖNIG [1964].

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Notation Index

$\mathbb{N}$ $\mathbb{Z}$ $\mathbb{Q}$ $\mathbb{R}$ $\mathbb{C}$ the usual number systems

$B^{\perp}$ the annihilator of the set B in a dual system

$B^{\times}$ the set of invertible elements of the algebra B

Subject Index