Appendix A Financial Derivatives

A1 Investment and Risk

Basic markets in which money is invested trade in particular with

equities (stocks),
bonds, and
commodites.

Front pages of *The Financial Times* or *The Wall Street Journal* open with charts informing about the trading in these key markets. Such charts symbolize and summarize myriads of buys and sales, and of individual gains and losses. The assets bought in the markets are collected and held in the portfolios of investors.

An easy way to buy or sell an asset is a spot contract, which is an agreement on the price assuming delivery on the same date. Typical examples are furnished by the trading of stocks on an exchange, where the spot price is paid the same day. On the spot markets, gain or loss, or risks are clearly visible. The spot contracts are contrasted with those contracts that agree today ($t = 0$) to sell or buy an asset for a certain price at a certain *future time* ($t = T$). Historically, the first objects traded in this way have been commodities, such as agricultural products, metals, or oil. For example, a farmer may wish to sell in advance the crop expected for the coming season. Later, such trading extended to stocks, currencies and other financial instruments. Today there is a virtually unlimited variety of contracts on objects and their future state, from credit risks to weather prediction.

The future price of the underlying asset is usually unknown, it may move up or down in an unexpected way. For example, scarcity of a product will result in higher prices. Or the prices of stocks may decline sharply. But the agreement must fix a price today, for an exchange of asset and payment that will happen in weeks or months. At maturity, the spot price usually differs from the agreed price of the contract. The difference between spot price and contract price may be significant. Hence contracts into the future are risky. Investors and portfolio managers hope their shares and markets perform well, and are concerned of risks that might weigh on their assets.

Different investments vary in their degree of uncertainty. The price of a stock may fall, and the company might default. The issuer of a bond may fail to meet the obligations in that he does not pay coupons or even fails to repay the principal amount. Some commodities like agricultural produce may spoil.

Financial risk of assets is defined as the degree of uncertainty of their return. *Market risks* are those risks that cannot be diversified away. Market risks are contrary to *default risks (credit risks).*

No investment is really free of risks. But bonds can come close to the idealization of being riskless. If the seller of a bond has top ratings, then the return of a bond at maturity can be considered safe, and its value is known today with certainty. Such a bond is regarded as "riskless asset." The rate earned on a riskless asset is the *risk-free interest rate.* To avoid the complication of re-investing coupons, *zero-coupon bonds* are considered. The interest rate, denoted r, depends on the time to maturity T. The interest rate r is the continuously compounded interest which makes an initial investment S_0 grow to $S_0 e^{rT}$. We shall often assume that $r > 0$ is constant throughout that time period. A candidate for r is the LIBOR[1], which can be found in the financial press. In the mathematical finance literature, the term "bond" is used as synonym for a risk-free investment. Examples of bonds in real bond markets that come close to our idealized risk-free bond are provided by Treasury bills, which are short-term obligations of the US government, and by the long-term Treasury notes. See [Hull00] for further introduction, and consult for instance *The Wall Street Journal* for market diaries.

All other assets are risky, with equities being the most prominent examples. *Hedging* is possible to protect against financial loss. Many hedging instruments have been developed. Since these financial instruments depend on the particular asset that is to be hedged, they are called *derivatives.* Main types of derivatives are *futures, forwards, options,* and *swaps*[2]. They are explained below in some more detail. Tailoring and pricing derivates is the core of *financial engineering.* Hedging with derivatives is the way to bound financial risks and to protect investments.

The risks will play an important role in fixing the terms of the agreements, and in designing strategies for compensation.

A2 Financial Derivatives

Derivatives are instruments to assist and regulate agreements on transactions of the future. Derivatives can be traded on specialized exchanges.

Futures and **forwards** are agreements between two parties to buy or sell an asset at a certain time in the future for a certain delivery price. Both parties make a binding commitment, there is nothing to choose at a later time.

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[1] London Interbank Offered Rate

[2] A comprehensive glossary of financial terms is provided by www.bloomberg.com/analysis

For forwards no premiums are required and no money changes hands until maturity. A basic difference between futures and forwards is that futures contracts are traded on exchanges and are more formalized, whereas forwards are traded in the over-the-counter market (OTC). Also the OTC market usually involves financial institutions. Large exchanges on which futures contracts are traded are the Chicago Board of Trade (CBOT), the Chicago Mercantile Exchange (CME), and the Eurex.

Options are *rights* to buy or sell underlying assets for an *exercise price (strike)*, which is fixed by the terms of the option contract. That is, the purchaser of the option is *not obligated* to buy or sell the asset. The decision will be based on the payoff, which is contingent on the underlying asset's behavior. The buying or selling of the underlying asset by exercising the option at a future date ($t = T$) must be distinguished from the purchase of the option (at $t = 0$, say), for which a premium ist paid. After the Chicago Board of Options Exchange (CBOE) opened in 1973, the volume of the trading with options has grown dramatically. Options are discussed in more detail in Section 1.1.

Swaps are contracts regulating an exchange of cash flows at different future times. A common type of swap is the *interest-rate swap*, in which two parties exchange interest payments periodically, typically fixed-rate payments for floating-rate payments. Counterparty A agrees to pay to counterparty B a fixed interest rate on some notional principal, and in return party B agrees to pay party A interest at a floating rate on the same notional principal. The principal itself is not exchanged. Each of the parties borrows the money at his market. The interest payment is received from the counterparty and paid to the lending bank. Since the interest payments are in the same currency, the counterparties only exchange the interest differences. The *swap rate* is the fixed-interest rate fixed such that the deal (initially) has no value to either party ("par swap"). For a *currency swap*, the two parties exchange cash flows in different currencies.

An important application of derivatives is **hedging**. Hedging means to eliminate or limit risks. For example, consider an investor who owns shares and wants protection against a possible decline of the price below a value K in the next three months. The investor could buy put options on this stock with strike K and a maturity that matches his three months time horizon. Since the investor can exercise his puts when the share price falls below K, it is guaranteed that the stock can be sold at least for the price K during the life time of the option. With this strategy the value of the stock is protected. The premium paid when purchasing the put option plays the role of an insurance premium. — Hedging is intrinsic for calls. The writer of a call must hedge his position to avoid being hit by rising asset prices. Generally speaking, options and other derivatives facilitate the transfer of financial risks.

What kind of principle is so powerful to serve as basis for a fair valuation of derivatives? The concept is **arbitrage**, or rather the assumption that arbitrage is not possible in an idealized market. Arbitrage means the existence of a portfolio, which requires no investment initially, and which with guarantee makes no loss but very likely a gain at maturity. Or shorter: arbitrage is a self-financing trading strategy with zero initial value and positive terminal value.

If an arbitrage profit becomes known, arbitrageurs will take advantage and try to lock in.[3] This makes the arbitrage profits shrink. In an idealized market, informations spread rapidly and arbitrage opportunites become apparent. So arbitrage cannot last for long. Hence, in efficient markets at most very small arbitrage opportunities are observed in practice. For the modeling of financial markets this leads to postulate the **no-arbitrage principle**: One assumes an idealized market such that arbitrage is ruled out. Arguments based on the no-arbitrage principle resemble indirect proofs in mathematics: Suppose a certain financial situation. If this assumed scenario enables constructing an arbitrage opportunity, then there is a conflict to the no-arbitrage principle. Consequently, the assumed scenario is impossible. See Appendix A3 for an example.

For valuing derivatives one compares the return of the risky financial investment with the return of an investment that is free of risk. For the comparison, one calculates the gain the same initial capital would yield when invested in bonds. To compare properly, one chooses a bond with time horizon T matching the terms of the derivative that is to be priced. Then, by the no-arbitrage principle, the risky investment should have the same price as the equivalent risk-free strategy. The construction and choice of derivatives to optimize portfolios and protect against extreme price movements is the essence of financial engineering.

The pricing of options is an ambitious task and requires sophisticated algorithms. Since this book is devoted to computational tools, mainly concentrating on options, the features of options are part of the text (Section 1.1 for standard options, and Section 6.1 for exotic options). This text will not enter further the discussion of forwards, futures, and swaps, with one exception: We choose the forward as an example (below) to illustrate the concept of arbitrage. For a detailed discussion of futures, forwards and swaps we refer to the literature, for instance to [Hull00], [BaR96], [MR97], [Wi98], [Shi99], [Lyuu02].

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[3] This assumes that investors prefer more to less, the basis for a rational pricing theory [Mer73].

A3 Forwards and the No-Arbitrage Principle

As stated above, a forward is a contract between two parties to buy or sell an asset to be delivered at a certain time T in the future for a certain delivery price K. The time the parties agree on the forward contract (fixing T and K) is set to $t_0 = 0$. Since no premiums and no money change hands until maturity, the initial value of a forward is zero.

The party with the *long position* agrees to buy the underlying asset; the other party assumes the *short position* and agrees to sell the asset.

For the subsequent explanations S_t denotes the price of the asset in the time interval $0 \le t \le T$. To fix ideas, we assume just one interest rate r for borrowing or lending risk-free money over the time period $0 \le t \le T$. By the definition of the forward, at time of maturity T the party with the long position pays K to get the asset, which is then worth S_T.

Arbitrage Arguments

As will be shown next, the no-arbitrage principle enforces the forward price to be

$$K = S_0 e^{rT} . \tag{A3.1}$$

Thereby it is assumed that the asset does not produce any income (dividends) and does not cost anything until $t = T$.

Let us see how the no-arbitrage principle is invoked. We ask what the fair price K of a forward is at time $t = 0$, when the terms of a forward are settled. Then the spot price of the asset is S_0.

> Assume first $K > S_0 e^{rT}$. Then an arbitrage strategy exists as follows: At $t = 0$ borrow S_0 at the interest rate r, buy the asset, and enter into a forward contract to sell the asset for the price K at $t = T$. When the time instant T has arrived, the arbitrageur completes the strategy by selling the asset $(+K)$ and by repaying the loan $(-S_0 e^{rT})$. The result is a riskless profit of $K - S_0 e^{rT} > 0$. This contradicts the no-arbitrage principle, so $K - S_0 e^{rT} \le 0$ must hold.
>
> Suppose next the complementary situation $K < S_0 e^{rT}$. In this case an investor who owns the asset[4] would sell it, invest the proceeds at interest rate r for the time period T, and enter a forward contract to buy the asset at $t = T$. In the end there would be a riskless profit of $S_0 e^{rT} - K > 0$. The conflict with the no-arbitrage principle implies $S_0 e^{rT} - K \le 0$.
>
> Combining the two inequalities $\le$ and $\ge$ proves the equality. [$S_0 e^{r_1 T} \le K \le S_0 e^{r_2 T}$ in case of different rates $0 \le r_1 \le r_2$ for lending or borrowing]

One of the many applications of forwards is to hedge risks caused by foreign exchange.

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[4] otherwise: *short sale*, selling a security the seller does not own.

Example (hedging against exchange rate moves)
A U.S. corporation will receive one million euro in three months (on December 25), and wants to hedge against exchange rate moves. The corporation contacts a bank ("today" on September 25) to ask for the forward foreign exchange quotes. The three-month forward exchange rate is that \$1.1428 will buy one euro, says the bank.[5] Why this? For completeness, on that day the spot rate is \$1.1457. If the corporation and the bank enter into the corresponding forward contract on September 25, the corporation is obligated to sell one million euro to the bank for \$1,142,800 on December 25. The bank then has a long forward contract on euro, and the corporation is in the short position.

Let us summarize the terms of the forward:
asset: one million euro
asset price S_t: the value of the asset in US \$ $(S_0 = \$1,145,700)$
maturity $T = 1/4$ (three months)
delivery price K: \$1,142,800 (forward price)

To understand the forward price in the above example, we need to generalize the basic forward price $S_0 e^{rT}$ to a situation where the asset produces income. In the foreign-exchange example, the asset earns the foreign interest rate, which we denote δ. To agree on a forward contract, $Ke^{-rT} = S_0 e^{-\delta T}$, so

$$K = S_0 e^{(r-\delta)T} \ . \tag{A3.2}$$

(See [Hull00].) On that date of the example the three-month interest rate in the U.S. was $r = 1\%$, and in the euro world $\delta = 2\%$. So

$$S_0 e^{(r-\delta)T} = 1145700 e^{-0.01 \frac{1}{4}} = 1142800$$

which explains the three-month forward exchange rate of the example.

A4 The Black-Scholes Equation

The Classical Equation

This appendix applies Itô's lemma to derive the Black-Scholes equation. The first basic assumption is a geometric Brownian motion of the stock price. According to Model 1.13 the price S obeys the linear stochastic differential equation (1.33)

$$dS = \mu S \, dt + \sigma S \, dW \tag{A4.1}$$

with constant μ and σ. Further consider a portfolio consisting at time t of α_t shares of the asset with value S_t, and of β_t shares of the bond with value B_t. The bond is assumed riskless with

[5] September 25, 2003

$$dB = rBdt \ . \tag{A4.2}$$

At time t the value of the portfolio is

$$\Pi_t := \alpha_t S_t + \beta_t B_t \ . \tag{A4.3}$$

The portfolio is supposed to hedge a European option with value V_t, and payoff V_T at maturity T. So we assume *replication*,

$$\Pi_T = V_T = \text{ payoff } \ . \tag{A4.4}$$

The European option cannot be traded before maturity; neither any investment is required in $0 < t < T$ for holding the option nor is there any payout stream. To compare the values of V_t and Π_t, and to apply no-arbitrage arguments, the portfolio should have an equivalent property. Suppose the portfolio is "closed" for $0 < t < T$ in the sense that no money is injected into or removed from the portfolio. This amounts to the *self-financing property*

$$d\Pi_t = \alpha_t dS_t + \beta_t dB_t \ . \tag{A4.5}$$

That is, changes in the value of Π_t can only be due to changes in S or B.

Now the no-arbitrage principle is invoked. Replication (A4.4) and self-financing (A4.5) imply

$$\Pi_t = V_t \quad \text{for all} \ \ t \text{ in } 0 \le t \le T \ , \tag{A4.6}$$

because both investments have the same payout stream. So the replicating and self-financing portfolio is equivalent to the risky option. This has consequences for the quantites α_t and β_t of stocks and bonds, which are continuously readjusted.

Assuming a sufficiently smooth value function $\Pi_t = V(S,t)$, we infer from Itô's lemma (Section 1.8)

$$d\Pi = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW \ . \tag{A4.7}$$

On the other hand, substitute (A4.1) and (A4.2) into (A4.5) and obtain another version of $d\Pi$, namely

$$d\Pi = (\alpha \mu S + \beta r B)dt + \alpha \sigma S dW \ . \tag{A4.8}$$

Because of uniqueness, the coefficients of both versions must match. Comparing the dW coefficients leads to the hedging strategy

$$\alpha_t = \frac{\partial V(S_t,t)}{\partial S} \ . \tag{A4.9}$$

Matching the dt coefficients gives a relation for β, in which the stochastic $\alpha \mu S$ terms drop out. The βB term is replaced via (A4.3) and (A4.6), which amounts to

$$S\frac{\partial V}{\partial S} + \beta B = V \ .$$

This results in the renowned Black-Scholes equation (1.2),

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

The terminal condition is given by (A4.4).

Choosing in (A4.9) the *delta hedge* $\Delta(S,t) := \alpha = \frac{\partial V}{\partial S}$ provides a dynamic strategy to eliminate the risk that lies in stochastic fluctuations and in the unknown drift μ of the underlying asset. In this sense the modeling of V is risk neutral. The only remaining parameter reflecting stochastic behavior in the Black-Scholes equation is the volatility σ. Note that in the above derivation the standard understanding of constant coefficients μ, σ, r was actually not used. In fact the Black-Scholes equation holds also for time-varying deterministic functions $\mu(t), \sigma(t), r(t)$ ($\longrightarrow$ Exercise 1.19). For reference see, for example, [BaR96], [Du96], [Irle98], [HuK00], [Ste01].

The Solution and the Greeks

The **delta** $\Delta = \frac{\partial V}{\partial S}$ plays a crucial role for hedging portfolios. The corresponding number of units of the underlying asset makes the portfolio (A4.3) riskless. As will be shown next, there is a simple analytic formula for Δ in case of European options. (In reality, hedging must be done in discrete time.)

The Black-Scholes equation has a closed-form solution. For a European call with continuous dividend yield δ as in (4.1) (in Section 4.1) the formulas are

$$d_1 := \frac{\log \frac{S}{K} + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{A4.10a}$$

$$d_2 := d_1 - \sigma\sqrt{T-t} = \frac{\log \frac{S}{K} + \left(r - \delta - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{A4.10b}$$

$$V_{\mathrm{C}}(S,t) = Se^{-\delta(T-t)}F(d_1) - Ke^{-r(T-t)}F(d_2), \tag{A4.10c}$$

where F denotes the standard normal cumulative distribution (compare Exercise 1.3 or Appendix D2). The value $V_{\mathrm{P}}(S,t)$ of a put is obtained by applying the put-call parity on (A4.10c), see Exercise 1.1. For a continuous dividend yield δ as in (4.1) the put-call parity of European options is

$$V_{\mathrm{P}} = V_{\mathrm{C}} - Se^{-\delta(T-t)} + Ke^{-r(T-t)} \tag{A4.11a}$$

from which

$$V_{\mathrm{P}} = -Se^{-\delta(T-t)}F(-d_1) + Ke^{-r(T-t)}F(-d_2) \tag{A4.11b}$$

follows.

For nonconstant but known coefficient functions $\sigma(t), r(t), \delta(t)$, the closed-form solution is modified by introducing integral mean values [Kwok98], [Øk98], [Wi98], [Zag02]. For example, replace the term $r(T-t)$ by the more general term $\int_t^T r(s)ds$, and replace

$$\sigma\sqrt{T-t} \quad \longrightarrow \quad \left(\int_t^T \sigma^2(s)ds\right)^{1/2}$$

Differentiating the Black-Scholes formula gives delta, $\Delta = \frac{\partial V}{\partial S}$, as

$$\begin{aligned} \Delta &= F(d_1) && \text{for a European call,} \\ \Delta &= F(d_1) - 1 && \text{for a European put.} \end{aligned} \tag{A4.12}$$

The delta of (A4.9) is the most prominent example of the "Greeks." Also other derivatives of V are denoted by Greek sounding names:

$$\text{gamma} = \frac{\partial^2 V}{\partial S^2}, \quad \text{theta} = \frac{\partial V}{\partial t}, \quad \text{vega} = \frac{\partial V}{\partial \sigma}, \quad \text{rho} = \frac{\partial V}{\partial r}$$

Analytic expressions can be obtained by differentiating (A4.10). The Greeks are important for a sensitivity analysis. A derivation of the Black-Scholes formula (A4.10) can be found in the literature; for references see [WDH96], [Shi99]. Essential parts of the derivation can also be collected from this book; see for instance Exercise 3.9 or Exercise 1.8.

Hedging a Portfolio in Case of a Jump Process

Next consider a jump-diffusion process as described in Section 1.9, summarized by equation (1.52). The portfolio is the same as above, see (A4.3), and the same assumptions such as replication and self-financing apply. Itô's lemma is applied in a piecewise fashion on the time intervals between jumps. Accordingly (A4.7) is modified by adding the jumps in V with jumps sizes

$$\Delta V := V(S_{\tau^+}, \tau) - V(S_{\tau^-}, \tau)$$

for all jump instances τ_j. Consequently the term $\Delta V dJ$ is added to (A4.7). On the other hand, (1.52) leads to add the term $\alpha(q-1)SdJ$ to (A4.8). Comparing coefficients of the dW terms in both expressions of Π again implies the hedging strategy (A4.9), namely $\alpha = \frac{\partial V}{\partial S}$. This allows to shorten both versions of Π by subtracting equal terms. Let us denote the resulting values of the reduced portfolios by $\tilde{\Pi}$. Then (A4.7) leads to

$$\tilde{\Pi} = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + (V(qS,t) - V(S,t))dJ$$

and (A4.8) leads to

$$\tilde{\Pi} = \left(rV - rS\frac{\partial V}{\partial S} \right) dt + \frac{\partial V}{\partial S}(q-1)SdJ$$

(The reader may check.)

Different from the analysis leading to the classical Black-Scholes equation, $d\tilde{\Pi}$ is not deterministic and it does not make sense to equate both versions. The risk can not be perfectly hedged away to zero in the case of jump-diffusion processes. Following [Mer76], we apply the expectation operator over the random variable q to both versions of $\tilde{\Pi}$. Denote this expectation E, with

$$\mathsf{E}(X) = \int_{-\infty}^{\infty} x f_q(x) dx \tag{A4.13}$$

in case q_t has a density f_q that obeys $q > 0$. The expectations of both versions of $\mathsf{E}(\tilde{\Pi})$ can be equated. The result is

$$\begin{aligned} 0 = & \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \right) dt \\ & + \mathsf{E}\left([V(qS,t) - V(S,t) - (q-1)S\frac{\partial V}{\partial S}] \, dJ \right) . \end{aligned}$$

Since all stochastic terms are assumed independent, the second part of the equation is

$$\mathsf{E}[...]\mathsf{E}(dJ).$$

Using from (1.50)

$$\mathsf{E}(dJ) = \lambda dt$$

and the abbreviation

$$c := \mathsf{E}(q-1)$$

this second part of the equation becomes

$$\{\mathsf{E}(V(qS,t)) - V(S,t) - cS\frac{\partial V}{\partial S}\} \, \lambda dt \; .$$

The integral $c = \mathsf{E}(q-1)$ does not depend on V. This number c can be calculated via (A4.13) as soon as a distribution for q is stipulated. For instance, one may assume a lognormal distribution, with relevant parameters fitted from marked data. (The parameters are not the same as those in (1.47).) With the precalculated number c, the resulting differential equation can be ordered into

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \lambda c)S\frac{\partial V}{\partial S} - (\lambda + r)V + \lambda \mathsf{E}(V(qS,t)) = 0 \tag{A4.14}$$

Note that the last term is an integral taken over the unknown solution function $V(S,t)$. So the resulting equation is a partial integro-differential equation (PIDE). The standard Black-Scholes PDE is included for $\lambda = 0$. The integral can be discretized, for example, by the means of the composite trapezoidal rule (⟶ Appendix C1). A further discussion requires a model for the process q_t, see for example [Mer76], [Wi98], [Tsay02], [ConT04]. For computational approaches see [AnA00], [MaPS02].

A5 Early-Exercise Curve

This appendix briefly discusses properties of the early-exercise curve S_f of standard American put and call options, compare Section 4.5.1. The following holds for the

Put:

(1) S_f is continuously differentiable for $0 \le t < T$.
(2) S_f is nondecreasing.
(3) A lower bound is

$$\begin{aligned} &S_\mathrm{f}(t) > \frac{\lambda_2}{\lambda_2 - 1} K \ \text{, where} \\ &\lambda_2 = \frac{1}{2\sigma^2}\left\{ -\left(r - \delta - \frac{\sigma^2}{2}\right) - \sqrt{\left(r - \delta - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}\right\} \end{aligned} \tag{A5.1}$$

(4) An upper bound for $t < T$ is given by (4.23P),

$$S_\mathrm{f}(t) < \lim_{\substack{t\to T \\ t<T}} S_\mathrm{f}(t) = \min\left(K, \frac{r}{\delta}K\right) \ .$$

For proofs of (1) see [MR97], [Kwok98]. For the smoothness of the value function $V(S,t)$ on the continuation region, see [MR97]. Monotonicity of $V(S,t)$ with respect to time implies (2), as shown for instance in [Kwok98].

The monotonicity of S_f leads to conclude that a lower bound is obtained by $T \to \infty$. This limiting case is the perpetual option, compare Exercise 4.8. Specifically for $\delta = 0$, λ_2 simplifies, and the lower bound is $K\frac{q}{1+q}$, where $q := \frac{2r}{\sigma^2}$. For an illustration of a long horizon $T = 40$ see Figure A.1. Simple calculus shows that λ_2 is the same as the λ_2 in Exercise 4.8.

Here we give a **proof of property (4)**. For $t = T$ the value V_P^am equals the payoff, $V_\mathrm{P}^\mathrm{am}(S,T) = K - S$ for $S < K$. Substitute this into the Black-Scholes equation gives[6]

$$\frac{\partial V}{\partial t} + 0 - (r-\delta)S - rV = 0 \quad ,$$

or

$$\frac{\partial V(S,T)}{\partial t} = rK - \delta S \ .$$

[6] Recall the context: V means V_P^am.

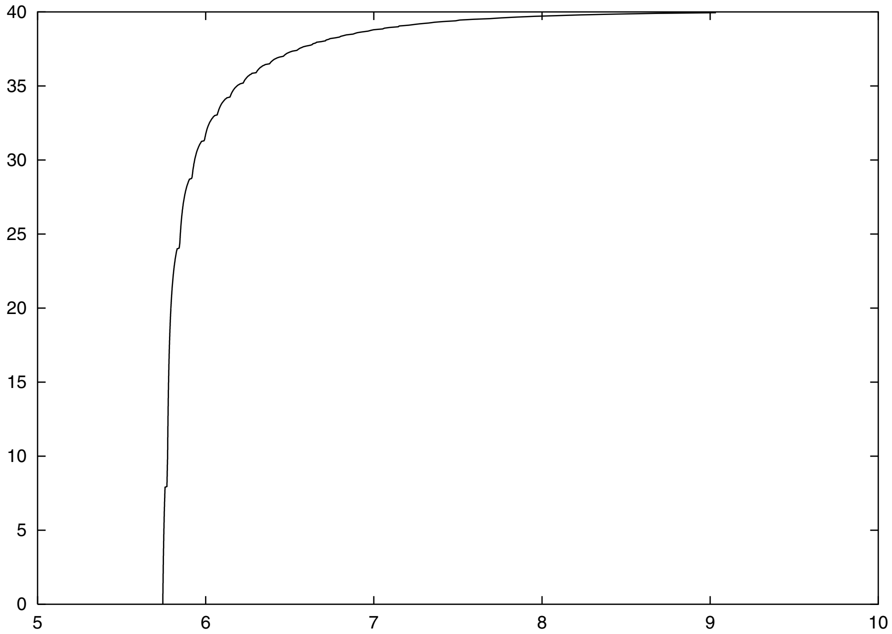

Fig. A.1. Early-exercise curve of an American put with $K = 10$, $r = 0.06$, $\sigma = 0.3$, $\delta = 0$, which leads to $\lambda_2 = -\frac{4}{3}$ and a lower bound of $\frac{4}{7}K$ (nonsmoothed output of a finite-difference calculation)

Observe that

$$\frac{\partial V(S,T)}{\partial t} \leq 0$$

because otherwise for t close to T a contradiction to $V \geq$payoff results. Hence, for $t = T$ and $S < K$,

$$rK - \delta S \leq 0 \quad , \quad S \geq \frac{r}{\delta}K \ .$$

This makes sense only for $\delta > r$. We conclude

$$\frac{r}{\delta}K < S < K \quad \text{implies} \quad \frac{\partial V(S,T)}{\partial t} < 0 \ ,$$
$$\text{and } V > \text{payoff} \quad \text{for } t \approx T \ .$$

On the other hand, for

$$0 < S < \frac{r}{\delta}K \ , \ \frac{\partial V(S,T)}{\partial t} = 0 \ \text{ and } \ V = \text{payoff for } t \approx T \ .$$

For convenience denote

$$S_\text{f}(T) := \lim_{\substack{t \to T \\ t < T}} S_\text{f}(t) \ .$$

Where is $S_f(T)$ located on the maturity line $t = T$? Recall V =payoff for $S < S_f$ and V >payoff for $S > S_f$. Assume first $S_f(T) > \frac{r}{\delta}K$. Then for S in the area $\frac{r}{\delta}K < S < S_f(T)$ and $t \approx T$ we would simultaneously have V >payoff (since $S > \frac{r}{\delta}K$) and V =payoff (since $S < S_f(t)$), which is a contradiction.

Assume next $S_f(T) < \frac{r}{\delta}K$. Then for S inbetween, the contradiction is analogous. So the conclusion is

$$S_f(T) = \frac{r}{\delta}K \quad \text{for } \delta > r \ .$$

Finally we discuss the case $\delta \le r$. By simple geometrical arguments, $S_f(T) > K$ cannot happen. Assume $S_f(T) < K$. Then for $S_f(T) < S < K$ and $t \approx T$

$$\underbrace{dV}_{\le 0} = \underbrace{rK - \delta S}_{>0} \underbrace{dt}_{>0}$$

leads to a contradiction. So

$$S_f(T) = K \quad \text{for } \delta \le r \ .$$

Both assertions are summarized to

$$\lim_{\substack{t \to T \\ t < T}} S_f(t) = \min\left(K, \frac{r}{\delta}K\right)$$

We conclude with listing the properties of an American

Call:

(1) S_f is continuously differentiable for $0 \le t < T$.

(2) S_f is nonincreasing.

(3) An upper bound is

$$\begin{aligned} &S_f(t) < \frac{\lambda_1}{\lambda_1 - 1}K \text{ , where} \\ &\lambda_1 = \frac{1}{2\sigma^2}\left\{ -\left(r - \delta - \frac{\sigma^2}{2}\right) + \sqrt{\left(r - \delta - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r} \right\} \end{aligned} \tag{A5.2}$$

(4) A lower bound for $t < T$ is given by (4.23C),

$$S_f(t) > \max\left(K, \frac{r}{\delta}K\right) \ .$$

Derivations are analogous as in the case of the American put. We note from properties (4) two extreme cases for $t \to T$:

$$\begin{array}{llll} \text{put}: & r \to 0 & \Rightarrow & S_f \to 0 \\ \text{call}: & \delta \to 0 & \Rightarrow & S_f \to \infty \ . \end{array}$$

The second assertion is another clue that for a call early exercise will never be optimal when no dividends are paid ($\delta = 0$).

By the way, the **symmetry** of the above properties is reflected by

$$\begin{aligned}&S_{\mathrm{f,call}}(t;r,\delta)S_{\mathrm{f,put}}(t;\delta,r)=K^2\\&V_{\mathrm{C}}^{\mathrm{am}}(S,T-t;K,r,\delta)=V_{\mathrm{P}}^{\mathrm{am}}(K,T-t;S,\delta,r)\ .\end{aligned} \tag{A5.3}$$

This put-call symmetry is derived in [Kwok98].

Appendix B Stochastic Tools

B1 Essentials of Stochastics

This appendix lists some basic instruments and notations of probability theory and statistics. For further foundations we refer to the literature, for example, [Fe50], [Fisz63], [Bi79], [Mik98], [JaP03], [Shr04].

Let Ω be a *sample space*. In our context Ω is mostly uncountable, for example, $\Omega = \mathbb{R}$. A subset of Ω is an *event* and an element $\omega \in \Omega$ is a sample point. The sample space Ω represents all possible scenarios. Classes of subsets of Ω must satisfy certain requirements to be useful for probability. One assumes that such a class $\mathcal{F}$ of events is a σ-*algebra* or a σ-*field*[1]. That is, $\Omega \in \mathcal{F}$, and $\mathcal{F}$ is closed under the formation of complements and countable unions. In our finance scenario, $\mathcal{F}$ represents the space of events that are observable in a market. If t denotes time, all informations available until t can be regarded as a σ-algebra $\mathcal{F}_t$. Then it is natural to assume a *filtration* —that is, $\mathcal{F}_t \subseteq \mathcal{F}_s$ for $t < s$.

The sets in $\mathcal{F}$ are also called *measurable sets*. A measure on these sets is the probability measure P, a real-valued function taking values in the interval $[0,1]$ with the three axioms

$$\mathsf{P}(A) \geq 0 \ \text{ for all events } A \in \mathcal{F} \ , \qquad \mathsf{P}(\Omega) = 1 \ ,$$
$$\mathsf{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathsf{P}(A_i) \ \text{ for any sequence of disjoint } A_i \in \mathcal{F} \ .$$

The tripel $(\Omega, \mathcal{F}, \mathsf{P})$ is called a *probability space*. An assertion is said to hold *almost everywhere* (P–a.e.) if it is wrong with probability 0.

A real-valued function X on Ω is called **random variable** if the sets

$$\{X \leq x\} := \{\omega \in \Omega \ : \ X(\omega) \leq x\} = X^{-1}((-\infty, x])$$

are measurable for all $x \in \mathbb{R}$. That is, $\{X \leq x\} \in \mathcal{F}$. This book does not explicitly indicate the dependence on the sample space Ω. We write X instead of $X(\omega)$, or X_t or $X(t)$ instead of $X_t(\omega)$ when the random variable depends on a parameter t.

---

[1] This notation with σ is not related with volatility.

For $x \in \mathbb{R}$ a **distribution function** $F(x)$ of X is defined by the probability P that $X \le x$,

$$F(x) := \mathsf{P}(X \le x). \tag{B1.1}$$

Distributions are nondecreasing, right-continuous, and satisfy the limits $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1$. Every absolutely continuous distribution F has a derivative almost everywhere, which is called **density function**. For all $x \in \mathbb{R}$ a density function f has the properties $f(x) \ge 0$ and

$$F(x) = \int_{-\infty}^{x} f(t)dt \ . \tag{B1.2}$$

To stress the dependence on X, the distribution is also written F_X and the density f_X. If X has a density f then the kth *moment* is defined as

$$\mathsf{E}(x^k) := \int_{-\infty}^{\infty} x^k f(x)dx = \int_{-\infty}^{\infty} x^k dF(x) \ , \tag{B1.3}$$

provided the integrals exist. The most important moment of a distribution is the **expected value** or **mean**

$$\mu := \mathsf{E}(X) := \int_{-\infty}^{\infty} x f(x)dx \ . \tag{B1.4}$$

The **variance** is defined as the second central moment

$$\sigma^2 := \mathsf{Var}(X) := \mathsf{E}((X-\mu)^2) \ = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx \ . \tag{B1.5}$$

A consequence is

$$\sigma^2 = \mathsf{E}(X^2) - \mu^2 \ .$$

The expectation depends on the underlying probability measure P, which is sometimes emphasized by writing E_P. Here and in the sequel we assume that the integrals exist. The square root $\sigma = \sqrt{\mathsf{Var}(X)}$ is the *standard deviation* of X. For $\alpha, \beta \in \mathbb{R}$ and two random variables X, Y on the same probability space, expectation and variance satisfy

$$\begin{aligned} &\mathsf{E}(\alpha X + \beta Y) = \alpha \mathsf{E}(X) + \beta \mathsf{E}(Y) \\ &\mathsf{Var}(\alpha X + \beta) = \mathsf{Var}(\alpha X) = \alpha^2 \mathsf{Var}(X). \end{aligned} \tag{B1.6}$$

The *covariance* of two random variables X and Y is

$$\mathsf{Cov}(X,Y) := \mathsf{E}\left((X - \mathsf{E}(X))(Y - \mathsf{E}(Y))\right) = \mathsf{E}(XY) - \mathsf{E}(X)\mathsf{E}(Y),$$

from which

$$\mathsf{Var}(X \pm Y) = \mathsf{Var}(X) + \mathsf{Var}(Y) \pm 2\mathsf{Cov}(X,Y) \tag{B1.7}$$

follows. More general, the covariance between the components of a *vector* X is the matrix

$$\mathsf{Cov}(X) = \mathsf{E}[(X - \mathsf{E}(X))(X - \mathsf{E}(X))^{tr}] = \mathsf{E}(XX^{tr}) - \mathsf{E}(X)\mathsf{E}(X)^{tr}, \quad \text{(B1.8)}$$

where the expectation E is applied to each component. The diagonal carries the variances of the components X_i. Back to the scalar world: Two random variables X and Y are called *independent* if

$$\mathsf{P}(X \le x, Y \le y) = \mathsf{P}(X \le x)\mathsf{P}(Y \le y).$$

For independent random variables X and Y the equations

$$\begin{aligned} \mathsf{E}(XY) &= \mathsf{E}(X)\mathsf{E}(Y), \\ \mathsf{Var}(X+Y) &= \mathsf{Var}(X) + \mathsf{Var}(Y) \end{aligned}$$

are valid; analogous assertions hold for more than two independent random variables.

Normal distribution (Gaussian distribution): The density of the normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \quad \text{(B1.9)}$$

$X \sim \mathcal{N}(\mu, \sigma^2)$ means: X is normally distributed with expectation μ and variance σ^2. An implication is $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$, which is the *standard* normal distribution, or $X = \sigma Z + \mu \sim \mathcal{N}(\mu, \sigma^2)$. The values of the corresponding distribution function $F(x)$ can be approximated by analytic expressions (⟶ Appendix D2) or numerically (⟶ Exercise 1.3). For multidimensional Gaussian, see Section 2.3.3.

Uniform distribution over an interval $a \le x \le b$:

$$f(x) = \frac{1}{b-a} \quad \text{for} \quad a \le x \le b \;;\; f = 0 \quad \text{elsewhere.} \quad \text{(B1.10)}$$

The uniform distribution has expected value $\frac{1}{2}(a+b)$ and variance $\frac{1}{12}(b-a)^2$. If the uniform distribution is considered over a higher-dimensional domain $\mathcal{D}$, then the value of the density is is the inverse of the volume of $\mathcal{D}$,

$$f = \frac{1}{\text{vol}(\mathcal{D})} \cdot 1_{\mathcal{D}}$$

For example, on a unit disc we have $f = 1/\pi$.

Estimates of mean and variance of a normally distributed random variable X from a sample of M realizations $x_1, ..., x_M$ are given by

$$\begin{aligned} \hat{\mu} &:= \frac{1}{M} \sum_{k=1}^{M} x_k \\ \hat{s}^2 &:= \frac{1}{M-1} \sum_{k=1}^{M} (x_k - \hat{\mu})^2 \end{aligned} \tag{B1.11}$$

These expressions of the sample mean $\hat{\mu}$ and the sample variance $\widehat{s}^2$ satisfy $\mathsf{E}(\hat{\mu}) = \mu$ and $\mathsf{E}(\hat{s}^2) = \sigma^2$. That is, $\hat{\mu}$ and $\hat{s}^2$ are unbiased estimates. For the computation see Exercise 1.4, or [PTVF92].

Central Limit Theorem: Suppose $X_1, X_2, ...$ are independent and identically distributed (i.i.d.) random variables, and $\mu := \mathsf{E}(X_i)$, $S_n := \sum_{i=1}^{n} X_i$, $\sigma^2 = \mathsf{E}(X_i - \mu)^2$. Then for each a

$$\lim_{n\to\infty} \mathsf{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le a\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-z^2/2} dz. \tag{B1.12}$$

The **weak law of large numbers** states that for all $\epsilon > 0$

$$\lim_{n\to\infty} \mathsf{P}\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) = 0,$$

and the strong law says $\mathsf{P}(\lim_n \frac{S_n}{n} = \mu) = 1$.

For a **discrete probability space** the sample space Ω is countable. The expectation and the variance of a discrete random variable X with realizations x_i are given by

$$\begin{aligned} \mu &= \mathsf{E}(X) = \sum_{\omega\in\Omega} X(\omega)\mathsf{P}(\omega) = \sum_i x_i \mathsf{P}(X = x_i) \\ \sigma^2 &= \sum_i (x_i - \mu)^2 \mathsf{P}(X = x_i) \end{aligned} \tag{B1.13}$$

Occasionally, the underlying probability measure P is mentioned in the notation. For example, a Bernoulli experiment[2] with $\Omega = \{\omega_1, \omega_2\}$ and $\mathsf{P}(\omega_1) = p$ has expectation

$$\mathsf{E}_{\mathsf{P}}(X) = pX(\omega_1) + (1-p)X(\omega_2).$$

The probability that for n Bernoulli trials the event ω_1 occurs exactly k times, is

$$\mathsf{P}(X = k) = b_{n,p}(k) := \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } 0 \le k \le n\ . \tag{B1.14}$$

[2] repeated independent trials, where only two possible outcomes are possible for each trial, such as tossing a coin

The *binomial coefficient* defined as

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

states in how many ways k elements can be chosen out of a population of size n. For the **binomial distribution** $b_{n,p}(k)$ the mean is $\mu = np$, and the variance $\sigma^2 = np(1-p)$. The probability that event ω_1 occurs at least M times is

$$\mathsf{P}(X \geq M) = B_{n,p}(M) := \sum_{k=M}^{n} \binom{n}{k} p^k (1-p)^{n-k} \ . \tag{B1.15}$$

This follows from the axioms of the probability measure.

For the **Poisson distribution** the probability that an event occurs exactly k times within a specified (time) interval is given by

$$\mathsf{P}(X = k) = \frac{a^k}{k!} e^{-a} \quad \text{for } k = 0, 1, 2, \ldots \tag{B1.16}$$

and a constant $a > 0$. Its mean and variance are both a.

Convergence in the mean: A sequence X_n is said to converge in the (square) mean to X, if $\mathsf{E}(X_n^2) < \infty$, $\mathsf{E}(X^2) < \infty$ and if

$$\lim_{n\to\infty} \mathsf{E}((X - X_n)^2) = 0.$$

A notation for convergence in the mean is

$$\text{l.i.m.}_{n\to\infty} X_n = X.$$

B2 Advanced Topics

General Itô Formula

Let $dX_t = a(.)dt + b(.)dW_t$, where X_t is n-dimensional, $a(.)$ too, and $b(.)$ $(n \times m)$ matrix and W_t m-dimensional, with uncorrelated components, see (1.42). Let g be twice continuously differentiable, defined for (X, t) with values in $\mathbb{R}$. Then $g(X, t)$ is an Itô process with

$$dg = \left[\frac{\partial g}{\partial t} + g_x^{tr} a + \frac{1}{2} \text{ trace } (b^{tr} g_{xx} b) \right] dt + g_x^{tr} b \, dW_t \ . \tag{B2.1}$$

g_x is the gradient vector of the first-order partial derivatives with respect to x, and g_{xx} is the matrix of the second-order derivatives, all evaluated at (X, t). The matrix $b^{tr} g_{xx} b$ is $m \times m$. (Recall that the trace of a matrix is the sum of the diagonal elements.)

(B2.1) is derived via Taylor expansion. The linear terms $g_x^{tr} dX$ are straightforward. The quadratic terms are

$$\frac{1}{2} dX^{tr} g_{xx} dX \quad ,$$

from which the order dt terms remain

$$\frac{1}{2}(bdW)^{tr} g_{xx} bdW = \frac{1}{2} dW^{tr} b^{tr} g_{xx} bdW =: \frac{1}{2} dW^{tr} AdW \ .$$

These remaining terms are

$$\frac{1}{2} \text{ trace } (A) dt \ .$$

A matrix manipulation shows that the elements of $b^{tr} g_{xx} b$ are

$$\sum_{i=1}^{n} \sum_{j=1}^{n} g_{x_i x_j} b_{il} b_{jk} \quad \text{for } l, k = 1, \ldots, m$$

This is different from $bb^{tr} g_{xx}$, but the traces are equal:

$$\text{trace } (b^{tr} g_{xx} b) = \text{ trace } (bb^{tr} g_{xx}) = \sum_{i,j} \frac{\partial^2 g}{\partial x_i \partial x_j} \underbrace{\sum_{k=1}^{m} b_{ik} b_{jk}}_{=: c_{ij}} \ .$$

See also [Øk98].

Exercise: Let X be vector and Y scalar, where $dX = a_1 dt + b_1 dW$, $dY = a_2 dt + b_2 dW$, and consider $g(X, Y) := XY$. Show

$$\begin{aligned} d(XY) &= Y dX + X dY + dX dY \\ &= (X a_2 + Y a_1 + b_1 b_2) dt + (X b_2 + Y b_1) dW \ . \end{aligned} \tag{B2.2}$$

Application:

$$dS = rSdt + \sigma S d\hat{W} \ \Rightarrow \ d(e^{-rt} S) = e^{-rt} \sigma S d\hat{W} \tag{B2.3}$$

for any Wiener process $\hat{W}$.

Filtration of a Brownian motion

$$\mathcal{F}_t^W := \sigma\{W_s \mid 0 \leq s \leq t\} \tag{B2.4}$$

Here $\sigma\{.\}$ denotes the smallest σ-algebra containing the sets put in braces. $\mathcal{F}_t^W$ is a model of the information available at time t, since it includes every event based on the history of W_s, $0 \leq s \leq t$. The null sets $\mathcal{N}$ are included in the sense $\mathcal{F}_t := \sigma(\mathcal{F}_t^W \cup \mathcal{N})$ ("augmented").

Conditional Expectation

We recall conditional expectation because it is required for martingales. Let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$.

$\mathsf{E}(X|\mathcal{G})$ is defined to be the (unique) $\mathcal{G}$-measurable random variable Y with the property

$$\mathsf{E}(XZ) = \mathsf{E}(YZ)$$

for all $\mathcal{G}$-measurable Z (such that $\mathsf{E}(XZ) < \infty$). This is the conditional expectation of X given $\mathcal{G}$. Or, following [Doob53], an equivalent definition is via

$$\int_A \mathsf{E}(Y|\mathcal{G}) d\mathsf{P} = \int_A Y d\mathsf{P} \quad \text{for all} \quad A \in \mathcal{G} \ .$$

In case $\mathsf{E}(X|Y)$, set $\mathcal{G} = \sigma(Y)$.

For properties of conditional expectation see, for example, [Mik98], [Shr04].

Martingales

Assume the standard scenario $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathsf{P})$ with a filtration $\mathcal{F}_t \subset \mathcal{F}$.

Definition: $\mathcal{F}_t$**-Martingale** M_t is a process, which is "adapted" (that is, $\mathcal{F}_t$-measurable), $\mathsf{E}(|M_t|) < \infty$, and

$$\mathsf{E}(M_t|\mathcal{F}_s) = M_s \quad (\mathsf{P}\text{-a.s.}) \text{ for } s \le t \ . \tag{B2.5}$$

The martingale property means that at time instant s with given information set $\mathcal{F}_s$ all variations of M_t for $t > s$ are unpredictable; M_s is the best forecast. The SDE of a martingale has no drift term.

Examples:

any Wiener process W_t ,

$W_t^2 - t$ for any Wiener process W_t ,

$\exp(\lambda W_t - \frac{1}{2}\lambda^2 t)$ for any $\lambda \in \mathbb{R}$ and any Wiener process W_t ,

$J_t - \lambda t$ for any Poisson process J_t with intensity λ.

For martingales, see for instance [Doob53], [Ne96], [Øk98], [Shi99], [Pro04], [Shr04].

For an adapted process γ define a process Z_t^γ by

$$Z_t^\gamma := \exp\left(-\frac{1}{2}\int_0^t \gamma_s^2 ds - \int_0^t \gamma_s dW_s\right) \ . \tag{B2.6}$$

Since $Z_0 = 1$, the integral equation

$$\log Z_t = \log Z_0 - \frac{1}{2}\int_0^t \gamma_s^2 ds - \int_0^t \gamma_s dW_s$$

follows, which is the SDE

$$d(\log Z_t) = (0 - \frac{1}{2}\gamma_t^2)dt - \gamma_t dW_t \ .$$

This is the Itô SDE for $\log Z_t$ when Z solves the drift-free $dZ_t = -Z_t\gamma_t dW_t$, $Z_0 = 1$. In summary, Z_t is the unique Itô process such that $dZ_t = -Z_t\gamma dW_t$, $Z_0 = 1$. Let Z^γ be a martingale. From the martingale properties, $\mathsf{E}(Z_T^\gamma) = 1$. Hence the Radon-Nikodym framework assures that an equivalent probability measure $\mathsf{Q}(\gamma)$ can be defined by

$$\frac{d\mathsf{Q}(\gamma)}{d\mathsf{P}} = Z_T^\gamma \quad \text{or} \quad \mathsf{Q}(A) := \int_A Z_T^\gamma d\mathsf{P} \tag{B2.7}$$

Girsanov's Theorem

Suppose a process γ is such that Z^γ is a martingale. Then

$$W_t^\gamma := W_t + \int_0^t \gamma_s ds \tag{B2.8}$$

is a Brownian motion and martingale under $\mathsf{Q}(\gamma)$.

B3 State-Price Process

Normalizing

A fundamental result of Harrison and Pliska [HP81] states that the existence of a martingale implies an arbitrage-free market. This motivates searching for a martingale. Since martingales have no drift term, we attempt to construct SDEs without drift.

Definition: A scalar positive Itô process Y_t with the property that the product Y_tX_t has zero drift is called **state-price process** or *pricing kernel* or *deflator* for X_t.

The importance of state-price processes is highlighted by the following theorem.

Theorem: Assume that for X_t a state-price process Y_t exists, b is self-financing, and $Yb^{trp}X$ is bounded below. Then
(a) $Yb^{trp}X$ is a martingale, and
(b) the market does not admit self-financing arbitrage strategies.
([Nie99], p.148)

Sketch of Proof:
(a) Y is a state-price process, hence there exists σ such that $d(Y_tX_t) = \sigma dW_t$ (zero drift). By Itô's lemma,

$$d(Yb^{tr}X) = Yd(b^{trp}X) + dYb^{tr}X + dYd(b^{tr}X) \ .$$

(B2.2) and self-financing imply

$$\begin{aligned} d(Yb^{tr}X) &= Yb^{tr}dX + dYb^{tr}X + dYb^{tr}dX \\ &= b^{tr}[YdX + dYX + dYdX] \\ &= b^{tr}d(XY) = b^{tr}\sigma dW =: \hat{\sigma}dW \ , \end{aligned}$$

hence zero drift of $Yb^{tr}X$.

It remains to show that $Yb^{tr}X$ is a martingale.
Because of the boundedness, $\tilde{Z} := Yb^{tr}X - c$ is a positive scalar Itô process for some c, with zero drift. For every such process there is a $\tilde{\gamma}$ such that $\tilde{Z}$ has the form

$$\tilde{Z}_t = \tilde{Z}_0 Z_t^{\tilde{\gamma}} \ .$$

Hence $Yb^{tr}X = \tilde{Z} + c$ has the same properties as $Z^{\tilde{\gamma}}$, namely it is a supermartingale. The final step is to show $\mathsf{E}(Z_t)$ =constant. Now Q is defined via (B2.7). (The last arguments are from martingale theory.)

(b) Assume arbitrage in the sense

$$\begin{aligned} &b_0^{tr}X_0 = 0 \ , \ \mathsf{P}(b_t^{tr}X_t \geq 0) = 1 \\ &\mathsf{P}(b_t^{tr}X_t > 0) > 0 \quad \text{for some fixed } t \ . \end{aligned}$$

For that t:

$$b^{tr}X > 0 \quad \Rightarrow \quad Yb^{tr}X > 0$$

Now $\mathsf{E}_{\mathsf{Q}}(Yb^{tr}X) > 0$ is intuitive. This amounts to

$$\mathsf{E}_{\mathsf{Q}}(Yb^{tr}X \mid \mathcal{F}_0) > 0$$

Because it is a martingale, $Y_0 b_0^{tr} X_0 > 0$. This contradicts $b_0^{tr}X_0 = 0$, so the market is free of arbitrage.

Existence of a State-Price Process

In order to discuss the existence of a state-price process we investigate the drift term of the product $Y_t X_t$. To this end take X as satisfying the vector SDE

$$dX = \mu^X dt + \sigma^X dW \ .$$

The coefficient functions μ^X and σ^X may vary with X. If no confusion arises, we drop the superscript X. Recall ($\longrightarrow$ Exercise 1.18) that each scalar positive Itô process must satisfy

$$dY = Y\alpha dt + Y\beta dW$$

for some α and β, where β and W can be vectors (β a one-row matrix). Without loss of generality, we take the SDE for Y in the form

$$dY = -rY dt - Y\gamma dW \ . \tag{B3.1}$$

(We leave the choice of the one-row matrix γ still open.) Itô's lemma (B2.1) allows to calculate the drift of YX. By (B2.2) the result is the vector

$$Y(\mu - rX - \sigma\gamma^{tr}) \ .$$

Hence Y is a state-price process for X if and only if

$$\mu^X - rX = \sigma^X \gamma^{tr} \tag{B3.2}$$

holds. This is a system of n equations for the m components of γ.

Special case geometric Brownian motion: For scalar $X = S$ and W, $\mu^X = \mu S$, $\sigma^X = \sigma S$, (B3.2) reduces to

$$\mu - r = \sigma\gamma \ .$$

Given μ, σ, r, the equation (B3.2) determines γ. (As explained in Section 1.7.3, γ is called the market price of risk.)

Discussion whether (B3.2) admits a (unique) solution:

Case I: unique solution: The market is complete. Further results below.

Case II: no solution: The market admits arbitrage.

Case III: multiple solutions: no arbitrage, but there are contingent claims that cannot be hedged; the market is said to be incomplete.

A solution of (B3.2) for full rank of the matrix σ is given by

$$\gamma^* := (\mu - rX)^{tr}(\sigma\sigma^{tr})^{-1}\sigma \ ,$$

which satisfies minimal length $\gamma^*\gamma^{*tr} \leq \gamma\gamma^{tr}$ for any other solution γ of (B3.2), see [Nie99].

Note that (B3.2) provides zero drift of YX but is not sufficient for YX to be a martingale. But it is "almost" a martingale; a small additional conditon suffices. Those trading strategies b are said to be *admissible* if $Yb^{tr}X$ is a martingale. (Sufficient is that $Yb^{tr}X$ be bounded below, such that it can not become arbitrarily negative. This rules out the "doubling strategy." For our purpose, we may consider the criterion as technical. [Gla04] on p.551: "It is common in applied work to assume that" a solution to an SDE with no drift term is a martingale.) There is ample literature on these topics; we just name [RY91], [BaR96], [Du96], [MR97], [Nie99].

Application: Derivative Pricing Formula for European Options

Let X_t be a vector price process, and b a self-financing trading strategy such that a European claim C is replicated. That is, for $V_t = b_t^{tr} X_t$ the payoff is reached: $V_T = b_T^{tr} X_T = C$. (Compare Appendix A4 for this argument.) We conclude from the above Theorem and from (B2.5)

$$Y_t b_t^{tr} X_t = \mathsf{E}_{\mathsf{Q}}(Y_T b_T^{tr} X_T \mid \mathcal{F}_t) \ ,$$

or

$$V_t = \frac{1}{Y_t} \mathsf{E}_{\mathsf{Q}}(Y_T C \mid \mathcal{F}_t) \ .$$

Specifically for $t = 0$ the relation $\mathsf{E}_{\mathsf{Q}}(Y_T C \mid \mathcal{F}_0) = \mathsf{E}_{\mathsf{Q}}(Y_T C)$ holds, see [HuK00] p.136. This gives the value of European options as

$$V_0 = \frac{1}{Y_0} \mathsf{E}_{\mathsf{Q}}(Y_T C) \ .$$

This result is basic for Monte Carlo simulation, compare Subsection 3.5.1. Y_t represents a discounting process, for example, e^{-rt}. (Other discounting processes are possible, as long as they are tradable. They are called *numeraires*.) For a variable interest rate r_s,

$$V_t = \mathsf{E}_{\mathsf{Q}}(\exp(-\int_t^T r_s ds) C \mid \mathcal{F}_t)$$

In the special case r and γ constant, $Z_t = \exp(-\frac{1}{2}\gamma^2 t - \gamma W_t)$ and

$$\frac{V(t)}{e^{rt}} = \mathsf{E}_{\mathsf{Q}} \left(\frac{C}{e^{rT}} \mid \mathcal{F}_t \right)$$
$$\Rightarrow V(t) = e^{-r(T-t)} \mathsf{E}_{\mathsf{Q}}(C \mid \mathcal{F}_t) \ .$$

Appendix C Numerical Methods

C1 Basic Numerical Tools

This appendix briefly describes numerical methods used in this text. For additional information and detailed discussion we refer to the literature, for example to [Sc89], [HH91], [PTVF92], [SB96], [GV96], [QSS00].

Interpolation

Suppose $n+1$ pairs of numbers $(x_i, y_i),\ i = 0, 1, ..., n$ are given. These points in the (x, y)-plane are to be connected by a curve. An interpolating function $\Phi(x)$ satisfies

$$\Phi(x_i) = y_i \quad \text{for} \quad i = 0, 1, ..., n.$$

Depending on the choice of the class of Φ we distinguish different types of interpolation. A prominent example is furnished by polynomials,

$$\Phi(x) = P_n(x) = a_0 + a_1 x + ... + a_n x^n;$$

the degree n matches the number $n+1$ of points. The evaluation of a polynomial is done by the *nested multiplication* given by

$$P_n(x) = (...((a_n x + a_{n-1})x + a_{n-2})x + ... + a_1)x + a_0,$$

which is also called *Horner's method.* In case many points are given, the interpolation with one polynomial is generally not advisable since the high degree goes along with strong oscillations. A piecewise approach is preferred where low-degree polynomials are defined locally on one or more subintervals $x_i \leq x \leq x_{i+1}$ such that globally certain smoothness requirements are met. The simplest example is obtained when the points (x_i, y_i) are joined by straight-line segments in the order $x_0 < x_1 < ... < x_n$. The resulting *polygon* is globally continuous and linear over each subinterval. For the error of polygon approximation of a function we refer to Lemma 5.9. A C^2-smooth interpolation is given by the cubic *spline* using locally defined third-degree polynomials

$$S_i(x) := a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \quad \text{for} \quad x_i \leq x < x_{i+1}$$

that interpolate the points and are $\mathcal{C}^2$-smooth at the nodes x_i.

Interpolation is applied for graphical illustration, numerical integration, and for solving differential equations. Generally interpolation is used to approximate functions.

Rational Approximation

Rational approximation is based on

$$\Phi(x) = \frac{a_0 + a_1 x + ... + a_n x^n}{b_0 + b_1 x + ... + b_m x^m}. \tag{C1.1}$$

Rational functions are advantageous in that they can approximate functions with poles. If the function that is to be approximated has a pole at $x = \xi$, then ξ must be zero of the denominator of Φ.

Integration

Approximating the definite integral

$$\int_a^b f(x)dx$$

is a classic problem of numerical analysis. Simple approaches replace the integral by

$$\int_a^b P_m(x)dx,$$

where the polynomial $P_m(x)$ approximates the function $f(x)$. The resulting formulas are called *quadrature* formulas. For example, an equidistant partition of the interval $[a, b]$ into m subintervals defines nodes x_i and support points $(x_i, f(x_i))$, $i = 0, \ldots, m$ for interpolation. After integrating the resulting polynomial $P_m(x)$ the *Newton-Cotes formulas* result. The simplest case $m = 1$ defines the *trapezoidal rule*.

A partition of the interval can be used more favorably. Applying the trapezoidal rule in each of n subintervals of length

$$h = \frac{b-a}{n}$$

leads to the composite formula of the *trapezoidal sum*

$$T(h) = h\left[\frac{f(a)}{2} + f(a+h) + ... + f(b-h) + \frac{f(b)}{2}\right]. \tag{C1.2}$$

The error of $T(h)$ satisfies a quadratic expansion

$$T(h) = \int_a^b f(x)dx + c_1 h^2 + c_2 h^4 + ...,$$

with a number of terms depending on the differentiability of f, and with constants c_i independent of h. This asymptotic expansion is fundamental for the high accuracy that can be achieved by *extrapolation*. Extrapolation evaluates $T(h)$ for a few h, for example, obtained by h_0, $h_1 = \frac{h_0}{2}$, $h_i = \frac{h_{i-1}}{2}$. Based on the values $T_i := T(h_i)$, an interpolating polynomial $\tilde{T}(h^2)$ is calculated with $\widetilde{T}(0)$ serving as approximation to the exact value $T(0)$ of the integral.

The error behavior reflected by the above expansion can be simplified to

$$|T(h) - \int_a^b f(x)dx| \leq ch^2 \ ,$$

or written even shorter with the Landau symbol:

The error is of the order $O(h^2)$.

Zeros of Functions

The aim is to calculate a zero x^* of a function $f(x)$. An approximation is constructed in an iterative manner. Starting from some suitable initial guess x_0 a sequence $x_1, x_2, \ldots$ is calculated such that the sequence converges to x^*. Newton's method calculates the iterates by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

In the vector case a system of linear equations needs to be solved in each step,

$$Df(x_k)(x_{k+1} - x_k) = -f(x_k), \tag{C1.3}$$

where Df denotes the Jacobian matrix of all first-order partial derivatives.

Example from Finance

Suppose a three-year bond with a principal of \$100 that pays a 6% coupon annually. Further assume zero rates of 5.8% for the first year, 6.3% for a two-year investment, and 6.4% for the three-year maturity. Then the *present value* (sum of all discounted future cashflows) is

$$6e^{-0.058} + 6e^{-0.063*2} + 106e^{-0.064*3} = 98.434$$

The *yield to maturity* (YTM) is the percentage rate of return y of the bond, when it is bought for the present value and is held to maturity. The YTM for the above example is the zero y of the cubic equation

$$0 = 98.434 - 6e^{-y} - 6e^{-2y} - 106e^{-3y}$$

which is 0.06384, or 6.384%, obtained with one iteration of Newton's method (C1.3), when started with 0.06 .

Convergence

Convergence is not guaranteed for any arbitrary choice of x_0. There are modifications and alternatives to Newton's method. Different methods are distinguished by their convergence speed. In the scalar case, *bisection* is a safe but slowly converging method. Newton's method for sufficiently regular problems shows fast convergence *locally.* That is, the error decays quadratically in a neighborhood of x^*,

$$\|x_{k+1} - x^*\| \le C\|x_k - x^*\|^p \quad \text{for } p = 2$$

for some constant C. This holds for an arbitrary vector norm $\|x\|$ such as

$$\begin{aligned} \|x\|_2 &:= \left(\sum_i x_i^2\right)^{1/2} && \textit{(Euclidian norm)} \\ \|x\|_\infty &:= \max_i |x_i| && \textit{(maximum norm)}, \end{aligned} \tag{C1.4}$$

$i = 1, \dots, n$ for $x \in \mathbb{R}^n$.

The derivative $f'(x_k)$ can be approximated by difference quotients. If the difference quotient is based on $f(x_k)$ and $f(x_{k-1})$, in the scalar case, the *secant method* results. The secant method is generally faster than Newton's method if the speed is measured with respect to costs in evaluating $f(x)$ or $f'(x)$.

Gerschgorin's Theorem

A criterion for localizing the eigenvalues of a matrix $A = (a_{ij})$ is given by Gerschgorin's theorem: Each eigenvalue lies in the union of the discs

$$\mathcal{D}_j := \{z \;\text{ complex and }\; |z - a_{jj}| \le \sum_{\substack{k=1 \\ k \ne j}}^{n} |a_{jk}|\}$$

$(j = 1, ..., n)$. The centers of the discs $\mathcal{D}_j$ are the diagonal elements of A and the radii are given by the off-diagonal row sums (absolute values).

Triangular Decomposition

Let L denote a lower-triangular matrix (where the elements l_{ij} satisfy $l_{ij} = 0$ for $i < j$) and R an upper-triangular matrix ($r_{ij} = 0$ for $i > j$); the diagonal elements of L satisfy $l_{11} = ... = l_{nn} = 1$. Matrices $A,\ L,\ R$ are supposed to be of size $n \times n$ and vectors $x,\ b, ...$ have n components. Frequently, numerical methods must solve one or more systems of linear equations

$$Ax = b.$$

A well-known direct method to solve this system is Gaussian elimination. First, in a "forward"-phase, an equivalent system

$$Rx = \widehat{b}$$

is calculated. Then, in a "backward"-phase starting with the last component x_n, all components of x are calculated one by one in the order $x_n, x_{n-1}, \dots, x_1$. Gaussian elimination requires $\frac{2}{3}n^3 + O(n^2)$ arithmetic operations for full matrices A. With this count of $O(n^3)$, Gaussian elimination must be considered as an expensive endeavor, and is prohibitive for large values of n. (For alternatives, see iterative methods below in Appendix A5.) The forward phase of Gaussian elimination is equivalent to an *LR-decomposition*. This means the factorization into the product of two triangular matrices L, R in the form

$$PA = LR.$$

Here P is a permutation matrix arranging for the exchange of rows that corresponds to the pivoting of the Gaussian algorithm. The LR-decomposition exists for all nonsingular A. After the LR-decomposition is calculated, only two equations with triangular matrices need to be solved,

$$Ly = Pb \quad \text{and} \quad Rx = y.$$

Tridiagonal Matrices

For tridiagonal matrices the LR-decomposition specializes to an algorithm that requires only $O(n)$ operations, which is inexpensive. Since several of the matrices in this book are tridiagonal, we include the algorithm. Let the tridiagonal system $Ax = b$ be in the form

$$\begin{pmatrix} \alpha_1 & \beta_1 & & & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma_{n-1} & \alpha_{n-1} & \beta_{n-1} \\ 0 & & & \gamma_n & \alpha_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} \tag{C1.5}$$

Starting the Gaussian elimination with the first row to produce zeros in the subdiagonal during a forward loop, the algorithm is as follows:

$$\left| \begin{array}{l} \hat{\alpha}_1 := \alpha_1, \ \hat{b}_1 := b_1 \\ \text{(forward loop)} \quad \text{for } i = 2, \dots, n: \\ \qquad \hat{\alpha}_i = \alpha_i - \beta_{i-1} \dfrac{\gamma_i}{\hat{\alpha}_{i-1}}, \quad \hat{b}_i = b_i - \hat{b}_{i-1} \dfrac{\gamma_i}{\hat{\alpha}_{i-1}} \\ x_n := \dfrac{\hat{b}_n}{\hat{\alpha}_n} \\ \text{(backward loop)} \quad \text{for } i = n-1, \dots, 1: \\ \qquad x_i = \dfrac{1}{\hat{\alpha}_i} (\hat{b}_i - \beta_i x_{i+1}) \end{array} \right| \tag{C1.6}$$

Here the "new" elements of the equivalent triangular system are indicated with a "hat;" the necessary checks for nonsingularity ($\hat{\alpha}_{i-1} \neq 0$) are omitted.

The algorithm (C1.6) needs about $8n$ operations. If one would start Gaussian elimination from the last row and produces zeros in the superdiagonal, an RL-decomposition results. The reader may wish to formulate the related backward/forward algorithm as an exercise.

Cholesky Decomposition

For *positive-definite* matrices A (means symmetric or Hermitian and $x^H Ax > 0$ for all $x \neq 0$) there is exactly one lower-triangular matrix L with positive diagonal elements such that

$$A = LL^H .$$

Here a normalization of the diagonal elements of L is not required. For real matrices A also L is real, hence $A = LL^{tr}$. (Hint: The Hermitian matrix A^H of A is defined as $\bar{A}^{tr}$, where $\bar{A}$ means elementwise complex conjugate.) For a computer program of Cholesky decomposition see [PTVF92].

C2 Iterative Methods for $Ax = b$

The system of linear equations $Ax = b$ in $\mathbb{R}^n$ can be written

$$Mx = (M - A)x + b,$$

where M is a suitable matrix. For nonsingular M the system $Ax = b$ is equivalent to the fixed-point equation

$$x = (I - M^{-1}A)x + M^{-1}b,$$

which leads to the iteration

$$x^{(k+1)} = (\underbrace{I - M^{-1}A}_{=:B})x^{(k)} + M^{-1}b. \tag{C2.1}$$

The computation of $x^{(k+1)}$ is done by solving the system of equations $Mx^{(k+1)} = (M - A)x^{(k)} + b$. Subtracting the fixed-point equation and applying Lemma 4.2 shows

$$\text{convergence} \quad \Longleftrightarrow \rho(B) < 1;$$

$\rho(B)$ is the spectral radius of matrix B. For this convergence criterion there is a sufficient criterion that is easy to check. Natural matrix norms satisfy $\|B\| \geq \rho(B)$. Hence $\|B\| < 1$ implies convergence. Application to the matrix norms

$$\|B\|_\infty = \max_i \sum_{j=1}^{n} |b_{ij}|,$$

$$\|B\|_1 = \max_j \sum_{i=1}^{n} |b_{ij}|,$$

produces sufficient convergence criteria: The iteration converges if

$$\sum_{j=1}^{n} |b_{ij}| < 1 \quad \text{for } 1 \leq i \leq n$$

or if

$$\sum_{i=1}^{n} |b_{ij}| < 1 \quad \text{for } 1 \leq j \leq n.$$

By obvious reasons these criteria are called row sum criterion and column sum criterion. The *preconditioner* matrix M is constructed such that rapid convergence of (C2.1) is achieved. Further, the structure of M must be simple so that the linear system is easily solved for $x^{(k+1)}$.

Simple examples are obtained by additive splitting of A into the form $A = D - L - U$, with

D diagonal matrix
L strict lower-triangular matrix
U strict upper-triangular matrix

Jacobi's Method

Choosing $M := D$ implies $M - A = L + U$ and establishes the iteration

$$Dx^{(k+1)} = (L + U)x^{(k)} + b.$$

By the above convergence criteria a strict diagonal dominance of A is sufficient for the convergence of Jacobi's method.

Gauß–Seidel Method

Here the choice is $M := D - L$. This leads via $M - A = U$ to the iteration

$$(D - L)x^{(k+1)} = Ux^{(k)} + b.$$

SOR (Successive Overrelaxation)

The SOR method can be seen as a modification of the Gauß-Seidel method, where a *relaxation parameter* ω_{R} is introduced and chosen in a way that speeds up the convergence:

$$M := \frac{1}{\omega_{\text{R}}} D - L \implies M - A = \left(\frac{1}{\omega_{\text{R}}} - 1 \right) D + U$$

$$\left(\frac{1}{\omega_{\text{R}}} D - L \right) x^{(k+1)} = \left(\left(\frac{1}{\omega_{\text{R}}} - 1 \right) D + U \right) x^{(k)} + b$$

The SOR-method can be written as follows:

$$\begin{cases} B_{\mathrm{R}} := \left(\frac{1}{\omega_{\mathrm{R}}} D - L\right)^{-1} \left(\left(\frac{1}{\omega_{\mathrm{R}}} - 1\right) D + U\right) \\ x^{(k+1)} = B_{\mathrm{R}} x^{(k)} + \left(\frac{1}{\omega_{\mathrm{R}}} D - L\right)^{-1} b \end{cases}$$

The Gauß–Seidel method is obtained as special case for $\omega_{\mathrm{R}} = 1$.

Choosing ω_{R}

The difference vectors $d^{(k+1)} := x^{(k+1)} - x^{(k)}$ satisfiy

$$d^{(k+1)} = B_{\mathrm{R}} d^{(k)}. \tag{$*$}$$

This is the power method for eigenvalue problems. Hence the $d^{(k)}$ converge to the eigenvector of the dominant eigenvalue $\rho(B_{\mathrm{R}})$. Consequently, if $(*)$ converges then

$$d^{(k+1)} = B_{\mathrm{R}} d^{(k)} \approx \rho(B_{\mathrm{R}}) d^{(k)}.$$

Then $|\rho(B_{\mathrm{R}})| \approx \frac{\|d^{(k+1)}\|}{\|d^{(k)}\|}$ for arbitrary vector norms. There is a class of matrices A with

$$\rho(B_{\mathrm{GS}}) = (\rho(B_{\mathrm{J}}))^2, \; B_{\mathrm{J}} := D^{-1}(L+U)$$

$$\omega_{\mathrm{opt}} = \frac{2}{1 + \sqrt{1 - \rho(B_{\mathrm{J}})^2}},$$

see [Va62], [SB96]. Here B_{J} denotes the iteration matrix of the Jacobi method and B_{GS} that of the Gauß-Seidel method. For matrices A of that kind a few iterations with $\omega_{\mathrm{R}} = 1$ suffice to estimate the value $\rho(B_{\mathrm{GS}})$, which in turn gives an approximation to ω_{opt}. With our experience with Cryer's projected SOR applied to the valuation of options (Section 4.6) the simple strategy $\omega_{\mathrm{R}} = 1$ is frequently recommendable.

This appendix has merely introduced classic iterative solvers, which are stationary in the sense that the preconditioner matrix M does not vary with k. For an overview on advanced nonstationary iterative methods see [Ba94].

C3 Function Spaces

Let real-valued functions u, v, w be defined on $\mathcal{D} \subseteq \mathbb{R}^n$. We assume that $\mathcal{D}$ is a *domain*. That is, $\mathcal{D}$ is open, bounded and connected. The space of continuous functions is denoted $\mathcal{C}^0(\mathcal{D})$ or $\mathcal{C}(\mathcal{D})$. The functions in $\mathcal{C}^k(\mathcal{D})$ are k times continuously differentiable: All partial derivatives up to order k exist and are continuous on $\mathcal{D}$. The sets $\mathcal{C}^k(\mathcal{D})$ are examples of function spaces. Functions in $\mathcal{C}^k(\bar{\mathcal{D}})$ have in addition bounded and uniformly continuous derivatives and consequently can be extended to $\bar{\mathcal{D}}$.

Apart from being distinguished by differentiability, functions are also characterized by their integrability. The proper type of integral is the Lebesgue integral. The space of square-integrable functions is

$$\mathcal{L}^2(\mathcal{D}) := \left\{ v : \int_{\mathcal{D}} v^2 \, dx < \infty \right\}. \tag{C3.1}$$

For example, $v(x) = x^{-1/4} \in \mathcal{L}^2(0,1)$ but $v(x) = x^{-1/2} \notin \mathcal{L}^2(0,1)$. More general, for $p > 0$ the $\mathcal{L}^p$-spaces are defined by

$$\mathcal{L}^p(\mathcal{D}) := \left\{ v : \int_{\mathcal{D}} |v(x)|^p \, dx < \infty \right\}.$$

For $p \geq 1$ these spaces have several important properties [Ad75]. For example,

$$\|v\|_p := \left(\int_{\mathcal{D}} |v(x)|^p dx \right)^{1/p} \tag{C3.2}$$

is a norm.

In order to establish the existence of integrals such as

$$\int_a^b uv \, dx, \quad \int_a^b u'v' \, dx$$

we might be tempted to use a simple approach, defining a function space

$$\mathcal{H}^1(a,b) := \left\{ u \in \mathcal{L}^2(a,b) : \ u' \in \mathcal{L}^2(a,b) \right\}, \tag{C3.3}$$

with $\mathcal{D} = (a,b)$. But a classical derivative u' may not exist for $u \in \mathcal{L}^2$ or needs not be square integrable. What is needed is a weaker notion of derivative.

Weak Derivatives

In $\mathcal{C}^k$-spaces classical derivatives are defined in the usual way. For $\mathcal{L}^2$-spaces *weak derivatives* are defined. For motivation let us review standard integration by parts

$$\int_a^b uv' \, dx = -\int_a^b u'v \, dx, \tag{C3.4}$$

which is correct for all $u, v \in \mathcal{C}^1(a,b)$ with $v(a) = v(b) = 0$. For $u \notin \mathcal{C}^1$ the equation (C3.4) can be used to define a weak derivative u' provided smoothness is transferred to v. For this purpose define

$$\mathcal{C}_0^\infty(\mathcal{D}) := \{ v \in \mathcal{C}^\infty(\mathcal{D}) : \ \text{supp}(v) \text{ is a compact subset of } \mathcal{D} \}.$$

$v \in \mathcal{C}_0^\infty(\mathcal{D})$ implies $v = 0$ at the boundary of $\mathcal{D}$. For $\mathcal{D} \subseteq \mathbb{R}^n$ one uses the multi-index notation

$$\alpha := (\alpha_1, ..., \alpha_n), \quad \alpha_i \in \mathbb{N} \cup \{0\}$$

with

$$|\alpha| := \sum_{i=1}^{n} \alpha_i .$$

Then the partial derivative of order $|\alpha|$ is defined as

$$D^\alpha v := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} v(x_1, ..., x_n).$$

If a $w \in \mathcal{L}^2$ exists with

$$\int_{\mathcal{D}} u D^\alpha v \; dx = (-1)^{|\alpha|} \int_{\mathcal{D}} wv \; dx \quad \text{for all} \;\; v \in \mathcal{C}_0^\infty(\mathcal{D}),$$

the weak derivative of u with multindex α is defined by $D^\alpha u := w$.

Sobolev Spaces

The definition (C3.3) is meaningful if u' is considered as weak derivative in the above sense. More general, one defines the *Sobolev spaces*

$$\mathcal{H}^k(\mathcal{D}) := \left\{ v \in \mathcal{L}^2(\mathcal{D}) : \; D^\alpha v \in \mathcal{L}^2(\mathcal{D}) \quad \text{für} \;\; |\alpha| \le k \right\} . \tag{C3.5}$$

The index $_0$ specifies the subspace of $\mathcal{H}^1$ that consists of those functions that vanish at the boundary of $\mathcal{D}$. For example,

$$\mathcal{H}_0^1(a, b) := \left\{ v \in \mathcal{H}^1(a, b) : \; v(a) = v(b) = 0 \right\} .$$

The Sobolev spaces $\mathcal{H}^k$ are equipped with the norm

$$\|v\|_k := \left(\sum_{|\alpha| \le k} \int_{\mathcal{D}} |D^\alpha v|^2 \; dx \right)^{1/2} , \tag{C3.6}$$

which is the sum of $\mathcal{L}^2$-norms of (C3.2). For the special case discussed in Chapter 5 with $k = 1$, $n = 1$, $\mathcal{D} = (a, b)$, the norm is

$$\|v\|_1 := \left(\int_a^b (v^2 + (v')^2) \; dx \right)^{1/2} .$$

Embedding theorems state which function spaces are subsets of other function spaces. In this way, elements of Sobolev spaces can be characterized and distinguished with respect to smoothness and integrability. For instance, the space $\mathcal{H}^1$ includes those functions that are globally continuous on all of $\mathcal{D}$ and its boundary and are *piecewise* $\mathcal{C}^1$-functions.

Hilbert Spaces

The function spaces $\mathcal{L}^2$ and $\mathcal{H}^k$ have numerous properties. Here we just mention that both spaces are *Hilbert spaces.* Hilbert spaces have an inner product

(,) such that the space is *complete* with respect to the norm $\|v\| := \sqrt{(v,v)}$. In complete spaces every Cauchy sequence converges. In Hilbert spaces the *Schwarzian inequality*

$$|(u,v)| \le \|u\| \; \|v\| \tag{C3.7}$$

holds. Examples of Hilbert spaces and their inner products are

$$\mathcal{L}^2(\mathcal{D}) \text{ with } (u,v)_0 := \int_{\mathcal{D}} u(x)v(x)\; dx$$

$$\mathcal{H}^k(\mathcal{D}) \text{ with } (u,v)_k := \sum_{|\alpha|\le k} (D^\alpha u, D^\alpha v)_0$$

For further discussion of function spaces we refer, for instance, to [Ad75], [KA64], [Ha92], [Wl87].

Appendix D Complementary Material

This appendix lists useful formula without further explanation.

D1 Bounds for Options

The following bounds can be derived based on arbitrage arguments, see [Mer73], [CR85], [In87], [Kwok98], [Hull00]. If neither the subscript $_\text{C}$ nor $_\text{P}$ is listed, the inequality holds for both put and call. If neither the $^\text{eur}$ nor the $^\text{am}$ is listed, the inequality holds for both American and European options. We always assume $r > 0$.

a) Bounds valid for both American and European options, no matter whether dividends are paid or not:

$$\begin{array}{ccccc} 0 & \leq & V_\text{C}(S_t, t) & \leq & S_t \\ 0 & \leq & V_\text{P}(S_t, t) & \leq & K \end{array}$$

$$\begin{aligned} V^\text{eur}(S_t, t) &\leq V^\text{am}(S_t, t) \\ S_t - K &\leq V_\text{C}^\text{am}(S_t, t) \\ K - S_t &\leq V_\text{P}^\text{am}(S_t, t) \\ V_\text{P}^\text{eur}(S_t, t) &\leq K e^{-r(T-t)} \end{aligned}$$

b) Bounds valid provided that no dividend is paid for $0 \leq t \leq T$:

$$\begin{aligned} S_t - K e^{-r(T-t)} &\leq V_\text{C}^\text{eur}(S_t, t) \\ K e^{-r(T-t)} - S_t &\leq V_\text{P}^\text{eur}(S_t, t) \end{aligned}$$

$V^\text{am} \geq V^\text{eur}$ allows to improve the lower the bound on V_C^am to

$$S_t - K e^{-r(T-t)} \leq V_\text{C}^\text{am}(S_t, t) \; .$$

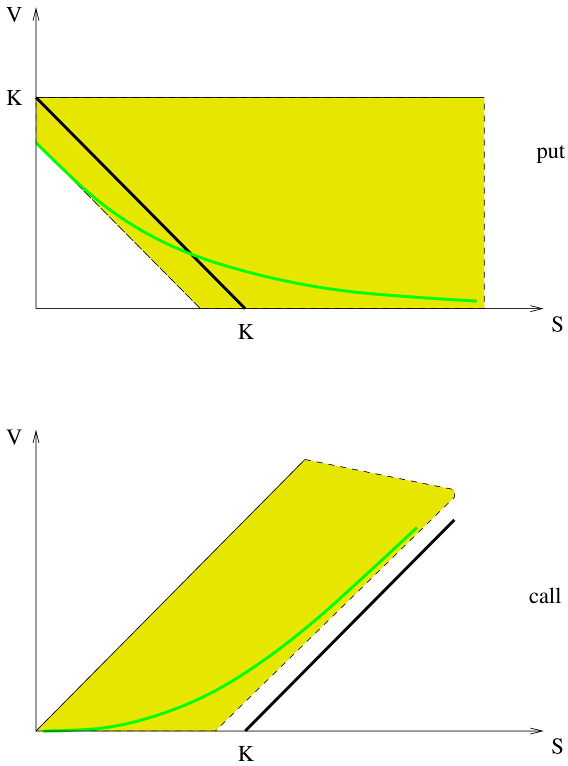

Fig. D.1. Bounding curves for the value of put and call options

c) Monotonicity:
Monotonicity with respect to S:

$$\begin{array}{ll} V_{\mathrm{C}}(S_1,t) < V_{\mathrm{C}}(S_2,t) & \text{for } S_1 < S_2 \\ V_{\mathrm{P}}(S_1,t) > V_{\mathrm{P}}(S_2,t) & \text{for } S_1 < S_2, \end{array}$$

which implies

$$\frac{\partial V_{\mathrm{C}}}{\partial S} > 0 \ , \ \frac{\partial V_{\mathrm{P}}}{\partial S} < 0 \ .$$

Monotonicity of American options with respect to time:

$$\begin{array}{ll} V_{\mathrm{C}}^{\mathrm{am}}(S,t_1) \geq V_{\mathrm{C}}^{\mathrm{am}}(S,t_2) & \text{for } t_1 < t_2 \\ V_{\mathrm{P}}^{\mathrm{am}}(S,t_1) \geq V_{\mathrm{P}}^{\mathrm{am}}(S,t_2) & \text{for } t_1 < t_2, \end{array}$$

which implies

$$\frac{\partial V^{\mathrm{am}}}{\partial t} \leq 0 \ .$$

Options are convex with respect to K and with respect to S.

To express monotonicity with respect to the strike K or to the time to expiration T, we indicate dependencies by writing $V(S,t;T,K)$, where we only quote the parameter that is changed.

$$\begin{array}{lcll} V^{\text{am}}(\,.\,;T_1) & \le & V^{\text{am}}(\,.\,;T_2) & \text{for } T_1 < T_2 \\ V_{\text{C}}(\,.\,;K_1) & \ge & V_{\text{C}}(\,.\,;K_2) & \text{for } K_1 < K_2 \\ V_{\text{P}}(\,.\,;K_1) & \le & V_{\text{P}}(\,.\,;K_2) & \text{for } K_1 < K_2 \end{array}$$

d) Put-call parity relation for American options:

$$Ke^{-r(T-t)} + V_{\text{C}}^{\text{am}}(S,t) \le S + V_{\text{P}}^{\text{am}}(S,t).$$

This holds no matter whether dividends are paid or not. If the asset pays no dividends, then also the upper bound

$$S + V_{\text{P}}^{\text{am}}(S,t) - V_{\text{C}}^{\text{am}}(S,t) \le K$$

holds.

D2 Approximation Formula

Distribution Function of the Standard Normal Distribution

$$f(x) := \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$F(x) := \int_{-\infty}^{x} f(t)\, dt$$

Let us define

$$z := \frac{1}{1 + 0.2316419x}$$

and the coefficients

$$\begin{array}{ll} a_1 = 0.319381530 & a_4 = -1.821255978 \\ a_2 = -0.356563782 & a_5 = 1.330274429 \\ a_3 = 1.781477937. & \end{array}$$

Then

$$F(x) = 1 - f(x)\left(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5\right) + \varepsilon(x)\ ,$$

for $0 \le x < \infty$ with an absolute error ε bounded by

$$|\varepsilon(x)| < 7.5 * 10^{-8}$$

(see [AS68]). Hence we have the approximating formula

$$F(x) \approx 1 - f(x)z((((a_5 z + a_4)z + a_3)z + a_2)z + a_1) \ ,$$

which requires 17 arithmetic operations and the evaluation of the exponential function to obtain an accuracy of about 7 decimals. For $x < 0$ apply $F(x) = 1 - F(-x)$. Higher accuracy can be achieved with quadrature methods ($\longrightarrow$ Exercise 1.3).

Inversion Formula

A FORTRAN code for the inversion of the normal distribution can be found in

`http://lib.stat.cmu.edu/apstat/111.`

(Many other codes relevant for statistical computation can be obtained via the `.../apstat` page.) Here we report the formula of [Moro95] to approximate the inverse function of the standard normal distribution

$$F(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt \ .$$

That is, we calculate $x = G(u)$ such that $G(u) \approx F^{-1}(u)$. The interval $0 < u < 1$ is truncated to $10^{-12} \leq u \leq 1 - 10^{-12}$. Symmetry with respect to $(x, u) = (0, 0.5)$ is exploited. The interval is subdivided into two relevant parts, namely

$$0.08 < u < 0.92 \quad \text{and} \quad 0.92 \leq u \leq 1 - 10^{-12} \ .$$

The part $10^{-12} \leq u \leq 0.08$ is obtained by symmetry. For each of the two subintervals an appropriate approximation is given. In the middle part of the interval a rational approximation in the form

$$(u - 0.5) \frac{\sum_{j=0}^{3} a_j (u - 0.5)^{2j}}{1 + \sum_{j=0}^{3} b_j (u - 0.5)^{2j}}$$

is used, whereas the tails are approximated by a polynomial in $\log(-\log r)$, where $10^{-12} \leq r \leq 0.08$.

Algorithm (inversion of the standard normal distribution)

input: u, drawn from $\mathcal{U}(0,1)$
$y := u - 0.5$
in case $|y| < 0.42$:
  $r := y^2$
  $x := y \frac{((a_3 r + a_2) r + a_1) r + a_0}{(((b_3 r + b_2) r + b_1) r + b_0) r + 1}$
in case $|y| \geq 0.42$:
  $r := u$, in case $y > 0$ set $r := 1 - u$
  $r := \log(-\log r)$
  $x := c_0 + r(c_1 + r(c_2 + r(c_3 + r(c_4 + r(c_5 + r(c_6 + r(c_7 + rc_8)))))))$
  in case $y < 0$ set $x := -x$
output: x

The coefficients of the above algorithm are given by[3]

$a_0 = 2.50662823884,$
$a_1 = -18.61500062529,$
$a_2 = 41.39119773534,$
$a_3 = -25.44106049637$

$b_0 = -8.47351093090,$
$b_1 = 23.08336743743,$
$b_2 = -21.06224101826,$
$b_3 = 3.13082909833$

$c_0 = 0.3374754822726147,$
$c_1 = 0.9761690190917186,$
$c_2 = 0.1607979714918209,$
$c_3 = 0.0276438810333863,$
$c_4 = 0.0038405729373609,$
$c_5 = 0.0003951896511919,$
$c_6 = 0.0000321767881768,$
$c_7 = 0.0000002888167364,$
$c_8 = 0.0000003960315187$

The rational approximation formula for $|y| < 0.42$ (that is, $0.08 < u < 0.92$) is reported to have a largest absolute error of $3 \cdot 10^{-9}$.

D3 Software

A dedicated computer person will program the mathematics such that the resulting codes run with utmost possible speed. Such a person will probably use compilers like C, C++, or FORTRAN to create production codes, where the speed counts.

[3] These digits are listed in [Moro95].

But there are packages available that make programming, implementing, testing, and graphics more comfortable. For example, MATLAB offers a platform for scientific computation and numerical experiments. This software package is widely used to develop algorithms, and is assisted by good graphics.

Programs related to finance have been published in the literature. For MATLAB codes see [Hig04], for MATHEMATICA codes see [Sto03]. For elementary computations, spreadsheets are also used.

For partial differential equations, the finite-element program PDE2D is recommended. This package is available via the University of Texas, El Paso. The PREMIA project offers codes via `www-rocq.inria.fr/mathfi` . For further hints and test algorithms see the platform `www.compfin.de` .

References

[AS68] M. Abramowitz, I. Stegun: Handbook of Mathematical Functions. With Formulas, Graphs, and Mathematical Tables. Dover Publications, New York (1968).

[Ad75] R.A. Adams: Sobolev Spaces. Academic Press, New York (1975).

[AiC97] F. AitSahlia, P. Carr: American options: A comparison of numerical methods. In [RT97] (1997) p. 67-87.

[AnA00] L. Andersen, J. Andreasen: Jump diffusion process: Volatility smile fitting and numerical methods for option pricing. Review Derivatives Research **4** (2000) 231-262.

[AB97] L.B.G. Andersen, R. Brotherton-Ratcliffe: The equity option volatility smile: an implicit finite-difference approach. J. Computational Finance **1**,2 (1997/1998) 5–38.

[AnéG00] T. Ané, H. Geman: Order flow, transaction clock, and normality of asset returns. J. of Finance **55** (2000) 2259-2284.

[Ar74] L. Arnold: Stochastic Differential Equations (Theory and Applications). Wiley, New York (1974).

[ArDEH99] P. Artzner, F. Delbaen, J.-M. Eber, D. Heath: Coherent measures of risk. Math. Finance **9** (1999) 203-228.

[BaS01] I. Babuška, T. Strouboulis: The Finite Element Method and its Reliability. Oxford Science Publications, Oxford (2001).

[BaR94] C.A. Ball, A. Roma: Stochastic volatility option pricing. J. Financial Quantitative Analysis **29** (1994) 589-607.

[Bar97] G. Barles: Convergence of numerical schemes for degenerate parabolic equations arising in finance theory. in [RT97] (1997) 1-21.

[BaN97] O.E. Barndorff-Nielsen: Processes of normal inverse Gaussian type. Finance & Stochastics **2** (1997) 41–68.

[BaW87] G. Barone-Adesi, R.E. Whaley: Efficient analytic approximation of American option values. J. Finance **42** (1987) 301-320.

[BaP96] J. Barraquand, T. Pudet: Pricing of American path-dependent contingent claims. Mathematical Finance **6** (1996) 17–51.

[Ba94] R. Barrett et al.: Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods. SIAM, Philadelphia (1994).

[BaR96] M. Baxter, A. Rennie: Financial Calculus. An Introduction to Derivative Pricing. Cambridge University Press, Cambridge (1996).

[Beh00] E. Behrends: Introduction to Markov Chains. Vieweg, Braunschweig (2000).

[Ben84] A. Bensoussan: On the theory of option pricing. Acta Applicandae Math. **2** (1984) 139-158.

[Bi79] P. Billingsley: Probability and Measure. John Wiley, New York (1979).

[BV00] G.I. Bischi, V. Valori: Nonlinear effects in a discrete-time dynamic model of a stock market. Chaos, Solitons and Fractals **11** (2000) 2103-2121.

[BS73] F. Black, M. Scholes: The pricing of options and corporate liabilities. J. Political Economy **81** (1973) 637–659.

[Blo86] E.C. Blomeyer: An analytic approximation for the American put price for options with dividends. J. Financial Quantitative Analysis **21** (1986) 229-233.

[BP00] J.-P. Bouchaud, M. Potters: Theory of Financial Risks. From Statistical Physics to Risk Management. Cambridge Univ. Press, Cambridge (2000).

[Bo98] N. Bouleau: Martingales et Marchés Financiers. Edition Odile Jacob (1998).

[BoM58] G.E.P. Box, M.E. Muller: A note on the generation of random normal deviates. Annals Math.Statistics **29** (1958) 610-611.

[BBG97] P. Boyle, M. Broadie, P. Glasserman: Monte Carlo methods for security pricing. J. Economic Dynamics and Control **21** (1997) 1267-1321.

[BTT00] M.-E. Brachet, E. Taflin, J.M. Tcheou: Scaling transformation and probability distributions for time series. Chaos, Solitons and Fractals **11** (2000) 2343-2348.

[Br91] R. Breen: The accelerated binomial option pricing model. J. Financial and Quantitative Analysis **26** (1991) 153–164.

[BrS77] M. Brennan, E. Schwartz: The valuation of American put options. J. of Finance **32** (1977) 449-462.

[BrS02] S.C. Brenner, L.R. Scott: The Mathematical Theory of Finite Element Methods. Second Edition. Springer, New York (2002).

[Br94] R.P. Brent: On the periods of generalized Fibonacci recurrences. Math. Comput. **63** (1994) 389–401.

[BrD97] M. Broadie, J. Detemple: Recent advances in numerical methods for pricing derivative securities. in [RT97] (1997) 43-66.

[BrG97] M. Broadie, P. Glasserman: Pricing American-style securities using simulation. J. Economic Dynamics and Control **21** (1997) 1323-1352.

[BrG04] M. Broadie, P. Glasserman: A stochastic mesh method for pricing high-dimensional American options. J. Computational Finance **7**,4 (2004) 35-72.

[BH98] W.A. Brock, C.H. Hommes: Heterogeneous beliefs and routes to chaos in a simple asset pricing model. J. Economic Dynamics and Control **22** (1998) 1235–1274.

[BuJ92] D.S. Bunch, H. Johnson: A simple and numerically efficient valuation method for American puts using a modified Geske-Johnson approach. J. Finance **47** (1992) 809-816.

[CaMO97] R.E. Caflisch, W. Morokoff, A. Owen: Valuation of mortgaged-backed securities using Brownian bridges to reduce effective dimension. J. Computational Finance **1**,1 (1997) 27-46.

[CaF95] P. Carr, D. Faguet: Fast accurate valuation of American options. Working paper, Cornell University (1995).

[CaM99] P. Carr, D.B. Madan: Option valuation using the fast Fourier transform. J. Computational Finance **2**,4 (1999) 61-73.

[Cash84] J.R. Cash: Two new finite difference schemes for parabolic equations. SIAM J.Numer.Anal. **21** (1984) 433-446.

[CDG00] C. Chiarella, R. Dieci, L. Gardini: Speculative behaviour and complex asset price dynamics. Proceedings Urbino 2000, Ed.: G.I. Bischi (2000).

[CW83] K.L. Chung, R.J. Williams: Introduction to Stochastic Integration. Birkhäuser, Boston (1983).

[Ci91] P.G. Ciarlet: Basic Error Estimates for Elliptic Problems. in: Handbook of Numerical Analysis, Vol. II (Eds. P.G. Ciarlet, J.L. Lions) Elsevier/North-Holland, Amsterdam (1991) 19–351.

[CL90] P. Ciarlet, J.L. Lions: Finite Difference Methods (Part 1) Solution of equations in $\mathbb{R}^n$. North-Holland Elsevier, Amsterdam (1990).

[CoLV02] T.F. Coleman, Y. Li, Y. Verma: A Newton method for American option pricing. J. Computational Finance **5**,3 (2002) 51-78.

[ConT04] R. Cont, P. Tankov: Financial Modelling with Jump Processes. Chapman & Hall, Boca Raton (2004).

[CRR79] J.C. Cox, S. Ross, M. Rubinstein: Option pricing: A simplified approach. Journal of Financial Economics **7** (1979) 229–264.

[CR85] J.C. Cox, M. Rubinstein: Options Markets. Prentice Hall, Englewood Cliffs (1985).

[Cra84] J. Crank: Free and Moving Boundary Problems. Clarendon Press, Oxford (1984).

[CN47] J.C. Crank, P. Nicolson: A practical method for numerical evaluation of solutions of partial differential equations of the heat-conductive type. Proc. Cambr. Phil. Soc. **43** (1947) 50–67.

[Cr71] C. Cryer: The solution of a quadratic programming problem using systematic overrelaxation. SIAM J. Control **9** (1971) 385–392.

[CKO01] S. Cyganowski, P. Kloeden, J. Ombach: From Elementary Probability to Stochastic Differential Equations with MAPLE. Springer (2001).

[Dai00] M. Dai: A closed-form solution for perpetual American floating strike lookback options. J. Computational Finance **4**,2 (2000) 63-68.

[DaJ03] R.-A. Dana, M. Jeanblanc: Financial Markets in Continuous Time. Springer, Berlin (2003).

[DH99] M.A.H. Dempster, J.P. Hutton: Pricing American stock options by linear programming. Mathematical Finance **9** (1999) 229–254.

[DeHR98] M.A.H. Dempster, J.P. Hutton, D.G. Richards: LP valuation of exotic American options exploiting structure. J. Computational Finance **2**,1 (1998) 61-84.

[Deu02] H.-P. Deutsch: Derivatives and Internal Models. Palgrave, Houndmills (2002).

[Dev86] L. Devroye: Non–Uniform Random Variate Generation. Springer, New York (1986).

[DBG01] R. Dieci, G.-I. Bischi, L. Gardini: From bi-stability to chaotic oscillations in a macroeconomic model. Chaos, Solitons and Fractals **12** (2001) 805-822.

[Doob53] J.L. Doob: Stochastic Processes. John Wiley, New York (1953).

[Dowd98] K. Dowd: Beyond Value at Risk: The New Science of Risk Management. Wiley & Sons, Chichester (1998).

[Du96] D. Duffie: Dynamic Asset Pricing Theory. Second Edition. Princeton University Press, Princeton (1996).

[EK95] E. Eberlein, U. Keller: Hyperbolic distributions in finance. Bernoulli **1** (1995) 281–299.

[EO82] C.M. Elliott, J.R. Ockendon: Weak and Variational Methods for Moving Boundary Problems. Pitman, Boston (1982).

[ElK99] R.J. Elliott, P.E. Kopp: Mathematics of Financial Markets. Springer, New York (1999).

[EKM97] P. Embrechts, C. Klüppelberg, T. Mikosch: Modelling Extremal Events. Springer, Berlin (1997).

[Epps00] T.W. Epps: Pricing Derivative Securities. World Scientific, Singapore (2000).

[Fe50] W. Feller: An Introduction to Probability Theory and its Applications. Wiley, New York (1950).

[Fi96] G.S. Fishman: Monte Carlo. Concepts, Algorithms, and Applications. Springer, New York (1996).

[Fisz63] M. Fisz: Probability Theory and Mathematical Statistics. John Wiley, New York (1963).

[FöS02] H. Föllmer, A. Schied: Stochastic Finance: An Introduction to Discrete Time. de Gruyter, Berlin (2002).

[FV02] P.A. Forsyth, K.R. Vetzal: Quadratic convergence of a penalty method for valuing American options. SIAM J. Sci. Comp. **23** (2002) 2095-2122.

[FVZ99] P.A. Forsyth, K.R. Vetzal, R. Zvan: A finite element approach to the pricing of discrete lookbacks with stochastic volatility. Applied Math. Finance **6** (1999) 87–106.

[FoLLLT99] E. Fournié, J.-M. Lasry, J. Lebuchoux, P.-L. Lions, N. Touzi: An application of Malliavin calculus to Monte Carlo methods in finance. Finance & Stochastics **3** (1999) 391-412.

[FHH04] J. Franke, W. Härdle, C.M. Hafner: Statistics of Financial Markets. Springer, Berlin (2004).

[Fr71] D. Freedman: Brownian Motion and Diffusion. Holden Day, San Francisco (1971).

[Fu01] M.C. Fu (et al): Pricing American options: a comparison of Monte Carlo simulation approaches. J. Computational Finance **4**,3 (2001) 39-88.

[FuST02] G. Fusai, S. Sanfelici, A. Tagliani: Practical problems in the numerical solution of PDEs in finance. Rend. Studi Econ. Quant. 2001 (2002) 105-132.

[Gem00] H. Geman et al. (Eds.): Mathematical Finance. Bachelier Congress 2000. Springer, Berlin (2002).

[Ge98] J.E. Gentle: Random Number Generation and Monte Carlo Methods. Springer, New York (1998)

[GeJ84] R. Geske, H.E. Johnson: The American put option valued analytically. J.Finance **39** (1984) 1511-1524.

[GeG98] T. Gerstner, M. Griebel: Numerical integration using sparse grids. Numer. Algorithms **18** (1998) 209-232.

[GeG03] T. Gerstner, M. Griebel: Dimension-adaptive tensor-product quadrature. Computing **70**,4 (2003).

[GiRS96] W.R. Gilks, S. Richardson, D.J. Spiegelhalter (Eds.): Markov Chain Monte Carlo in Practice. Chapman & Hall, Boca Raton (1996).

[Gla04] P. Glasserman: Monte Carlo Methods in Financial Engineering. Springer, New York (2004).

[GV96] G. H. Golub, C. F. Van Loan: Matrix Computations. Third Edition. The John Hopkins University Press, Baltimore (1996).

[GK01] L. Grüne, P.E. Kloeden: Pathwise approximation of random ODEs. BIT **41** (2001) 710-721.

[Ha85] W. Hackbusch: Multi-Grid Methods and Applications. Springer, Berlin (1985).

[Ha92] W. Hackbusch: Elliptic Differential Equations: Theory and Numerical Treatment. Springer Series in Computational Mathematics **18**, Berlin, Springer (1992).

[Häg02] O. Häggström: Finite Markov Chains and Algorithmic Applications. Cambridge University Press, Cambridge (2002).

[HNW93] E. Hairer, S.P. Nørsett, G. Wanner: Solving Ordinary Differential Equations I. Nonstiff Problems. Springer, Berlin (1993).

[Ha60] J.H. Halton: On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals. Numer. Math. **2** (1960) 84–90.

[HH64] J.M. Hammersley, D.C. Handscomb: Monte Carlo Methods. Methuen, London (1964).

[HH91] G. Hämmerlin, K.-H. Hoffmann: Numerical Mathematics. Springer, Berlin (1991).

[HP81] J.M. Harrison, S.R. Pliska: Martingales and stochastic integrals in the theory of continuous trading. Stoch. Processes and their Applications **11** (1981) 215-260.

[Hei05] P. Heider: A condition number for the integral representation of American options. Report Mathem. Inst., Univ. Köln (2005).

[Hig01] D.J. Higham: An algorithmic introduction to numerical solution of stochastic differential equations. SIAM Review **43** (2001) 525-546.

[Hig04] D.J. Higham: An Introduction to Financial Option Valuation. Cambridge, Univ. Press, Cambridge (2004).

[HiMS04] N. Hilber, A.-M. Matache, C. Schwab: Sparse Wavelet Methods for Option Pricing under Stochastic Volatility. Report, ETH-Zürch (2004).

[HPS92] N. Hofmann, E. Platen, M. Schweizer: Option pricing under incompleteness and stochastic volatility. Mathem. Finance **2** (1992) 153–187.

[HoP02] P. Honoré, R. Poulsen: Option pricing with EXCEL. in [Nie02].

[Hull00] J.C. Hull: Options, Futures, and Other Derivatives. Fourth Edition. Prentice Hall International Editions, Upper Saddle River (2000).

[HuK00] P.J. Hunt, J.E. Kennedy: Financial Derivatives in Theory and Practice. Wiley, Chichester (2000).

[IkT04] S. Ikonen, J. Toivanen: Pricing American options using LU decomposition. Report Univ. Jyvaskyla (2004).

[In87] J.E. Ingersoll: Theory of Financial Decision Making. Rowmann and Littlefield, Savage (1987).

[Int05] R. Int-Veen: Avoiding numerical dispersion in option valuation. Computing and Visualization in Science. to appear (2005).

[Irle98] A. Irle: Finanzmathematik — Die Bewertung von Derivaten. Teubner, Stuttgart 1998.

[IK66] E. Isaacson, H.B. Keller: Analysis of Numerical Methods. John Wiley, New York (1966).

[JaP03] J. Jacod, P. Protter: Probability Essentials. Second Edition. Springer, Berlin (2003).

[Jam92] F. Jamshidian: An analysis of American options. Review of Futures Markets **11** (1992) 72-80.

[JiD04] L. Jiang, M. Dai: Convergence of binomial tree method for European/American path-dependent options. SIAM J. Numerical Analysis **42** (2004) 1094-1109.

[Joh83] H.E. Johnson: An analytic approximation for the American put price. J. Financial Quantitative Analysis **18** (1983) 141-148.

[KMN89] D. Kahaner, C. Moler, S. Nash: Numerical Methods and Software. Prentice Hall Series in Computational Mathematics, Englewood Cliffs (1989).

[KaN00] R. Kangro, R. Nicolaides: Far field boundary conditions for Black-Scholes equations. SIAM J.Numer.Anal. **38** (2000) 1357-1368.

[KA64] L.W. Kantorovich, G.P. Akilov: Functional Analysis in Normed Spaces. Pergamon Press, Elmsford (1964).

[KS91] I. Karatzas, S.E. Shreve: Brownian Motion and Stochastic Calculus. Second Edition. Springer Graduate Texts, New York (1991).

[KS98] I. Karatzas, S.E. Shreve: Methods of Mathematical Finance. Springer, New York (1998).

[Kat95] H.M. Kat: Pricing Lookback options using binomial trees: An evaluation. J. Financial Engineering **4** (1995) 375–397.

[Kl01] T.R. Klassen: Simple, fast and flexible pricing of Asian options. J. Computational Finance **4**,3 (2001) 89-124.

[KP92] P.E. Kloeden, D. Platen: Numerical Solution of Stochastic Differential Equations. Springer, Berlin (1992).

[KPS94] P.E. Kloeden, E. Platen, H. Schurz: Numerical Solution of SDE Through Computer Experiments. Springer, Berlin (1994).

[Kn95] D. Knuth: The Art of Computer Programming, Vol 2. Addison–Wiley, Reading (1995).

[Korn01] R. Korn, E. Korn: Option Pricing and Portfolio Optimization. American Mathem.Soc., Providence 2001.

[Kr97] D. Kröner: Numerical Schemes for Conservation Laws. Wiley Teubner, Chichester (1997).

[Kwok98] Y.K. Kwok: Mathematical Models of Financial Derivatives. Springer, Singapore (1998).

[La91] J.D. Lambert: Numerical Methods for Ordinary Differential Systems. The Initial Value Problem. John Wiley, Chichester (1991).

[LL96] D. Lamberton, B. Lapeyre: Introduction to Stochastic Calculus Applied to Finance. Chapman & Hall, London (1996).

[La99] K. Lange: Numerical Analysis for Statisticians. Springer, New York (1999).

[LEc99] P. L'Ecuyer: Tables of linear congruential generators of different sizes and good lattice structure. Mathematics of Computation **68** (1999) 249-260.

[Lehn02] J. Lehn: Random Number Generators. GAMM-Mitteilungen **25** (2002) 35-45.

[LonS01] F.A. Longstaff, E.S. Schwartz: Valuing American options by simulation: a simple least-squares approach. Review Financial Studies **14** (2001) 113-147.

[Los01] C.A. Los: Computational Finance: A Scientific Perspective. World Scientific, Singapore (2001).

[Lux98] T. Lux: The socio-economic dynamics of speculative markets: interacting agents, chaos, and the fat tails of return distributions. J. Economic Behavior & Organization **33** (1998) 143-165.

[Lyuu02] Y.-D. Lyuu: Financial Engineering and Computation. Principles, Mathematics, Algorithms. Cambridge University Press, Cambridge (2002).

[MaM86] L.W. MacMillan: Analytic approximation for the American put option. Advances in Futures and Options Research **1** (1986) 119-139.

[MRGS00] R. Mainardi, M. Roberto, R. Gorenflo, E. Scalas: Fractional calculus and continuous-time finance II: the waiting-time distribution. Physica A **287** (2000) 468-481.

[Man99] B.B. Mandelbrot: A multifractal walk down Wall Street. Scientific American, Febr. 1999, 50–53.

[MCFR00] M. Marchesi, S. Cinotti, S. Focardi, M. Raberto: Development and testing of an artificial stock market. Proceedings Urbino 2000, Ed. I.-G. Bischi (2000).

[Mar78] W. Margrabe: The value of an option to exchange one asset for another. J. Finance **33** (1978) 177-186.
[Ma68] G. Marsaglia: Random numbers fall mainly in the planes. Proc. Nat. Acad. Sci. USA **61** (1968) 23–28.
[MaB64] G. Marsaglia, T.A. Bray: A convenient method for generating normal variables. SIAM Rev. **6** (1964) 260-264.
[Mas99] M. Mascagni: Parallel pseudorandom number generation. SIAM News **32**, 5 (1999).
[MaPS02] A.-M. Matache, T. von Petersdorff, C. Schwab: Fast deterministic pricing of options on Lévy driven assets. Report 2002-11, Seminar for Applied Mathematics, ETH Zürich (2002).
[MaN98] M. Matsumoto, T. Nishimura: Mersenne Twister: A 623-dimensionally equidistributed uniform pseudorandom number generator. ACM Transactions on Modeling and Computer Simulations **8** (1998) 3-30.
[Mayo00] A. Mayo: Fourth order accurate implicit finite difference method for evaluating American options. Proceedings of Computational Finance, London (2000).
[McW01] L.A. McCarthy, N.J. Webber: Pricing in three-factor models using icosahedral lattices. J. Computational Finance **5**,2 (2001/02) 1-33.
[MeVN02] A.V. Mel'nikov, S.N. Volkov, M.L. Nechaev: Mathematics of Financial Obligations. Amer. Math. Soc., Providence (2002).
[Mer73] R.C. Merton: Theory of rational option pricing. Bell J. Economics and Management Science **4** (1973) 141-183.
[Mer76] R. Merton: Option pricing when underlying stock returns are discontinous. J. Financial Economics **3** (1976) 125-144.
[Me90] R.C. Merton: Continuous-Time Finance. Blackwell, Cambridge (1990).
[Mik98] T. Mikosch: Elementary Stochastic Calculus, with Finance in View. World Scientific, Singapore (1998).
[Mi74] G.N. Mil'shtein: Approximate integration of stochastic differential equations. Theory Prob. Appl. **19** (1974) 557–562.
[Moe76] P. van Moerbeke: On optimal stopping and free boundary problems. Archive Rat.Mech.Anal. **60** (1976) 101-148.
[Moro95] B. Moro: The full Monte. Risk **8** (1995) 57–58.
[MC94] W.J. Morokoff, R.E. Caflisch: Quasi-random sequences and their discrepancies. SIAM J. Sci. Comput. **15** (1994) 1251–1279.
[Mo98] W.J. Morokoff: Generating quasi-random paths for stochastic processes. SIAM Review **40** (1998) 765–788.
[Mo96] K.W. Morton: Numerical Solution of Convection-Diffusion Problems. Chapman & Hall, London (1996).
[MR97] M. Musiela, M. Rutkowski: Martingale Methods in Financial Modelling. (Second Edition 2005) Springer, Berlin (1997).
[Ne96] S.N. Neftci: An Introduction to the Mathematics of Financial Dervatives. Academic Press, San Diego (1996).
[New97] N.J. Newton: Continuous-time Monte Carlo methods and variance reduction. in [RT97] (1997) 22-42.
[Ni78] H. Niederreiter: Quasi-Monte Carlo methods and pseudo-random numbers. Bull. Am. Math. Soc. **84** (1978) 957–1041.
[Ni92] H. Niederreiter: Random Number Generation and Quasi–Monte Carlo Methods. Society for Industrial and Applied Mathematics, Philadelphia (1992).

[Ni95] H. Niederreiter, P. Jau–Shyong Shiue (Eds.): Monte Carlo and Quasi–Monte Carlo Methods in Scientific Computing. Proceedings of a Conference at the University of Nevada, Las Vegas, Nevada, USA, June 23–25, 1994. Springer, New York (1995).

[NiST02] B.F. Nielsen, O. Skavhaug, A. Tveito: Penalty and front-fixing methods for the numrical solution of American option problems. J. Computational Finance **5**,4 (2002) 69-97.

[Nie99] L.T. Nielsen: Pricing and Hedging of Derivative Securities. Oxford University Press, Oxford (1999).

[Nie02] S. Nielsen (Ed.): Programming Languages and Systems in Computational Economics and Finance. Kluwer, Amsterdam (2002).

[Øk98] B. Øksendal: Stochastic Differential Equations. Springer, Berlin (1998).

[Oo03] C.W. Oosterlee: On multigrid for linear complementarity problems with application to American-style options. Electronic Transactions on Numerical Analysis **15** (2003) 165-185.

[PT96] A. Papageorgiou, J.F. Traub: New results on deterministic pricing of financial derivatives. Columbia University Report CUCS-028-96 (1996).

[PT95] S. Paskov, J. Traub: Faster valuation of financial derivatives. J. Portfolio Management **22** (1995) 113–120.

[PT83] R. Peyret, T.D. Taylor: Computational Methods for Fluid Flow. Springer, New York (1983).

[Pl99] E. Platen: An introduction to numerical methods for stochastic differential equations. Acta Numerica (1999) 197-246.

[Pli97] S.R. Pliska: Introduction to Mathematical Finance. Discrete Time Models. Blackwell, Malden (1997).

[PFVS00] D. Pooley, P.A. Forsyth, K. Vetzal, R.B. Simpson: Unstructured meshing for two asset barrier options. Appl. Mathematical Finance **7** (2000) 33-60.

[PTVF92] W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery: Numerical Recipes in FORTRAN. The Art of Scientific Computing. Second Edition. Cambridge University Press, Cambridge (1992).

[Pro04] P.E. Protter: Stochastic Integration and Differential Equations. Springer, Berlin (2004).

[QSS00] A. Quarteroni, R. Sacco, F. Saleri: Numerical Mathematics. Springer, New York (2000).

[Ran84] R. Rannacher: Finite element solution of diffusion problems with irregular data. Numer. Math. **43** (1984) 309-327.

[Re96] R. Rebonato: Interest-Rate Option Models: Understanding, Analysing and Using Models for Exotic Interest-Rate Options. John Wiley & Sons, Chichester (1996).

[RY91] D. Revuz, M. Yor: Continuous Martingales and Brownian Motion. Springer, Berlin 1991.

[RiW02] C. Ribeiro, N. Webber: Valuing path dependent options in the Variance-Gamma model by Monte Carlo with a gamma bridge. Working paper, City University, London (2002).

[RiW03] C. Ribeiro, N. Webber: A Monte Carlo method for the normal inverse Gaussian option valuation model using an inverse Gaussian bridge. J. Computational Finance **7**,2 (2003/04) 81-100.

[Ri87] B.D. Ripley: Stochastic Simulation. Wiley Series in Probability and Mathematical Statistics, New York (1987).

[Ro00] L.C.G. Rogers: Monte Carlo valuation of American options. Manuscript, University of Bath (2000).

[RT97] L.C.G. Rogers, D. Talay (Eds.): Numerical Methods in Finance. Cambridge University Press, Cambridge (1997).
[Ru94] M. Rubinstein: Implied binomial trees. J. Finance **69** (1994) 771-818.
[Ru81] R.Y. Rubinstein: Simulation and the Monte Carlo Method. Wiley, New York (1981).
[Rup04] D. Ruppert: Statistics and Finance. An Introduction. Springer, New York (2004).
[SM96] Y. Saito, T. Mitsui: Stability analysis of numerical schemes for stochastic differential equations. SIAM J. Numer. Anal. **33** (1996) 2254–2267.
[Sa01] K. Sandmann: Einführung in die Stochastik der Finanzmärkte. Second edition. Springer, Berlin (2001).
[Sato99] K.-I. Sato: Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, Cambridge (1999).
[SH97] J.G.M. Schoenmakers, A.W. Heemink: Fast Valuation of Financial Derivatives. J. Comp. Finance **1** (1997) 47–62.
[Sc80] Z. Schuss: Theory and Applications of Stochastic Differential Equations. Wiley Series in Probability and Mathematical Statistics, New York (1980).
[Sc89] H.R. Schwarz: Numerical Analysis. John Wiley & Sons, Chichester (1989).
[Sc91] H.R. Schwarz: Methode der finiten Elemente. Teubner, Stuttgart (1991).
[Se94] R. Seydel: Practical Bifurcation and Stability Analysis. From Equilibrium to Chaos. Second Edition. Springer Interdisciplinary Applied Mathematics Vol. 5, New York (1994).
[Shi99] A.N. Shiryaev: Essentials of Stochastic Finance. Facts, Models, Theory. World Scientific, Singapore (1999).
[Shr04] S.E. Shreve: Stochastic Calculus for Finance. Springer, New York (2004).
[Sm78] G.D. Smith: Numerical Solution of Partial Differential Equations: Finite Difference Methods. Second Edition. Clarendon Press, Oxford (1978).
[Smi97] C. Smithson: Multifactor options. Risk **10**,5 (1997) 43-45.
[SM94] J. Spanier, E.H. Maize: Quasi-random methods for estimating integrals using relatively small samples. SIAM Review **36** (1994) 18–44.
[Sta01] D. Stauffer: Percolation models of financial market dynamics. Advances in Complex Systems **4** (2001) 19–27.
[Ste01] J.M. Steele: Stochastic Calculus and Financial Applications. Springer, New York (2001).
[SWH99] M. Steiner, M. Wallmeier, R. Hafner: Baumverfahren zur Bewertung diskreter Knock-Out-Optionen. OR Spektrum **21** (1999) 147–181.
[SB96] J. Stoer, R. Bulirsch: Introduction to Numerical Analysis. Springer, Berlin (1996).
[SW70] J. Stoer, C. Witzgall: Convexity and Optimization in Finite Dimensions I. Springer, Berlin (1970).
[Sto03] S. Stojanovic: Computational Financial Mathematics using MATHEMATICA. Birkhäuser, Boston (2003).
[St86] G. Strang: Introduction to Applied Mathematics. Wellesley, Cambridge (1986).
[SF73] G. Strang, G. Fix: An Analysis of the Finite Element Method. Prentice-Hall, Englewood Cliffs (1973).
[Sw84] P.K. Sweby: High resolution schemes using flux limiters for hyperbolic conservation laws. SIAM J. Numer. Anal. **21** (1984) 995–1011.

[TR00] D. Tavella, C. Randall: Pricing Financial Instruments. The Finite Difference Method. John Wiley, New York (2000).

[Te95] S. Tezuka: Uniform Random Numbers: Theory and Practice. Kluwer Academic Publishers, Dordrecht (1995).

[Th95] J.W. Thomas: Numerical Partial Differential Equations: Finite Difference Methods. Springer, New York (1995).

[Th99] J.W. Thomas: Numerical Partial Differential Equations. Conservation Laws and Elliptic Equations. Springer, New York (1999).

[Top00] J. Topper: Finite element modeling of exotic options. OR Proceedings (2000) 336–341.

[Top05] J. Topper: Financial Engineering with Finite Elements. Wiley (2005).

[TW92] J.F. Traub, H. Wozniakowski: The Monte Carlo algorithm with a pseudo-random generator. Math. Computation **58** (1992) 323–339.

[TOS01] U. Trottenberg, C. Oosterlee, A. Schüller: Multigrid. Academic Press, San Diego (2001).

[Tsay02] R.S. Tsay: Analysis of Financial Time Series. Wiley, New York (2002).

[Va62] R.S. Varga: Matrix Iterative Analysis. Prentice Hall, Englewood Cliffs (1962).

[Vi81] R. Vichnevetsky: Computer Methods for Partial Differential Equations. Volume I. Prentice-Hall, Englewood Cliffs (1981).

[We01] P. Wesseling: Principles of Computational Fluid Dynamics. Springer, Berlin (2001).

[Wi98] P. Wilmott: Derivatives. John Wiley, Chichester (1998).

[WDH96] P. Wilmott, J. Dewynne, S. Howison: Option Pricing. Mathematical Models and Computation. Oxford Financial Press, Oxford (1996).

[Wl87] J. Wloka: Partial Differential Equations. Cambridge University Press, Cambridge (1987).

[Zag02] R. Zagst: Interest-Rate Management. Springer, Berlin (2002).

[Zi77] O.C. Zienkiewicz: The Finite Element Method in Engineering Science. McGraw-Hill, London (1977).

[ZhWC04] Y.-I. Zhu, X. Wu, I.-L. Chern: Derivative Securities and Difference Methods. Springer (2004).

[ZFV98] R. Zvan, P.A. Forsyth, K.R. Vetzal: Robust numerical methods for PDE models of Asian options. J. Computational Finance **1**,2 (1997/98) 39–78.

[ZvFV98a] R. Zvan, P.A. Forsyth, K.R. Vetzal: Penalty methods for American options with stochastic volatility. J. Comp. Appl. Math. **91** (1998) 199-218.

[ZFV99] R. Zvan, P.A. Forsyth, K.R. Vetzal: Discrete Asian barrier options. J. Computational Finance **3**,1 (1999) 41–67.

[ZVF00] R. Zvan, K.R. Vetzal, P.A. Forsyth: PDE methods for pricing barrier options. J. Econ. Dynamics & Control **24** (2000) 1563-1590.

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