APPENDIX I

LINEAR SPACES, LINEAR OPERATORS

A linear scalarproduct space Φ is (-not necessarily a space of functions but-) a set of mathematical (imagined) objects for which three operations are defined.

I Addition of elements of Φ, which has the following properties:

(i) For $h,f,g \in \Phi$ $\quad f + g \in \Phi$ and

$f + g = g + f$; $(f + g) + h = f + (g + h) = f + g + h$

(ii) There exists an element $0 \in \Phi$ such that

$$f + 0 = f$$

(iii) $h + f = g$ has a unique solution $h \stackrel{\text{def}}{=} g - f$

II Multipliciation by complex numbers, which has the following properties

$\alpha(f + g) = \alpha f + \alpha g$, for $f,g \in \Phi$, $\alpha \in \mathbb{C}$ = complex numbers

$(\alpha + \beta)f = \alpha f + \beta f \qquad \alpha,\beta \in \mathbb{C}$

$\alpha(\beta f) = (\alpha\beta)f$; $0f = 0$

III Scalar product of two elements $\phi,\psi \in \Phi$ i.e. with every pair of elements $\phi,\psi \in \Phi$ there is associated a complex number (ϕ,ψ) with the properties

(i) $(\phi,\psi) = \overline{(\psi,\phi)}$

(ii) $(\alpha\phi_1 + \beta\phi_2,\psi) = \bar{\alpha}(\phi_1,\psi) + \bar{\beta}(\phi_2,\psi)$

(iii) $(\phi,\phi) \geq 0$ with $(\phi,\phi) = 0$ only for $\phi = 0$

An linear operator A is (-not necessarily a differential operation but-) a map from (a subset of) Φ into Φ which has the property:

$$A(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 Af_1 + \alpha_2 Af_2$$

$$\alpha_1,\alpha_2 \in \mathbb{C};\ f_1,f_2 \in \Phi$$

APPENDIX II

ALGEBRA

A set $\mathcal{A}$ is an (associative) algebra with unit element iff

(i) $\mathcal{A}$ is a linear space over $\mathbb{C}$.

(ii) For every pair $A, B \in \mathcal{A}$, there is defined a product $AB \in \mathcal{A}$ such that

$$(AB)C = A(BC)$$

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

$$(\alpha A)B = A(\alpha B) = \alpha AB$$

(iii) There exists an element $1 \in \mathcal{A}$ such that $1A = A$ for all $A \in \mathcal{A}$.

A set K of elements of $\mathcal{A}$ is called a system of generators iff the smallest closed subalgebra with unit which contains K coincides with $\mathcal{A}$. The element **1** is not to be called a generator.

Let the elements of K be called X_i $i = 1,2,..n$. Then each element of $\mathcal{A}$ can be written

$$\text{(1)} \quad A = C^o + \sum_{i=1}^{n} C^i X_i + \sum_{i,j}^{n} C^{ij} X_i X_j + \ldots.$$

$$C^o, C^i \ldots \in \mathbb{C}$$

We restrict ourselves to the case that n is finite and we will not discuss topologies of $\mathcal{A}$, i.e. we assume that the above sums for every A are arbitrarily large but finite. Defining algebraic relations are relations among the generators

(2) $P(X_i) = 0$

where $P(x_i)$ is a polynomial with complex coefficients of n variables x_i. An element $B \in \mathcal{A}$

(3) $B = b^o + \Sigma b^i X_i + \Sigma b^{ij} X_i X_j + \ldots.$

$b^i \ldots \in \mathbb{C}$

is equal to the element A iff (3) can be brought into the same form (1) with the same coefficients $c^o, c^i, c^i \ldots.$ by the use of the defining relations (2).

The best known examples of associative algebras are the enveloping algebras $\varepsilon(G)$ of Lie groups G.

An enveloping algebra is the associative algebra generated by $X_1, X_2 \ldots X_n$ in which the multiplication is defined by the commutation relations

(4) $P(X_i) = X_i X_j - X_j X_i - \sum_{k=1}^{n} c_{ij}{}^k X_k = 0$

$c_{ij}{}^k$ are called the structure constants of the Lie group G. The commutation relations (4) do in general not suffice to fully define an enveloping algebra of linear operators in given linear topological spaces. I.e., there are several algebras of linear operators in linear topological spaces that fulfill a given commutation relation. In order to specify a particular algebra of operators one has to require additional algebraic relations and other properties (such as ΣX_i^2 be e.s.a.).

APPENDIX III

LINEAR OPERATORS IN HILBERT SPACES

We recall here a few of the definitions and important theorems from the theory of linear operators in Hilbert space.

Let $A: K \to K$ be a linear but not necessarily bounded operator on a Hilbert space K and let $D(A)$ be dense in K. Then A^*, the <u>adjoint</u> of A, is defined in K by

$$(A^*f,h) = (f,Ah).$$

The domain $D(A^*)$ is the set of all vectors $f \in K$ such that $(f,Ah) = (z,h)$ holds for all $h \in D(A)$; the vector z is then uniquely defined and $z = A^*f$.

An operator A on K is called <u>symmetric</u> if $D(A)$ is dense in K and $A \subset A^*$ (i.e., $D(A) \subset D(A^*)$ and $Af = A^*f$ for every $f \in D(A)$); it is called <u>self-adjoint</u> if $D(A)$ is dense in K and $A = A^*$.

An operator A on K is called <u>closed</u> if the relations

$$\lim_{n \to \infty} f_n = f, \quad \lim_{n \to \infty} Af_n = g, \quad f_n \in D(A)$$

imply $f \in D(A)$ and $Af = g$. Closedness is a weaker condition than continuity since, if an operator A on K is continuous, then

$$\lim_{n \to \infty} f_n = f \quad \text{for} \quad f_n \in D(A)$$

implies that the sequence $\{Af_n\}$ converges, while if it only closed, then the convergence of the sequence $\{f_n\}$ for $f_n \in D(A)$ does not imply the convergence of the sequence $\{Af_n\}$. However, if A is closed then, in particular, it has the property that two sequences $\{Af_n\}$ and $\{Ag_n\}$ cannot converge to different limits if the corresponding sequences $\{f_n\}$ and $\{g_n\}$ converge to the same limit. A^* is always closed.

If A is not closed, it is sometimes possible to find an extension of A which is closed. An operator A <u>admits of a closure</u> $\bar{A}$ iff the relations

$$f_n \in D(A), \quad f'_n \in D(A), \quad \lim_{n \to \infty} f_n = f, \quad \lim_{n \to \infty} f'_n = f,$$

$$\lim Af_n = g, \ \lim Af'_n = g'$$

imply $g = g'$. In this case $D(\bar{A})$ consists of all $f \in K$ for which there exists a sequence $\{f_n\} \in D(A)$ which satisfies the conditions:

i) $\lim f_n = f$ and ii) $\{Af_n\}$ converges. Then by definition

$$\bar{A}f = \lim Af_n.$$

An operator A on K is called <u>essentially self-adjoint</u> if $\bar{A}$ is self-adjoint. Physical observable are assumed to be represented by essentially self-adjoint operators.

An operator A on K is called <u>Hermitian</u> if A is bounded and self-adjoint. An operator A on K is called <u>unitary</u>

if $||Af|| = ||f||$ for all $f \in K$. A unitary operator satisfies the relations $A^*A = AA^* = 1$.

Let M be a closed subspace of a Hilbert space K. An operator P which associates with each $f \in K$ its projection f_1 on M, $P:K \to M$ is called a projection operator on M. Projection operators are linear, Hermitian (i.e., $P^*=P$, $D(P)=K$) and idempotent (i.e., $P^2 = P$); and every linear, Hermitian, idempotent operator with $D(P) = K$ is a projection operator. Two projection operators P_1 and P_2 are called orthogonal iff $P_1P_2 = 0$; in this case P_1K and P_2K are orthogonal, i.e., for every $h_1 \in P_1K$ and every $h_2 \in P_2K$, we have $(h_1,h_2) = 0$.

APPENDIX IV

PROOF OF LEMMA (14) SECTION II BY INDUCTION

(A1) From the c.r. follows: $(N+1)a^+ = a^+(N+2)$

$$a(N+1) = (N+2)a$$

(14) is true for p = 1 because

$$(\phi, a(N+1)a^+\phi) = (\phi, aa^+(N+2)\phi) = (\phi,(N+1)(N+2)\phi)$$

$$= (\phi,(N+1)^2\phi) + (\phi,(N+1)\phi) \leq 2(\phi,(N+1)^2\phi)$$

because $(\phi,(N+1)\phi) \leq (\phi_1(N+1)^2\phi)$

Assume (14) is true for p = q: i.e.

(A2) $(\phi,a(N+1)^q a^+\phi) \leq k(\phi,(N+1)^{q+1}\phi)$ for every ϕ

and calculate

$$(\phi,a(N+1)^{q+1}a^+\phi) = (\phi,a(N+1)(N+1)^{q-1}(N+1)a^+\phi) =$$

because of (A1)

$$=(\phi,(N+2)\ a(N+1)^{q-1}\ a^+(N+2)\phi)$$

$$\leq k((N+2)\phi,\ (N+1)^{q-1}(N+2)\phi)$$

because (A2) is valid for every ϕ, in particular for $\psi = (N+2)\phi$

$$\leq k[((N+1)\phi,(N+1)^{q-1}(N+1)\phi) + (\phi,(N+1)^q\phi) + ((N+1)\phi,(N+1)^{q-1}\phi)$$

$$+ (\phi,(N+1)^{q-1}\phi)]$$

$$\leq 4\cdot k\ (\phi,(N+1)^{q+1}\phi)$$

because of (6a).

Consequently (14) has been shown to be also fulfilled for p = q+1 and therefore it is generally true.

APPENDIX V

We will show in this Appendix that the defining assumptions, i.e. (I,1)(I,2)(I,3) and the τ_Φ-continuity of P and Q, will lead to the Schrödinger representation in the Schwartz-space S.[31)33)]

We choose one of the energy eigenstates Φ_n, n = 1,2,...n and calculate (using $a\Phi_n = \sqrt{n}\,\Phi_{n-1}$, $a^+\Phi_n = \sqrt{n+1}\,\Phi_{n+1}$):

$$(1)\qquad \langle Q\Phi_n|x\rangle = \frac{\hbar}{2m\omega}(\sqrt{n}\,\langle\Phi_{n-1}|x\rangle + \sqrt{n+1}\,\langle\Phi_{n+1}|x\rangle)$$
$$\|$$
$$\bar{x}\langle\Phi_n|x\rangle$$

for n = 0 one obtains

$$(1)\qquad \sqrt{2}\,\frac{\bar{x}}{\sqrt{\hbar/m\omega}}\langle\Phi_o|x\rangle = \sqrt{1}\,\langle\Phi_1|x\rangle$$

(1) is a recurrence relation for $\langle\Phi_n|x\rangle$, which can be brought into a well known form by introducing

$$(2)\quad y = \frac{\bar{x}}{\sqrt{\hbar/m\omega}},\ f_n(y) = \sqrt{2^n n!}\,\frac{\langle\Phi_n|x\rangle}{\langle\Phi_o|x\rangle}\ (\text{assuming } \langle\Phi_o|x\rangle \neq 0)$$

(1) is then written

$$(1')\qquad f_{n+1}(y) = 2yf_n(y) - 2n\,f_{n-1}(y)\qquad n=1,2,...$$

$$(1')\qquad f_1(y) = 2y\,f_o(y);\ f_o(y) = 1$$

(1') are the well known recurrence relations for the Hermite polynomials and have solutions for any complex y.

Thus we conclude that for any complex value x there is an antilinear (not necessarily continuous) functional $|x\rangle = F_x$ which is a generalized eigenvector.

The functionals $\langle\Phi_n|x\rangle$ are given by

$$(3_x) \quad \langle\Phi_n|x\rangle = \frac{1}{\sqrt{2^n n!}} \langle\Phi_o|x\rangle \, H_n\left(\frac{x}{\sqrt{\hbar/m\omega}}\right), \quad x \in \mathbb{C}$$

As the operator Q is e.s.a. (as a consequence of (I,3)) the spectrum of Q must be real[28]. Thus the spectrum is only a subset of generalized eigenvalues and in the spectral decomposition

$$(4_x) \quad \Phi_n = \int d\mu(x) \; |x\rangle\langle x|\Phi_n\rangle$$

only those generalized eigenvectors and those functionals $\langle x|\Phi_n\rangle = \overline{\langle\Phi_n|x\rangle}$ appear for which x is real. To find the spectrum of Q then means to determine which real values x appear in (4_x).

If we consider Φ_n (or any $\phi \in \Phi$) as functional at the generalized eigenvector $F_{x'} = |x'\rangle \in \Phi^X$, x' ε spectrum Q, then, according to (III,12), we obtain from (4_x):

$$(5_x) \quad \tilde{\Phi}_n(F_{x'}) = \langle x'|\Phi_n\rangle = \int d\mu(x) \; \langle x'|x\rangle\langle x|\Phi_n\rangle$$

Thus $d\mu(x) \langle x'|x\rangle$ must be the Dirac measure, i.e. the distribution $\langle x'|x\rangle$ defined by (5_x) must have the property of the Dirac δ- "functions" (where X is continuous):

$$(6_x) \quad d\mu(x) \; \langle x'|x\rangle = dx\delta(x'-x)$$

We consider now the generalized eigenvectors of the operator P:

$$P^X|p\rangle = P^X F_p = \bar{p} F_p$$

or $\langle P\phi|p\rangle = \bar{p}\langle\phi|p\rangle$ for every $\phi \in \Phi$.

Using $P = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}\,(a-a^+)$ one can proceed in complete analogy to the procedure for Q following eq. (1). One obtains that the set of generalized eigenvalues of P is the complex plane and

$$(3_p)\quad \langle\Phi_n|p\rangle = i^n \frac{1}{\sqrt{2^n n!}} \langle\Phi_o|p\rangle\, H_n\left(\frac{p}{\sqrt{\hbar m\omega}}\right)$$

In analogy to the case for Q, also in the spectral decomposition of Φ_n with respect to the operator P:

$$(4_p)\quad \Phi_n = \int d\mu(p)\ |p\rangle\langle p|\Phi_n\rangle$$

only those $\langle p|\Phi_n\rangle = \overline{\langle\Phi_n|p\rangle}$ of (3_p) occur for which p is real.

If we consider $\Phi_n \in \Phi$ as the functional on the space Φ^X at the element $F_{p'} \in \Phi^X$, then

$$(5_p)\quad \langle p'|\tilde{\phi}\rangle = \int d\mu(p)\ \langle p'|p\rangle\langle p|\tilde{\phi}\rangle$$

so that we conclude

$$(6_p)\quad d\mu(p)\ \langle p'|p\rangle = dp\ \delta(p'-p)$$

The analogy between the p- and x- spectral decomposition as expressed e.g. by the analogy between (3_x) and (3_p)

we should have expected as the assumptions we started with, (I,1) and (I,3) are symmetric in Q and P.

We now calculate the scalar product of Φ_n and Φ_m using (4_x) with (3_x) and (4_p) with (3p):

$$(7)\quad \delta_{nm} = (\Phi_n,\Phi_m) = \int_X d\mu(x)\ \langle\Phi_n|x\rangle\langle x|\Phi_m\rangle$$

$$= \frac{1}{\sqrt{2^m m!}}\ \frac{1}{\sqrt{2^n n!}} \int_X d\mu(x)|\langle\Phi_o|x\rangle|^2\ H_n\left(\frac{x}{\sqrt{\hbar/m\omega}}\right)\overline{H_m\left(\frac{x}{\sqrt{\hbar/m\omega}}\right)}$$

$$= \int_{\text{spectrum } P} d\mu(p)\ \langle\Phi_n|p\rangle\langle p|\Phi_m\rangle$$

$$= \frac{1}{\sqrt{2^m m!}}\ \frac{1}{\sqrt{2^n n!}} \int_{(\text{spectrum } P)} d\mu(p)|\langle\Phi_o|p\rangle|^2\ H_n\left(\frac{P}{\sqrt{\hbar m\omega}}\right)\ \overline{H_m\left(\frac{P}{\sqrt{\hbar m\omega}}\right)}$$

Comparing this with the orthogonality relations for the Hermite polynomials

$$(8)\qquad \frac{1}{n!\,2^n\sqrt{\pi}} \int_{-\infty}^{+\infty} dy\ e^{-y^2}\ H_m(y)H_n(y) = \delta_{mn}$$

and taking into account that the Hermite polynomials are only orthogonal polynomials if associated with the interval $-\infty < y < +\infty$ and the weight e^{-y^2} (one can define $H_n(y)$ by (8) and derive (1') for real y) we conclude:

$$(9_x)\qquad d\mu(x)|\langle\Phi_o|x\rangle|^2 = \sqrt{\frac{m\omega}{\hbar\pi}}\ e^{-\frac{m\omega}{\hbar.2}x^2}\,dx$$

(10_x) spectrum $Q = X = \{x|-\infty<x<+\infty\}$

and

$$(9_p) \quad d\mu(p)|\langle\Phi_o|p\rangle|^2 = \sqrt{\frac{1}{m\omega\hbar\ \pi}}\ e^{\frac{p^2}{-m\omega\hbar\cdot 2}}\ dp$$

$$(10_p) \quad \text{spectrum } P = \{p|-\infty<p<+\infty\}$$

If we agree to normalize the generalized eigenvectors such that

$$(11_x) \quad \langle x'|x\rangle = \delta(x'-x)$$

$$(11_p) \quad \langle p'|p\rangle = \delta(p'-p)$$

then according to (6_x) and (6_p)

$$(12_x) \quad d\mu(x) = dx$$

$$(12_p) \quad d\mu(p) = dp$$

and

$$(13_x) \quad \langle\Phi_n|x\rangle = \sqrt[4]{\frac{m\omega}{\hbar}}\ \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\ H_n\left(\frac{x}{\sqrt{\hbar/m\omega}}\right)\ e^{-\frac{m\omega}{\hbar 2}x^2}$$

$$(13_p) \quad \langle\Phi_n|p\rangle = i^n\ \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\ \sqrt[4]{\frac{1}{m\omega\hbar}}\ H_n\left(\frac{p}{\sqrt{\hbar m\omega}}\right)\ e^{-\frac{p^2}{m\omega\hbar . 2}}$$

Thus far we know the matrix elements of Q in the basis of generalized eigenvectors of Q

$$(10_x') \quad \langle x|Q|\phi\rangle = \overline{\langle Q\phi|x\rangle} = \overline{\langle\phi|Q^X|x\rangle} = x\langle x|\phi\rangle$$

and the matrix elements of P in the basis of generalized eigenvectors of P

$(10'_p)$ $\langle p|P|\phi\rangle = p\,\langle p|\phi\rangle$

We now want to calculate the matrix elements of P in the basis of generalized eigenvectors of Q and the matrix elements of Q in the basis of generalized eigenvectors of P. In order to do this, we will consider Φ_n as a functional at the generalized eigenvector $F_p \in \Phi^X$, $p \in$ spectrum P, and use the spectral decomposition (4_x):

(14_x) $\langle p|\Phi_n\rangle = \int dx \langle p|x\rangle\langle x|\Phi_n\rangle$

and then as a functional at the generalized eigenvector $F_x \in \Phi^X$, $x \in$ Spectrum Q, and use the spectral decomposition (4_p):

(14_p) $\langle x|\Phi_n\rangle = \int dp\, \langle x|p\rangle\langle p|\Phi_n\rangle$

$\langle x|\Phi_n\rangle$ and $\langle p|\Phi_n\rangle$ in (14) are given by (13_x) and (13_p) respectively.

The Hermite polynomials have the following property:

$$(15) \qquad i^n\, e^{-\eta^2/2}\, H_n(\eta) = \int_{-\infty}^{+\infty} d\xi\, \frac{e^{i\xi\eta}}{\sqrt{2\pi}}\, e^{-\xi^2/2}\, H_n(\xi)$$

Inserting (13_p) and (13_x) into this relation it follows

$$(16_x) \qquad \langle\Phi_n|p\rangle = \int_{-\infty}^{+\infty} dx\, \frac{e^{\frac{ixp}{\hbar}}}{\sqrt{2\pi\hbar}}\, \langle\Phi_n|x\rangle$$

or taking the complex conjugate

$$(\overline{16_x}) \qquad \langle p|\Phi_n\rangle = \int dx \, \frac{e^{\frac{-ixp}{\hbar}}}{\sqrt{2\pi\hbar}} \langle x|\Phi_n\rangle$$

Comparing $(\overline{16_x})$ with (14_x) we find that the distributions $\langle p|x\rangle$ are given by:

$$(17_x) \qquad \langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{-ixp}{\hbar}}$$

In the same way one obtains from (15) and (14_p)

$$(17_p) \qquad \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ixp/\hbar}$$

(17_x) and (17_p) together give

$$(18) \qquad \langle x|p\rangle = \overline{\langle p|x\rangle}$$

We emphasize that (18) does not follow from the "hermiticity property of the "scalar product" $\langle x|p\rangle$" but is a very particular property of the operators P and Q (and--as a consequence thereof--of the Fourier transformation).

For the general case of two arbitrary systems of generalized eigenvectors $|\lambda\rangle$; $A^X|\lambda\rangle = \bar{\lambda}|\lambda\rangle$

and $|\beta\rangle$; $B^X|\beta\rangle = \bar{\beta}|\beta\rangle$

it is not always true that

$$\langle\lambda|\beta\rangle = \langle\overline{\beta|\lambda}\rangle$$

though (III,12) holds always.

It is now simple to calculate the matrix element of P in the basis of generalized eigenvectors of Q, using (17_p)

$$<x|P|\phi> = \int dp\ p\ <x|p><p|\phi>$$

$$= \int dp\ \frac{\hbar}{i}\frac{d}{dx} <x|p><p|\phi>$$

$$(19_x) \qquad <x|P|\phi> = \frac{\hbar}{i}\frac{d}{dx} <x|\phi>$$

In the same way using (17_x) one obtains

$$(19_p) \qquad <p|Q|\phi> = -\frac{\hbar}{i}\frac{d}{dp} <p|\phi>$$

It is now easy to see what the realization of Φ by the functions $<x|\phi>$ (or $<p|\phi>$) is: As every power Q^n and P^m of the operators P and Q are defined on Φ, the functions $<x|\phi> = \phi(x)$ must fulfill:

$$(\phi, Q^n\tilde{\phi}) = \int dx\ x^n |<x|\tilde{\phi}>|^2 < \infty$$

(20)

$$(\phi, P^m\tilde{\phi}) = (\frac{\hbar}{i})^m \int dx\ \overline{<x|\phi>}\ \frac{d^m}{dx^m} <x|\tilde{\phi}> < \infty$$

$$(\phi, Q^nP^m\phi) = (\frac{\hbar}{i})^m \int dx\ \overline{<x|\phi>}\ x^n\ \frac{d^m}{dx^n} <x|\tilde{\phi}> < \infty$$

As all algebraic expressions in P and Q are continuous operators in Φ the sequence $\phi_\nu(x) = <x|\phi_\nu>$, which is the realization of the convergent sequence $\phi_\nu \xrightarrow{\tau_\Phi} \phi$, converges to $\phi(x) = <x|\phi>$ if the $x^n \frac{d}{dx^m} \phi_\nu(x)$ converge uniformly on every bounded region to $x^n \frac{d}{dx^m} \phi(x)$.

The space of functions $\phi(x)$ that fulfill (20) and for which the convergence is defined in the above described way is the space of test functions S (Schwartz space).

Concluding this appendix we emphasize that in deriving (19) and (10) we have made use of (I,3), because by (I,3) we were led to (1) and therewith to the Hermite polynomials, whose properties have been used extensively to establish (19) and (10).

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2) K. Maurin, General Eigenfunction Expansions and Unitary Representations of Topological Groups. Workawa 1968.

3) J. von Neumann, Mathematical Foundations of Quantum Mechanics; Springer Berlin 1932, Princeton University Press 1955.

4) P. A. M. Dirac, The Principles of Quantum Mechanics, Clarendon Press Oxford 1958.

5) J. E. Roberts, Journal Math. Phys. 7, 1097 (1966). Also J. P. Antoine J. Math. Phys. 10, 53, 2276(1969).

6) A. Bohm, Boulder Lectures in Theoretical Physics, Vol. 9A, 255(1966).

7) The explanation of the content of this basic assumption is the subject of this paper. All the unknown mathematical notions will be defined or described in the following sections .... Appendices I(Linear Space), II(Algebra) and III(Operators in Hilbert Space) contain further definitions which are not given in the paper.

8) The formulation given here is a special case of the construction of Rigged Hilbert Spaces for enveloping algebras of Lie groups which has been suggested in A. Bohm Journ. Math. Phys. 8, 1557(1966), Appendix; and independently by B. Nagel, College de France Lectures (1970). A detailed treatment of $\varepsilon(SU(1,1))$, which is slightly more complicated than the example treated here, has been given in G. Lindblad, B. Nagel, Ann. Inst. Poincaré Section A-(N.S.) 13, 27(1970).

9) See Appendix II

10) See Appendix I

11) See Appendix III

12) E. Nelson, Ann. Math. 70, 572(1959).

13) H. D. Doebner Proceedings of the 1966 Istambul Summer Institute.

14) In a scalar product space the norm $|| \quad ||$ is defined by $||\phi|| = \sqrt{(\phi,\phi)}$.

15) The construction of the completion $\bar{R}$ of the space R consists of the following: The elements of the space $\bar{R}$ are all possible Cauchy sequences $x = \{x_n\}$, $x_n \in R$, where two such sequences $x = \{x_n\}$ and $y = \{y_n\}$ are not considered distinct if $||x_n - y_n|| \to 0$ and the elements $x \in R$ are identified in $\bar{R}$ with the sequence $\{x,x,x,.....\}$. The

operations with these sequences are defined by

$$\alpha x = \{\alpha x_n\},\ x + y = \{x_n + y_n\},\ (x,y) = \lim_{n \to \infty} (x_n, y_n)$$

One can verify that $\bar{R}$ is then a complete scalar product space, i.e. a Hilbert space

16) That $(N+1)^p$ is e.s.a. can be proved in many ways. It also follows from the fact that $(N+1)^p$ is an elliptic element in the enveloping algebra of a group representation; see ref.20).

17) It is well known that P, Q and aa^+ cannot be continuous operators with respect to τ_H and are, therefore, not defined on the whole Hilbert space.

18) Let $\phi_\nu \overset{\tau_\Phi}{\to} \phi$ i.e. $(\phi_\nu - \phi) \overset{\tau_\Phi}{\to} 0$. Let us assume that A was not defined on ϕ. As A is a continuous operator $A(\phi_\nu - \phi) \overset{\tau_\Phi}{\to} 0$. Therefore we can define A on ϕ by $A\phi = \tau_\Phi\text{-lim}\, A\phi_\nu$.

19) The definition of closed and closable operators is given in Appendix III. P,Q,H are closable because they are symmetric and a symmetric operator admits a closure.

20) E. Nelson, W. F. Stinespring, Amer. Journ. Math. 81, 547, (1959).

21) The original definition of nuclearity for countably normed spaces (Grothendieck, Gelfand and Kostyuchenko) is: Φ is nuclear iff for any m there is an n such that

the mapping $\Phi_n \to \Phi_m$ is nuclear i.e. has the form

$\Phi_n \ni \phi \to \sum_{k=1}^{\infty} \lambda_k (\phi,\phi_k)_n \psi_k$ where Φ_i is the completion

of Ψ with respect to the norm $|| \quad ||_i$ and $\{\phi_k\}$,$\{\psi_k\}$ are orthonormal systems in the spaces Φ_n and Φ_m respectively, $\lambda_k > 0$ and $\Sigma\lambda_k < \infty$.

22) J. E. Roberts Commun. Math. Phys. 3, 98(1966).

23) A. A. Kirillov, Dokl. Akad. Nauk. SSSR 130, 966(1960).

24) A. Bohm, Appendix B of Journ. Math. Phys. 8, 1557(1967).

25) B. Nagel, Lecture Notes, College de France (1970).

26) For a proof we refer to Gelfand, Shilov, ref. 1.) Vol. II, Ch. I, Sect. 5, 6 and Gelfand, Vilenkin, ref. 1.) Vol. IV, Ch. I, Sect. 3, 4.

27) This statement holds for countably Hilbert spaces if τ^X is the strong topology and in nuclear spaces — where strong and weak topology coincide — also for the weak topology See ref. 25.

28) For a simple proof of this statement see e.g. Liusternik Sobolev, Elements of Functional Analysis §31 or Akhiezer, Glazman, Theory of Linear Operators in Hilbert Space §43.

29) For a proof of the Nuclear Spectral Theorem see ref. 1.) and ref. 2.)

30) J. Dixmier, Comp. Math. 13, 263, (1958) and references thereof.

31) P. Kristensen, L. Meljbo, E. Thue Poulsen, Commun. Math. Phys. 1, 175, (1965).

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