

# List of Figures

- 1.1 The Weber triangle ..... 7
- 1.2 Transportation cost surface ..... 8
- 1.3 Isodapans for transportation cost ..... 9
- 1.4 A sort of Varignon machine ..... 10
- 1.5 Forces in balance ..... 11
- 1.6 Pick's construction ..... 12
- 1.7 Angles in the weight triangle ..... 13
- 1.8 Change of prelocated point with no resulting change in optimal location ..... 15
- 1.9 Profit surface over the location triangle ..... 20
- 1.10 Contours of constant profits as dependent on location ..... 21
- 1.11 Radial routes and circular price contours ..... 26
- 1.12 Hyperbolic routes and price contours ..... 27
- 1.13 Profit landscape with four location optima ..... 29
  
- 2.1 Arc length element ..... 32
- 2.2 Central field of logspiral paths ..... 40
- 2.3 Another central field ..... 44
- 2.4 Fermat's Principle ..... 48
- 2.5 Caustic formed by refraction of traffic ..... 49
- 2.6 Stereographic projection ..... 54
- 2.7 Flow lines and isovectors ..... 55
- 2.8 Lengths of arc and secant ..... 56
- 2.9 Coordinate transformation ..... 60
- 2.10 The change of area ..... 61
  
- 3.1 Flow along hyperbolas with tangent field vectors ..... 68
- 3.2 Line integral and Green's Theorem ..... 72
- 3.3 Constructive method for finding price lines ..... 90

3.4	Price contours and orthogonal trajectories .....	91
3.5	Price surface and gradient lines .....	92
4.1	Market areas according to Launhardt .....	102
4.2	Launhardt's "funnels" .....	103
4.3	Market areas with equal freight rates .....	104
4.4	Construction for boundary curve .....	108
4.5	Triangular tessellation .....	109
4.6	Quadratic tessellation .....	110
4.7	Hexagonal tessellation .....	111
4.8	Rhombic dodecahedron .....	113
4.9	Nested triangles .....	115
4.10	Nested squares .....	116
4.11	Nested hexagons .....	117
4.12	Nested hexagons, areal scaling 3 .....	118
4.13	Nested hexagons, areal scaling 4 .....	119
4.14	Nested hexagons, areal scaling 7 .....	120
4.15	Nested deformed hexagons, areal scaling 7 .....	121
4.16	Monopolistic price policies .....	133
4.17	Local price variation over the region around the $i$ :th firm .....	136
4.18	Global price variation with two firms on the interval $[-1, 1]$ ...	143
4.19	Price undercutting and relocation by the first firm .....	151
4.20	Profit curves and competition regimes .....	152
4.21	Market areas and price structure in rectangular lattice of firms and a diagonal Manhattan metric .....	155
4.22	Market area of the central firm at different mill prices below and above the equilibrium price .....	158
4.23	Market area of the central firm when it changes location horizontally to the right .....	164
4.24	Market area of the central firm when it changes location horizontally to the right and infinitesimally up .....	165
4.25	Cost functions for public utility .....	169
4.26	Dependence of market radius on transportation cost .....	172
4.27	Circular market areas in a hexagonal lattice .....	180
4.28	Circular market areas in a square lattice .....	181
4.29	Hexagonal tessellation of market areas .....	184
4.30	Square tessellation of market areas .....	185
5.1	Von Thünen rings in land use .....	190
5.2	Land rent pattern in von Thünen's theory .....	193

5.3	Topological equivalents: saddle-node flow .....	205
5.4	Splitting a monkey saddle in two saddles .....	206
5.5	Splitting a heteroclinic saddle connection .....	207
5.6	The basic structurally stable grid .....	209
5.7	Flow fitted to the square grid .....	210
5.8	Square land rent landscape .....	211
5.9	Removing saddle-node pairs of singularities .....	214
5.10	Transforming to hexagonal / triangular grid .....	214
5.11	Triangular tessellation element .....	215
5.12	Square tessellation element .....	216
5.13	Hexagonal tessellation element .....	217
5.14	The elliptic umblic catastrophe .....	220
5.15	The degenerate global hexagonal pattern .....	221
5.16	First global bifurcation .....	222
5.17	Second global bifurcation .....	223
5.18	The hexagonal / triangular rent landscape .....	224
5.19	Fractal land rent landscape .....	226
6.1	The segments of a chord .....	234
6.2	Traffic distribution on a circular disk .....	236
6.3	Logistic growth functions .....	253
6.4	The solution curves to Hotelling's equation .....	256
6.5	Evolution curves for population $p$ over time $t$ .....	262
6.6	Phase space for population $p$ (horizontal) and its space derivative $\partial p / \partial x$ (vertical) .....	264
6.7	Stack of solution curves population $p$ (vertical) as functions of space variable $x$ (horizontal), with $K$ increasing from 0 to 0.5 (foreground to background) .....	265
6.8	Travelling population wave front function at various time points .....	268
7.1	Development of choppy spatial growth .....	276
7.2	Stationary solution to spatial growth model .....	277
7.3	Node lines associated with one single eigenvalue .....	284
7.4	Node lines by mixing modes for an eigenvalue .....	285
8.1	Manhattan metric .....	295
8.2	Minkowski metrics .....	296
8.3	Metric for triangular grid .....	298
8.4	Metric for hexagonal grid .....	299

8.5	Selection of ring-radial routes .....	301
8.6	Ring-radial distance metric .....	302
8.7	Combined transportation modes, airports and surface (Euclidean) .....	303
8.8	Combined transportation modes, airports and surface Manhattan) .....	304
8.9	Combined transportation modes, 25 airports .....	305
8.10	Combined highway and pedestrian paths .....	306
8.11	Finite mesh highways and pedestrian routes .....	307
8.12	Larger cost difference in transportation mode .....	308
8.13	Feeding lines to square network .....	312
8.14	Feeding lines to triangular network .....	313
8.15	Stable triangular grid .....	315
8.16	Euclidean metric of a real network .....	317
8.17	Space filling Peano curve .....	319
8.18	Radial roads and feeding sectors .....	321
8.19	Construction of the optimal collection lines .....	323
8.20	Total transportation cost as dependent on the number of radials, for different freight rates, and approximations .....	326
8.21	Bifurcation of a radial road segment .....	329
8.22	Bifurcation of a radial road segment .....	330
8.23	Curved bifurcation of radial .....	337
8.24	Accessibility as volume according to Euclidean Metric .....	340
8.25	Accessibility as volume according to Manhattan Metric .....	341

# Author Index

## A

Angel, S. 66, 231, 242, 269  
Arnol'd, V. I. 203, 204

## B

Beckmann, M. J. 30, 86, 88, 89,  
93, 99, 122, 123, 126, 127,  
132, 163, 186, 189, 198,  
199, 225, 227, 231, 269,  
271, 286, 292, 293  
Bessel, F. W. 285  
Boltzmann, L. 245  
Bos, H. C. 122  
Braudel, F. 173, 187, 344

## C

Christaller, W. 103, 105, 112, 115,  
119, 120, 121, 187, 190  
Cournot, A. 134, 135, 151, 187

## E

Edgeworth, F. Y. 134  
von Euler, L. 34, 37, 38, 40, 43,  
46, 47, 48, 49, 54, 57, 58,  
59, 65, 66, 80, 83, 84, 85,  
86, 87, 88, 107, 131, 336

## F

de Fermat, P. 46, 49, 293  
Foureier, J. 225, 226, 275, 277, 278,  
283  
Frisch, R. 167, 168, 187

## G

Gauss, C. F. 51, 75, 76, 82, 95,  
98, 254, 275, 290  
Green, G. 74, 75

## H

Harrod, R. 271, 278, 292  
Heckscher, E. F. 343  
Hicks, J. R. 271, 279  
Hotelling, H. 134, 135, 136, 139,  
142, 146, 147, 153, 154,  
187, 252, 254, 255, 257,  
259, 261, 266, 269  
Hyman, G. M. 66, 231, 242, 269

## J

Jacobi, C. G. J. 62

## K

Keynes, J. M. 271

**L**

- Lagrange, J. L. 58, 59, 65, 86, 87,  
88, 107, 248, 250, 279  
Laplace, P. S. 78, 93, 95, 254, 272,  
274  
Launhardt, W. 5, 17, 30, 101, 103,  
154, 156, 164, 165, 187, 190,  
300  
Legendre, A.-M. 37, 285  
Leontief, W. 7, 17  
Lösch, A. 101, 103, 104, 105, 112,  
114, 115, 119, 120, 187,  
190, 222

**M**

- Malthus, T. R. 252, 257  
Mandelbrot, B. B. 318, 342  
Maxwell, J. C. 245  
Minkowski, H. 297, 299  
Mosler, K. C. 66, 300, 328, 335,  
342

**O**

- Ohlin, B. 343

**P**

- Palander, T. F. 30, 49, 50, 66, 311  
Peano, G. 320  
Peixoto, M. M. 208  
Phillips 271, 279, 280, 292  
Pick, G. 11, 12, 13  
Plateau, J. A. F. 83  
Poincaré, H. 105, 314, 315  
Ponsard, C. 99

**S**

- Samuelson, P. A. 88, 99, 204, 227,  
271, 279, 286, 292  
Sierpinski, W. 118

Skellam, J. G. 252

Smale, S. 208

von Stackelberg, H. 49, 50, 66, 151,  
311

**T**

- von Thünen, J. H. 189, 190, 192,  
193, 194, 197, 199, 212,  
213, 224, 225, 227, 343  
Tinbergen, J. 122

**V**

- Wardrop, J. G. 51, 66  
Varignon, P. 11, 50  
Vaughan, R. 269  
Weber, A. 5, 8, 13, 16, 17, 21,  
23, 30, 50, 190  
Wilson, A. G. 245, 247, 251, 269

**Z**

- Zipf, G. K. 229, 230, 236, 269

# Subject Index

## A

accessibility 309, 311, 312, 313  
adjustment speed 279, 287  
arc length 32, 34, 39, 45, 53, 58,  
67, 88  
arcs 293, 294, 331  
autonomous expenditures 277  
average variable cost 168

## B

balanced regional growth 277  
balancing factors 249, 250  
Beckmann's Flow Model 86, 93, 198,  
199, 225, 231, 286  
beehive 105  
Bénard Convection 105  
Bessel Functions 285  
bifurcation 172, 173, 174, 175, 181,  
183, 184, 224, 328  
bifurcation angle 332, 333, 334, 335  
bifurcation, curved 335  
bifurcation manifold 220, 221, 224  
bifurcation point 332, 333, 334, 335,  
336  
Bonnard's Fundamental Theorem 51  
boundary conditions 81, 85, 91, 96,  
106, 336  
business cycles 279  
butterfly catastrophe 218

## C

calculus of variations 35, 36, 41, 80  
Catastrophe Theory 167, 218, 219  
caustic 50  
central field 41, 42  
central place hierarchy 114  
CES production functions 297  
chaos 278  
characteristic equation 201, 281  
Chladny Plates 284  
Cobb-Douglas Production Function  
18, 21, 22, 27, 28, 194, 195,  
196  
collection lines 322  
commuting 230, 246, 251  
compactness 106  
comparative advantages 192  
conformal map 51, 61  
congestion 6, 31, 35, 41, 92, 212,  
239  
conjugate points 41  
conservation equation 69  
constant land rent contours 193  
constant price contours 25, 26, 89, 90  
constraints 58, 65, 83  
constraints, integral 66, 86  
constraints, local 66, 86  
construction and maintenance cost  
327, 328, 333, 335

coordinate change 39, 47, 59, 127,  
207, 219, 333  
coordinate lines 61  
coordinates, polar 39, 63, 64, 65,  
127, 179, 309, 310, 322  
correspondence principle 204  
cultivation rings 212  
cultivation zones 193, 197, 212  
curve integral 71, 98  
cusp 218

**D**

demand function 170, 176  
destination 230, 231, 247, 251, 306  
determinant, of characteristic equation  
203  
detour 31, 309, 310, 311, 313, 317  
differential geometry 51  
diffusion 252, 254, 255, 272  
diffusivity 255  
dimension, fractal 318, 319  
dimension, topological 319  
dipole 55  
discriminating monopolist 131  
disequilibrium 286  
distance, Euclidean 6, 7, 25  
divergence 75, 76, 77, 78, 82, 87,  
94, 95, 232  
dodekahedra, rhombic 112, 114  
dot product 75, 95

**E**

edge-to-face ratio 316  
edges 314, 316  
Eigenfunctions 283  
Eigenvalue Problem 281  
eigenvalues 202, 203, 219, 274, 282  
elasticity of supply 167  
element of arc 88  
element of area 83  
elliptic umblic catastrophe 218, 225  
endpoint conditions 35, 42  
energy, potential 10  
entropy 247, 251

envelope 89  
equilibrium 286  
equilibrium of traffic 239  
Euler-Poincaré Index 105, 314, 315  
Euler's Equation 34, 37, 38, 40, 43,  
46, 47, 48, 49, 54, 57, 58,  
59, 65, 66, 80, 83, 84, 85,  
86, 87, 88, 107, 131, 336  
excess demand 68, 69, 70, 76, 77,  
79, 87, 94, 96, 200, 286, 287,  
290  
excess supply 68, 77  
exports 98, 272, 273, 279, 280

**F**

faces 314, 316  
feeding lines 311, 328  
feeding sector 311, 312  
Fermat's Principle 46, 49, 293  
flow 25, 31, 68, 69, 71, 75, 77, 87,  
88, 89, 92, 200, 206, 211,  
212, 231, 287, 288, 293  
flow lines 67, 90  
Fourier's Theorem 275  
Fourier Coefficients 277, 283  
Fourier Expansion 225, 226, 278  
fractal 118, 225, 226, 318, 319, 320  
frequency 282  
Fundamental Lemma of the Calculus of  
Variations 37, 83  
Fundamental Theorem of Calculus  
70, 74

**G**

Gauss's Integral Theorem 75, 76, 82,  
95, 98, 254, 275, 290  
Gauss's Teorema Egregium 51  
genus number 314  
geodesics 50, 55  
gradient 23, 24, 25, 26, 27, 48, 78,  
84, 88, 95, 198, 199, 200,  
202, 219, 254, 286, 287  
gradient field 23, 26  
graph 208



gravity model 229, 236, 245, 251  
 Green's Theorem 74, 75  
 grid, square 211, 215, 223, 224, 314  
 growth 252, 253, 255, 271, 282  
 growth, logistic 253

## H

habitats 256  
 Harrod-Domar Growth Model 271, 278  
 heteroclinic saddle connection 207, 218, 219, 220  
 hexagonal /triangular tessellation 213, 215, 217, 222, 223, 224  
 homoclinic saddle connection 215  
 Hotelling's Migration Model 252  
 Hotelling's Oligopoly Model 142

## I

immigration 254  
 imports 272, 278, 279  
 incidence angle 311  
 integration by parts 36  
 interaction 230, 231, 251  
 interregional trade equilibrium 69, 76, 88, 200  
 interregional trade multiplier 271, 285  
 investments 271, 272, 273, 279, 280  
 isodapans 9  
 Isoperimetric Problem 59, 63, 66, 336  
 isotropic approximation 317

## J

Jacobian 62  
 junction 306, 307, 309

## L

Lagrange Multiplier 58, 86, 87, 88, 248, 250, 279  
 Lagrangean 59, 65, 86, 87, 107, 248  
 land rent 193, 198, 199, 219, 226, 227  
 land rent landscape 224

land rent maximum 211  
 land rent minimum 211  
 Laplace's Differential Equation 93  
 Laplacian 78, 93, 95, 254, 272, 274  
 Launhardt's Funnels 103  
 Law of Cosines 13, 235  
 Law of Sines 330  
 Legendre Conditions 37  
 Legendre Polynomials 285  
 length of curve 32  
 Leontief fixed proportions 7, 17  
 linear systems 201  
 link 305  
 logarithmic spiral 41, 48

## M

marginal cost pricing 168  
 market area 101, 104, 105, 116, 118, 129, 173, 186, 212  
 market boundary 101, 102  
 market radius 129, 132, 170, 171, 173, 174, 175, 176, 181, 183, 184, 185  
 market radius, maximal 173  
 market radius, minimal 173  
 Maxwell-Boltzmann Energy Distribution 245  
 metric 296, 303, 309, 310  
 metric, Euclidean 6, 295, 296, 297, 305, 311  
 metric, Manhattan 31, 153, 296, 297, 304, 320  
 metric, Minkowski 297, 299  
 metric, non-Euclidean 23  
 migration 252  
 minimal surface 81, 83, 85  
 modes, multiple 309  
 monkey saddle 206, 207, 218, 219, 220, 221, 222, 223  
 monkey saddle, periodic 222  
 monopoly mill price 127, 128, 129, 130, 133  
 multinomial coefficients 247  
 multiplier-accelerator model 271

**N**

nesting 114, 115, 116, 117, 225, 303  
 network 293, 294, 306, 316, 317, 318  
 network, central 320  
 network, collection or discharge 300  
 network, square 299, 312  
 network, triangular 298, 299, 311, 312, 313, 315  
 networks, nested 311  
 nodal lines 283, 284  
 nodes 201, 203, 206, 207, 208, 210, 213, 215, 223, 294, 305  
 nonlinear processes 278  
 number of radials 300

**O**

oligopoly 134, 135, 142, 153, 158, 160, 162, 164  
 optics 31, 49  
 optimal scale of operations 114, 168  
 origin 230, 231, 233, 247, 251, 306  
 oscillation 282

**P**

Palander-Stackelberg Law 49, 50, 311  
 parameterization 33, 34, 57, 71, 74, 127  
 Peano Curve 320  
 Peixoto's Characterization Theorem 208  
 perturbation 204, 207  
 phase portrait 206, 207  
 Pick's Construction 11, 12, 13  
 Plateau's Problem 83  
 polar coordinates 77, 324  
 polygon, regular 104, 105, 107, 183, 185  
 population growth 252, 254  
 population, sustainable 253, 255  
 potential 23, 25, 202, 218  
 Potential Theory 93

price discrimination, perfect 127, 128, 131, 133, 134  
 Principle of Acceleration 272, 279  
 public utility 167, 168, 169

**R**

radials 321, 322, 323, 327  
 radials, number of 326, 327, 328  
 Radiolarian 105  
 rank-size relation 122, 125  
 refraction 323  
 refraction angle 325  
 refraction index 49  
 returns to scale, constant 194  
 returns to scale, decreasing 18, 254  
 roads, radial 322  
 routes, optimal 31

**S**

saddle 201, 203, 206, 207, 208, 209, 211, 213, 215, 220, 221, 222  
 savings 272, 273, 279  
 scaling 320  
 sector of collection 324  
 self similarity 118, 318  
 separation of variables 252, 273, 280  
 service capacity 172  
 shortest path 34  
 Sierpinski Triangle 118  
 singular point 203, 206, 207, 208, 209  
 singularities 203, 211, 213  
 sink 69, 209, 210, 211, 213, 215, 217, 220, 221, 222, 223, 232, 321, 322  
 Snell's Law 46, 49  
 source 69, 209, 210, 211, 213, 215, 217, 220, 221, 222, 223, 232, 321, 322  
 space filling curve 318, 320  
 specialization 192, 194, 199, 225  
 specialization theorem 199, 225  
 specialization zones 211

spirals 201, 202  
 stability of equilibrium 288  
 stability, structural 201, 203, 204,  
     207, 208, 211, 217, 218,  
     219, 225, 251, 314  
 statistical mechanics 245  
 stereographic projection 54  
 Stirling's Formula 248  
 structural change 218  
 structures, global 221  
 substitutability 7  
 substitutes, perfect 17  
 superposition principle 275, 277  
 surface tension 85

## T

Taylor Series, truncated 167  
 terminal costs 305  
 tessellation 105, 111, 112, 114, 115,  
     116, 117, 118, 216, 299, 314  
 tessellation, square 216, 217  
 tessellation, triangular 314  
 tessellation, triangular/hexagonal 224  
 topological equivalence 204, 205,  
     206, 207, 208, 210, 219  
 topology, algebraic 314  
 trace 203  
 trade 272, 286  
 trade, interregional 77, 278  
 traffic 230, 231, 233, 236, 239  
 trajectories 90, 200, 201, 203, 206,  
     207, 208, 209, 211, 213, 215  
 transversality 45, 113, 193, 212,  
     225, 314  
 transverse 113  
 trip matrix 246, 247, 248  
 trips, commuting 246, 247

## U

undercutting 148, 151, 160, 161  
 uniform delivery price 127, 128, 132,  
     133, 134  
 uniqueness of equilibrium 93

## V

Varignon's Machine 11, 50  
 wave front 49  
 Weber Triangle 8, 16  
 Weber's Problem  
     5, 13, 17, 21, 23, 50  
 vector analysis 28, 74, 78  
 vector field 67, 76, 78  
 vertex-to-face ratio 316  
 vertices 314, 316  
 Wilson's Entropy Model 245  
 volume of flow 69, 79, 95  
 von Thünen's Model 189, 197

# Glossary of Formulas

## Law of Sines:

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

where  $a$ ,  $b$  and  $c$  are the sides of a triangle and  $\alpha$ ,  $\beta$  and  $\gamma$  are the angles opposite to these sides.

## Law of Cosines:

$$a^2 + b^2 - 2ab \cos \gamma = c^2$$

with Pythagoras's Theorem  $a^2 + b^2 = c^2$  as special case when  $\gamma = \pi / 2$

## Cosine and Sine for Sum and Difference:

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

## Special cases:

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha$$

$$\sin(2\alpha) = 2 \sin \alpha \cos \alpha$$

## Trigonometric Identity:

$$\cos^2 \alpha + \sin^2 \alpha = 1$$

**Gradient of Scalar Field**  $\lambda(x, y)$ :

$$\nabla \lambda = \left( \frac{\partial \lambda}{\partial x}, \frac{\partial \lambda}{\partial y} \right)$$

**Divergence of Vector Field**

$\phi = (\phi_1(x, y), \phi_2(x, y))$ :

$$\nabla \cdot \phi = \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y}$$

**Dot product of vectors**

$\phi = (\phi_1(x, y), \phi_2(x, y))$  and  $\psi = (\psi_1(x, y), \psi_2(x, y))$ :

$$\phi \cdot \psi = \phi_1 \psi_1 + \phi_2 \psi_2$$

**Tangent and Normal**

to parameterized curve  $(x(t), y(t))$ :

$$\mathbf{t} = (x'(t), y'(t))$$

$$\mathbf{n} = (y'(t), -x'(t))$$

The dot product of normal and tangent is zero:

$$\mathbf{t} \cdot \mathbf{n} = 0$$

**Laplacian:**

$$\nabla^2 \lambda = \nabla \cdot \nabla \lambda = \frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2}$$

**Gauss's Integral Theorem:**

$$\iint_R \nabla \cdot \phi dx dy = \oint_C \phi \cdot \mathbf{n} ds$$

**Euler's Equation:**

$$F_y - \frac{d}{dx} F_{y'} = 0$$

minimizes the integral

$$\int_a^b F(x, y, y') dx$$

with respect to the function  $y(x)$ ,  $a < x < b$

**Special Case**

when  $x$  is not included:

$$F - y' F_{y'} = c$$

**Euler's Equation in 2D**

$$F_u - \frac{\partial}{\partial x} F_p - \frac{\partial}{\partial y} F_q = 0$$

minimizes the integral

$$\iint_R F(x, y, u, p, q) dx dy$$

with respect to  $u(x, y)$  on  $R$  where

$$p = \frac{\partial u}{\partial x} \quad \text{and} \quad q = \frac{\partial u}{\partial y}$$

**Jacobian Determinant**for change of coordinates  $x = f(u, v)$ ,  $y = g(u, v)$ :

$$dxdy = |J|dudv$$

with

$$J = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}$$

**Special case for polar coordinates**

$$x = r \cos \theta, y = r \sin \theta$$

$$dxdy = r drd\theta$$

**Beckmann's Equations:**

$$k \frac{\phi}{|\phi|} = \nabla \lambda$$

minimizes

$$\iint_R k |\phi| dxdy$$

subject to the constraint

$$\nabla \cdot \phi + z = 0$$

**Market Area and Transportation Work:**

$$A(P_n) = \iint_{P_n} dydx = nR^2 \cos \frac{\pi}{n} \sin \frac{\pi}{n}$$

$$T(P_n) = \iint_{P_n} \sqrt{x^2 + y^2} dydx = \frac{nR^3}{3} \left( \cos \frac{\pi}{n} \sin \frac{\pi}{n} + \cos^3 \frac{\pi}{n} \ln \tan \left( \frac{\pi}{4} + \frac{\pi}{2n} \right) \right)$$

are the market area of a regular  $n$ -sided polygon with radius  $R$ . In the limiting case of a circle:  $\lim_{n \rightarrow \infty} A(P_n) = \pi R^2$  and  $\lim_{n \rightarrow \infty} T(P_n) = \frac{2}{3} \pi R^3$ . Special cases:

$$n = 3 \quad A = \frac{\sqrt{3}}{4} R^2 \quad T = \left( \frac{\sqrt{3}}{4} + \frac{\ln(2 + \sqrt{3})}{8} \right) R^3$$

$$n = 4 \quad A = \frac{2}{3} R^2 \quad T = \left( \frac{2}{3} + \frac{\sqrt{2} \ln(1 + \sqrt{2})}{3} \right) R^3$$

$$n = 6 \quad A = \frac{\sqrt{3}}{2} R^2 \quad T = \left( \frac{\sqrt{3}}{2} + \frac{3\sqrt{3} \ln(3)}{8} \right) R^3$$



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