Appendix A
Polytropic Stars

Following the definition of a polytropic process in thermodynamics, a star is said to be polytropic when its equation of state has the form

$$p = K\rho^{\Gamma}, \tag{A.1}$$

where K is the *polytropic constant* and Γ is the *polytropic exponent*, usually also written as

$$\Gamma = 1 + \frac{1}{n}, \tag{A.2}$$

where n is the *polytropic index*. To avoid confusion with the mode overtone n, defined in Sect. 2.4, the polytropic exponent Γ is mostly used in the main chapters, but the polytropic index n will be preferred throughout this appendix.

A.1 Nonrotating Polytropes

We write the equation of hydrostatic equilibrium for a nonrotating star [Eq. (2.1.5) with $\Omega = 0$] as

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r}\right) = -4\pi G\rho,$$

where we used the fact that

$$\boldsymbol{g} = -\nabla\Phi = -\frac{GM_r}{r^2}\boldsymbol{e}_r,$$

$\boldsymbol{g}$ being the local gravitational acceleration, and replaced M_r from Eq. (2.3.42). Substituting Eqs. (A.1) and (A.2), we get

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$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n, \tag{A.1.1}$$

where the dimensionless variables θ and ξ are defined from the relations

$$\rho = \rho_c\theta^n \tag{A.1.2}$$

and

$$r = \left[\frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1}\right]^{1/2}\xi \tag{A.1.3}$$

respectively, ρ_c being the central density of the star. Equation (A.1.1) is the well-known *Lane-Emden equation*, for a polytrope with index n.[1]

At the centre of the star ($\xi \to 0$), the boundary conditions are (Shapiro and Teukolsky 1983, Sect. 3.3)

$$\theta(0) = 1 \tag{A.1.4}$$

and

$$\theta'(0) = 0, \tag{A.1.5}$$

where the prime denotes differentiation with respect to ξ. Using these, we can integrate Eq. (A.1.1) and show that $\theta(\xi)$, known as Emden's function, always has a finite root ξ_1 for $n < 5$. Hence, ξ_1 corresponds to the surface of the star, namely

$$R = \left[\frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1}\right]^{1/2}\xi_1,$$

whereas, using Eq. (2.3.42), the mass is found as

$$M = 4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2}\rho_c^{\frac{3-n}{2n}}\xi_1^2\left|\theta'(\xi_1)\right|. \tag{A.1.6}$$

The Lane-Emden equation admits an analytic solution for the two extremes, namely, $n = 0$, which describes a homogeneous star with $\rho = \rho_c$ everywhere (see Sect. 2.5), and $n = 5$, known as the Roche model, for which the mass is concentrated towards the centre of a star with an infinite radius. Thus, density and pressure decrease faster towards the surface as the polytropic index increases, i.e., as the equation of state gets softer (see Sect. 1.3). An additional case with an analytic solution is the $n = 1$ polytrope, where Emden's function is given by

$$\theta(\xi) = \frac{\sin\xi}{\xi} \tag{A.1.7}$$

[1] The variable ξ used here should not be confused with the displacement vector $\boldsymbol{\xi}$ or its components, used throughout the main chapters. We did not choose a different variable, because of the popularity of the Lane-Emden formalism in this particular notation.

and its first root is $\xi_1 = \pi$, for which $\theta'(\xi_1) = -1/\pi$.

A.2 Rotating Polytropes

The Lane-Emden formalism was extended for rotationally and tidally distorted polytropes by Chandrasekhar (1933a, b, c, d). We will be concerned with the first case, for which the equation of hydrostatic equilibrium (2.1.5) takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial x}\left(\frac{1-x^2}{\rho}\frac{\partial p}{\partial x}\right) = -4\pi G\rho + 2\Omega^2,$$

where $x = \cos\theta$. Like in the case of nonrotating polytropes, we write it in the dimensionless form

$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2\frac{\partial \Theta}{\partial \xi}\right) + \frac{1}{\xi^2}\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial \Theta}{\partial x}\right] = -\Theta^n + \tilde{\Omega}^2, \tag{A.2.1}$$

where ξ is given by Eq. (A.1.3), Θ is defined as

$$\rho = \rho_c \Theta^n, \tag{A.2.2}$$

and the angular velocity Ω is normalised as

$$\tilde{\Omega}^2 = \frac{\Omega^2}{2\pi G\rho_c}.$$

Expanding Θ in terms of $\tilde{\Omega}$ we get

$$\Theta = \theta + \Psi\tilde{\Omega}^2 + \mathcal{O}\left(\tilde{\Omega}^4\right), \tag{A.2.3}$$

where θ is the nonrotating solution, obtained from Eq. (A.1.1). We can replace Eq. (A.2.3) in Eq. (A.2.1) to get an equation for the correction function Ψ, namely

$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2\frac{\partial \Psi}{\partial \xi}\right) + \frac{1}{\xi^2}\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial \Psi}{\partial x}\right] = -n\theta^{n-1}\Psi + 1. \tag{A.2.4}$$

Then, we expand Ψ in terms of the Legendre polynomials P_i (e.g., see Abramowitz and Stegun 1972, Chap. 8] and, after a series of arguments and calculations (which can be found in Chandrasekhar 1933a), we obtain

$$\Psi = \psi_0(\xi) + A_2\psi_2(\xi)P_2(x), \tag{A.2.5}$$

where the functions ψ_0 and ψ_2 satisfy the equations

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\psi_0}{\mathrm{d}\xi}\right) = -n\theta^{n-1}\psi_0 + 1 \tag{A.2.6}$$

and

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\psi_2}{\mathrm{d}\xi}\right) = \left(-n\theta^{n-1} + \frac{6}{\xi^2}\right)\psi_2, \tag{A.2.7}$$

and

$$A_2 = -\frac{5}{6}\frac{\xi_1^2}{3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)}, \tag{A.2.8}$$

with ξ_1 still denoting the first root of Emden's function $\theta(\xi)$.

Near the centre of the star ($\xi \to 0$), ψ_0 and ψ_2 behave like

$$\psi_0(\xi) = \frac{\xi^2}{6} + \mathcal{O}\left(\xi^4\right) \tag{A.2.9}$$

and

$$\psi_2(\xi) = \xi^2 + \mathcal{O}\left(\xi^4\right). \tag{A.2.10}$$

Thus, we can integrate Eqs. (A.2.6) and (A.2.7) to obtain the solution Θ for a rotationally distorted (to second order) polytrope. Then, the surface of the star is found from $\Theta\left(\xi_0\right) = 0$, or

$$\xi_0 = \xi_1 + \frac{\tilde{\Omega}^2}{|\theta'(\xi_1)|}\left[\psi_0(\xi_1) + A_2\psi_2(\xi_1)P_2(x)\right].$$

This shows that, to second order in rotation, the star expands (first correction term) and changes its shape to an oblate spheroid (second correction term). At the equator $P_2(0) = -1/2$ and at the poles $P_2(1) = 1$, so, using Eq. (A.2.8), the oblateness of the star f is found as

$$f = \frac{R_e - R_p}{R_e} = \frac{5}{4}\frac{\tilde{\Omega}^2}{|\theta'(\xi_1)|}\frac{\xi_1\psi_2(\xi_1)}{3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)}, \tag{A.2.11}$$

where R_e is the equatorial and R_p the polar radius. The mass of the star can also be obtained as

$$M = 4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2}\rho_c^{\frac{3-n}{2n}}\xi_1^2\left|\theta'\left(\xi_1\right)\right|\left[1 + \tilde{\Omega}^2\frac{\xi_1/3 - \psi_0'(\xi_1)}{|\theta'(\xi_1)|}\right]. \tag{A.2.12}$$

Comparing Eqs. (A.1.6) and (A.2.12) we see that, if the rotating polytrope has the same central density ρ_c with its nonrotating counterpart, then its mass is larger by a factor of

$$\frac{M(\Omega)}{M(0)} = 1 + \tilde{\Omega}^2 \frac{\xi_1/3 - \psi_0'(\xi_1)}{|\theta'(\xi_1)|}, \tag{A.2.13}$$

where $M(0)$ and $M(\Omega)$ are the masses of the nonrotating and the rotating star respectively. If instead we demand that the rotating polytrope have the same mass with the nonrotating one, then its central density changes by a factor of

$$\frac{\rho_c(\Omega)}{\rho_c(0)} = 1 - \tilde{\Omega}^2 \frac{2n}{3-n} \frac{\xi_1/3 - \psi_0'(\xi_1)}{|\theta'(\xi_1)|}, \tag{A.2.14}$$

where $\rho_c(0)$ and $\rho_c(\Omega)$ are the central densities of the nonrotating and the rotating polytrope respectively. Hence, if $n < 3$ ($n > 3$) the central density of the rotating polytrope is lower (greater) than the central density of its nonrotating counterpart with the same mass (for the subtleties related to the $n = 3$ polytrope and the details of the calculations above, see Chandrasekhar 1933a).

Like with the Lane-Emden equation, there are analytic solutions for ψ_0 and ψ_2, for the $n = 0,\ 1$ and 5 polytropes (although the $n = 0$ and 5 cases need to be handled with caution; see Chandrasekhar 1933d). For $n = 1$, the solutions are given by

$$\psi_0(\xi) = 1 - \theta(\xi) = 1 - \frac{\sin\xi}{\xi} \tag{A.2.15}$$

and

$$\psi_2(\xi) = 15\sqrt{\frac{\pi}{2\xi}} J_{5/2}(\xi) = -15 \frac{3\xi\cos\xi + (\xi^2 - 3)\sin\xi}{\xi^3}, \tag{A.2.16}$$

where J_i are the Bessel functions of the first kind (see, for example, Abramowitz and Stegun 1972, Chaps. 9 and 10]. For $\xi_1 = \pi$ the solutions above give $\psi_0(\xi_1) = 1$, $\psi_0'(\xi_1) = 1/\pi$, $\psi_2(\xi_1) = 45/\pi^2$, and $\psi_2'(\xi_1) = 15(1 - 9/\pi^2)/\pi$. Tables with numerical results on other polytropic indices can be found in Chandrasekhar (1933a) and Chandrasekhar and Lebovitz (1962).

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Appendix B
Polar Mode Rotational Corrections

B.1 First-Order Corrections

The first-order rotational corrections to the eigenfrequencies ω and eigenfunctions $\boldsymbol{\xi}$ of polar modes can be found from Eq. (2.6.9), namely

$$-\omega_\alpha^{(0)2}\boldsymbol{\xi}_\alpha^{(1)} + \boldsymbol{\mathcal{C}}^{(0)}\left(\boldsymbol{\xi}_\alpha^{(1)}\right) - 2\omega_\alpha^{(0)}\omega_\alpha^{(1)}\boldsymbol{\xi}_\alpha^{(0)} + i\omega_\alpha^{(0)}\boldsymbol{\mathcal{B}}^{(1)}\left(\boldsymbol{\xi}_\alpha^{(0)}\right) = \mathbf{0}. \qquad \text{(B.1.1)}$$

Following Unno et al. (1989, § 19), we will expand $\boldsymbol{\xi}_\alpha^{(0)}$ and $\boldsymbol{\xi}_\alpha^{(1)}$ in terms of the eigenfunctions $\boldsymbol{\xi}_\alpha$ of the nonrotating star, as

$$\boldsymbol{\xi}_\alpha^{(0)} = \sum_{m=-l}^{l} c_{\alpha m}^{(0)}\boldsymbol{\xi}_\alpha \qquad \text{(B.1.2)}$$

and

$$\boldsymbol{\xi}_\alpha^{(1)} = \sum_{\substack{\beta \\ \beta\neq\alpha}} c_{\alpha\beta}^{(1)}\boldsymbol{\xi}_\beta. \qquad \text{(B.1.3)}$$

The zeroth-order eigenfunction $\boldsymbol{\xi}_\alpha^{(0)}$ cannot be simply taken equal to $\boldsymbol{\xi}_\alpha$, because of the degeneracy of the eigenfrequencies with respect to m in the nonrotating limit (see Sect. 2.4). As a result, the $2l+1$ eigenfunctions with the same degree l and overtone n correspond to the same eigenfrequency ω_α. So, we have to consider the zeroth-order eigenfunction as a linear combination of all the degenerate eigenfunctions with the same l and n, but different m (see, for instance, Mathews and Walker 1970, Chap. 10). In Eq. (B.1.2), mode $\boldsymbol{\xi}_\alpha$ is associated with the triplet (n, l, m), with the summation changing only m.

The first-order correction to the eigenfunction $\boldsymbol{\xi}_\alpha^{(1)}$ is taken to be the linear combination of all modes (including axial modes) of the nonrotating star (except for $\boldsymbol{\xi}_\alpha$

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itself), where the correction coefficients $c^{(1)}_{\alpha\beta}$ are $\mathcal{O}(\Omega)$. In Eq. (B.1.3), the "quantum numbers" of the mode $\boldsymbol{\xi}_\beta$ are (n', l', m').

Replacing Eqs. (B.1.2) and (B.1.3) in Eq. (B.1.1) and taking the inner product, defined by Eq. (2.3.55), with $\boldsymbol{\xi}_\gamma$ [corresponding to (n'', l'', m'')], we get

$$\left(-\omega_\alpha^{(0)2} + \omega_\gamma^{(0)2}\right) c^{(1)}_{\alpha\gamma} I_\gamma - 2\omega_\alpha^{(0)}\omega_\alpha^{(1)}\delta_{n''n}\delta_{l''l} I_\alpha c^{(0)}_{\alpha m''} + i\omega_\alpha^{(0)} \sum_m c^{(0)}_{\alpha m}\langle \boldsymbol{\xi}_\gamma, \boldsymbol{\mathcal{B}}^{(1)}\left(\boldsymbol{\xi}_\alpha\right)\rangle = 0, \tag{B.1.4}$$

with I_α defined in Eq. (2.3.55). For $(n'', l'') = (n, l)$ we obtain

$$\sum_m \mathcal{B}^{(1)}_{m''m} c^{(0)}_{\alpha m} = \omega_\alpha^{(1)} c^{(0)}_{\alpha m''}, \tag{B.1.5}$$

where

$$\mathcal{B}^{(1)}_{m''m} = \frac{i}{2I_\alpha}\langle \boldsymbol{\xi}_{m''}, \boldsymbol{\mathcal{B}}^{(1)}\left(\boldsymbol{\xi}_m\right)\rangle, \tag{B.1.6}$$

with $\boldsymbol{\xi}_m$ and $\boldsymbol{\xi}_{m''}$ corresponding to mode $\boldsymbol{\xi}_\alpha$ with (n, l, m) and (n, l, m'') respectively.

Equation (B.1.5) represents the m''-component of the matrix equation

$$B^{(1)} c^{(0)}_\alpha = \omega^{(1)}_\alpha c^{(0)}_\alpha, \tag{B.1.7}$$

$B^{(1)}$ being the $(2l+1)\times(2l+1)$ matrix with components $\mathcal{B}^{(1)}_{m''m}$. The eigenvalues of this matrix give the first-order eigenfrequency corrections (one for each value of m) of the mode $\boldsymbol{\xi}_\alpha$, whereas its eigenvectors give the components of the zeroth-order eigenfunction $\boldsymbol{\xi}^{(0)}_\alpha$.

If we now set $(n'', l'') = (n', l') \neq (n, l)$ in Eq. (B.1.4), we get for the first-order correction coefficients

$$c^{(1)}_{\alpha\beta} = \frac{i\omega_\alpha^{(0)}}{I_\beta\left(\omega_\alpha^{(0)2} - \omega_\beta^{(0)2}\right)} \sum_m c^{(0)}_{\alpha m}\langle \boldsymbol{\xi}_\beta, \boldsymbol{\mathcal{B}}^{(1)}\left(\boldsymbol{\xi}_\alpha\right)\rangle. \tag{B.1.8}$$

We will proceed with the evaluation of the matrix $B^{(1)}$. Taking the angular velocity along the z axis $(\theta = 0)$, namely

$$\boldsymbol{\Omega} = (\Omega\cos\theta, -\Omega\sin\theta, 0)\,,$$

then, replacing the polar mode eigenfunction (2.3.19) in Eq. (B.1.6) and using the spherical harmonic orthogonality relation (2.3.14), it becomes

$$\mathcal{B}^{(1)}_{m''m} = \delta_{m''m}\frac{m\Omega}{I_\alpha}\int_0^R \left(2\xi_r\xi_h + \xi_h^2\right)\rho r^2 \mathrm{d}r.$$

The equation above shows that the matrix $B^{(1)}$ is diagonal, so its components are its eigenvalues and they are given by

$$\omega_\alpha^{(1)} = mC_1\Omega, \tag{B.1.9}$$

where, using Eq. (2.3.56),

$$C_1 = \frac{\displaystyle\int_0^R \left(2\xi_r\xi_h + \xi_h^2\right)\rho r^2 \mathrm{d}r}{\displaystyle\int_0^R \left[\xi_r^2 + l(l+1)\xi_h^2\right]\rho r^2 \mathrm{d}r}. \tag{B.1.10}$$

Then, for a certain eigenvalue (i.e., for a specific m), Eq. (B.1.7) can be written as

$$\begin{pmatrix} \mathcal{B}^{(1)}_{-l,-l} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \mathcal{B}^{(1)}_{-l+1,-l+1} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{B}^{(1)}_{m,m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \mathcal{B}^{(1)}_{l,l} \end{pmatrix} \begin{pmatrix} c^{(0)}_{\alpha,-l} \\ c^{(0)}_{\alpha,-l+1} \\ \vdots \\ c^{(0)}_{\alpha,m} \\ \vdots \\ c^{(0)}_{\alpha,l} \end{pmatrix} = \omega_\alpha^{(1)} \begin{pmatrix} c^{(0)}_{\alpha,-l} \\ c^{(0)}_{\alpha,-l+1} \\ \vdots \\ c^{(0)}_{\alpha,m} \\ \vdots \\ c^{(0)}_{\alpha,l} \end{pmatrix}.$$

So, the eigenvector $c_\alpha^{(0)}$ corresponding to the eigenvalue $\omega_\alpha^{(1)}$ is

$$c_\alpha^{(0)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \leftarrow \text{ Position } -l \\ \leftarrow \text{ Position } -l+1 \\ \vdots \\ \leftarrow \text{ Position } m \\ \vdots \\ \leftarrow \text{ Position } l \end{matrix}$$

Consequently, the zeroth-order eigenfunction (B.1.2) is given by a single spherical harmonic, namely

$$\boldsymbol{\xi}_\alpha^{(0)} = \boldsymbol{\xi}_\alpha, \tag{B.1.11}$$

and Eq. (B.1.8) becomes

$$c_{\alpha\beta}^{(1)} = \frac{i\omega_\alpha^{(0)}}{I_\beta\left(\omega_\alpha^{(0)2} - \omega_\beta^{(0)2}\right)}\langle \boldsymbol{\xi}_\beta, \boldsymbol{\mathcal{B}}^{(1)}\left(\boldsymbol{\xi}_\alpha\right)\rangle. \tag{B.1.12}$$

Using these results, we can also derive the first-order inner product of two rotationally corrected eigenfunctions, $\boldsymbol{\xi}_\alpha(\Omega) = \boldsymbol{\xi}_\alpha^{(0)} + \boldsymbol{\xi}_\alpha^{(1)} + \mathcal{O}\left(\Omega^2\right)$, as (Schenk et al. 2001)

$$\langle \boldsymbol{\xi}_\alpha(\Omega), \boldsymbol{\xi}_\beta(\Omega) \rangle = \frac{i}{\omega_\alpha^{(0)} + \omega_\beta^{(0)}} \langle \boldsymbol{\xi}_\alpha, \boldsymbol{\mathcal{B}}^{(1)} \left(\boldsymbol{\xi}_\beta\right) \rangle + \mathcal{O}\left(\Omega^2\right), \tag{B.1.13}$$

for $\alpha \neq \beta$, where we used the fact that operator $\boldsymbol{\mathcal{B}}$ is anti-Hermitian[2] (Lynden-Bell and Ostriker 1967). From this relation one can see that the rotationally corrected eigenfunctions do not necessarily satisfy the orthogonality condition (2.3.55); Eq. (2.6.3) should be used instead as an orthogonality relation.

An Alternative Approach

From Eqs. (B.1.3) and (B.1.12) we see that the actual computation of the first-order corrections to the eigenfunctions can be cumbersome, since it is an expansion over all the modes of the nonrotating star. In practice, due to the form of $c_{\alpha\beta}^{(1)}$, only neighbouring modes with similar eigenfrequencies are considered, with the contribution from the rest of the modes being negligible.

However, there is an alternative way to obtain the first-order corrections to the eigenfunctions, first presented by Hansen et al. (1978) in the Cowling approximation (see Sect. 2.3.4) and then extended for the general case by Saio (1981). First, from Eqs. (B.1.3) and (2.3.53) we notice that axial modes do not contribute to the radial component $\xi_{\alpha,r}^{(1)}$ of $\boldsymbol{\xi}_\alpha^{(1)}$. From the radial component of Eq. (B.1.1) one can further show that

$$\int \xi_{\alpha,r}^{(1)} \xi_{\beta,r}^{(0)*} \rho \mathrm{d}^3\boldsymbol{r} = 0,$$

for $\alpha \neq \beta$. Therefore, $\xi_r^{(1)}$ is proportional to a single spherical harmonic Y_l^m, namely

$$\xi_r^{(1)}(r, \theta, \phi) = \xi_r^{(1)}(r) Y_l^m(\theta, \phi) \tag{B.1.14}$$

(we will omit the subscript α from now on, for simplicity). Then, using Eqs. (B.1.14) and (2.3.19), we take the θ and ϕ components of Eq. (B.1.1) to get

$$\xi_\theta^{(1)}(r, \theta, \phi) = \frac{1}{\omega^{(0)2} r} \left\{ \left[\chi^{(1)}(r) - \frac{2\omega^{(1)}}{\omega^{(0)}} \chi^{(0)}(r) \right] \frac{\partial}{\partial\theta} + \frac{2m\Omega}{\omega^{(0)}} \chi^{(0)}(r) \cot\theta \right\} Y_l^m(\theta, \phi) \tag{B.1.15}$$

and

$$\xi_\phi^{(1)}(r, \theta, \phi) = \left\{ \frac{1}{\omega^{(0)2} r} \left[\chi^{(1)}(r) - \frac{2\omega^{(1)}}{\omega^{(0)}} \chi^{(0)}(r) \right] \frac{1}{\sin\theta} \frac{\partial}{\partial\phi} + \frac{2i\Omega}{\omega^{(0)}} \left[\xi_r^{(0)}(r) \sin\theta + \frac{\chi^{(0)}(r)}{\omega^{(0)2} r} \cos\theta \frac{\partial}{\partial\theta} \right] \right\} Y_l^m(\theta, \phi), \tag{B.1.16}$$

where the variable χ is defined as

[2]Namely, it satisfies the relation $\langle \boldsymbol{\xi}, \boldsymbol{\mathcal{B}} \cdot \boldsymbol{\xi}' \rangle = -\langle \boldsymbol{\mathcal{B}} \cdot \boldsymbol{\xi}, \boldsymbol{\xi}' \rangle$, for any $\boldsymbol{\xi}, \boldsymbol{\xi}'$ on the space of complex vector functions, with respect to the inner product $\langle \boldsymbol{\xi}, \boldsymbol{\xi}' \rangle = \int \boldsymbol{\xi} \cdot \boldsymbol{\xi}' \rho \mathrm{d}^3\boldsymbol{r}$.

$$\chi(r,\theta,\phi) = \frac{\delta p}{\rho} + \delta\Phi. \tag{B.1.17}$$

It was shown in Sect. 2.3 that $\delta p^{(0)}$ and $\delta\Phi^{(0)}$ are proportional to Y_l^m. By replacing Eq. (B.1.3) in the perturbed continuity equation (2.2.4), the perturbed Poisson equation (2.2.6), and the perturbed equation of state (2.2.7), we see that the same applies to $\delta p^{(1)}$ and $\delta\Phi^{(1)}$, which means that χ can be expanded as

$$\chi(r,\theta,\phi) = \left[\chi^{(0)}(r) + \chi^{(1)}(r)\right] Y_l^m(\theta,\phi) + \mathcal{O}\left(\Omega^2\right).$$

It is obvious from Eq. (2.3.20) that $\chi^{(0)}(r) = \omega^{(0)2} r \xi_h^{(0)}(r)$. Comparing Eqs. (B.1.15) and (B.1.16) with Eq. (2.3.19) we see that the rotationally corrected eigenfunctions of polar modes do not follow the angular dependence of polar modes any more.

Given that $\omega^{(0)}$, $\xi_r^{(0)}$, and $\chi^{(0)}$ have been found from the integration of Eqs. (2.3.16)–(2.3.18) and their boundary conditions (2.3.24)–(2.3.27), whereas $\omega^{(1)}$ can be obtained from Eq. (B.1.9), it only remains to calculate $\xi_r^{(1)}$ and $\chi^{(1)}$. Applying a procedure similar to that followed for zeroth-order quantities in Sect. 2.3, we find a system of differential equations and boundary conditions for $\xi_r^{(1)}$, $\delta p^{(1)}$, and $\delta\Phi^{(1)}$. Using the dimensionless formulation presented in Sect. 2.3.3, they are written as

$$x\frac{\mathrm{d}y_1^{(1)}}{\mathrm{d}x} = \left(V_g - 3\right) y_1^{(1)} + \left[\frac{l(l+1)}{c_1\tilde{\omega}^{(0)2}} - V_g\right] y_2^{(1)} + V_g y_3^{(1)} + \frac{2m\Omega}{\omega^{(0)}}\left\{y_1^{(0)} + \left[1 - \frac{\omega^{(1)}}{m\Omega} l(l+1)\right]\frac{y_2^{(0)}}{c_1\tilde{\omega}^{(0)2}}\right\}, \tag{B.1.18}$$

$$x\frac{\mathrm{d}y_2^{(1)}}{\mathrm{d}x} = \left(c_1\tilde{\omega}^{(0)2} - A^*\right) y_1^{(1)} + \left(A^* - U + 1\right) y_2^{(1)} - A^* y_3^{(1)} + \frac{2m\Omega}{\omega^{(0)}}\left(\frac{\omega^{(1)}}{m\Omega} c_1\tilde{\omega}^{(0)2} y_1^{(0)} - y_2^{(0)}\right), \tag{B.1.19}$$

$$x\frac{\mathrm{d}y_3^{(1)}}{\mathrm{d}x} = (1 - U)\, y_3^{(1)} + y_4^{(1)}, \tag{B.1.20}$$

$$x\frac{\mathrm{d}y_4^{(1)}}{\mathrm{d}x} = U A^* y_1^{(1)} + U V_g y_2^{(1)} + \left[l(l+1) - U V_g\right] y_3^{(1)} - U y_4^{(1)}, \tag{B.1.21}$$

whereas the boundary conditions at the centre ($x \to 0$) and the surface ($x \to 1$) are

$$l y_3^{(1)} - y_4^{(1)} = 0, \tag{B.1.22}$$

$$c_1\tilde{\omega}^{(0)2} y_1^{(1)} - l y_2^{(1)} + \frac{2m\Omega}{\omega^{(0)}}\left(\frac{\omega^{(1)}}{m\Omega} - \frac{1}{l}\right) c_1\tilde{\omega}^{(0)2} y_1^{(0)} = 0, \tag{B.1.23}$$

and

$$U y_1^{(1)} + (l+1) y_3^{(1)} + y_4^{(1)} = 0, \tag{B.1.24}$$

$$\left(1 - \frac{4 + c_1 \tilde{\omega}^{(0)2}}{V}\right) y_1^{(1)} + \left[\frac{l(l+1)}{c_1 \tilde{\omega}^{(0)2} V} - 1\right] y_2^{(1)} + \left(1 - \frac{l+1}{V}\right) y_3^{(1)}$$
$$+ \frac{2m\Omega}{\omega^{(0)} V} \left\{\left(1 - \frac{\omega^{(1)}}{m\Omega} c_1 \tilde{\omega}^{(0)2}\right) y_1^{(0)} + \left[1 - \frac{\omega^{(1)}}{m\Omega} l(l+1) + c_1 \tilde{\omega}^{(0)2}\right] \frac{y_2^{(0)}}{c_1 \tilde{\omega}^{(0)2}}\right\} = 0, \tag{B.1.25}$$

respectively. Now, Eqs. (B.1.18)–(B.1.25) can be solved as a boundary value problem for $y_1^{(1)}$, $y_2^{(1)}$, $y_3^{(1)}$, and $y_4^{(1)}$, from which $\xi_r^{(1)}$ and $\chi^{(1)}$ are obtained. It should be noted that the solutions are proportional to $m\Omega$, so they may be found for one value of m and then rescaled for the rest.

B.2 Second-Order Corrections

In order to obtain the second-order rotational corrections to the eigenfrequencies ω and eigenfunctions $\boldsymbol{\xi}$ of polar modes we need to use Eq. (2.6.10), namely

$$\begin{aligned} &-\omega_\alpha^{(0)2} \boldsymbol{\xi}_\alpha^{(2)} + \boldsymbol{\mathcal{C}}^{(0)} \left(\boldsymbol{\xi}_\alpha^{(2)}\right) - 2\omega_\alpha^{(0)} \omega_\alpha^{(1)} \boldsymbol{\xi}_\alpha^{(1)} + i\omega_\alpha^{(0)} \boldsymbol{\mathcal{B}}^{(1)} \left(\boldsymbol{\xi}_\alpha^{(1)}\right) \\ &- 2\omega_\alpha^{(0)} \omega_\alpha^{(2)} \boldsymbol{\xi}_\alpha^{(0)} - \omega_\alpha^{(1)2} \boldsymbol{\xi}_\alpha^{(0)} + i\omega_\alpha^{(1)} \boldsymbol{\mathcal{B}}^{(1)} \left(\boldsymbol{\xi}_\alpha^{(0)}\right) + \boldsymbol{\mathcal{C}}^{(2)} \left(\boldsymbol{\xi}_\alpha^{(0)}\right) = \mathbf{0}. \end{aligned} \tag{B.2.1}$$

The situation in this case gets much more complicated than with the calculation of $\mathcal{O}(\Omega)$ corrections, because, at second order in Ω, the equilibrium configuration is distorted by the centrifugal force [see Eq. (2.1.5)]. Hence, the various equilibrium quantities (i.e., density, pressure, and gravitational potential) do not depend only on the radial coordinate r, but also on the colatitude coordinate θ.

For the rigorous derivation of second-order corrections, the reader is referred to Saio (1981), whose basic steps we are going to reproduce in this section.[3] A similar formulation was also developed by Smeyers and Denis (1971), who applied it in a homogeneous, compressible star (see Sect. 2.5).

[3]In addition to second-order rotational effects on the polar modes of a star, Saio (1981) also considered the influence of tidal deformations from a companion star, whose orbital motion is parallel and synchronous to the rotation of the first star. These will be ignored in the current analysis.

B.2.1 Generic Formulation

In the presence of centrifugal acceleration, which is a second-order rotational effect, the star is deformed into an oblate spheroid. We define a new coordinate system (a, θ, ϕ), rotating with the star, where the radial coordinate r of spherical polar coordinates is replaced with a. Then, a distorted equipotential surface can be described as

$$r = a\left[1 + \varepsilon(a, \theta)\right], \tag{B.2.2}$$

with a remaining constant on each equipotential surface. If the central density of the nonrotating, spherical star, $\rho_{c,\,\mathrm{sp}}$, changes due to rotation by $\Delta\rho_c \equiv \rho_c - \rho_{c,\,\mathrm{sp}}$ (not to be confused with the Lagrangian perturbation Δ, which will not be used in this section), then the density on the rotating star can be expressed as

$$\rho\left[a(1+\varepsilon)\right] = \rho_{\mathrm{sp}}(a)\frac{\rho_c}{\rho_{c,\,\mathrm{sp}}} = \rho_{\mathrm{sp}}(a)\left(1 + \frac{\Delta\rho_c}{\rho_{c,\,\mathrm{sp}}}\right), \tag{B.2.3}$$

where $\rho_{\mathrm{sp}}(a)$ is the density on a sphere, with radius a, on the nonrotating star. Since density and pressure depend only on a, they can simply be written as

$$\rho(a) = \rho_{\mathrm{sp}}(a) + \Delta\rho(a) \tag{B.2.4}$$

and likewise for the pressure.

In this notation, Eq. (B.2.1) becomes (omitting the mode index α from now on, for simplicity)

$$\begin{aligned}
&-\left(2\omega^{(2)}\omega^{(0)} + \omega^{(1)2}\right)\boldsymbol{\xi}^{(0)} - \omega^{(0)2}\left[\boldsymbol{\xi}^{(2)} + 2\varepsilon\boldsymbol{\xi}^{(0)} + a\xi_a^{(0)}\nabla\varepsilon + \boldsymbol{e}_a a\left(\boldsymbol{\xi}^{(0)}\cdot\nabla\right)\varepsilon\right] \\
&-2\omega^{(1)}\omega^{(0)}\boldsymbol{\xi}^{(1)} + \nabla\chi^{(2)} - \boldsymbol{e}_a\left(\frac{p\Gamma_1}{\rho}A\right)_{\mathrm{sp}}\left[\nabla\cdot\boldsymbol{\xi}^{(2)} + \left(\boldsymbol{\xi}^{(0)}\cdot\nabla\right)\left(3\varepsilon + a\frac{\partial\varepsilon}{\partial a}\right)\right] \\
&-i\left[\omega^{(0)}\boldsymbol{\mathcal{B}}^{(1)}\left(\boldsymbol{\xi}^{(1)}\right) + \omega^{(1)}\boldsymbol{\mathcal{B}}^{(1)}\left(\boldsymbol{\xi}^{(0)}\right)\right] - \Delta\left(\frac{p\Gamma_1}{\rho}A\right)\left(\nabla\cdot\boldsymbol{\xi}^{(0)}\right)\boldsymbol{e}_a = \mathbf{0}.
\end{aligned} \tag{B.2.5}$$

The displacement vector in the nonrotating limit, $\boldsymbol{\xi}^{(0)} = \left(\xi_a^{(0)}, \xi_\theta^{(0)}, \xi_\phi^{(0)}\right)$, admits the form

$$\boldsymbol{\xi}^{(0)}(a, \theta, \phi) = \left[\xi_a^{(0)}(a),\ \xi_h^{(0)}(a)\frac{\partial}{\partial\theta},\ \xi_h^{(0)}(a)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]Y_l^m(\theta, \phi), \tag{B.2.6}$$

which is the same as in Eq. (2.3.19), with r replaced by a (we are not going to separate the spherical harmonics yet though). Accordingly, $\boldsymbol{e}_a$ is the unit vector in the direction of increasing a. The nabla operator is defined as in spherical polar coordinates, but with a substituting r. Finally, the Schwarzschild discriminant A and the variable χ are given by Eqs. (2.3.8) and (B.1.17) respectively, with a replacing r.

Like for the case of first-order rotational corrections, we expand the second-order corrections as

$$\boldsymbol{\xi}_\alpha^{(2)} = \sum_\beta c_{\alpha\beta}^{(2)} \boldsymbol{\xi}_\beta^{(0)} \tag{B.2.7}$$

(we temporarily reintroduce mode indices here). With the help of the (expressed in the notation above) perturbed continuity equation (2.2.4), perturbed Poisson equation (2.2.6), and perturbed equation of state (2.2.7), we can also obtain an expansion for $\chi_\alpha^{(2)}$. Then, replacing $\boldsymbol{\xi}_\alpha^{(2)}$ and $\chi_\alpha^{(2)}$ in Eq. (B.2.5) and taking the inner product, defined by Eq. (2.3.55), with $\boldsymbol{\xi}_\alpha^{(0)}$, it becomes (omitting again the index α)

$$\begin{aligned}\left(2\omega^{(2)}\omega^{(0)} - \omega^{(1)2}\right) I^{(0)} &= -\omega^{(0)} \int \left[2\omega^{(1)}\boldsymbol{\xi}^{(1)} + i\boldsymbol{\mathcal{B}}^{(1)}\left(\boldsymbol{\xi}^{(1)}\right)\right] \cdot \boldsymbol{\xi}^{(0)*} \rho_{\rm sp} \mathrm{d}^3\boldsymbol{r} \\ &\quad -2\omega^{(0)2} \int \left[\varepsilon \boldsymbol{\xi}^{(0)} \cdot \boldsymbol{\xi}^{(0)*} + \mathrm{Re}\left(a \xi_a^{(0)} \boldsymbol{\xi}^{(0)*} \cdot \nabla\varepsilon\right)\right] \rho_{\rm sp} \mathrm{d}^3\boldsymbol{r} \\ &+ \int \left(\frac{p\Gamma_1}{\rho}\right)_{\rm sp} \left[\xi_a^{(0)*}\left(\frac{1}{\Gamma_1}\frac{\mathrm{d}\ln p}{\mathrm{d}a}\right)_{\rm sp} + \nabla\cdot\boldsymbol{\xi}^{(0)*}\right] \left(\boldsymbol{\xi}^{(0)}\cdot\nabla\right)\left(3\varepsilon + a\frac{\partial\varepsilon}{\partial a}\right)\rho_{\rm sp}\mathrm{d}^3\boldsymbol{r} \\ &- \int \left[\xi_a^{(0)*}\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}a}\right)_{\rm sp} + \nabla\cdot\boldsymbol{\xi}^{(0)*}\right]\Psi^{(2)}\rho_{\rm sp}\mathrm{d}^3\boldsymbol{r} - \int \boldsymbol{\xi}^{(0)*}\cdot\boldsymbol{\mathcal{D}}^{(2)}\left(\boldsymbol{\xi}^{(0)}\right)\rho_{\rm sp}\mathrm{d}^3\boldsymbol{r}.\end{aligned} \tag{B.2.8}$$

Here, $I^{(0)}$ is defined by Eq. (2.3.55) as the inner product between zeroth-order eigenfunctions and Re denotes the real part of a complex quantity. $\Psi^{(2)}$ is part of the solution of the (corrected to second order in Ω) perturbed Poisson equation (2.2.6) and is given by

$$\begin{aligned}\Psi^{(2)} &= G \iiint \frac{\left\{\nabla\cdot\left[\rho_{\rm sp}\boldsymbol{\xi}^{(0)}\left(3\varepsilon + \frac{\partial\varepsilon}{\partial a}\right)\right]\right\}'}{|\boldsymbol{r}-\boldsymbol{r}'|}(a')^2\sin\theta'\mathrm{d}a'\mathrm{d}\theta'\mathrm{d}\phi' \\ &+ G\iiint\left[\nabla\cdot\left(\rho_{\rm sp}\boldsymbol{\xi}^{(0)}\right)\right]'\left[\left(\varepsilon a\frac{\partial}{\partial a} + \varepsilon' a'\frac{\partial}{\partial a'}\right)\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right](a')^2\sin\theta'\mathrm{d}a'\mathrm{d}\theta'\mathrm{d}\phi',\end{aligned}$$

where $\boldsymbol{r}$ is the position vector and the primed variables (a', θ', ϕ') are the integration variables, with the rest of the primed quantities being functions of these variables. Finally,

$$\begin{aligned}\boldsymbol{\mathcal{D}}^{(2)}\left(\boldsymbol{\xi}^{(0)}\right) &= \nabla\left[\frac{1}{\rho_{\rm sp}}\Delta\left(\frac{p\Gamma_1}{\rho}\right)\nabla\cdot\left(\rho_{\rm sp}\boldsymbol{\xi}^{(0)}\right) - \Delta\left(\frac{p\Gamma_1}{\rho}A\right)\xi_a^{(0)} - \frac{\Delta\rho_c}{\rho_{c,\rm sp}}\chi^{(0)}\right] \\ &\quad + \Delta\left(\frac{p\Gamma_1}{\rho}A\right)\left(\nabla\cdot\boldsymbol{\xi}^{(0)}\right)\boldsymbol{e}_a,\end{aligned} \tag{B.2.9}$$

which contains all the terms associated with the difference operator $\varDelta$, defined above.

We can now calculate the second-order rotational corrections to the eigenfrequencies, $\omega^{(2)}$, for a given form of the deformation function ε. All the terms on the right-hand side of Eqs. (B.2.8) are related to the distortion of the star, induced by the centrifugal force, except the first term, which is due to the Coriolis force and requires the knowledge of the first-order rotational corrections to the eigenfunctions. For these we will use Eqs. (B.1.14)–(B.1.16) (with r replaced by a), which we substitute, together with Eq. (B.2.6), in the Coriolis term, to get

$$\begin{aligned}
-\omega^{(0)}\int\left[2\omega^{(1)}\boldsymbol{\xi}^{(1)}+i\boldsymbol{\mathcal{B}}^{(1)}\left(\boldsymbol{\xi}^{(1)}\right)\right]\cdot\boldsymbol{\xi}^{(0)*}\rho_{\rm sp}{\rm d}^3\boldsymbol{r} = \\
&-2\omega^{(1)}\omega^{(0)}\int_0^R\left[\xi_a^{(1)}\xi_a^{(0)}+\frac{l(l+1)}{\omega^{(0)4}a^2}\chi^{(0)}\chi^{(1)}\right]\rho_{\rm sp}a^2{\rm d}a \\
&+2m\Omega\omega^{(0)}\int_0^R\frac{1}{\omega^{(0)2}a}\left[\chi^{(1)}\left(\xi_a^{(0)}+\frac{\chi^{(0)}}{\omega^{(0)2}a}\right)+\chi^{(0)}\xi_a^{(1)}\right]\rho_{\rm sp}a^2{\rm d}a \\
&-4m\Omega\omega^{(1)}\int_0^R\left(\frac{\chi^{(0)}}{\omega^{(0)2}a}\right)^2\rho_{\rm sp}a^2{\rm d}a-4\omega^{(1)2}\int_0^R\xi_a^{(0)2}\rho_{\rm sp}a^2{\rm d}a \\
&+4\Omega^2\int_0^R\left\{\frac{2}{3}\left(1-\mathcal{I}_{l,m}\right)\xi_a^{(0)2}-2\mathcal{I}_{l,m}\xi_a^{(0)}\frac{\chi^{(0)}}{\omega^{(0)2}a}\right. \\
&\left.+\left(\frac{\chi^{(0)}}{\omega^{(0)2}a}\right)^2\left[\frac{l(l+1)}{3}\left(2\mathcal{I}_{l,m}+1\right)-2\mathcal{I}_{l,m}\right]\right\}\rho_{\rm sp}a^2{\rm d}a, \qquad \text{(B.2.10)}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}_{l,m} &= \iint Y_l^m(\theta,\phi)Y_l^{m*}(\theta,\phi)P_2(\cos\theta)\sin\theta{\rm d}\theta{\rm d}\phi \\
&= \frac{3}{2(2l+1)}\left[\frac{(l+1)^2-m^2}{2l+3}+\frac{l^2-m^2}{2l-1}\right]-\frac{1}{2}, \qquad \text{(B.2.11)}
\end{aligned}$$

with P_l denoting the Legendre polynomials (see, for example, Abramowitz and Stegun 1972, Chap. 8). Here, $\xi_a^{(0)}$, $\xi_a^{(1)}$, $\chi^{(0)}$, and $\chi^{(1)}$ depend only on a; as discussed earlier, zeroth-order quantities are found from Eqs. (2.3.16)–(2.3.18) and (2.3.24)–(2.3.27) and first-order ones from Eqs. (B.1.18)–(B.1.25). Finally, $\omega^{(1)}$ is given by Eq. (B.1.9).

B.2.2 Application to Polytropes

We are going to apply the generic formalism presented above in polytropes (see Appendix A). In order to simplify the process, we will assume that central density and pressure changes, induced by rotation, have a negligible effect on the calculation of second-order eigenfrequency corrections, i.e.,

$$\Delta\rho_c = \Delta p_c = 0.$$

This choice makes the rotating star have a larger mass than its nonrotating counterpart [see Appendix A.2 and specifically Eq. (A.2.13); cf. also Christensen-Dalsgaard and Thompson 1999]. Based on Eqs. (B.2.3) and (B.2.4), this implies

$$\rho(a) = \rho_{\rm sp}(a),$$

which means that the density on an equipotential surface, corresponding to a, on the rotating star coincides with the density on a sphere, with radius a, on the nonrotating star. Then, from Eq. (B.2.9) we see that

$$\boldsymbol{\mathcal{D}}^{(2)}\left(\boldsymbol{\xi}^{(0)}\right) = \mathbf{0}.$$

In order to obtain an expression for the deformation function ε, we need to use the results obtained in Appendix A.2. To avoid conflict between the Lane-Emden variable ξ and the displacement vector $\boldsymbol{\xi}$, we will replace the former with ζ. In our current notation, we may represent ζ as

$$\zeta = \bar{\zeta}(1+\varepsilon), \tag{B.2.12}$$

where $\bar{\zeta}$ is the Lane-Emden analogue of a, namely, from Eqs. (A.1.3), (B.2.2) and (B.2.12),

$$a = \left[\frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1}\right]^{1/2}\bar{\zeta}.$$

Then, according to Eq. (B.2.3), the solution $\theta(\zeta)$ of the Lane-Emden equation (A.1.1) is related to the solution $\Theta(\zeta)$ of the equation for rotating polytropes (A.2.1) by

$$\theta(\bar{\zeta}) = \Theta\left[\bar{\zeta}(1+\varepsilon)\right],$$

from which we get

$$\varepsilon(a,\theta) = \frac{1}{3}\frac{\Omega^2}{GM/R^3}\left(\frac{a}{R}\right)^3\frac{M}{M_r}\left[\alpha(a) - \beta(a)P_2(\cos\theta)\right], \tag{B.2.13}$$

where

$$\alpha(a) = \frac{6\psi_0(\bar{\zeta})}{\bar{\zeta}^2}$$

and

$$\beta(a) = -\frac{6A_2\psi_2(\bar{\zeta})}{\bar{\zeta}^2},$$

with Chandrasekhar's functions, ψ_0 and ψ_2, obtained from Eqs. (A.2.6) and (A.2.7), and A_2 given by Eq. (A.2.8). Also, M_r is defined in Eq. (2.3.42), with r replaced by a, whereas M and R denote the mass and radius of the (nonrotating) star respectively.

We can now substitute Eq. (B.2.13) in Eq. (B.2.8), to get (omitting the subscript sp)

$$\begin{aligned}
\frac{\omega^{(2)}}{\omega^{(0)}} I^{(0)} &= \frac{1}{2}\left(\frac{\omega^{(1)}}{\omega^{(0)}}\right)^2 I^{(0)} - \frac{1}{2\omega^{(0)}}\int\left[2\omega^{(1)}\boldsymbol{\xi}^{(1)} + i\boldsymbol{\mathcal{B}}^{(1)}\left(\boldsymbol{\xi}^{(1)}\right)\right]\cdot\boldsymbol{\xi}^{(0)*}\rho \mathrm{d}^3\boldsymbol{r} \\
&- S\int_0^R\left\{\left[(4-U)\,\xi_a^{(0)2} + l(l+1)\left(\frac{\chi^{(0)}}{\omega^{(0)2}a}\right)^2\right]\alpha + \xi_a^{(0)2}\frac{\mathrm{d}\alpha}{\mathrm{d}\ln a}\right\} w\mathrm{d}a \\
&+ D\mathcal{I}_{l,m}\int_0^R\left\{\left[(4-U)\,\beta + \frac{\mathrm{d}\beta}{\mathrm{d}\ln a}\right]\xi_a^{(0)2} + 3\beta\xi_a^{(0)}\frac{\chi^{(0)}}{\omega^{(0)2}a}\right. \\
&\qquad\left. + [l(l+1)-3]\left(\frac{\chi^{(0)}}{\omega^{(0)2}a}\right)^2\beta\right\} w\mathrm{d}a \\
&- \frac{S}{2\omega^{(0)2}}\int_0^R\left(\chi^{(0)} - \delta\Phi^{(0)}\right)\xi_a^{(0)} f(\alpha)\frac{w}{a}\mathrm{d}a \\
&+ \frac{D\mathcal{I}_{l,m}}{2\omega^{(0)2}}\int_0^R\left(\chi^{(0)} - \delta\Phi^{(0)}\right)\left\{\xi_a^{(0)} f(\beta) + 3\left[(6-U)\,\beta + \frac{\mathrm{d}\beta}{\mathrm{d}\ln a}\right]\frac{\chi^{(0)}}{\omega^{(0)2}a}\right\}\frac{w}{a}\mathrm{d}a \\
&- \frac{4\pi G}{2\omega^{(0)2}(2l+1)}\int_0^R\left[\xi_a^{(0)}A - \frac{\rho}{\Gamma_1 p}\left(\chi^{(0)} - \delta\Phi^{(0)}\right)\right] \\
&\qquad\times\left\{\frac{1}{a^{l+1}}\int_0^a (a')^{l+2}\left[\mathbb{S}(\alpha)S - \mathbb{D}(\beta)D\mathcal{I}_{l,m}\right]\mathrm{d}a'\right. \\
&\qquad\left. + a^l\int_a^R\frac{1}{(a')^{l-1}}\left[\mathbb{S}(\alpha)S - \mathbb{D}(\beta)D\mathcal{I}_{l,m}\right]\mathrm{d}a'\right\}\rho a^2\mathrm{d}a, \qquad \text{(B.2.14)}
\end{aligned}$$

where[4]

$$w = c_1\rho a^2,$$
$$S = D = \frac{1}{3}\frac{\Omega^2}{GM/R^3},$$

[4]In the original work of Saio (1981), the parameter D also includes the effects of tidal deformations from a companion star and is not the same as parameter S.

and

$$
\begin{aligned}
f\begin{pmatrix}\alpha\\ \beta\end{pmatrix} &= \left[2\,(3-U)^2 + \left(A^* + V_g\right)U + (9-2U)\,\frac{\mathrm{d}}{\mathrm{d}\ln a} + \frac{\mathrm{d}^2}{\mathrm{d}(\ln a)^2}\right]\begin{pmatrix}\alpha\\ \beta\end{pmatrix},\\
\mathbb{S}(\alpha) &= \left[\!\!\left[c_1\rho\left\{\left[\xi_a^{(0)}A - \frac{\rho}{\Gamma_1 p}\left(\chi^{(0)} - \delta\Phi^{(0)}\right)\right]\left[(6-U)\,\alpha + \frac{\mathrm{d}\alpha}{\mathrm{d}\ln a}\right] + \frac{\xi_a^{(0)}}{a}f(\alpha)\right\}\right]\!\!\right]_{a=a'},\\
\mathbb{D}(\beta) &= \mathbb{S}(\beta) + \left\{\frac{3c_1\rho}{a}\frac{\chi^{(0)}}{\omega^{(0)2}a}\left[(6-U)\,\beta + \frac{\mathrm{d}\beta}{\mathrm{d}\ln a}\right]\right\}_{a=a'}.
\end{aligned}
$$

The variables $c_1,\ U,\ A^*$, and V_g are defined in Sect. 2.3.3 (with a replacing r), $\mathcal{I}_{l,m}$ is given by Eq. (B.2.11), and the second term on the right-hand side of Eq. (B.2.14) was evaluated in Eq. (B.2.10).

We may now write the second-order eigenfrequency correction in the form

$$
\omega^{(2)} = C_2\omega^{(0)}\left(\frac{\Omega}{\omega^{(0)}}\right)^2. \tag{B.2.15}
$$

The parameter C_2 can be decomposed as

$$
C_2 = X_1 + m^2 Y_1 + X_2 + m^2 Y_2 + Z, \tag{B.2.16}
$$

where the term $X_1 + m^2 Y_1$ corresponds to the effects of the Coriolis force, included in the first two terms on the right-hand side of Eq. (B.2.14), whereas $X_2 + m^2 Y_2$ and Z are due to the deformation of the star and comprise terms proportional to D and S respectively.

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Appendix C
Polar Mode Growth/Damping Rates

We will evaluate the polar mode growth/damping rate γ, including contributions from gravitational waves, shear viscosity, and bulk viscosity, namely

$$\gamma = \gamma_{\rm GW} + \gamma_{\rm SV} + \gamma_{\rm BV},$$

or, using Eq. (3.6.7),

$$\gamma = \frac{1}{2E}\left[\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\rm GW} + \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\rm SV} + \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\rm BV}\right],$$

where the mode energy E is given by Eq. (2.6.4) (evaluated at unit amplitude) and its rate of change due to gravitational waves, shear viscosity, and bulk viscosity is defined in Eqs. (3.5.1), (3.6.1), (3.6.4), respectively. We will consider mode eigenfunctions as obtained in the nonrotating limit, due to their simple spherical harmonic dependence, which makes the various integrals analytically tractable.

C.1 Gravitational Waves

From Eq. (3.5.1) we have

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\rm GW} = -\sum_{l_{\rm min}}^{\infty} N_l\,\omega\,(\omega - m\Omega)^{2l+1}\left(|\delta D_l^m|^2 + |\delta J_l^m|^2\right), \qquad \text{(C.1.1)}$$

with the constant N_l given by Eq. (3.5.3), and the mass and current multipoles, δD_l^m and δJ_l^m, defined in Eqs. (3.5.4) and (3.5.5) respectively. In the nonrotating limit, polar mode eigenfunctions are given by Eq. (2.3.19), which means that their current multipoles are zero, namely

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$$\delta J_l^m = \frac{2i\omega}{c(l+1)} \int \left(\xi_r Y_l^m \boldsymbol{e}_r + \xi_h r \nabla Y_l^m\right) \cdot \left(r\boldsymbol{e}_r \times \nabla Y_l^{*m}\right) \rho r^l \mathrm{d}^3\boldsymbol{r} = 0, \quad \text{(C.1.2)}$$

where we used the fact that $\delta \boldsymbol{v} = i\omega\boldsymbol{\xi}$ and $\boldsymbol{v} = \boldsymbol{0}$. Mass multipoles are accordingly written as

$$\delta D_l^m = \int_0^R \delta\rho(r) r^{l+2} \mathrm{d}r, \quad \text{(C.1.3)}$$

where the orthogonality of spherical harmonics (2.3.14) was used. The (Eulerian) density perturbation can be written as a function of $\boldsymbol{\xi}$ by using the perturbed continuity equation (2.2.4) as

$$\delta\rho(r, \theta, \phi) = -\rho\nabla \cdot \boldsymbol{\xi} - \xi_r(r) Y_l^m(\theta, \phi) \frac{\mathrm{d}\rho}{\mathrm{d}r},$$

with the divergence of the displacement vector given by

$$\nabla \cdot \boldsymbol{\xi} = \left[\frac{\mathrm{d}\xi_r}{\mathrm{d}r} + \frac{2}{r}\xi_r - l(l+1)\frac{\xi_h}{r}\right] Y_l^m(\theta, \phi), \quad \text{(C.1.4)}$$

where Eq. (2.3.13) was used. Alternatively, using Eq. (2.3.7), $\delta\rho$ is expressed in terms of the dimensionless variables defined in Sect. 2.3.3 as

$$\delta\rho(r) = \rho\left[V_g(y_2 - y_3) + A^* y_1\right].$$

Note that, since in the nonrotating star there can be no instability, the eigenfrequency ω in Eq. (C.1.1) cannot be evaluated in the nonrotating limit.

C.2 Shear Viscosity

According to Eq. (3.6.1),

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{SV}} = -\int 2\eta\, \delta\sigma^{ab} \delta\sigma_{ab}^* \mathrm{d}^3\boldsymbol{r}, \quad \text{(C.2.1)}$$

with the shear tensor $\delta\sigma^{ab}$ defined in Eq. (3.6.2) and the shear viscosity coefficient η given by Eq. (3.6.3). An expression for Eq. (C.2.1) in terms of the components of the polar mode eigenfunction (2.3.19) can be found in Cutler et al. (1990), namely[5]

[5]Cutler et al. (1990) give this expression in terms of the covariant components of the displacement vector, which, in their notation, are

$$\xi_a = \left[\frac{W(r)}{r} Y_l^m \nabla_a r - V(r) \nabla_a Y_l^m\right] r^l e^{i\omega t}.$$

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{SV}} = -2\omega^2 \frac{\tilde{\eta}}{T^2} \int_0^R \left\{ \frac{3}{2}\alpha_1^2 + 2l(l+1)\alpha_2^2 + l(l+1)\left[\frac{1}{2}l(l+1) - 1\right]\xi_h^2 \right\} \rho^{9/4}\mathrm{d}r, \tag{C.2.2}$$

where

$$\alpha_1 = \frac{r^2}{3}\left\{\frac{2}{r}\left[\frac{\mathrm{d}\xi_r}{\mathrm{d}r} - \frac{\xi_r}{r}\right] + l(l+1)\frac{\xi_h}{r^2}\right\}$$

and

$$\alpha_2 = -\frac{r}{2}\left[\frac{\mathrm{d}\xi_h}{\mathrm{d}r} - \frac{\xi_h}{r} + \frac{\xi_r}{r}\right].$$

Moreover, the eigenfrequency ω is evaluated in the nonrotating limit and the viscosity coefficient η is rescaled, for convenience, as

$$\tilde{\eta} = \frac{\eta}{\rho^{9/4}T^{-2}} = 347\ \mathrm{g}^{-5/4}\,\mathrm{cm}^{23/4}\,\mathrm{s}^{-1}\,\mathrm{K}^2.$$

Using the dimensionless variables of Sect. 2.3.3, Eq. (C.2.2) becomes

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{SV}} = -2\omega^2 \frac{\tilde{\eta}}{T^2} \int_0^R \left\{ \frac{3}{2}\alpha_1^2 + 2l(l+1)\alpha_2^2 + l(l+1)\left[\frac{1}{2}l(l+1) - 1\right]\left(\frac{ry_2}{c_1\tilde{\omega}^2}\right)^2 \right\} \rho^{9/4}\mathrm{d}r,$$

with

$$\alpha_1 = \frac{2}{3}r^2\frac{\mathrm{d}y_1}{\mathrm{d}r} + \frac{1}{3}l(l+1)\frac{ry_2}{c_1\tilde{\omega}^2}$$

and

Since we are working with spherical coordinates, these components do not coincide with the physical components $\xi_{(a)}$, i.e., the components expressed in the $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$ basis. To get the latter, we have to use the transformation $\xi^a = \xi_{(a)}/\sqrt{g_{aa}}$, or $\xi_a = g_{ab}\xi^b = g_{ab}\xi_{(b)}/\sqrt{g_{bb}}$, where the metric tensor of flat space is expressed in spherical coordinates as

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

Then, in our notation,

$$\boldsymbol{\xi} = \left(\frac{W}{r}r^l, -\frac{V}{r}r^l\frac{\partial}{\partial\theta}, -\frac{V}{r}r^l\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)Y_l^m e^{i\omega t}.$$

Comparing to Eq. (2.3.19), this means that $\xi_r = Wr^{l-1}$ and $\xi_h = -Vr^{l-1}$, from which Eq. (C.2.2) can be obtained. Furthermore, our parameters α_1 and α_2 differ from the ones defined in Cutler et al. (1990) by a factor r^{l-1}.

$$\alpha_2 = -\frac{r}{2}\left\{\frac{1}{c_1\tilde{\omega}^2}\left[r\frac{\mathrm{d}y_2}{\mathrm{d}r} - y_2(3-U)\right] + y_1\right\}.$$

C.3 Bulk Viscosity

The energy rate due to bulk viscosity is given by Eq. (3.6.4), i.e.,

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{BV}} = -\int \zeta\delta\sigma\delta\sigma^* \mathrm{d}^3\boldsymbol{r}, \tag{C.3.1}$$

with the expansion scalar $\delta\sigma$ defined in Eq. (3.6.5) and the bulk viscosity coefficient ζ given by Eq. (3.6.6). Using $\delta\boldsymbol{v} = i\omega\boldsymbol{\xi}$, we get

$$\delta\sigma = i\omega\nabla\cdot\boldsymbol{\xi},$$

where the divergence of the displacement vector is given by Eq. (C.1.4). We further rescale the bulk viscosity coefficient ζ as

$$\tilde{\zeta} = \frac{\zeta}{\rho^2\omega^{-2}T^6} = 6\times 10^{-59}\ \mathrm{g}^{-1}\,\mathrm{cm}^5\,\mathrm{s}^{-3}\,\mathrm{K}^{-6},$$

to get

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{BV}} = -\tilde{\zeta}T^6\int_0^R\left[\frac{\mathrm{d}\xi_r}{\mathrm{d}r} + \frac{2}{r}\xi_r - l(l+1)\frac{\xi_h}{r}\right]^2\rho^2 r^2\mathrm{d}r, \tag{C.3.2}$$

where the orthogonality of spherical harmonics (2.3.14) was used. Alternatively, we can use the perturbed continuity equation (2.2.4) to get

$$\nabla\cdot\boldsymbol{\xi} = -\frac{1}{\rho}\left[\delta\rho(r)Y_l^m + \xi_r(r)Y_l^m\frac{\mathrm{d}\rho}{\mathrm{d}r}\right],$$

which, with the help of Eqs. (2.3.7) and (2.3.8), can be expressed in terms of the dimensionless variables of Sect. 2.3.3 as

$$\nabla\cdot\boldsymbol{\xi} = V_g(y_1 - y_2 + y_3)Y_l^m.$$

C.4 Mode Energy

The mode energy E is given by Eq. (2.6.4) (evaluated at unit amplitude), namely

$$E = \omega \left[2\omega \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle - \langle \boldsymbol{\xi}, i\boldsymbol{\mathcal{B}}(\boldsymbol{\xi}) \rangle \right]. \tag{C.4.1}$$

Since we consider the eigenfunctions in the nonrotating limit, $\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle$ is the moment of inertia I of the perturbation, given by Eq. (2.3.56). Also, in Appendix B.1 we proved that

$$\omega^{(1)} = \frac{1}{2I} \langle \boldsymbol{\xi}, i\boldsymbol{\mathcal{B}}(\boldsymbol{\xi}) \rangle,$$

where $\omega^{(1)}$ is the first-order rotational correction to the mode eigenfrequency. Thus, Eq. (C.4.1) is written as

$$E = 2I\omega \left(\omega - \omega^{(1)} \right). \tag{C.4.2}$$

In Eq. (C.4.2) the eigenfrequency ω has to include rotational corrections. If the eigenfrequency is also evaluated in the nonrotating limit, the mode energy is simply given by Eq. (2.3.58) (evaluated at unit amplitude).

Reference

1. Cutler, C., Lindblom, L., & Splinter, R. J. (1990). Damping times for neutron star oscillations. *The Astrophysical Journal*, *363*, 603–611. https://doi.org/10.1086/169370.

Appendix D
Equations of Motion

D.1 Equation of Motion for Quadratic Perturbations

The equation of motion for quadratic perturbations, in terms of the velocity $\boldsymbol{v} \equiv \delta \boldsymbol{v}$, is easily derived by differentiating the perturbed Euler equation (4.1.4) with respect to time. Then,

$$\ddot{\boldsymbol{v}} + \boldsymbol{\mathcal{B}}(\dot{\boldsymbol{v}}) + \boldsymbol{\mathcal{C}}(\boldsymbol{v}) + \boldsymbol{\mathcal{N}} = \boldsymbol{0}, \tag{D.1.1}$$

where

$$\boldsymbol{\mathcal{B}}(\boldsymbol{v}) = 2\boldsymbol{\Omega} \times \boldsymbol{v}, \tag{D.1.2}$$

$$\boldsymbol{\mathcal{C}}(\boldsymbol{v}) = \frac{1}{\rho}\nabla\left(\frac{\partial \delta_1 p}{\partial t}\right) - \frac{\nabla p}{\rho^2}\frac{\partial \delta_1 \rho}{\partial t} + \nabla\left(\frac{\partial \delta_1 \Phi}{\partial t}\right), \tag{D.1.3}$$

and

$$\boldsymbol{\mathcal{N}} = \frac{\partial}{\partial t}\left[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \frac{\nabla \delta_2 p}{\rho} + \delta_1\left(\frac{1}{\rho}\right)\nabla \delta_1 p + \delta_2\left(\frac{1}{\rho}\right)\nabla p + \nabla \delta_2 \Phi\right]. \tag{D.1.4}$$

We will attempt to derive expressions for the perturbations in terms of the velocity and first-order terms. From the perturbed continuity equation (4.1.3), distinguishing first- and second-order terms, we get

$$\frac{\partial \delta_1 \rho}{\partial t} = -\rho \nabla\cdot\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\rho \tag{D.1.5}$$

and

$$\frac{\partial \delta_2 \rho}{\partial t} = -\delta_1\rho \nabla\cdot\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\delta_1\rho. \tag{D.1.6}$$

Accordingly, the perturbed Poisson equation (4.1.5) gives

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$$\nabla^2 \delta_1 \Phi = 4\pi G \delta_1 \rho$$

and

$$\nabla^2 \delta_2 \Phi = 4\pi G \delta_2 \rho,$$

whose solutions are

$$\delta_1 \Phi = -G \int \frac{\delta_1 \rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}^3 \boldsymbol{r}'$$

and

$$\delta_2 \Phi = -G \int \frac{\delta_2 \rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}^3 \boldsymbol{r}',$$

or, differentiating with respect to time and using Eqs. (D.1.5) and (D.1.6),

$$\frac{\partial \delta_1 \Phi}{\partial t} = G \int \frac{\nabla' \cdot (\rho \boldsymbol{v})}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}^3 \boldsymbol{r}' \tag{D.1.7}$$

and

$$\frac{\partial \delta_2 \Phi}{\partial t} = G \int \frac{\nabla' \cdot (\delta_1 \rho \boldsymbol{v})}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}^3 \boldsymbol{r}', \tag{D.1.8}$$

where ∇' denotes differentiation with respect to $\boldsymbol{r}'$.

In order to derive the perturbed equation of state (4.1.6), we applied a Lagrangian perturbation to the equation of state (2.1.4), which, to quadratic order, gives

$$\Delta p = \left(\frac{\partial p}{\partial \rho}\right)_\mu \Delta \rho + \frac{1}{2} \left(\frac{\partial^2 p}{\partial \rho^2}\right)_\mu (\Delta_1 \rho)^2,$$

considering frozen or adiabatic perturbations ($\Delta\mu \approx 0$; see Sect. 2.2). After some manipulation, we get Eq. (4.1.6), which, making use of Eq. (4.1.2), can be decomposed into first- and second-order Eulerian perturbations, namely

$$\frac{\partial \delta_1 p}{\partial t} = -\boldsymbol{v} \cdot \nabla p - p \Gamma_1 \nabla \cdot \boldsymbol{v} \tag{D.1.9}$$

and

$$\frac{\partial \delta_2 p}{\partial t} = \nabla \cdot \boldsymbol{v} \left\{ \boldsymbol{\xi} \cdot \nabla (p \Gamma_1) + p \Gamma_1 \left[\Gamma_1 + \left(\frac{\partial \ln \Gamma_1}{\partial \ln \rho} \right)_\mu \right] \nabla \cdot \boldsymbol{\xi} \right\} - \boldsymbol{v} \cdot \nabla \delta_1 p. \tag{D.1.10}$$

Finally, an expression for the Eulerian perturbation of $1/\rho$ can be obtained as

$$\frac{1}{\rho+\delta\rho} = \frac{1}{\rho\left(1+\frac{\delta\rho}{\rho}\right)} = \frac{1}{\rho}\left[1-\frac{\delta\rho}{\rho}+\left(\frac{\delta_1\rho}{\rho}\right)^2+\mathcal{O}\left(\boldsymbol{\xi}^3\right)\right],$$

which implies

$$\delta_1\left(\frac{1}{\rho}\right) = -\frac{\delta_1\rho}{\rho^2} \tag{D.1.11}$$

and

$$\delta_2\left(\frac{1}{\rho}\right) = -\frac{\delta_2\rho}{\rho^2}+\frac{(\delta_1\rho)^2}{\rho^3}. \tag{D.1.12}$$

Using the equations above, Eq. (D.1.4) can be reduced to Eq. (4.1.10), i.e.,

$$\begin{aligned}\boldsymbol{\mathcal{N}} = {} & \frac{\partial}{\partial t}\left[\nabla\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}\right)-\boldsymbol{v}\times(\nabla\times\boldsymbol{v})-\frac{\delta_1\rho}{\rho^2}\nabla\delta_1 p+\frac{(\delta_1\rho)^2}{2\rho^3}\nabla p\right] \\ & +\frac{1}{\rho}\nabla\left[\!\left[\nabla\cdot\boldsymbol{v}\left\{\boldsymbol{\xi}\cdot\nabla\left(p\Gamma_1\right)+p\Gamma_1\left[\Gamma_1+\left(\frac{\partial\ln\Gamma_1}{\partial\ln\rho}\right)_\mu\right]\nabla\cdot\boldsymbol{\xi}\right\}-\boldsymbol{v}\cdot\nabla\delta_1 p\right]\!\right] \\ & +\frac{\nabla p}{\rho}\left(\boldsymbol{v}\cdot\nabla\right)\left(\frac{\delta_1\rho}{\rho}\right)+G\nabla\left[\int\frac{\nabla'\cdot(\delta_1\rho\boldsymbol{v})}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'\right],\end{aligned} \tag{D.1.13}$$

where the quadratic velocity term was rewritten using the vectorial identity

$$\nabla\left(\boldsymbol{A}\cdot\boldsymbol{B}\right) = \left(\boldsymbol{A}\cdot\nabla\right)\boldsymbol{B}+\left(\boldsymbol{B}\cdot\nabla\right)\boldsymbol{A}+\boldsymbol{A}\times\left(\nabla\times\boldsymbol{B}\right)+\boldsymbol{B}\times\left(\nabla\times\boldsymbol{A}\right).$$

D.2 Amplitude Equation of Motion

In order to derive the equation of motion for the amplitude (4.1.13) we have to replace the velocity expansion (4.1.11) in the equation of motion for quadratic perturbations (4.1.7) [or (D.1.1)].

Starting from the linear terms and using the eigenvalue equation for the perturbations (2.2.12), we get

$$\begin{aligned}\ddot{\boldsymbol{v}}+\boldsymbol{\mathcal{B}}(\dot{\boldsymbol{v}})+\boldsymbol{\mathcal{C}}(\boldsymbol{v}) = {} & \sum_\alpha\left[\left(i\omega_\alpha\ddot{Q}_\alpha-\omega_\alpha^2\dot{Q}_\alpha\right)\boldsymbol{\xi}_\alpha e^{i\omega_\alpha t}+\left(-i\omega_\alpha\ddot{Q}_\alpha^*-\omega_\alpha^2\dot{Q}_\alpha^*\right)\boldsymbol{\xi}_\alpha^* e^{-i\omega_\alpha t}\right] \\ & -\sum_\alpha\left[\dot{Q}_\alpha\boldsymbol{\mathcal{C}}(\boldsymbol{\xi}_\alpha)e^{i\omega_\alpha t}+\dot{Q}_\alpha^*\boldsymbol{\mathcal{C}}(\boldsymbol{\xi}_\alpha^*)e^{-i\omega_\alpha t}\right].\end{aligned} \tag{D.2.1}$$

Now, from Eq. (2.6.2) it is implied that

$$\sum_{\alpha} \left(\dot{Q}_\alpha \boldsymbol{\xi}_\alpha e^{i\omega_\alpha t} + \dot{Q}^*_\alpha \boldsymbol{\xi}^*_\alpha e^{-i\omega_\alpha t} \right) = \mathbf{0}. \tag{D.2.2}$$

Hence, since $\boldsymbol{\mathcal{C}}$ is a linear operator, the second sum in Eq. (D.2.1) vanishes. In order to isolate a single Q_α, we need to use the mode orthogonality condition (2.6.3). We add and subtract terms containing the eigenfrequency $\omega_\beta \neq \omega_\alpha$, to obtain

$$\begin{aligned}
\ddot{\boldsymbol{v}} + \boldsymbol{\mathcal{B}}(\dot{\boldsymbol{v}}) + \boldsymbol{\mathcal{C}}(\boldsymbol{v}) = i \sum_{\alpha} \Big\{ & \left[\left(\omega_\beta + \omega_\alpha \right) \boldsymbol{\xi}_\alpha - i \boldsymbol{\mathcal{B}}(\boldsymbol{\xi}_\alpha) \right] \left(\ddot{Q}_\alpha + i\omega_\alpha \dot{Q}_\alpha \right) e^{i\omega_\alpha t} \\
& + \left[\left(\omega_\beta - \omega_\alpha \right) \boldsymbol{\xi}^*_\alpha - i \boldsymbol{\mathcal{B}}(\boldsymbol{\xi}^*_\alpha) \right] \left(\ddot{Q}^*_\alpha - i\omega_\alpha \dot{Q}^*_\alpha \right) e^{-i\omega_\alpha t} \Big\} \\
-i \sum_{\alpha} \Big\{ & \left[\omega_\beta \boldsymbol{\xi}_\alpha - i \boldsymbol{\mathcal{B}}(\boldsymbol{\xi}_\alpha) \right] \left(\ddot{Q}_\alpha + i\omega_\alpha \dot{Q}_\alpha \right) e^{i\omega_\alpha t} \\
& + \left[\omega_\beta \boldsymbol{\xi}^*_\alpha - i \boldsymbol{\mathcal{B}}(\boldsymbol{\xi}^*_\alpha) \right] \left(\ddot{Q}^*_\alpha - i\omega_\alpha \dot{Q}^*_\alpha \right) e^{-i\omega_\alpha t} \Big\}.
\end{aligned} \tag{D.2.3}$$

Differentiating Eq. (D.2.2) with respect to time, we get

$$\sum_{\alpha} \left(\ddot{Q}_\alpha \boldsymbol{\xi}_\alpha e^{i\omega_\alpha t} + i\omega_\alpha \dot{Q}_\alpha \boldsymbol{\xi}_\alpha e^{i\omega_\alpha t} + \ddot{Q}^*_\alpha \boldsymbol{\xi}^*_\alpha e^{-i\omega_\alpha t} - i\omega_\alpha \dot{Q}^*_\alpha \boldsymbol{\xi}^*_\alpha e^{-i\omega_\alpha t} \right) = \mathbf{0}, \tag{D.2.4}$$

which implies that the second sum in Eq. (D.2.3) vanishes.

Taking the inner product of the remaining terms with $\boldsymbol{\xi}_\beta$, then, based on the orthogonality relation (2.6.3), all terms for which $\alpha \neq \beta$ vanish. Hence,

$$i b_\alpha \left(\ddot{Q}_\alpha + i\omega_\alpha \dot{Q}_\alpha \right) e^{i\omega_\alpha t} = -\langle \boldsymbol{\xi}_\alpha, \boldsymbol{\mathcal{N}} \rangle,$$

or

$$\ddot{Q}_\alpha + i\omega_\alpha \dot{Q}_\alpha = \frac{i}{b_\alpha} \langle \boldsymbol{\xi}_\alpha, \boldsymbol{\mathcal{N}} \rangle e^{-i\omega_\alpha t}. \tag{D.2.5}$$

We shall further assume that the amplitude Q_α changes on a time scale much larger than the mode oscillation period (Dziembowski 1982; see Sect. 4.2.1), i.e.,

$$|\dot{Q}_\alpha| \ll \omega_\alpha |Q_\alpha|, \tag{D.2.6}$$

so second-order derivatives of Q can be ignored. Then, Eq. (D.2.5) takes the form of Eq. (4.1.12), namely

$$\dot{Q}_\alpha = \frac{1}{\omega_\alpha b_\alpha} \langle \boldsymbol{\xi}_\alpha, \boldsymbol{\mathcal{N}} \rangle e^{-i\omega_\alpha t}. \tag{D.2.7}$$

We will now proceed with replacing the velocity expansion (4.1.11) into the quadratic terms $\boldsymbol{\mathcal{N}}$, for which an expression was derived in Appendix D.1. For instance, the first term of Eq. (4.1.10) [or (D.1.13)] becomes

$$\frac{\partial}{\partial t}\left[\nabla\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}\right)\right] = -\sum_{\beta}\sum_{\gamma}\omega_\beta\omega_\gamma\Big[Q_\beta\left(\dot{Q}_\gamma + i\omega_\gamma Q_\gamma\right)\nabla\left(\boldsymbol{\xi}_\beta\cdot\boldsymbol{\xi}_\gamma\right)e^{i(\omega_\beta+\omega_\gamma)t}$$
$$-Q_\beta\left(\dot{Q}^*_\gamma - i\omega_\gamma Q^*_\gamma\right)\nabla\left(\boldsymbol{\xi}_\beta\cdot\boldsymbol{\xi}^*_\gamma\right)e^{i(\omega_\beta-\omega_\gamma)t}$$
$$-Q^*_\beta\left(\dot{Q}_\gamma + i\omega_\gamma Q_\gamma\right)\nabla\left(\boldsymbol{\xi}^*_\beta\cdot\boldsymbol{\xi}_\gamma\right)e^{i(-\omega_\beta+\omega_\gamma)t}$$
$$+Q^*_\beta\left(\dot{Q}^*_\gamma - i\omega_\gamma Q^*_\gamma\right)\nabla\left(\boldsymbol{\xi}^*_\beta\cdot\boldsymbol{\xi}^*_\gamma\right)e^{-i(\omega_\beta+\omega_\gamma)t}\Big].$$

Due to Eq. (D.2.6), the terms involving derivatives of Q will be ignored. The rest of the quadratic terms can be expanded accordingly, using the equations of Appendix D.1. Then, taking the inner product of $\boldsymbol{\mathcal{N}}$ with $\boldsymbol{\xi}_\alpha$, we obtain Eq. (4.1.13), i.e.,

$$\dot{Q}_\alpha(t) = \frac{i}{b_\alpha}\sum_{\beta}\sum_{\gamma}\Big[F_{\alpha\beta\gamma}Q_\beta Q_\gamma e^{i(-\omega_\alpha+\omega_\beta+\omega_\gamma)t} + F_{\alpha\bar{\beta}\gamma}Q^*_\beta Q_\gamma e^{i(-\omega_\alpha-\omega_\beta+\omega_\gamma)t}$$
$$+F_{\alpha\beta\bar{\gamma}}Q_\beta Q^*_\gamma e^{i(-\omega_\alpha+\omega_\beta-\omega_\gamma)t} + F_{\alpha\bar{\beta}\bar{\gamma}}Q^*_\beta Q^*_\gamma e^{i(-\omega_\alpha-\omega_\beta-\omega_\gamma)t}\Big], \tag{D.2.8}$$

where F is the *coupling coefficient*, generally given by

$$F_{\alpha\beta\gamma} = \frac{1}{i\omega_\alpha}\langle\boldsymbol{\xi}_\alpha, \boldsymbol{\mathcal{N}}(\boldsymbol{\xi}_\beta, \boldsymbol{\xi}_\gamma)\rangle. \tag{D.2.9}$$

A bar over an index means that the corresponding mode eigenfunction in $\boldsymbol{\mathcal{N}}$ has to be complex conjugated and its eigenfrequency sign reversed. The explicit form of the coupling coefficient is

$$F_{\alpha\beta\gamma} = \frac{1}{\omega_\alpha}\left(\omega_\beta S_{\alpha\beta\gamma} + \omega_\gamma S_{\alpha\gamma\beta}\right), \tag{D.2.10}$$

where

$$S_{\alpha\beta\gamma} = \int\Big[\!\!\Big[\rho\omega_\beta\omega_\gamma\big[-\nabla\left(\boldsymbol{\xi}_\beta\cdot\boldsymbol{\xi}_\gamma\right) + \boldsymbol{\xi}_\beta\times\left(\nabla\times\boldsymbol{\xi}_\gamma\right) + \boldsymbol{\xi}_\gamma\times\left(\nabla\times\boldsymbol{\xi}_\beta\right)\big]$$
$$-\frac{1}{\rho}\big[\nabla\cdot\left(\rho\boldsymbol{\xi}_\beta\right)\nabla\left(\boldsymbol{\xi}_\gamma\cdot\nabla p + p\Gamma_1\nabla\cdot\boldsymbol{\xi}_\gamma\right) +$$
$$\nabla\cdot\left(\rho\boldsymbol{\xi}_\gamma\right)\nabla\left(\boldsymbol{\xi}_\beta\cdot\nabla p + p\Gamma_1\nabla\cdot\boldsymbol{\xi}_\beta\right)\big]$$
$$+\nabla\cdot\left(\rho\boldsymbol{\xi}_\beta\right)\nabla\cdot\left(\rho\boldsymbol{\xi}_\gamma\right)\frac{\nabla p}{\rho^2} - \left\{\boldsymbol{\xi}_\beta\cdot\nabla\left[\frac{\nabla\cdot\left(\rho\boldsymbol{\xi}_\gamma\right)}{\rho}\right]\right\}\nabla p$$
$$-G\rho\nabla\left\{\int\frac{\nabla'\cdot\left[\boldsymbol{\xi}_\beta\nabla\cdot\left(\rho\boldsymbol{\xi}_\gamma\right)\right]}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'\right\}$$
$$+\nabla\Big\{\boldsymbol{\xi}_\beta\cdot\nabla\left(\boldsymbol{\xi}_\gamma\cdot\nabla p + p\Gamma_1\nabla\cdot\boldsymbol{\xi}_\gamma\right) + \left(\nabla\cdot\boldsymbol{\xi}_\beta\right)\boldsymbol{\xi}_\gamma\cdot\nabla\left(p\Gamma_1\right)$$

$$+ \, p\Gamma_1 \left[\Gamma_1 + \left(\frac{\partial \ln \Gamma_1}{\partial \ln \rho} \right)_\mu \right] \left(\nabla \cdot \boldsymbol{\xi}_\beta \right) \left(\nabla \cdot \boldsymbol{\xi}_\gamma \right) \Bigg\} \Bigg]\!\Bigg] \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3 \boldsymbol{r}. \quad \text{(D.2.11)}$$

Reference

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Appendix E
Polar Mode Coupling Coefficient

Following Dziembowski (1982), we are going to derive an expression for the coupling coefficient (D.2.10) in the nonrotating limit, assuming that the resonant coupled triplet (see Sect. 4.2.1) consists of polar modes, i.e., modes whose eigenfunctions are given by Eq. (2.3.19). This will simplify the calculation greatly, due to the simple spherical harmonic dependence of the eigenfunctions. In this sense, we will find an expression for the *zeroth-order component* (with respect to rotation) of the coupling coefficient.

For the sake of generality though, we will keep assuming that the eigenfrequencies which appear in the formula for the coupling coefficient (D.2.10) have been obtained for the case of a rotating star. This way, we take into account rotational corrections in the zeroth-order component of the coupling coefficient only through the eigenfrequencies. Following the notation of Sect. 2.6.2, we will denote the eigenfrequency in the nonrotating limit, whenever needed, as $\omega^{(0)}$.

E.1 Parametrised Form

For notational convenience, we define the following variables:

$$\mu_\alpha = -\boldsymbol{\xi}_\alpha \cdot \nabla p - p\Gamma_1 \nabla \cdot \boldsymbol{\xi}_\alpha, \tag{E.1.1}$$

$$\lambda_\alpha = \frac{1}{\rho} \nabla \cdot \left(\rho \boldsymbol{\xi}_\alpha\right), \tag{E.1.2}$$

$$w_\alpha = G \int \frac{\rho \lambda_\alpha}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}^3 \boldsymbol{r}', \tag{E.1.3}$$

and

$$\chi = \Gamma_1 + \left(\frac{\partial \ln \Gamma_1}{\partial \ln \rho}\right)_\mu. \tag{E.1.4}$$

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Then, the eigenvalue equation for the perturbations in the nonrotating limit, given by Eq. (2.3.54) [or (2.6.8)], can be expressed as

$$\omega_\alpha^{(0)2}\boldsymbol{\xi}_\alpha = \boldsymbol{\mathcal{C}}(\boldsymbol{\xi}_\alpha) = \frac{\nabla\mu_\alpha}{\rho} + \frac{\nabla p}{\rho}\lambda_\alpha + \nabla w_\alpha, \tag{E.1.5}$$

where we used Eq. (2.2.11) and various first-order relations from Appendix D.1. The coupling coefficient $F_{\alpha\beta\gamma}$, defined by Eq. (D.2.10), is written as

$$\omega_\alpha F_{\alpha\beta\gamma} = \omega_\beta S_{\alpha\beta\gamma} + \omega_\gamma S_{\alpha\gamma\beta}, \tag{E.1.6}$$

where

$$\begin{aligned} S_{\alpha\beta\gamma} = \int \Bigg\{ & \underbrace{-\rho\omega_\beta\omega_\gamma\nabla\left(\boldsymbol{\xi}_\beta\cdot\boldsymbol{\xi}_\gamma\right)}_{T_1} \underbrace{+\rho\omega_\beta\omega_\gamma\boldsymbol{\xi}_\beta\times\left(\nabla\times\boldsymbol{\xi}_\gamma\right) + \rho\omega_\beta\omega_\gamma\boldsymbol{\xi}_\gamma\times\left(\nabla\times\boldsymbol{\xi}_\beta\right)}_{T_2} \\ & \underbrace{+\lambda_\beta\nabla\mu_\gamma + \lambda_\gamma\nabla\mu_\beta + \lambda_\beta\lambda_\gamma\nabla p}_{T_3} \underbrace{-\left(\boldsymbol{\xi}_\beta\cdot\nabla\lambda_\gamma\right)\nabla p}_{T_4} \underbrace{-G\rho\nabla\left[\int\frac{\nabla'\cdot\left(\rho\boldsymbol{\xi}_\beta\lambda_\gamma\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}^3\boldsymbol{r}'\right]}_{T_5} \\ & \underbrace{-\nabla\left(\boldsymbol{\xi}_\beta\cdot\nabla\mu_\gamma\right)}_{T_6} \underbrace{+\nabla\left[\left(\nabla\cdot\boldsymbol{\xi}_\beta\right)\boldsymbol{\xi}_\gamma\cdot\nabla\left(p\Gamma_1\right)\right]}_{T_7} \underbrace{+\nabla\left[p\Gamma_1\chi\left(\nabla\cdot\boldsymbol{\xi}_\beta\right)\left(\nabla\cdot\boldsymbol{\xi}_\gamma\right)\right]}_{T_8} \Bigg\}\cdot\boldsymbol{\xi}_\alpha^*\mathrm{d}^3\boldsymbol{r}. \end{aligned} \tag{E.1.7}$$

We are going to study each term of $\omega_\alpha F_{\alpha\beta\gamma}$ separately (including the corresponding terms from $S_{\alpha\gamma\beta}$). Also, in the following calculation, we will neglect the eigenfrequency detuning $\Delta\omega$, i.e., we shall consider $\omega_\alpha \approx \omega_\beta + \omega_\gamma$.

Term 1

$$T_1 = -(\omega_\beta+\omega_\gamma)\int\rho\omega_\beta\omega_\gamma\nabla\left(\boldsymbol{\xi}_\beta\cdot\boldsymbol{\xi}_\gamma\right)\cdot\boldsymbol{\xi}_\alpha^*\mathrm{d}^3\boldsymbol{r} = -\omega_\alpha\int\rho\omega_\beta\omega_\gamma\nabla\left(\boldsymbol{\xi}_\beta\cdot\boldsymbol{\xi}_\gamma\right)\cdot\boldsymbol{\xi}_\alpha^*\mathrm{d}^3\boldsymbol{r}.$$

Using Gauss's theorem, we get

$$T_1 = -\omega_\alpha\omega_\beta\omega_\gamma\left(\oint\boldsymbol{\xi}_\beta\cdot\boldsymbol{\xi}_\gamma\rho\boldsymbol{\xi}_\alpha^*\cdot\mathrm{d}\boldsymbol{S} - \int\boldsymbol{\xi}_\beta\cdot\boldsymbol{\xi}_\gamma\lambda_\alpha^*\rho\mathrm{d}^3\boldsymbol{r}\right),$$

where $\mathrm{d}\boldsymbol{S}$ is the differential normal area vector at the stellar surface. Since the density ρ vanishes at the surface, the first term is neglected (for similar reasons, all surface integrals will be neglected from now on). So, finally,

$$T_1 = \omega_\alpha\omega_\beta\omega_\gamma K_{\alpha\beta\gamma}, \tag{E.1.8}$$

where

$$K_{\alpha\beta\gamma} = \int\boldsymbol{\xi}_\beta\cdot\boldsymbol{\xi}_\gamma\lambda_\alpha^*\rho\mathrm{d}^3\boldsymbol{r}. \tag{E.1.9}$$

Term 2

$$T_2 = (\omega_\beta + \omega_\gamma) \int \rho\omega_\beta\omega_\gamma \left[\boldsymbol{\xi}_\beta \times \left(\nabla \times \boldsymbol{\xi}_\gamma\right) + \boldsymbol{\xi}_\gamma \times \left(\nabla \times \boldsymbol{\xi}_\beta\right)\right] \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r},$$

or

$$T_2 = \omega_\alpha\omega_\beta\omega_\gamma \left(N_{\alpha\beta\gamma} + N_{\alpha\gamma\beta}\right), \tag{E.1.10}$$

where

$$N_{\alpha\beta\gamma} = \int \rho \left[\boldsymbol{\xi}_\beta \times \left(\nabla \times \boldsymbol{\xi}_\gamma\right)\right] \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r}. \tag{E.1.11}$$

Term 3

$$T_3 = (\omega_\beta + \omega_\gamma) \int \left(\lambda_\beta \nabla\mu_\gamma + \lambda_\gamma \nabla\mu_\beta + \lambda_\beta\lambda_\gamma \nabla p\right) \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r},$$

or

$$T_3 = \omega_\alpha J_{\alpha\beta\gamma}, \tag{E.1.12}$$

where

$$J_{\alpha\beta\gamma} = \int \left(\lambda_\beta \nabla\mu_\gamma + \lambda_\gamma \nabla\mu_\beta + \lambda_\beta\lambda_\gamma \nabla p\right) \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r}. \tag{E.1.13}$$

Term 4

$$T_4 = -\omega_\beta \int \left(\boldsymbol{\xi}_\beta \cdot \nabla\lambda_\gamma\right) \nabla p \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r} - \omega_\gamma \int \left(\boldsymbol{\xi}_\gamma \cdot \nabla\lambda_\beta\right) \nabla p \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r}. \tag{E.1.14}$$

Term 5

$$T_5 = -\omega_\beta \underbrace{\int G\rho\nabla \left[\int \frac{\nabla' \cdot \left(\rho\boldsymbol{\xi}_\beta\lambda_\gamma\right)}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}^3\boldsymbol{r}'\right] \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r}}_{T_{5a}} - \omega_\gamma \underbrace{\int G\rho\nabla \left[\int \frac{\nabla' \cdot \left(\rho\boldsymbol{\xi}_\gamma\lambda_\beta\right)}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}^3\boldsymbol{r}'\right] \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r}}_{T_{5b}}.$$

Using Gauss's theorem, we get

$$T_{5\mathrm{a}} = G \oint \left[\int \frac{\nabla' \cdot \left(\rho\boldsymbol{\xi}_\beta\lambda_\gamma\right)}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}^3\boldsymbol{r}'\right] \rho\boldsymbol{\xi}_\alpha^* \cdot \mathrm{d}\boldsymbol{S} - G \iint \frac{\nabla' \cdot \left(\rho\boldsymbol{\xi}_\beta\lambda_\gamma\right)}{|\boldsymbol{r} - \boldsymbol{r}'|} \rho\lambda_\alpha^* \mathrm{d}^3\boldsymbol{r}' \mathrm{d}^3\boldsymbol{r}.$$

Neglecting the first term and taking into account the symmetry of $1/|\boldsymbol{r} - \boldsymbol{r}'|$, we obtain

$$T_{5\mathrm{a}} = -\int \nabla \cdot \left(\rho \boldsymbol{\xi}_\beta \lambda_\gamma\right) \left[\int \frac{G\rho\lambda_\alpha^*}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{d}^3\boldsymbol{r}'\right] \mathrm{d}^3\boldsymbol{r} = -\int \nabla \cdot \left(\rho \boldsymbol{\xi}_\beta \lambda_\gamma\right) w_\alpha^* \mathrm{d}^3\boldsymbol{r}.$$

Using Gauss's theorem one more time, $T_{5\mathrm{a}}$ gives

$$T_{5\mathrm{a}} = -\oint \rho \lambda_\gamma w_\alpha^* \boldsymbol{\xi}_\beta \cdot \mathrm{d}\boldsymbol{S} + \int \rho \lambda_\gamma \boldsymbol{\xi}_\beta \cdot \nabla w_\alpha^* \mathrm{d}^3\boldsymbol{r}.$$

Neglecting the surface integral and using Eq. (E.1.5) to eliminate ∇w_α^* from the remaining term, the equation above becomes

$$T_{5\mathrm{a}} = \int \omega_\alpha^{(0)2} \rho \lambda_\gamma \boldsymbol{\xi}_\beta \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r} - \int \lambda_\gamma \boldsymbol{\xi}_\beta \cdot \nabla \mu_\alpha^* \mathrm{d}^3\boldsymbol{r} - \int \lambda_\alpha^* \lambda_\gamma \boldsymbol{\xi}_\beta \cdot \nabla p \mathrm{d}^3\boldsymbol{r}.$$

The corresponding expression for T_{5b} is the same, with the indices β and γ interchanged. So, finally, T_5 can be written as

$$\begin{aligned} T_5 = & \underbrace{-\omega_\alpha^{(0)2}\left(\omega_\beta K_{\gamma\alpha\beta} + \omega_\gamma K_{\beta\alpha\gamma}\right)}_{T_{5.1}} \\ & \underbrace{+\omega_\beta \int \lambda_\gamma \boldsymbol{\xi}_\beta \cdot \nabla \mu_\alpha^* \mathrm{d}^3\boldsymbol{r} + \omega_\gamma \int \lambda_\beta \boldsymbol{\xi}_\gamma \cdot \nabla \mu_\alpha^* \mathrm{d}^3\boldsymbol{r}}_{T_{5.2}} \\ & \underbrace{+\omega_\beta \int \lambda_\alpha^* \lambda_\gamma \boldsymbol{\xi}_\beta \cdot \nabla p \mathrm{d}^3\boldsymbol{r} + \omega_\gamma \int \lambda_\alpha^* \lambda_\beta \boldsymbol{\xi}_\gamma \cdot \nabla p \mathrm{d}^3\boldsymbol{r}}_{T_{5.3}}, \end{aligned} \tag{E.1.15}$$

with $K_{\alpha\beta\gamma}$ defined in Eq. (E.1.9).

Term 6

$$T_6 = -\omega_\beta \int \nabla\left[\boldsymbol{\xi}_\beta \cdot \nabla \mu_\gamma\right] \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r} - \omega_\gamma \int \nabla\left[\boldsymbol{\xi}_\gamma \cdot \nabla \mu_\beta\right] \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r}.$$

Applying Gauss's theorem, the first integral becomes

$$\int \nabla\left[\boldsymbol{\xi}_\beta \cdot \nabla \mu_\gamma\right] \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r} = \oint \boldsymbol{\xi}_\beta \cdot \nabla \mu_\gamma \boldsymbol{\xi}_\alpha^* \cdot \mathrm{d}\boldsymbol{S} - \int \boldsymbol{\xi}_\beta \cdot \nabla \mu_\gamma \nabla \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r}.$$

Neglecting the surface integral, the remaining terms give

$$T_6 = \omega_\beta \int \boldsymbol{\xi}_\beta \cdot \nabla \mu_\gamma \nabla \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r} + \omega_\gamma \int \boldsymbol{\xi}_\gamma \cdot \nabla \mu_\beta \nabla \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r}. \tag{E.1.16}$$

Term 7

$$T_7 = \omega_\beta \int \nabla \left[\left(\nabla \cdot \boldsymbol{\xi}_\beta \right) \boldsymbol{\xi}_\gamma \cdot \nabla \left(p\Gamma_1 \right) \right] \cdot \boldsymbol{\xi}^*_\alpha \mathrm{d}^3\boldsymbol{r}$$
$$+ \omega_\gamma \int \nabla \left[\left(\nabla \cdot \boldsymbol{\xi}_\gamma \right) \boldsymbol{\xi}_\beta \cdot \nabla \left(p\Gamma_1 \right) \right] \cdot \boldsymbol{\xi}^*_\alpha \mathrm{d}^3\boldsymbol{r}.$$

Using Gauss's theorem, the first term becomes

$$\int \nabla \left[\left(\nabla \cdot \boldsymbol{\xi}_\beta \right) \boldsymbol{\xi}_\gamma \cdot \nabla \left(p\Gamma_1 \right) \right] \cdot \boldsymbol{\xi}^*_\alpha \mathrm{d}^3\boldsymbol{r} = \oint \left(\nabla \cdot \boldsymbol{\xi}_\beta \right) \boldsymbol{\xi}_\gamma \cdot \nabla \left(p\Gamma_1 \right) \boldsymbol{\xi}^*_\alpha \cdot \mathrm{d}\boldsymbol{S}$$
$$- \int \left(\nabla \cdot \boldsymbol{\xi}_\beta \right) \boldsymbol{\xi}_\gamma \cdot \nabla \left(p\Gamma_1 \right) \nabla \cdot \boldsymbol{\xi}^*_\alpha \mathrm{d}^3\boldsymbol{r}.$$

With the surface integral neglected, we get

$$T_7 = -\omega_\beta \int \left(\nabla \cdot \boldsymbol{\xi}_\beta \right) \boldsymbol{\xi}_\gamma \cdot \nabla \left(p\Gamma_1 \right) \nabla \cdot \boldsymbol{\xi}^*_\alpha \mathrm{d}^3\boldsymbol{r}$$
$$-\omega_\gamma \int \left(\nabla \cdot \boldsymbol{\xi}_\gamma \right) \boldsymbol{\xi}_\beta \cdot \nabla \left(p\Gamma_1 \right) \nabla \cdot \boldsymbol{\xi}^*_\alpha \mathrm{d}^3\boldsymbol{r}. \quad \text{(E.1.17)}$$

Term 8

$$T_8 = \left(\omega_\beta + \omega_\gamma \right) \int \nabla \left[p\Gamma_1 \chi \left(\nabla \cdot \boldsymbol{\xi}_\beta \right) \left(\nabla \cdot \boldsymbol{\xi}_\gamma \right) \right] \cdot \boldsymbol{\xi}^*_\alpha \mathrm{d}^3\boldsymbol{r}.$$

Applying Gauss's theorem and neglecting the surface term, we obtain

$$T_8 = \omega_\alpha M_{\alpha\beta\gamma}, \quad \text{(E.1.18)}$$

where

$$M_{\alpha\beta\gamma} = -\int p\Gamma_1 \chi \left(\nabla \cdot \boldsymbol{\xi}^*_\alpha \right) \left(\nabla \cdot \boldsymbol{\xi}_\beta \right) \left(\nabla \cdot \boldsymbol{\xi}_\gamma \right) \mathrm{d}^3\boldsymbol{r}. \quad \text{(E.1.19)}$$

Terms 4, 5.2, 5.3, 6 and 7 are combined to give

$$T' = \omega_\beta L_{\beta\alpha\gamma} + \omega_\gamma L_{\gamma\alpha\beta}, \quad \text{(E.1.20)}$$

where

$$L_{\beta\alpha\gamma} = \int \big[- \boldsymbol{\xi}^*_\alpha \cdot \nabla p \boldsymbol{\xi}_\beta \cdot \nabla \lambda_\gamma + \boldsymbol{\xi}_\beta \cdot \nabla \mu^*_\alpha \lambda_\gamma + \lambda^*_\alpha \lambda_\gamma \boldsymbol{\xi}_\beta \cdot \nabla p$$
$$+ \left(\nabla \cdot \boldsymbol{\xi}^*_\alpha \right) \boldsymbol{\xi}_\beta \cdot \nabla \mu_\gamma - \left(\nabla \cdot \boldsymbol{\xi}^*_\alpha \right) \left(\nabla \cdot \boldsymbol{\xi}_\beta \right) \boldsymbol{\xi}_\gamma \cdot \nabla (p\Gamma_1) \big] \mathrm{d}^3\boldsymbol{r}. \quad \text{(E.1.21)}$$

Combining terms 1, 2, 3, 5.1, 8, and T' above, we finally get the more compact expression

$$\omega_\alpha F_{\alpha\beta\gamma} = \omega_\alpha\omega_\beta\omega_\gamma K_{\alpha\beta\gamma} - \omega_\alpha^{(0)2}\omega_\beta K_{\gamma\alpha\beta} - \omega_\alpha^{(0)2}\omega_\gamma K_{\beta\alpha\gamma} + \omega_\alpha\omega_\beta\omega_\gamma\left(N_{\alpha\beta\gamma} + N_{\alpha\gamma\beta}\right) + \omega_\alpha M_{\alpha\beta\gamma} + \omega_\alpha J_{\alpha\beta\gamma} + \omega_\beta L_{\beta\alpha\gamma} + \omega_\gamma L_{\gamma\alpha\beta}. \tag{E.1.22}$$

E.2 Angular Part

Henceforth, we are going to use the dimensionless formulation presented in Sect. 2.3.3. The polar mode eigenfunction (2.3.19) is written as

$$\boldsymbol{\xi}_\alpha = r y_{1,\alpha} Y_\alpha \boldsymbol{e}_r + \frac{r y_{2,\alpha}}{c_1 \tilde{\omega}_\alpha^{(0)2}} r \nabla_\perp Y_\alpha, \tag{E.2.1}$$

where

$$Y_\alpha \equiv Y_{l_\alpha}^{m_\alpha} \tag{E.2.2}$$

and $\nabla_\perp$ is the horizontal component of the gradient operator, defined by Eq. (2.3.3). Furthermore, in this notation,

$$w_\alpha = g r y_{3,\alpha} Y_\alpha \tag{E.2.3}$$

and

$$\frac{\partial w_\alpha}{\partial r} = g y_{4,\alpha} Y_\alpha. \tag{E.2.4}$$

Equations (E.2.1), (E.2.3) and (E.2.4) will be used throughout the following calculations without reference.

The horizontal component of Eq. (E.1.5) gives

$$\mu_\alpha = g\rho r z_\alpha Y_\alpha, \tag{E.2.5}$$

with

$$z_\alpha = y_{2,\alpha} - y_{3,\alpha}. \tag{E.2.6}$$

Using Eq. (E.2.5) in Eq. (E.1.1), we also get

$$\nabla\cdot\boldsymbol{\xi}_\alpha = V_g\left(y_{1,\alpha} - z_\alpha\right) Y_\alpha. \tag{E.2.7}$$

Substituting Eq. (E.2.7) in Eq. (E.1.2), we obtain

$$\lambda_\alpha = -\left(A^* y_{1,\alpha} + V_g z_\alpha\right) Y_\alpha. \tag{E.2.8}$$

Now, we will examine the terms of Eq. (E.1.22) individually and prove that their angular part is reduced to the integral

$$Z_{\alpha\beta\gamma} = \iint Y_\alpha^* Y_\beta Y_\gamma \sin\theta \mathrm{d}\theta \mathrm{d}\phi. \tag{E.2.9}$$

This is a known integral (e.g., see Sakurai and Napolitano 2011, Sect. 3.8) and is equal to

$$Z_{\alpha\beta\gamma} = \sqrt{\frac{(2l_\beta+1)(2l_\gamma+1)}{4\pi(2l_\alpha+1)}} \langle l_\beta l_\gamma 00 | l_\beta l_\gamma l_\alpha 0\rangle \langle l_\beta l_\gamma m_\beta m_\gamma | l_\beta l_\gamma l_\alpha m_\alpha \rangle, \tag{E.2.10}$$

where

$$\langle l_\beta l_\gamma m_\beta m_\gamma | l_\beta l_\gamma l_\alpha m_\alpha \rangle = \frac{\delta_{m_\alpha, m_\beta+m_\gamma}\sqrt{2l_\alpha+1}}{\sqrt{\left(l_\alpha+l_\beta+l_\gamma+1\right)!}} \prod_k \sqrt{(-l_k+l_{k'}+l_{k''})!}\sqrt{(l_k+m_k)!}\sqrt{(l_k-m_k)!}$$
$$\sum_{j=j_-}^{j_+} \frac{(-1)^j}{j!(-j-l_\alpha+l_\beta+l_\gamma)!(-j+l_\beta-m_\beta)!(-j+l_\gamma+m_\gamma)!(j+l_\alpha-l_\gamma+m_\beta)!(j+l_\alpha-l_\beta-m_\gamma)!}$$

are the *Clebsch-Gordan coefficients*, with

$$j_- = \max(-l_\alpha+l_\gamma-m_\beta, -l_\alpha+l_\beta+m_\gamma, 0) \quad \text{and} \quad j_+ = \min(l_\beta-m_\beta, l_\gamma+m_\gamma).$$

The index k in the product successively takes one of the values α, β, γ, whereas the indices k' and k'' take the values that come next and after next respectively, namely

$$(k, k', k'') = \begin{cases} (\alpha, \beta, \gamma) \\ (\beta, \gamma, \alpha) \\ (\gamma, \alpha, \beta)\,. \end{cases} \tag{E.2.11}$$

$\boldsymbol{K_{\alpha\beta\gamma}}$

Using Eq. (E.2.8), Eq. (E.1.9) becomes

$$K_{\alpha\beta\gamma} = -\int \left(A^* y_{1,\alpha} + V_g z_\alpha\right)\left[y_{1,\beta} y_{1,\gamma} Z_{\alpha\beta\gamma} + y_{2,\beta} y_{2,\gamma} \frac{X_{\alpha\beta\gamma}}{\left(c_1 \tilde{\omega}_\beta^{(0)} \tilde{\omega}_\gamma^{(0)}\right)^2}\right] \rho r^4 \mathrm{d}r,$$

where

$$X_{\alpha\beta\gamma} = \iint Y_\alpha^* \nabla_\perp Y_\beta \cdot \nabla_\perp Y_\gamma r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi.$$

Integrating by parts and making use of Eq. (2.3.13), we get

$$X_{\alpha\beta\gamma} + X_{\beta\alpha\gamma} = \Lambda_\gamma Z_{\alpha\beta\gamma},$$

where

$$\Lambda = l(l+1). \tag{E.2.12}$$

In a similar manner, we find

$$X_{\alpha\beta\gamma} + X_{\gamma\alpha\beta} = \Lambda_\beta Z_{\alpha\beta\gamma}$$

and

$$X_{\beta\alpha\gamma} + X_{\gamma\alpha\beta} = \Lambda_\alpha Z_{\alpha\beta\gamma}.$$

Thus,

$$X_{\alpha\beta\gamma} = \frac{-\Lambda_\alpha + \Lambda_\beta + \Lambda_\gamma}{2} Z_{\alpha\beta\gamma}. \tag{E.2.13}$$

So, $K_{\alpha\beta\gamma}$ is written as

$$K_{\alpha\beta\gamma} = -Z_{\alpha\beta\gamma} \int \left(A^* y_{1,\alpha} + V_g z_\alpha\right) \left[y_{1,\beta} y_{1,\gamma} + y_{2,\beta} y_{2,\gamma} \frac{\Lambda_\beta + \Lambda_\gamma - \Lambda_\alpha}{2\left(c_1 \tilde{\omega}_\beta^{(0)} \tilde{\omega}_\gamma^{(0)}\right)^2} \right] \rho r^4 \mathrm{d}r. \tag{E.2.14}$$

N$_{\alpha\beta\gamma}$

Taking the curl of Eq. (E.1.5), we get

$$\omega_\alpha^{(0)2} \nabla \times \boldsymbol{\xi}_\alpha = \nabla \left(\frac{1}{\rho}\right) \times \nabla \mu_\alpha + \nabla \lambda_\alpha \times \left(\frac{\nabla p}{\rho}\right),$$

or, using Eqs. (E.2.5) and (E.2.8),

$$\omega_\alpha^{(0)2} \nabla \times \boldsymbol{\xi}_\alpha = g A^* \left(z_\alpha - y_{1,\alpha}\right) \boldsymbol{e}_r \times \nabla_\perp Y_\alpha.$$

So, making use of the vectorial identity

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\, \boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\, \boldsymbol{C},$$

we obtain

$$\omega_\gamma^{(0)2} \boldsymbol{\xi}_\beta \times \left(\nabla \times \boldsymbol{\xi}_\gamma\right) = r g A^* \left(z_\gamma - y_{1,\gamma}\right) \left[\left(\nabla_\perp Y_\beta\right) \cdot \left(\nabla_\perp Y_\gamma\right) \frac{r y_{2,\beta}}{c_1 \tilde{\omega}_\beta^{(0)2}} \boldsymbol{e}_r - y_{1,\beta} Y_\beta \nabla_\perp Y_\gamma \right].$$

Finally, with the help of Eq. (E.2.13), Eq. (E.1.11) is written as

$$N_{\alpha\beta\gamma} = \frac{Z_{\alpha\beta\gamma}}{\omega_\gamma^{(0)2}} \int \left(z_\gamma - y_{1,\gamma}\right) \left[y_{1,\alpha} y_{2,\beta} \frac{\Lambda_\beta + \Lambda_\gamma - \Lambda_\alpha}{2c_1 \tilde{\omega}_\beta^{(0)2}} - y_{1,\beta} y_{2,\alpha} \frac{\Lambda_\gamma + \Lambda_\alpha - \Lambda_\beta}{2c_1 \tilde{\omega}_\alpha^{(0)2}} \right] g A^* \rho r^3 \mathrm{d}r. \tag{E.2.15}$$

$M_{\alpha\beta\gamma}$

Substituting Eq. (E.2.7) in Eq. (E.1.19), we get

$$M_{\alpha\beta\gamma} = -Z_{\alpha\beta\gamma} \int \prod_k \left(y_{1,k} - z_k\right) \chi V_g^2 g \rho r^3 \mathrm{d}r, \tag{E.2.16}$$

with k successively taking the values α, β, γ.

$J_{\alpha\beta\gamma}$

Equations (E.2.5) and (E.2.8) may be combined to give

$$\lambda_\alpha = -A^* y_{1,\alpha} Y_\alpha - \frac{\mu_\alpha}{p\Gamma_1}. \tag{E.2.17}$$

Using Eq. (E.2.17), Eq. (E.1.13) becomes

$$J_{\alpha\beta\gamma} = -\int \boldsymbol{\xi}_\alpha^* \cdot \left[A^* \left(y_{1,\beta} Y_\beta \nabla \mu_\gamma + y_{1,\gamma} Y_\gamma \nabla \mu_\beta\right) - \lambda_\beta \lambda_\gamma \nabla p + \frac{\nabla\left(\mu_\beta \mu_\gamma\right)}{p\Gamma_1} \right] \mathrm{d}^3\boldsymbol{r}.$$

Applying Gauss's theorem to the last term, we get

$$\begin{aligned} -\int \frac{\nabla\left(\mu_\beta \mu_\gamma\right)}{p\Gamma_1} \cdot \boldsymbol{\xi}_\alpha^* \mathrm{d}^3\boldsymbol{r} &= -\oint \frac{\mu_\beta \mu_\gamma}{p\Gamma_1} \boldsymbol{\xi}_\alpha^* \cdot \mathrm{d}\boldsymbol{S} + \int \frac{\mu_\beta \mu_\gamma}{p\Gamma_1} \left[\nabla \cdot \boldsymbol{\xi}_\alpha^* - \boldsymbol{\xi}_\alpha^* \cdot \nabla \ln(p\Gamma_1)\right] \mathrm{d}^3\boldsymbol{r} \\ &= Z_{\alpha\beta\gamma} \int z_\beta z_\gamma \left[-z_\alpha V_g + y_{1,\alpha} \left(V + V_g - \frac{\mathrm{d}\ln\Gamma_1}{\mathrm{d}\ln r}\right)\right] V_g g \rho r^3 \mathrm{d}r, \end{aligned}$$

where we used Eqs. (E.2.5) and (E.2.7), and the surface integral was neglected. Next, we calculate the radial component of $\nabla \mu_\alpha$ via Eq. (E.1.5), as

$$\begin{aligned} \frac{\partial \mu_\alpha}{\partial r} &= \rho \left[\omega_\alpha^{(0)2} r y_{1,\alpha} Y_\alpha + g\left(\lambda_\alpha - y_{4,\alpha} Y_\alpha\right)\right] \\ &= g \rho Y_\alpha \left[y_{1,\alpha}\left(c_1 \tilde{\omega}_\alpha^{(0)2} - A^*\right) - V_g z_\alpha - y_{4,\alpha}\right], \end{aligned} \tag{E.2.18}$$

where Eq. (E.2.8) was used. With the help of the relations above, and using Eqs. (E.2.8) and (E.2.13), we obtain the final expression for $J_{\alpha\beta\gamma}$, which is

$$\begin{aligned} J_{\alpha\beta\gamma} = Z_{\alpha\beta\gamma} \int \Bigg\{ & A^* y_{2,\alpha} \left(\frac{\Lambda_\beta - \Lambda_\gamma - \Lambda_\alpha}{2c_1 \tilde{\omega}_\alpha^{(0)2}} y_{1,\beta} z_\gamma + \frac{\Lambda_\gamma - \Lambda_\beta - \Lambda_\alpha}{2c_1 \tilde{\omega}_\alpha^{(0)2}} y_{1,\gamma} z_\beta \right) \\ & + A^* \left[y_{1,\alpha} y_{1,\beta} y_{1,\gamma} \left(A^* - c_1 \tilde{\omega}_\beta^{(0)2} - c_1 \tilde{\omega}_\gamma^{(0)2} \right) + y_{1,\alpha} \left(y_{1,\beta} y_{4,\gamma} + y_{1,\gamma} y_{4,\beta} \right) \right] \\ & + V_g \left[y_{1,\alpha} z_\beta z_\gamma \left(V - \frac{\mathrm{d}\ln\Gamma_1}{\mathrm{d}\ln r} \right) - V_g z_\alpha z_\beta z_\gamma \right] \Bigg\} g \rho r^3 \mathrm{d}r. \end{aligned} \tag{E.2.19}$$

$\boldsymbol{L_{\beta\alpha\gamma}}$

Applying Gauss's theorem to the first and fourth terms of Eq. (E.1.21), while neglecting the surface integrals, we get

$$L_{\beta\alpha\gamma} = \int \{\boldsymbol{\xi}_\beta \cdot \left[\lambda_\gamma \nabla \left(-p\Gamma_1 \nabla \cdot \boldsymbol{\xi}_\alpha^*\right) - \mu_\gamma \nabla \left(\nabla \cdot \boldsymbol{\xi}_\alpha^*\right)\right] - \mu_\gamma \left(\nabla \cdot \boldsymbol{\xi}_\alpha^*\right) \left(\nabla \cdot \boldsymbol{\xi}_\beta\right) \\ + \nabla p \cdot \left(\boldsymbol{\xi}_\alpha^* \lambda_\gamma \nabla \cdot \boldsymbol{\xi}_\beta + \lambda_\alpha^* \lambda_\gamma \boldsymbol{\xi}_\beta\right) - \nabla(p\Gamma_1) \cdot \boldsymbol{\xi}_\gamma \left(\nabla \cdot \boldsymbol{\xi}_\alpha^*\right) \left(\nabla \cdot \boldsymbol{\xi}_\beta\right) \} \mathrm{d}^3\boldsymbol{r},$$

where we used Eq. (E.2.5). Furthermore, with the help of Eq. (E.2.17), we obtain

$$L_{\beta\alpha\gamma} = \int \Big\{ p\Gamma_1 A^* y_{1,\gamma} Y_\gamma \boldsymbol{\xi}_\beta \cdot \nabla \left(\nabla \cdot \boldsymbol{\xi}_\alpha^*\right) + \nabla p \cdot \left(\boldsymbol{\xi}_\alpha^* \lambda_\gamma \nabla \cdot \boldsymbol{\xi}_\beta + \lambda_\alpha^* \lambda_\gamma \boldsymbol{\xi}_\beta\right) \\ - \mu_\gamma \left(\nabla \cdot \boldsymbol{\xi}_\alpha^*\right) \left(\nabla \cdot \boldsymbol{\xi}_\beta\right) - \nabla(p\Gamma_1) \cdot \left[\lambda_\gamma \left(\nabla \cdot \boldsymbol{\xi}_\alpha^*\right) \boldsymbol{\xi}_\beta + \left(\nabla \cdot \boldsymbol{\xi}_\alpha^*\right) \left(\nabla \cdot \boldsymbol{\xi}_\beta\right) \boldsymbol{\xi}_\gamma\right] \Big\} \mathrm{d}^3\boldsymbol{r}.$$

Next, we calculate the radial component of $\nabla \left(\nabla \cdot \boldsymbol{\xi}_\alpha\right)$. First, we take the derivative of Eq. (E.1.1) with respect to r and solve for $\partial \left(\nabla \cdot \boldsymbol{\xi}_\alpha\right)/\partial r$, i.e.,

$$\frac{\partial}{\partial r} \left(\nabla \cdot \boldsymbol{\xi}_\alpha\right) = \frac{1}{p\Gamma_1} \left[-\frac{\partial \mu_\alpha}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}r} \left(\boldsymbol{\xi}_\alpha \cdot \nabla p\right) - \frac{\mathrm{d}(p\Gamma_1)}{\mathrm{d}r} \nabla \cdot \boldsymbol{\xi}_\alpha \right].$$

We evaluate the first and third terms of the relation above by making use of Eqs. (E.2.18) and (E.2.7) respectively. From the second term we get

$$-\frac{\mathrm{d}}{\mathrm{d}r} \left(\boldsymbol{\xi}_\alpha \cdot \nabla p\right) = \rho g Y_\alpha \frac{\mathrm{d}}{\mathrm{d}r} \left(r y_{1,\alpha}\right) + y_{1,\alpha} Y_\alpha r \frac{\mathrm{d}(\rho g)}{\mathrm{d}r}.$$

Taking the divergence of $\boldsymbol{\xi}_\alpha$ via Eq. (E.2.1), we obtain an expression for $\mathrm{d}\left(r y_{1,\alpha}\right)/\mathrm{d}r$. So, finally,

$$\frac{\partial}{\partial r} \left(\nabla \cdot \boldsymbol{\xi}_\alpha\right) = \frac{V_g}{r} Y_\alpha \Bigg[y_{1,\alpha} \left(U - 4 - c_1 \tilde{\omega}_\alpha^{(0)2}\right) + \left(y_{1,\alpha} - z_\alpha\right) \left(V - \frac{\mathrm{d}\ln\Gamma_1}{\mathrm{d}\ln r}\right) \\ + \frac{\Lambda_\alpha}{c_1 \tilde{\omega}_\alpha^{(0)2}} y_{2,\alpha} + y_{4,\alpha} \Bigg].$$

Using the relation above, together with Eqs. (E.2.7) and (E.2.8), and appropriately rearranging the terms, the final expression we get for $L_{\beta\alpha\gamma}$ is

$$L_{\beta\alpha\gamma} = Z_{\alpha\beta\gamma} \int \Big\{ - y_{1,\alpha} y_{1,\beta} y_{1,\gamma} \left[\left(A^* + V_g\right)^2 + A^* \left(4 + c_1 \tilde{\omega}_\alpha^{(0)2} - U\right)\right] + A^* y_{4,\alpha} y_{1,\beta} y_{1,\gamma} \\ + V_g \sum_k y_{1,k} \left[\left(A^* + V_g\right) y_{1,k'} \left(y_{1,k''} - z_{k''}\right) - V_g \left(y_{1,k'} - z_{k'}\right) \left(y_{1,k''} - z_{k''}\right)\right]$$

$$+ V_g \left(V - \frac{\mathrm{d}\ln\Gamma_1}{\mathrm{d}\ln r} \right) \left(y_{1,\alpha} - z_\alpha \right) \left[y_{1,\beta} \left(y_{1,\gamma} - z_\gamma \right) + y_{1,\gamma} \left(y_{1,\beta} - z_\beta \right) - y_{1,\beta} y_{1,\gamma} \right]$$

$$+ A^* y_{1,\gamma} \left[\frac{\Lambda_1 + \Lambda_2 - \Lambda_3}{2 c_1 \tilde{\omega}_\beta^{(0)2}} y_{2,\beta} \left(y_{1,\alpha} - z_\alpha \right) + \frac{\Lambda_\alpha}{c_1 \tilde{\omega}_\alpha^{(0)2}} y_{2,\alpha} y_{1,\beta} \right]$$

$$+ V_g^2 \prod_k \left(y_{1,k} - z_k \right) \Big\} g \rho r^3 \mathrm{d}r, \tag{E.2.20}$$

with the indices k, k', and k'' behaving as in Eq. (E.2.11).

E.3 Radial Part

We will now proceed with the evaluation of $F_{\alpha\beta\gamma}$ from Eq. (E.1.22), based on the formulae derived in the previous section for the various parameters. Once again, we will study each term separately. For reasons explained later, we will divide $F_{\alpha\beta\gamma}$ by GM/R^3.

$\boldsymbol{K_{\alpha\beta\gamma}}$

Using Eq. (E.2.14), the combination of the K terms in Eq. (E.1.22) gives

$$\frac{R^3}{GM} \left[K_{\alpha\beta\gamma} \omega_\beta \omega_\gamma - K_{\gamma\alpha\beta} \frac{\omega_\alpha^{(0)2} \omega_\beta}{\omega_\alpha} - K_{\beta\alpha\gamma} \frac{\omega_\alpha^{(0)2} \omega_\gamma}{\omega_\alpha} \right] = \tilde{\mathcal{H}}_1,$$

where

$$\tilde{\mathcal{H}}_1 = Z_{\alpha\beta\gamma} \int \Big\{ - \sum_k \left(A^* y_{1,k} + V_g z_k \right) \Psi_k \times \left(\varpi_{k'} \varpi_{k''} y_{1,k'} y_{1,k''} + \frac{QC_k}{c_1^2} \psi_{k'} \psi_{k''} y_{2,k'} y_{2,k''} \right) \Big\} \rho r^4 \mathrm{d}r, \tag{E.3.1}$$

with

$$\varpi_k = \begin{cases} \tilde{\omega}_k & \text{for } k = \alpha \\ -\tilde{\omega}_k & \phantom{\text{for }} k = \beta, \gamma \end{cases} \tag{E.3.2}$$

and

$$QC_k = \frac{-\Lambda_k + \Lambda_{k'} + \Lambda_{k''}}{2 \varpi_{k'} \varpi_{k''}} \tag{E.3.3}$$

[Λ has been defined in Eq. (E.2.12)], whereas

$$\psi_k = \left(\frac{\tilde{\omega}_k}{\tilde{\omega}_k^{(0)}} \right)^2 \tag{E.3.4}$$

and

$$\Psi_k = \begin{cases} 1 & \text{for} \quad k = \alpha \\ 1/\psi_\alpha & \phantom{\text{for}} \quad k = \beta, \gamma. \end{cases} \tag{E.3.5}$$

The behaviour of the indices k, k', and k'' is explained in Eq. (E.2.11).

$\boldsymbol{N_{\alpha\beta\gamma}}$

From Eqs. (E.1.22) and (E.2.15) we obtain

$$\frac{R^3}{GM} N_{\alpha\beta\gamma}\omega_\beta\omega_\gamma = Z_{\alpha\beta\gamma}\frac{\tilde{\omega}_\beta}{\tilde{\omega}_\gamma}\psi_\gamma \int \frac{A^*}{c_1^2}\left(z_\gamma - y_{1,\gamma}\right) \times \left(y_{1,\alpha}y_{2,\beta}\psi_\beta \frac{\Lambda_\beta + \Lambda_\gamma - \Lambda_\alpha}{2\tilde{\omega}_\beta^2} - y_{1,\beta}y_{2,\alpha}\psi_\alpha \frac{\Lambda_\gamma + \Lambda_\alpha - \Lambda_\beta}{2\tilde{\omega}_\alpha^2}\right)\rho r^4 \mathrm{d}r.$$

The corresponding expression for $N_{\alpha\gamma\beta}\omega_\beta\omega_\gamma/(GM/R^3)$ is the same, with the indices β and γ interchanged. For later convenience, we will write the correction factor ψ_γ (but not ψ_α and ψ_β) as

$$\psi_\gamma = 1 + \Xi_\gamma,$$

where

$$\Xi_\gamma = \frac{\tilde{\omega}_\gamma^2 - \tilde{\omega}_\gamma^{(0)2}}{\tilde{\omega}_\gamma^{(0)2}}. \tag{E.3.6}$$

Doing the same for the equivalent factor ψ_β in $N_{\alpha\gamma\beta}\omega_\beta\omega_\gamma/(GM/R^3)$, we get for the N terms

$$\begin{aligned} \frac{R^3}{GM}\left(N_{\alpha\beta\gamma} + N_{\alpha\gamma\beta}\right)\omega_\beta\omega_\gamma = {} & Z_{\alpha\beta\gamma}\frac{\tilde{\omega}_\beta}{\tilde{\omega}_\gamma}\left(1 + \Xi_\gamma\right)\int \frac{A^*}{c_1^2}\left(z_\gamma - y_{1,\gamma}\right) \\ & \times \left(y_{1,\alpha}y_{2,\beta}\psi_\beta \frac{\Lambda_\beta + \Lambda_\gamma - \Lambda_\alpha}{2\tilde{\omega}_\beta^2} - y_{1,\beta}y_{2,\alpha}\psi_\alpha \frac{\Lambda_\gamma + \Lambda_\alpha - \Lambda_\beta}{2\tilde{\omega}_\alpha^2}\right)\rho r^4 \mathrm{d}r \\ & + Z_{\alpha\beta\gamma}\frac{\tilde{\omega}_\gamma}{\tilde{\omega}_\beta}\left(1 + \Xi_\beta\right)\int \frac{A^*}{c_1^2}\left(z_\beta - y_{1,\beta}\right) \\ & \times \left(y_{1,\alpha}y_{2,\gamma}\psi_\gamma \frac{\Lambda_\gamma + \Lambda_\beta - \Lambda_\alpha}{2\tilde{\omega}_\gamma^2} - y_{1,\gamma}y_{2,\alpha}\psi_\alpha \frac{\Lambda_\beta + \Lambda_\alpha - \Lambda_\gamma}{2\tilde{\omega}_\alpha^2}\right)\rho r^4 \mathrm{d}r. \end{aligned}$$

$\boldsymbol{M_{\alpha\beta\gamma}}$

Equations (E.1.22) and (E.2.16) give

$$\frac{R^3}{GM} M_{\alpha\beta\gamma} = -Z_{\alpha\beta\gamma}\int \frac{1}{c_1}\prod_k \left(y_{1,k} - z_k\right)\chi V_g^2 \rho r^4 \mathrm{d}r.$$

J $_{\alpha\beta\gamma}$

From Eqs. (E.1.22) and (E.2.19) we get

$$\frac{R^3}{GM}J_{\alpha\beta\gamma} = Z_{\alpha\beta\gamma}\int\Bigg\{\underbrace{\frac{A^*}{c_1^2}y_{2,\alpha}\psi_\alpha\left(\frac{\Lambda_\beta-\Lambda_\gamma-\Lambda_\alpha}{2\tilde{\omega}_\alpha^2}y_{1,\beta}z_\gamma+\frac{\Lambda_\gamma-\Lambda_\beta-\Lambda_\alpha}{2\tilde{\omega}_\alpha^2}y_{1,\gamma}z_\beta\right)}_{J_\Lambda}$$
$$+\frac{A^*}{c_1}\Bigg[\underbrace{y_{1,\alpha}y_{1,\beta}y_{1,\gamma}\left(A^*-c_1\frac{\tilde{\omega}_\beta^2}{\psi_\beta}-c_1\frac{\tilde{\omega}_\gamma^2}{\psi_\gamma}\right)}_{J_1}\underbrace{+y_{1,\alpha}\left(y_{1,\beta}y_{4,\gamma}+y_{1,\gamma}y_{4,\beta}\right)}_{J_2}\Bigg]$$
$$+\frac{V_g}{c_1}\Bigg[\underbrace{y_{1,\alpha}z_\beta z_\gamma\left(V-\frac{\mathrm{d}\ln\Gamma_1}{\mathrm{d}\ln r}\right)}_{J_3}\underbrace{-V_g z_\alpha z_\beta z_\gamma}_{J_4}\Bigg]\Bigg\}\rho r^4\mathrm{d}r.$$

L $_{\beta\alpha\gamma}$

With the help of Eq. (E.2.20), the L terms in Eq. (E.1.22) yield

$$\frac{R^3}{GM}\left[\frac{\omega_\beta}{\omega_\alpha}L_{\beta\alpha\gamma}+\frac{\omega_\gamma}{\omega_\alpha}L_{\gamma\alpha\beta}\right] = Z_{\alpha\beta\gamma}\int\frac{1}{c_1}\Bigg\{\underbrace{V_g^2\prod_k\left(y_{1,k}-z_k\right)}_{L_1}$$
$$\underbrace{+V_g\sum_k y_{1,k}\left[\left(A^*+V_g\right)y_{1,k'}\left(y_{1,k''}-z_{k''}\right)-V_g\left(y_{1,k'}-z_{k'}\right)\left(y_{1,k''}-z_{k''}\right)\right]}_{L_2}$$
$$\underbrace{+V_g\left(V-\frac{\mathrm{d}\ln\Gamma_1}{\mathrm{d}\ln r}\right)\left(y_{1,\alpha}-z_\alpha\right)\left[y_{1,\beta}\left(y_{1,\gamma}-z_\gamma\right)+y_{1,\gamma}\left(y_{1,\beta}-z_\beta\right)-y_{1,\beta}y_{1,\gamma}\right]}_{L_3}$$
$$\underbrace{-y_{1,\alpha}y_{1,\beta}y_{1,\gamma}\left[\left(A^*+V_g\right)^2+A^*\left(4+c_1\frac{\tilde{\omega}_\alpha^2}{\psi_\alpha}-U\right)\right]}_{L_4}\underbrace{+A^*y_{4,\alpha}y_{1,\beta}y_{1,\gamma}}_{L_5}$$
$$\left.\begin{aligned}&+A^*y_{1,\gamma}\left[\frac{\Lambda_\alpha+\Lambda_\beta-\Lambda_\gamma}{2c_1\tilde{\omega}_\alpha\tilde{\omega}_\beta}\psi_\beta y_{2,\beta}\left(y_{1,\alpha}-z_\alpha\right)+\frac{\Lambda_\alpha}{c_1\tilde{\omega}_\alpha^2}\psi_\alpha y_{2,\alpha}y_{1,\beta}\right]\\&+A^*y_{1,\beta}\frac{\Lambda_\gamma+\Lambda_\alpha-\Lambda_\beta}{2c_1\tilde{\omega}_\alpha\tilde{\omega}_\gamma}\psi_\gamma y_{2,\gamma}\left(y_{1,\alpha}-z_\alpha\right)\Bigg\}g\rho r^3\mathrm{d}r.\end{aligned}\right\}L_\Lambda$$

Adding the N terms to J_Λ and L_Λ, we get

$$\tilde{\mathcal{H}}_4 = Z_{\alpha\beta\gamma}\int\frac{A^*}{c_1^2}\sum_k\psi_k y_{2,k}\left(GC_k y_{1,k'}y_{1,k''}+QC_{k'}y_{1,k'}z_{k''}+QC_{k''}y_{1,k''}z_{k'}\right)\rho r^4\mathrm{d}r+N_{\mathrm{cor}}, \tag{E.3.7}$$

where

$$GC_k = \frac{\Lambda_k\varpi_k+\left(\Lambda_{k'}-\Lambda_{k''}\right)\left(\varpi_{k'}-\varpi_{k''}\right)}{2\varpi_k\varpi_{k'}\varpi_{k''}} \tag{E.3.8}$$

and

$$
\begin{aligned}
N_{\mathrm{cor}} =& Z_{\alpha\beta\gamma}\,\Xi_\beta \int \frac{A^*}{c_1^2}\left(z_\beta - y_{1,\beta}\right)\left(QC_\alpha y_{1,\alpha} y_{2,\gamma}\psi_\gamma - \frac{\varpi_\gamma}{\varpi_\alpha} QC_\gamma y_{1,\gamma} y_{2,\alpha}\psi_\alpha\right)\rho r^4 \mathrm{d}r \\
&+ Z_{\alpha\beta\gamma}\,\Xi_\gamma \int \frac{A^*}{c_1^2}\left(z_\gamma - y_{1,\gamma}\right)\left(QC_\alpha y_{1,\alpha} y_{2,\beta}\psi_\beta - \frac{\varpi_\beta}{\varpi_\alpha} QC_\beta y_{1,\beta} y_{2,\alpha}\psi_\alpha\right)\rho r^4 \mathrm{d}r.
\end{aligned}
\tag{E.3.9}
$$

Furthermore, $J_2 + L_5$ gives

$$
\tilde{\mathcal{H}}_{3.3} = Z_{\alpha\beta\gamma} \int \frac{A^*}{c_1} \sum_k y_{4,k} y_{1,k'} y_{1,k''} \rho r^4 \mathrm{d}r. \tag{E.3.10}
$$

Also, from $J_1 + L_4$, we obtain

$$
\tilde{\mathcal{H}}_{3.1} \underbrace{- Z_{\alpha\beta\gamma} \int \frac{V_g^2}{c_1} \prod_k y_{1,k} \rho r^4 \mathrm{d}r}_{R_1} \underbrace{- Z_{\alpha\beta\gamma} \int \frac{3 V_g A^*}{c_1} \prod_k y_{1,k} \rho r^4 \mathrm{d}r}_{R_2},
$$

where

$$
\tilde{\mathcal{H}}_{3.1} = Z_{\alpha\beta\gamma} \int \frac{A^*}{c_1} \left(V_g + U - 4 - c_1 \sum_k \frac{\varpi_k^2}{\psi_k}\right) \prod_k y_{1,k} \rho r^4 \mathrm{d}r. \tag{E.3.11}
$$

Adding the terms J_3, M, L_1, and L_3, we get

$$
\tilde{\mathcal{H}}_{2.2} \underbrace{+ Z_{\alpha\beta\gamma} \int \frac{V_g}{c_1} \left[V_g \prod_k \left(y_{1,k} - z_k\right) + \left(V - \frac{\mathrm{d}\ln\Gamma_1}{\mathrm{d}\ln r}\right) \prod_k z_k \right] \rho r^4 \mathrm{r}}_{R_3},
$$

where

$$
\tilde{\mathcal{H}}_{2.2} = Z_{\alpha\beta\gamma} \int \frac{V_g}{c_1} A_g \prod_k \left(y_{1,k} - z_k\right) \rho r^4 \mathrm{d}r, \tag{E.3.12}
$$

with

$$
A_g = -\frac{\mathrm{d}\ln\Gamma_1}{\mathrm{d}\ln r} - V_g \left(\frac{\partial \ln \Gamma_1}{\partial \ln \rho}\right)_\mu. \tag{E.3.13}
$$

Moreover, from $J_4 + R_1 + R_3$, we obtain

$$\tilde{\mathcal{H}}_{2.1} \underbrace{+Z_{\alpha\beta\gamma}\int \frac{V_g^2}{c_1}\sum_k \left(y_{1,k}z_{k'}z_{k''} - y_{1,k}y_{1,k'}z_{k''}\right)\rho r^4 \mathrm{d}r}_{R_4},$$

where

$$\tilde{\mathcal{H}}_{2.1} = Z_{\alpha\beta\gamma}\int \frac{V_g}{c_1}\left(V - 2V_g - \frac{\mathrm{d}\ln\Gamma_1}{\mathrm{d}\ln r}\right)\prod_k z_k \rho r^4 \mathrm{d}r. \tag{E.3.14}$$

Adding up the terms L_2, R_2, and R_4 also gives

$$\tilde{\mathcal{H}}_{3.2} = -Z_{\alpha\beta\gamma}\int \frac{A^*}{c_1}V_g \sum_k z_k y_{1,k'}y_{1,k''}\rho r^4 \mathrm{d}r. \tag{E.3.15}$$

Finally (!), collecting all the $\tilde{\mathcal{H}}$ terms, as defined by Eqs. (E.3.1), (E.3.7), (E.3.10)–(E.3.12), (E.3.14) and (E.3.15), we obtain the desired expression for the zeroth-order component of the coupling coefficient (corrected due to rotation only through the eigenfrequencies), for polar mode coupling, as

$$\begin{aligned}\tilde{\mathcal{H}} = Z_{\alpha\beta\gamma}\int \Bigg\{ &-\sum_k \left(A^* y_{1,k} + V_g z_k\right)\Psi_k \left(\varpi_{k'}\varpi_{k''}y_{1,k'}y_{1,k''} + \frac{QC_k}{c_1^2}\psi_{k'}\psi_{k''}y_{2,k'}y_{2,k''}\right)\\ &+ \frac{V_g}{c_1}\left[\left(V - 2V_g - \frac{\mathrm{d}\ln\Gamma_1}{\mathrm{d}\ln r}\right)\prod_k z_k + A_g\prod_k\left(y_{1,k} - z_k\right)\right]\\ &+ \frac{A^*}{c_1}\left[\left(V_g + U - 4 - c_1\sum_k \frac{\varpi_k^2}{\psi_k}\right)\prod_k y_{1,k} - V_g\sum_k z_k y_{1,k'}y_{1,k''} + \sum_k y_{4,k}y_{1,k'}y_{1,k''}\right]\\ &+ \frac{A^*}{c_1^2}\sum_k \psi_k y_{2,k}\left(GC_k y_{1,k'}y_{1,k''} + QC_{k'}y_{1,k'}z_{k''} + QC_{k''}y_{1,k''}z_{k'}\right)\Bigg\}\rho r^4\mathrm{d}r + N_{\mathrm{cor}}.\end{aligned} \tag{E.3.16}$$

For the reader who skipped the derivation of Eq. (E.3.16), we have used the dimensionless formulation of Sect. 2.3.3. The auxiliary parameters z_k, ϖ_k, QC_k, GC_k, and A_g are given by Eqs. (E.2.6), (E.3.2), (E.3.3), (E.3.8) and (E.3.13), respectively, and the indices k, k', and k'' behave according to Eq. (E.2.11). The angular part of the coupling coefficient has been denoted by $Z_{\alpha\beta\gamma}$, which is given by Eq. (E.2.9). Equation (E.3.16) is identical with Eq. (3.12) in Dziembowski (1982), with the exception of some variables which parametrise rotational corrections to the eigenfrequencies, namely, ψ_k, Ψ_k, and N_{cor}, given by Eqs. (E.3.4), (E.3.5) and (E.3.9), respectively. If eigenfrequency corrections are not considered (in which case we obtain the actual zeroth-order component of the coupling coefficient), then $\psi_k \rightarrow 1$, $\Psi_k \rightarrow 1$, and $N_{\mathrm{cor}} \rightarrow 0$.

Since, as mentioned in the beginning of the section, we have divided the coupling coefficient $F_{\alpha\beta\gamma}$ by GM/R^3 for this derivation, $\tilde{\mathcal{H}}$ is related to $F_{\alpha\beta\gamma}$ as

$$F_{\alpha\beta\gamma} = \frac{GM}{R^3}\tilde{\mathcal{H}} \equiv \mathcal{H}.$$

Also, it can be easily seen that Eq. (E.3.16) is invariant to the transformations

$$Y_\alpha \rightleftarrows Y_\beta, \quad y_{i,\alpha} \rightleftarrows y_{i,\beta}, \quad Y_\gamma \to Y_\gamma^*, \quad \tilde{\omega}_\gamma \to -\tilde{\omega}_\gamma,$$

and

$$Y_\alpha \rightleftarrows Y_\gamma, \quad y_{i,\alpha} \rightleftarrows y_{i,\gamma}, \quad Y_\beta \to Y_\beta^*, \quad \tilde{\omega}_\beta \to -\tilde{\omega}_\beta,$$

which proves that

$$F_{\alpha\beta\gamma} = F_{\beta\bar{\gamma}\alpha} = F_{\gamma\alpha\bar{\beta}} \equiv \mathcal{H}.$$

$\mathcal{H}$ has units of energy; the normalisation in Eq. (E.3.16) is useful when all quantities in the coupled triplet equations of motion (4.2.3) are normalised accordingly. Defining a dimensionless time $\tau = t\sqrt{GM/R^3}$, the equations of motion are written as

$$Q'_\alpha = \tilde{\gamma}_\alpha Q_\alpha + \frac{i\widetilde{\mathcal{H}}}{\tilde{b}_\alpha} Q_\beta Q_\gamma e^{-i\Delta\tilde{\omega}\tau}, \tag{E.3.17a}$$

$$Q'_\beta = \tilde{\gamma}_\beta Q_\beta + \frac{i\widetilde{\mathcal{H}}}{\tilde{b}_\beta} Q_\gamma^* Q_\alpha e^{i\Delta\tilde{\omega}\tau}, \tag{E.3.17b}$$

$$Q'_\gamma = \tilde{\gamma}_\gamma Q_\gamma + \frac{i\widetilde{\mathcal{H}}}{\tilde{b}_\gamma} Q_\alpha Q_\beta^* e^{i\Delta\tilde{\omega}\tau}, \tag{E.3.17c}$$

where $\tilde{\gamma} = \gamma/\sqrt{GM/R^3}$, $\tilde{b} = b/\sqrt{GM/R^3}$, the dimensionless eigenfrequency $\tilde{\omega}$ is defined in Eq. (2.3.41), and the prime denotes differentiation with respect to τ. Now, Eqs. (E.3.17) coincide with Eqs. (2.25) and (2.26) in Dziembowski (1982), except they have been generalised for the case of a rotating star.[6]

References

1. Dziembowski, W. (1982). Nonlinear mode coupling in oscillating stars. I. Second order theory of the coherent mode coupling. *Acta Astronomica*, *32*, 147–171. http://adsabs.harvard.edu/abs/1982AcA....32..147D.
2. Sakurai, J. J., & Napolitano, J. (2011). *Modern quantum mechanics* (2nd ed.). San Francisco: Addison-Wesley. http://adsabs.harvard.edu/abs/1985mqm..book.....S.

[6] Note that Dziembowski (1982) normalises with respect to $4\pi G\langle\rho\rangle$, where $\langle\rho\rangle$ is the mean density, instead of GM/R^3. He also uses a different convention for the spherical harmonics, which he normalises to 4π, instead of unity like us [see Eq. (2.3.14)]. Finally, he uses c_1 (in our notation) = $3(r/R)^3 M/M_r$, which is a factor of 3 larger than our c_1 [Eq. (2.3.39)].

Appendix F
Parametrically Unstable Mode Triplet

F.1 Parametric Instability Threshold

The amplitude equations of motion of the coupled triplet are given by Eqs. (4.2.12), namely

$$\dot{Q}_\alpha = \gamma_\alpha Q_\alpha + i\omega_\alpha \frac{\mathcal{H}}{E_{\rm unit}} Q_\beta Q_\gamma e^{-i\Delta\omega t}, \tag{F.1.1a}$$

$$\dot{Q}_\beta = \gamma_\beta Q_\beta + i\omega_\beta \frac{\mathcal{H}}{E_{\rm unit}} Q_\gamma^* Q_\alpha e^{i\Delta\omega t}, \tag{F.1.1b}$$

$$\dot{Q}_\gamma = \gamma_\gamma Q_\gamma + i\omega_\gamma \frac{\mathcal{H}}{E_{\rm unit}} Q_\alpha Q_\beta^* e^{i\Delta\omega t}. \tag{F.1.1c}$$

In order to derive the formula for the parametric instability threshold (4.3.1), we take the amplitude equations of motion for the daughter modes (F.1.1b) and (F.1.1c) and ask what the value of the parent mode's amplitude Q_α should be, in order for the daughters' amplitudes $Q_{\beta,\gamma}$ start growing. Setting

$$Q_{\beta,\gamma} = \widetilde{Q}_{\beta,\gamma} e^{i\Delta\omega t/2},$$

Eqs. (F.1.1b) and (F.1.1c) become

$$\dot{\widetilde{Q}}_\beta = \left(\gamma_\beta - i\frac{\Delta\omega}{2}\right)\widetilde{Q}_\beta + \frac{i\omega_\beta \mathcal{H}}{E_{\rm unit}} Q_\alpha \widetilde{Q}_\gamma^*$$

and

$$\dot{\widetilde{Q}}_\gamma^* = \left(\gamma_\gamma + i\frac{\Delta\omega}{2}\right)\widetilde{Q}_\gamma^* - \frac{i\omega_\gamma \mathcal{H}}{E_{\rm unit}} Q_\alpha^* \widetilde{Q}_\beta,$$

or, in matrix form,

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Springer Theses, https://doi.org/10.1007/978-3-319-98258-8

$$\begin{pmatrix} \dot{\widetilde{Q}}_\beta \\ \dot{\widetilde{Q}}^*_\gamma \end{pmatrix} = \begin{pmatrix} \gamma_\beta - i\,\Delta\omega/2 & i\,Q_\alpha\omega_\beta\mathcal{H}/E_{\text{unit}} \\ -i\,Q^*_\alpha\omega_\gamma\mathcal{H}/E_{\text{unit}} & \gamma_\gamma + i\,\Delta\omega/2 \end{pmatrix} \begin{pmatrix} \widetilde{Q}_\beta \\ \widetilde{Q}^*_\gamma \end{pmatrix},$$

with Q_α treated as an unknown constant. If T is the trace and d is the determinant of the system matrix, then its eigenvalues $\lambda_{1,\,2}$ can be found as

$$\lambda_{1,\,2} = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4d}\right),$$

or

$$\lambda_{1,\,2} = \frac{1}{2}\left[\gamma_\beta + \gamma_\gamma \pm \sqrt{\left(\gamma_\gamma - \gamma_\beta + i\,\Delta\omega\right)^2 + \frac{4\omega_\beta\omega\gamma\mathcal{H}^2}{E^2_{\text{unit}}}|Q_\alpha|^2}\,\right]. \tag{F.1.2}$$

For the system to admit a growing exponential solution, i.e., for the daughter modes to grow, the condition $\text{Re}(\lambda) > 0$ has to be satisfied, where Re denotes the real part, for at least one of the eigenvalues. Hence, at the onset of the parametric instability, $\text{Re}(\lambda) = 0$, or

$$\left(\gamma_\beta + \gamma_\gamma\right)^2 = \left[\text{Re}\left(\sqrt{\left(\gamma_\gamma - \gamma_\beta + i\,\Delta\omega\right)^2 + \frac{4\omega_\beta\omega_\gamma\mathcal{H}^2}{E^2_{\text{unit}}}|Q_\alpha|^2}\right)\right]^2. \tag{F.1.3}$$

We set the radicand in the expression above equal to the complex number $u + iv$. Then, let $\sqrt{u + iv} = x + iy$, or $u + iv = x^2 - y^2 + i2xy$. Distinguishing the real from the imaginary part and omitting y, we get

$$4x^4 - 4ux^2 - v^2 = 0.$$

From Eq. (F.1.3), we have

$$u = \left(\gamma_\gamma - \gamma_\beta\right)^2 - \Delta\omega^2 + \frac{4\omega_\beta\omega\gamma\mathcal{H}^2}{E^2_{\text{unit}}}|Q_\alpha|^2,$$
$$v = 2\left(\gamma_\gamma - \gamma_\beta\right)\Delta\omega,$$

and

$$x^2 = \left(\gamma_\beta + \gamma_\gamma\right)^2.$$

Thus, replacing u, v, and x in the quartic equation for x above, we obtain

$$|Q_\alpha|^2 = \frac{\gamma_\beta\gamma_\gamma}{\omega_\beta\omega_\gamma}\frac{E^2_{\text{unit}}}{\mathcal{H}^2}\left[1 + \left(\frac{\Delta\omega}{\gamma_\beta + \gamma_\gamma}\right)^2\right] \equiv |Q_{\text{PIT}}|^2. \tag{F.1.4}$$

Note the importance of the mode eigenfrequency signs here: if $\omega_\beta\omega_\gamma < 0$, then no parametric instability can occur. This is a result of the assumed resonance (4.2.1) between the parent and the daughters. If we perform the same analysis, for example, for mode β being the parent, then $\omega_\beta \approx \omega_\alpha - \omega_\gamma$, in which case $\omega_\alpha\omega_\gamma < 0$ is a *necessary* condition for parametric instability.

F.2 Equilibrium Solution

The amplitude equations of motion (4.2.12) [or (F.1.1)] admit an easy-to-obtain equilibrium solution. Expressing the complex amplitudes Q in terms of real amplitude and phase variables, we can introduce the variable transformation

$$Q_\alpha = \frac{1}{\sqrt{\omega_\beta\omega_\gamma}} \frac{E_{\text{unit}}}{\mathcal{H}} \varepsilon_\alpha e^{i\vartheta_\alpha}, \tag{F.2.1a}$$

$$Q_\beta = \frac{1}{\sqrt{\omega_\gamma\omega_\alpha}} \frac{E_{\text{unit}}}{\mathcal{H}} \varepsilon_\beta e^{i\vartheta_\beta}, \tag{F.2.1b}$$

$$Q_\gamma = \frac{1}{\sqrt{\omega_\alpha\omega_\beta}} \frac{E_{\text{unit}}}{\mathcal{H}} \varepsilon_\gamma e^{i\vartheta_\gamma}. \tag{F.2.1c}$$

Then, Eqs. (F.1.1) are written as

$$\dot{\varepsilon}_\alpha = \gamma_\alpha\varepsilon_\alpha + \varepsilon_\beta\varepsilon_\gamma \sin\varphi, \tag{F.2.2a}$$

$$\dot{\varepsilon}_\beta = \gamma_\beta\varepsilon_\beta - \varepsilon_\gamma\varepsilon_\alpha \sin\varphi, \tag{F.2.2b}$$

$$\dot{\varepsilon}_\gamma = \gamma_\gamma\varepsilon_\gamma - \varepsilon_\alpha\varepsilon_\beta \sin\varphi, \tag{F.2.2c}$$

and

$$\dot{\varphi} = \cos\varphi \left(\frac{\varepsilon_\beta\varepsilon_\gamma}{\varepsilon_\alpha} - \frac{\varepsilon_\gamma\varepsilon_\alpha}{\varepsilon_\beta} - \frac{\varepsilon_\alpha\varepsilon_\beta}{\varepsilon_\gamma} \right) + \Delta\omega, \tag{F.2.2d}$$

or, equivalently,

$$\dot{\varphi} = \cot\varphi \left(\frac{\dot{\varepsilon}_\alpha}{\varepsilon_\alpha} + \frac{\dot{\varepsilon}_\beta}{\varepsilon_\beta} + \frac{\dot{\varepsilon}_\gamma}{\varepsilon_\gamma} - \gamma \right) + \Delta\omega, \tag{F.2.2d$'$}$$

where

$$\varphi = \vartheta_\alpha - \vartheta_\beta - \vartheta_\gamma + \Delta\omega t \tag{F.2.3}$$

and

$$\gamma = \gamma_\alpha + \gamma_\beta + \gamma_\gamma. \tag{F.2.4}$$

Setting the time derivatives in Eqs. (F.2.2) to zero, we get

$$-\gamma_\alpha \varepsilon_\alpha = \varepsilon_\beta \varepsilon_\gamma \sin\varphi, \tag{F.2.5a}$$
$$\gamma_\beta \varepsilon_\beta = \varepsilon_\gamma \varepsilon_\alpha \sin\varphi, \tag{F.2.5b}$$
$$\gamma_\gamma \varepsilon_\gamma = \varepsilon_\alpha \varepsilon_\beta \sin\varphi, \tag{F.2.5c}$$

and

$$\cot\varphi = \kappa, \tag{F.2.5d}$$

where

$$\kappa = \frac{\Delta\omega}{\gamma}. \tag{F.2.6}$$

Then, combining Eqs. (F.2.5a)–(F.2.5c) in pairs and using the trigonometric identity

$$\frac{1}{\sin^2\varphi} = 1 + \cot^2\varphi = 1 + \kappa^2,$$

we obtain the equilibrium solution

$$\varepsilon_\alpha^2 = \gamma_\beta \gamma_\gamma \left(1 + \kappa^2\right), \tag{F.2.7a}$$
$$\varepsilon_\beta^2 = -\gamma_\gamma \gamma_\alpha \left(1 + \kappa^2\right), \tag{F.2.7b}$$
$$\varepsilon_\gamma^2 = -\gamma_\alpha \gamma_\beta \left(1 + \kappa^2\right), \tag{F.2.7c}$$

which, in terms of the original variables Q, admits the form (4.3.3).

Combining Eqs. (F.2.2b) and (F.2.2c) with Eq. (F.2.2a), we can further show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon_\alpha^2 + \varepsilon_\beta^2\right) = \gamma_\alpha \varepsilon_\alpha^2 + \gamma_\beta \varepsilon_\beta^2$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon_\alpha^2 + \varepsilon_\gamma^2\right) = \gamma_\alpha \varepsilon_\alpha^2 + \gamma_\gamma \varepsilon_\gamma^2,$$

or, restoring the original variables Q with the help of Eqs. (F.2.1),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\omega_\beta |Q_\alpha|^2 + \omega_\alpha |Q_\beta|^2\right) = \gamma_\alpha \omega_\beta |Q_\alpha|^2 + \gamma_\beta \omega_\alpha |Q_\beta|^2$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\omega_\gamma |Q_\alpha|^2 + \omega_\alpha |Q_\gamma|^2\right) = \gamma_\alpha \omega_\gamma |Q_\alpha|^2 + \gamma_\gamma \omega_\alpha |Q_\gamma|^2.$$

Adding the equations above and using $\omega_\beta + \omega_\gamma \approx \omega_\alpha$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|Q_\alpha|^2+|Q_\beta|^2+|Q_\gamma|^2\right)=\gamma_\alpha|Q_\alpha|^2+\gamma_\beta|Q_\beta|^2+\gamma_\gamma|Q_\gamma|^2, \tag{F.2.8}$$

which, multiplied by E_{unit}, gives the rate of change of the triplet energy (4.4.11).

Finally, we can incorporate the phases ϑ, as defined in Eqs. (F.2.1), in the harmonic time dependence of the modes, as

$$\boldsymbol{\xi}_k \propto e^{i(\omega_k t+\vartheta_k)},$$

where $k=\alpha,\beta,\gamma$. This implies that the eigenfrequency of the mode is shifted to

$$\omega_k' = \omega_k + \dot{\vartheta}_k. \tag{F.2.9}$$

From Eq. (F.2.2d) we already know that

$$\dot{\vartheta}_\alpha = \frac{\varepsilon_\beta\varepsilon_\gamma}{\varepsilon_\alpha}\cos\varphi,$$
$$\dot{\vartheta}_\beta = \frac{\varepsilon_\gamma\varepsilon_\alpha}{\varepsilon_\beta}\cos\varphi,$$
$$\dot{\vartheta}_\gamma = \frac{\varepsilon_\alpha\varepsilon_\beta}{\varepsilon_\gamma}\cos\varphi,$$

or, in compact form,

$$\dot{\vartheta}_k = \frac{\varepsilon_\alpha\varepsilon_\beta\varepsilon_\gamma}{\varepsilon_k^2}\cos\varphi. \tag{F.2.10}$$

In terms of the original variables Q, Eq. (F.2.10) is written as Eq. (4.4.13). Replacing the equilibrium values (F.2.7) and (F.2.5d) in Eq. (F.2.10), we get

$$\dot{\vartheta}_k = |\gamma_k|\sqrt{1+\kappa^2}\cos\varphi_0,$$

where

$$\cot\varphi_0 = \kappa.$$

We use the trigonometric identity

$$\cos^2\varphi = \frac{\cot^2\varphi}{1+\cot^2\varphi}$$

to obtain

$$\cos\varphi_0 = \pm\frac{|\kappa|}{\sqrt{1+\kappa^2}}.$$

We notice from Eq. (F.2.5a) that, in equilibrium, $\sin\varphi_0 < 0$, which means that, if $\kappa = \cot\varphi_0 > 0$, then $\cos\varphi_0 < 0$, and vice versa. In other words, $\mathrm{sgn}(\cos\varphi_0) = -\mathrm{sgn}(\kappa)$, where sgn is the sign function. So, finally, we obtain for the eigenfrequency shift in equilibrium

$$\dot{\vartheta}_k = -|\gamma_k|\kappa. \tag{F.2.11}$$

F.3 Linear Stability Analysis

We will linearise Eqs. (F.2.2) by imposing small perturbations about their equilibrium solutions (F.2.5). Denoting these perturbations by δ (not to be confused with a Eulerian perturbation), we get

$$\frac{\mathrm{d}\delta\varepsilon_\alpha}{\mathrm{d}t} - \gamma_\alpha\delta\varepsilon_\alpha = \delta\varepsilon_\beta\varepsilon_\gamma\sin\varphi + \varepsilon_\beta\delta\varepsilon_\gamma\sin\varphi + \varepsilon_\beta\varepsilon_\gamma\cos\varphi\delta\varphi.$$

Dividing by ε_α and using Eq. (F.2.5a), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta\varepsilon_\alpha}{\varepsilon_\alpha}\right) = -\gamma_\alpha\left(-\frac{\delta\varepsilon_\alpha}{\varepsilon_\alpha} + \frac{\delta\varepsilon_\beta}{\varepsilon_\beta} + \frac{\delta\varepsilon_\gamma}{\varepsilon_\gamma} + \kappa\delta\varphi\right). \tag{F.3.1a}$$

Note that ε_α above corresponds to the equilibrium solution and, hence, is a constant. In a similar manner, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta\varepsilon_\beta}{\varepsilon_\beta}\right) = -\gamma_\beta\left(\frac{\delta\varepsilon_\alpha}{\varepsilon_\alpha} - \frac{\delta\varepsilon_\beta}{\varepsilon_\beta} + \frac{\delta\varepsilon_\gamma}{\varepsilon_\gamma} + \kappa\delta\varphi\right), \tag{F.3.1b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta\varepsilon_\gamma}{\varepsilon_\gamma}\right) = -\gamma_\gamma\left(\frac{\delta\varepsilon_\alpha}{\varepsilon_\alpha} + \frac{\delta\varepsilon_\beta}{\varepsilon_\beta} - \frac{\delta\varepsilon_\gamma}{\varepsilon_\gamma} + \kappa\delta\varphi\right). \tag{F.3.1c}$$

The linearisation of Eq. (F.2.2d′) about its equilibrium (F.2.5d) yields

$$\frac{\mathrm{d}\delta\varphi}{\mathrm{d}t} - \kappa\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta\varepsilon_\alpha}{\varepsilon_\alpha} + \frac{\delta\varepsilon_\beta}{\varepsilon_\beta} + \frac{\delta\varepsilon_\gamma}{\varepsilon_\gamma}\right) = \gamma\left(1+\kappa^2\right)\delta\varphi,$$

where we used the fact that

$$\cot\left(\varphi + \delta\varphi\right) = \kappa - \left(1+\kappa^2\right)\delta\varphi + \mathcal{O}\left(\delta\varphi^2\right).$$

With the help of Eqs. (F.3.1a)–(F.3.1c), the equation above becomes

$$\frac{\mathrm{d}\delta\varphi}{\mathrm{d}t} = \kappa\sum_k \varGamma_k\frac{\delta\varepsilon_k}{\varepsilon_k} + \gamma\delta\varphi, \tag{F.3.1d}$$

where

$$\Gamma_k = 2\gamma_k - \gamma, \tag{F.3.2}$$

with the index k successively taking the values α, β, γ.

The matrix of the linear system (F.3.1) is

$$\boldsymbol{A} = \begin{pmatrix} \gamma_\alpha & -\gamma_\alpha & -\gamma_\alpha & -\kappa\gamma_\alpha \\ -\gamma_\beta & \gamma_\beta & -\gamma_\beta & -\kappa\gamma_\beta \\ -\gamma_\gamma & -\gamma_\gamma & \gamma_\gamma & -\kappa\gamma_\gamma \\ \kappa\Gamma_\alpha & \kappa\Gamma_\beta & \kappa\Gamma_\gamma & \gamma \end{pmatrix},$$

with the help of which we can find the system's characteristic polynomial, via the relation $|\boldsymbol{A} - \lambda\boldsymbol{I}| = 0$, where λ are the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{I}$ is the identity matrix. The polynomial has the form

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where

$$\begin{aligned} a_1 &= -2\gamma, \\ a_2 &= \gamma^2\left(1+\kappa^2\right) - 4\kappa^2\sum_k \gamma_k\gamma_{k'}, \\ a_3 &= 4\left(1+3\kappa^2\right)\prod_k \gamma_k, \end{aligned}$$

and

$$a_4 = -4\left(1+\kappa^2\right)\gamma\prod_k \gamma_k,$$

with the index k' taking the value that comes after k's value, as explained in Eq. (E.2.11).

Now, we can use the Routh-Hurwitz stability criteria (see, for instance, Horn and Johnson 1991, Sect. 2.3), in order to determine the behaviour of the system. First, we construct the Routh-Hurwitz matrix, using the polynomial coefficients, as

$$\boldsymbol{M} = \begin{pmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{pmatrix}.$$

Then, the stability criteria are given by

$$W_1 \equiv a_1 > 0, \tag{F.3.3}$$

$$W_2 \equiv \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} = a_1 a_2 - a_3 > 0, \tag{F.3.4}$$

$$W_3 \equiv \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} = a_3 W_2 - a_1^2 a_4 > 0, \tag{F.3.5}$$

and

$$W_4 \equiv |\boldsymbol{M}| = a_4 W_3 > 0. \tag{F.3.6}$$

Since $\gamma_{\beta,\gamma} < 0$, it can be easily shown that the second and fourth criteria are redundant and follow from the other ones. Indeed, if $W_1 > 0$ then a_4 is also positive, which, combined with $W_3 > 0$, makes the fourth criterion true. Also, $W_3 > 0$ yields $W_2 > a_1^2 a_4 / a_3$, but since $a_3 > 0$, the second criterion is also true. So, finally, from Eqs. (F.3.3) and (F.3.5), we obtain the stability conditions (4.4.1) and (4.4.2).

Reference

1. Horn, R. A., & Johnson, C. R. (1991). *Topics in matrix analysis*. Cambridge, England: Cambridge University Press. https://doi.org/10.1017/CBO9780511840371.

Appendix G
Coupling Spectrum

In the following table we present the *coupling spectrum* of the octupole ($l = m = 3$) f-mode, in a typical neutron star with $M \approx 1.4\,M_\odot$ and $R \approx 10\,\text{km}$, described by a polytropic equation of state with a polytropic exponent $\Gamma = 3$ and an adiabatic exponent $\Gamma_1 = 3.1$ (see Table 5.1). This is the coupling spectrum used to generate Figs. 5.2 (right) and 5.4. The following data is presented in the table, by column:

1. Angular velocity Ω, normalised to the Kepler limit Ω_K.
2. Temperature (decimal) logarithm $\log(T/1\,\text{K})$.

3–4. Daughter pair; the notation ${}^m_l f$ and ${}^m_l g_n$ is used for f- and g-modes respectively.

5. Triplet detuning $\Delta\tilde{\omega} = \Delta\omega/\sqrt{GM/R^3}$.
6. Coupling coefficient $\mathcal{H}$, normalised to $E_\text{unit} = Mc^2$.

7–9. Parent (α) and daughter (β, γ) growth/damping rates $\tilde{\gamma}_k = \gamma_k/\sqrt{GM/R^3}$.

10. Lowest stable parametric instability threshold $|Q_\text{PIT}|$.

For this model, the normalisation factor $\sqrt{GM/R^3}$ evaluates to $13\,416\ \text{rad}\,\text{s}^{-1} \approx 2\,\text{kHz}$.

For all models used in Chap. 5, the instability window in the $(\log T, \Omega/\Omega_\text{K})$ plane was divided into blocks with dimensions $(0.1, 0.002)$, forming a grid. For typical neutron star models, the maximum allowed detuning (see Sect. 5.1.3) was set to $\Delta\tilde{\omega}_\text{max} = 0.1$, whereas for supramassive neutron star models we chose $\Delta\tilde{\omega}_\text{max} = 0.2$.

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Springer Theses, https://doi.org/10.1007/978-3-319-98258-8

Table G.1 Coupling spectrum of the octupole f-mode in a typical neutron star (see text for model details)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
93.4	9.4	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-2.1×10^{-2}	3.3	6.1×10^{-14}	-3.4×10^{-4}	-2.5×10^{-13}	2.3×10^{-7}
93.4	9.5	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-2.1×10^{-2}	3.3	7.5×10^{-14}	-3.4×10^{-4}	-8.9×10^{-13}	4.4×10^{-7}
93.6	9.3	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-2.4×10^{-2}	3.3	8.1×10^{-14}	-3.4×10^{-4}	-9.6×10^{-14}	1.6×10^{-7}
93.6	9.4	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-2.4×10^{-2}	3.3	1.7×10^{-13}	-3.4×10^{-4}	-2.4×10^{-13}	2.6×10^{-7}
93.6	9.5	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-2.4×10^{-2}	3.3	1.8×10^{-13}	-3.4×10^{-4}	-8.9×10^{-13}	5.0×10^{-7}
93.6	9.6	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-2.4×10^{-2}	3.3	3.3×10^{-14}	-3.4×10^{-4}	-3.5×10^{-12}	9.8×10^{-7}
93.8	9.2	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-2.8×10^{-2}	3.4	7.5×10^{-14}	-3.4×10^{-4}	-7.8×10^{-14}	1.7×10^{-7}
93.8	9.3	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-2.8×10^{-2}	3.4	2.3×10^{-13}	-3.4×10^{-4}	-9.5×10^{-14}	1.8×10^{-7}
93.8	9.4	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-2.8×10^{-2}	3.4	3.1×10^{-13}	-3.4×10^{-4}	-2.4×10^{-13}	2.9×10^{-7}
93.8	9.5	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-2.8×10^{-2}	3.4	3.3×10^{-13}	-3.4×10^{-4}	-8.8×10^{-13}	5.5×10^{-7}
93.8	9.6	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-2.8×10^{-2}	3.4	1.8×10^{-13}	-3.4×10^{-4}	-3.5×10^{-12}	1.1×10^{-6}
94.0	9.1	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-3.1×10^{-2}	3.4	3.1×10^{-14}	-3.4×10^{-4}	-1.0×10^{-13}	2.1×10^{-7}
94.0	9.2	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-3.1×10^{-2}	3.4	2.8×10^{-13}	-3.4×10^{-4}	-7.7×10^{-14}	1.8×10^{-7}
94.0	9.3	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-3.1×10^{-2}	3.4	4.3×10^{-13}	-3.4×10^{-4}	-9.4×10^{-14}	2.0×10^{-7}
94.0	9.4	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-3.1×10^{-2}	3.4	5.2×10^{-13}	-3.4×10^{-4}	-2.4×10^{-13}	3.2×10^{-7}
94.0	9.5	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-3.1×10^{-2}	3.4	5.3×10^{-13}	-3.4×10^{-4}	-8.7×10^{-13}	6.1×10^{-7}
94.0	9.6	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-3.1×10^{-2}	3.4	3.8×10^{-13}	-3.4×10^{-4}	-3.4×10^{-12}	1.2×10^{-6}
94.2	9.1	$^{-4}_{4}f$	$^{7}_{7}g_{7}$	4.8×10^{-3}	−2.9	3.1×10^{-13}	-4.9×10^{-5}	-3.9×10^{-13}	2.0×10^{-7}
94.2	9.2	$^{-4}_{4}f$	$^{7}_{7}g_{7}$	4.8×10^{-3}	−2.9	5.6×10^{-13}	-4.9×10^{-5}	-3.4×10^{-13}	1.9×10^{-7}
94.2	9.3	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-3.5×10^{-2}	3.4	7.1×10^{-13}	-3.4×10^{-4}	-9.3×10^{-14}	2.2×10^{-7}
94.2	9.4	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-3.5×10^{-2}	3.4	8.0×10^{-13}	-3.4×10^{-4}	-2.4×10^{-13}	3.5×10^{-7}
94.2	9.5	$^{-3}_{3}f$	$^{6}_{6}g_{10}$	-3.5×10^{-2}	3.4	8.1×10^{-13}	-3.4×10^{-4}	-8.7×10^{-13}	6.7×10^{-7}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
94.2	9.6	${}^{-3}_{3}f$	${}^{6}_{6}g_{10}$	-3.5×10^{-2}	3.4	6.6×10^{-13}	-3.4×10^{-4}	-3.4×10^{-12}	1.3×10^{-6}
94.4	9.0	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	2.7×10^{-3}	−2.9	2.9×10^{-13}	-5.0×10^{-5}	-5.8×10^{-13}	1.4×10^{-7}
94.4	9.1	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	2.7×10^{-3}	−2.9	6.9×10^{-13}	-5.0×10^{-5}	-3.9×10^{-13}	1.1×10^{-7}
94.4	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	9.7×10^{-2}	−6.7	9.4×10^{-13}	-2.6×10^{-3}	-3.6×10^{-14}	1.0×10^{-7}
94.4	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	9.7×10^{-2}	−6.7	1.1×10^{-12}	-2.6×10^{-3}	-4.6×10^{-14}	1.2×10^{-7}
94.4	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	9.7×10^{-2}	−6.7	1.2×10^{-12}	-2.6×10^{-3}	-1.2×10^{-13}	1.9×10^{-7}
94.4	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	9.7×10^{-2}	−6.7	1.2×10^{-12}	-2.6×10^{-3}	-4.5×10^{-13}	3.6×10^{-7}
94.4	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	9.7×10^{-2}	−6.7	1.0×10^{-12}	-2.6×10^{-3}	-1.8×10^{-12}	7.1×10^{-7}
94.4	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	9.7×10^{-2}	−6.7	3.1×10^{-13}	-2.6×10^{-3}	-7.0×10^{-12}	1.4×10^{-6}
94.6	8.9	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	4.9×10^{-4}	−3.0	1.7×10^{-13}	-5.0×10^{-5}	-9.1×10^{-13}	3.1×10^{-8}
94.6	9.0	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	4.9×10^{-4}	−3.0	8.0×10^{-13}	-5.0×10^{-5}	-5.8×10^{-13}	2.5×10^{-8}
94.6	9.1	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	4.9×10^{-4}	−3.0	1.2×10^{-12}	-5.0×10^{-5}	-3.9×10^{-13}	2.0×10^{-8}
94.6	9.2	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	4.9×10^{-4}	−3.0	1.5×10^{-12}	-5.0×10^{-5}	-3.3×10^{-13}	1.9×10^{-8}
94.6	9.3	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	4.9×10^{-4}	−3.0	1.6×10^{-12}	-5.0×10^{-5}	-5.6×10^{-13}	2.4×10^{-8}
94.6	9.4	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	4.9×10^{-4}	−3.0	1.7×10^{-12}	-5.0×10^{-5}	-1.8×10^{-12}	4.3×10^{-8}
94.6	9.5	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	4.9×10^{-4}	−3.0	1.7×10^{-12}	-5.0×10^{-5}	-6.7×10^{-12}	8.4×10^{-8}
94.6	9.6	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	4.9×10^{-4}	−3.0	1.6×10^{-12}	-5.0×10^{-5}	-2.7×10^{-11}	1.7×10^{-7}
94.6	9.7	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	4.9×10^{-4}	−3.0	8.2×10^{-13}	-5.0×10^{-5}	-1.1×10^{-10}	3.3×10^{-7}
94.8	8.9	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	-1.9×10^{-3}	−3.0	8.6×10^{-13}	-5.0×10^{-5}	-9.0×10^{-13}	1.2×10^{-7}
94.8	9.0	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	-1.9×10^{-3}	−3.0	1.5×10^{-12}	-5.0×10^{-5}	-5.7×10^{-13}	9.4×10^{-8}
94.8	9.1	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	-1.9×10^{-3}	−3.0	1.9×10^{-12}	-5.0×10^{-5}	-3.8×10^{-13}	7.7×10^{-8}
94.8	9.2	${}^{-4}_{4}f$	${}^{7}_{7}g_{7}$	-1.9×10^{-3}	−3.0	2.2×10^{-12}	-5.0×10^{-5}	-3.3×10^{-13}	7.1×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
94.8	9.3	${}^{-4}_{4}f$	${}^{7}_{7}g_7$	-1.9×10^{-3}	−3.0	2.3×10^{-12}	-5.0×10^{-5}	-5.6×10^{-13}	9.3×10^{-8}
94.8	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	8.7×10^{-2}	−6.9	2.4×10^{-12}	-2.6×10^{-3}	-1.2×10^{-13}	1.6×10^{-7}
94.8	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	8.7×10^{-2}	−6.9	2.4×10^{-12}	-2.6×10^{-3}	-4.4×10^{-13}	3.1×10^{-7}
94.8	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	8.7×10^{-2}	−6.9	2.3×10^{-12}	-2.6×10^{-3}	-1.7×10^{-12}	6.2×10^{-7}
94.8	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	8.7×10^{-2}	−6.9	1.5×10^{-12}	-2.6×10^{-3}	-6.8×10^{-12}	1.2×10^{-6}
95.0	8.8	${}^{-5}_{5}f$	${}^{8}_{8}g_3$	2.4×10^{-4}	−4.9	7.9×10^{-13}	-7.5×10^{-6}	-4.2×10^{-12}	5.7×10^{-8}
95.0	8.9	${}^{-5}_{5}f$	${}^{8}_{8}g_3$	2.4×10^{-4}	−4.9	1.8×10^{-12}	-7.5×10^{-6}	-2.7×10^{-12}	4.6×10^{-8}
95.0	9.0	${}^{-5}_{5}f$	${}^{8}_{8}g_3$	2.4×10^{-4}	−4.9	2.4×10^{-12}	-7.5×10^{-6}	-1.7×10^{-12}	3.7×10^{-8}
95.0	9.1	${}^{-5}_{5}f$	${}^{8}_{8}g_3$	2.4×10^{-4}	−4.9	2.9×10^{-12}	-7.5×10^{-6}	-1.3×10^{-12}	3.2×10^{-8}
95.0	9.2	${}^{-5}_{5}f$	${}^{8}_{8}g_3$	2.4×10^{-4}	−4.9	3.1×10^{-12}	-7.5×10^{-6}	-1.7×10^{-12}	3.6×10^{-8}
95.0	9.3	${}^{-5}_{5}f$	${}^{8}_{8}g_3$	2.4×10^{-4}	−4.9	3.3×10^{-12}	-7.5×10^{-6}	-4.4×10^{-12}	5.8×10^{-8}
95.0	9.4	${}^{-5}_{5}f$	${}^{8}_{8}g_3$	2.4×10^{-4}	−4.9	3.4×10^{-12}	-7.5×10^{-6}	-1.6×10^{-11}	1.1×10^{-7}
95.0	9.5	${}^{-5}_{5}f$	${}^{8}_{8}g_3$	2.4×10^{-4}	−4.9	3.4×10^{-12}	-7.5×10^{-6}	-6.3×10^{-11}	2.2×10^{-7}
95.0	9.6	${}^{-5}_{5}f$	${}^{8}_{8}g_3$	2.4×10^{-4}	−4.9	3.2×10^{-12}	-7.5×10^{-6}	-2.5×10^{-10}	4.4×10^{-7}
95.0	9.7	${}^{-5}_{5}f$	${}^{8}_{8}g_3$	2.4×10^{-4}	−4.9	2.5×10^{-12}	-7.5×10^{-6}	-9.9×10^{-10}	8.8×10^{-7}
95.2	8.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	7.5×10^{-2}	−7.0	4.2×10^{-13}	-2.6×10^{-3}	-2.8×10^{-13}	2.1×10^{-7}
95.2	8.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	7.5×10^{-2}	−7.0	2.0×10^{-12}	-2.6×10^{-3}	-1.8×10^{-13}	1.7×10^{-7}
95.2	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	7.5×10^{-2}	−7.0	3.1×10^{-12}	-2.6×10^{-3}	-1.1×10^{-13}	1.3×10^{-7}
95.2	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	7.5×10^{-2}	−7.0	3.7×10^{-12}	-2.6×10^{-3}	-7.1×10^{-14}	1.1×10^{-7}
95.2	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	7.5×10^{-2}	−7.0	4.1×10^{-12}	-2.6×10^{-3}	-4.6×10^{-14}	8.6×10^{-8}
95.2	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	7.5×10^{-2}	−7.0	4.4×10^{-12}	-2.6×10^{-3}	-3.5×10^{-14}	7.5×10^{-8}
95.2	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	7.5×10^{-2}	−7.0	4.5×10^{-12}	-2.6×10^{-3}	-4.4×10^{-14}	8.5×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
95.2	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	7.5×10^{-2}	−7.0	4.6×10^{-12}	-2.6×10^{-3}	-1.2×10^{-13}	1.4×10^{-7}
95.2	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	7.5×10^{-2}	−7.0	4.7×10^{-12}	-2.6×10^{-3}	-4.3×10^{-13}	2.6×10^{-7}
95.2	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	7.5×10^{-2}	−7.0	4.5×10^{-12}	-2.6×10^{-3}	-1.7×10^{-12}	5.2×10^{-7}
95.2	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	7.5×10^{-2}	−7.0	3.7×10^{-12}	-2.6×10^{-3}	-6.7×10^{-12}	1.0×10^{-6}
95.2	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	7.5×10^{-2}	−7.0	6.3×10^{-13}	-2.6×10^{-3}	-2.7×10^{-11}	2.1×10^{-6}
95.4	8.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.9×10^{-2}	−7.1	2.1×10^{-12}	-2.6×10^{-3}	-2.8×10^{-13}	1.9×10^{-7}
95.4	8.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.9×10^{-2}	−7.1	3.7×10^{-12}	-2.6×10^{-3}	-1.8×10^{-13}	1.5×10^{-7}
95.4	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.9×10^{-2}	−7.1	4.8×10^{-12}	-2.6×10^{-3}	-1.1×10^{-13}	1.2×10^{-7}
95.4	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.9×10^{-2}	−7.1	5.4×10^{-12}	-2.6×10^{-3}	-7.0×10^{-14}	9.7×10^{-8}
95.4	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.9×10^{-2}	−7.1	5.8×10^{-12}	-2.6×10^{-3}	-4.6×10^{-14}	7.8×10^{-8}
95.4	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.9×10^{-2}	−7.1	6.1×10^{-12}	-2.6×10^{-3}	-3.4×10^{-14}	6.8×10^{-8}
95.4	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.9×10^{-2}	−7.1	6.3×10^{-12}	-2.6×10^{-3}	-4.4×10^{-14}	7.7×10^{-8}
95.4	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.9×10^{-2}	−7.1	6.4×10^{-12}	-2.6×10^{-3}	-1.2×10^{-13}	1.2×10^{-7}
95.4	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.9×10^{-2}	−7.1	6.4×10^{-12}	-2.6×10^{-3}	-4.3×10^{-13}	2.4×10^{-7}
95.4	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.9×10^{-2}	−7.1	6.2×10^{-12}	-2.6×10^{-3}	-1.7×10^{-12}	4.7×10^{-7}
95.4	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.9×10^{-2}	−7.1	5.4×10^{-12}	-2.6×10^{-3}	-6.7×10^{-12}	9.4×10^{-7}
95.4	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.9×10^{-2}	−7.1	2.3×10^{-12}	-2.6×10^{-3}	-2.7×10^{-11}	1.9×10^{-6}
95.6	8.6	${}^{-4}_{4}g_{7}$	${}^{7}_{7}g_{4}$	1.0×10^{-6}	4.0	1.8×10^{-12}	-1.3×10^{-12}	-6.0×10^{-12}	1.5×10^{-7}
95.6	8.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.3×10^{-2}	−7.2	4.4×10^{-12}	-2.6×10^{-3}	-2.8×10^{-13}	1.7×10^{-7}
95.6	8.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.3×10^{-2}	−7.2	6.0×10^{-12}	-2.6×10^{-3}	-1.7×10^{-13}	1.4×10^{-7}
95.6	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.3×10^{-2}	−7.2	7.1×10^{-12}	-2.6×10^{-3}	-1.1×10^{-13}	1.1×10^{-7}
95.6	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.3×10^{-2}	−7.2	7.7×10^{-12}	-2.6×10^{-3}	-7.0×10^{-14}	8.7×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{\rm unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{\rm PIT}\rvert$
95.6	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.3×10^{-2}	−7.2	8.1×10^{-12}	-2.6×10^{-3}	-4.5×10^{-14}	7.0×10^{-8}
95.6	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.3×10^{-2}	−7.2	8.4×10^{-12}	-2.6×10^{-3}	-3.4×10^{-14}	6.1×10^{-8}
95.6	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.3×10^{-2}	−7.2	8.6×10^{-12}	-2.6×10^{-3}	-4.4×10^{-14}	6.9×10^{-8}
95.6	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	6.3×10^{-2}	−7.2	8.7×10^{-12}	-2.6×10^{-3}	-1.2×10^{-13}	1.1×10^{-7}
95.6	9.5	${}^{-4}_{4}g_{7}$	${}^{7}_{7}g_{4}$	1.0×10^{-6}	4.0	8.7×10^{-12}	-1.9×10^{-12}	-2.7×10^{-11}	9.3×10^{-8}
95.6	9.6	${}^{-4}_{4}g_{7}$	${}^{7}_{7}g_{4}$	1.0×10^{-6}	4.0	8.5×10^{-12}	-4.5×10^{-12}	-1.1×10^{-10}	7.5×10^{-8}
95.6	9.7	${}^{-4}_{4}g_{7}$	${}^{7}_{7}g_{4}$	1.0×10^{-6}	4.0	7.7×10^{-12}	-1.5×10^{-11}	-4.2×10^{-10}	6.9×10^{-8}
95.6	9.8	${}^{-4}_{4}g_{7}$	${}^{7}_{7}g_{4}$	1.0×10^{-6}	4.0	4.6×10^{-12}	-5.7×10^{-11}	-1.7×10^{-9}	6.7×10^{-8}
95.8	8.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	5.9×10^{-13}	-2.6×10^{-3}	-6.9×10^{-13}	2.4×10^{-7}
95.8	8.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	4.8×10^{-12}	-2.6×10^{-3}	-4.3×10^{-13}	1.9×10^{-7}
95.8	8.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	7.4×10^{-12}	-2.6×10^{-3}	-2.7×10^{-13}	1.5×10^{-7}
95.8	8.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	9.1×10^{-12}	-2.6×10^{-3}	-1.7×10^{-13}	1.2×10^{-7}
95.8	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	1.0×10^{-11}	-2.6×10^{-3}	-1.1×10^{-13}	9.6×10^{-8}
95.8	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	1.1×10^{-11}	-2.6×10^{-3}	-6.9×10^{-14}	7.7×10^{-8}
95.8	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	1.1×10^{-11}	-2.6×10^{-3}	-4.5×10^{-14}	6.2×10^{-8}
95.8	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	1.1×10^{-11}	-2.6×10^{-3}	-3.4×10^{-14}	5.4×10^{-8}
95.8	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	1.2×10^{-11}	-2.6×10^{-3}	-4.3×10^{-14}	6.1×10^{-8}
95.8	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	1.2×10^{-11}	-2.6×10^{-3}	-1.2×10^{-13}	9.9×10^{-8}
95.8	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	1.2×10^{-11}	-2.6×10^{-3}	-4.2×10^{-13}	1.9×10^{-7}
95.8	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	1.2×10^{-11}	-2.6×10^{-3}	-1.7×10^{-12}	3.7×10^{-7}
95.8	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	1.1×10^{-11}	-2.6×10^{-3}	-6.6×10^{-12}	7.5×10^{-7}
95.8	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	5.6×10^{-2}	−7.3	7.6×10^{-12}	-2.6×10^{-3}	-2.6×10^{-11}	1.5×10^{-6}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
96.0	8.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	4.6×10^{-12}	-2.6×10^{-3}	-6.8×10^{-13}	2.1×10^{-7}
96.0	8.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	8.8×10^{-12}	-2.6×10^{-3}	-4.3×10^{-13}	1.7×10^{-7}
96.0	8.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	1.2×10^{-11}	-2.6×10^{-3}	-2.7×10^{-13}	1.3×10^{-7}
96.0	8.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	1.3×10^{-11}	-2.6×10^{-3}	-1.7×10^{-13}	1.0×10^{-7}
96.0	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	1.4×10^{-11}	-2.6×10^{-3}	-1.1×10^{-13}	8.3×10^{-8}
96.0	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	1.5×10^{-11}	-2.6×10^{-3}	-6.9×10^{-14}	6.6×10^{-8}
96.0	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	1.5×10^{-11}	-2.6×10^{-3}	-4.5×10^{-14}	5.3×10^{-8}
96.0	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	1.6×10^{-11}	-2.6×10^{-3}	-3.4×10^{-14}	4.6×10^{-8}
96.0	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	1.6×10^{-11}	-2.6×10^{-3}	-4.3×10^{-14}	5.3×10^{-8}
96.0	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	1.6×10^{-11}	-2.6×10^{-3}	-1.1×10^{-13}	8.5×10^{-8}
96.0	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	1.6×10^{-11}	-2.6×10^{-3}	-4.2×10^{-13}	1.6×10^{-7}
96.0	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	1.6×10^{-11}	-2.6×10^{-3}	-1.6×10^{-12}	3.2×10^{-7}
96.0	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	1.5×10^{-11}	-2.6×10^{-3}	-6.5×10^{-12}	6.5×10^{-7}
96.0	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.9×10^{-2}	−7.4	1.2×10^{-11}	-2.6×10^{-3}	-2.6×10^{-11}	1.3×10^{-6}
96.2	8.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.2×10^{-2}	−7.5	3.2×10^{-12}	-2.6×10^{-3}	-1.1×10^{-12}	2.2×10^{-7}
96.2	8.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.2×10^{-2}	−7.5	1.0×10^{-11}	-2.6×10^{-3}	-6.8×10^{-13}	1.8×10^{-7}
96.2	8.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.2×10^{-2}	−7.5	1.4×10^{-11}	-2.6×10^{-3}	-4.3×10^{-13}	1.4×10^{-7}
96.2	8.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.2×10^{-2}	−7.5	1.7×10^{-11}	-2.6×10^{-3}	-2.7×10^{-13}	1.1×10^{-7}
96.2	8.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.2×10^{-2}	−7.5	1.9×10^{-11}	-2.6×10^{-3}	-1.7×10^{-13}	8.8×10^{-8}
96.2	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.2×10^{-2}	−7.5	2.0×10^{-11}	-2.6×10^{-3}	-1.1×10^{-13}	7.0×10^{-8}
96.2	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.2×10^{-2}	−7.5	2.0×10^{-11}	-2.6×10^{-3}	-6.8×10^{-14}	5.6×10^{-8}
96.2	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	4.2×10^{-2}	−7.5	2.1×10^{-11}	-2.6×10^{-3}	-4.4×10^{-14}	4.5×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
96.2	9.2	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	4.2×10^{-2}	−7.5	2.1×10^{-11}	-2.6×10^{-3}	-3.3×10^{-14}	3.9×10^{-8}
96.2	9.3	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	4.2×10^{-2}	−7.5	2.1×10^{-11}	-2.6×10^{-3}	-4.3×10^{-14}	4.4×10^{-8}
96.2	9.4	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	4.2×10^{-2}	−7.5	2.1×10^{-11}	-2.6×10^{-3}	-1.1×10^{-13}	7.2×10^{-8}
96.2	9.5	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	4.2×10^{-2}	−7.5	2.1×10^{-11}	-2.6×10^{-3}	-4.2×10^{-13}	1.4×10^{-7}
96.2	9.6	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	4.2×10^{-2}	−7.5	2.1×10^{-11}	-2.6×10^{-3}	-1.6×10^{-12}	2.7×10^{-7}
96.2	9.7	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	4.2×10^{-2}	−7.5	2.0×10^{-11}	-2.6×10^{-3}	-6.5×10^{-12}	5.5×10^{-7}
96.2	9.8	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	4.2×10^{-2}	−7.5	1.7×10^{-11}	-2.6×10^{-3}	-2.6×10^{-11}	1.1×10^{-6}
96.2	9.9	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	4.2×10^{-2}	−7.5	4.2×10^{-12}	-2.6×10^{-3}	-1.0×10^{-10}	2.2×10^{-6}
96.4	8.4	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	1.0×10^{-11}	-2.6×10^{-3}	-1.1×10^{-12}	1.8×10^{-7}
96.4	8.5	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	1.7×10^{-11}	-2.6×10^{-3}	-6.7×10^{-13}	1.4×10^{-7}
96.4	8.6	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	2.2×10^{-11}	-2.6×10^{-3}	-4.2×10^{-13}	1.1×10^{-7}
96.4	8.7	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	2.4×10^{-11}	-2.6×10^{-3}	-2.7×10^{-13}	9.1×10^{-8}
96.4	8.8	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	2.6×10^{-11}	-2.6×10^{-3}	-1.7×10^{-13}	7.2×10^{-8}
96.4	8.9	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	2.7×10^{-11}	-2.6×10^{-3}	-1.1×10^{-13}	5.7×10^{-8}
96.4	9.0	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	2.8×10^{-11}	-2.6×10^{-3}	-6.8×10^{-14}	4.5×10^{-8}
96.4	9.1	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	2.8×10^{-11}	-2.6×10^{-3}	-4.4×10^{-14}	3.7×10^{-8}
96.4	9.2	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	2.8×10^{-11}	-2.6×10^{-3}	-3.3×10^{-14}	3.2×10^{-8}
96.4	9.3	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	2.9×10^{-11}	-2.6×10^{-3}	-4.2×10^{-14}	3.6×10^{-8}
96.4	9.4	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	2.9×10^{-11}	-2.6×10^{-3}	-1.1×10^{-13}	5.9×10^{-8}
96.4	9.5	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	2.9×10^{-11}	-2.6×10^{-3}	-4.1×10^{-13}	1.1×10^{-7}
96.4	9.6	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	2.9×10^{-11}	-2.6×10^{-3}	-1.6×10^{-12}	2.2×10^{-7}
96.4	9.7	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	3.5×10^{-2}	−7.7	2.8×10^{-11}	-2.6×10^{-3}	-6.4×10^{-12}	4.4×10^{-7}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
96.4	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	3.5×10^{-2}	−7.7	2.4×10^{-11}	-2.6×10^{-3}	-2.6×10^{-11}	8.9×10^{-7}
96.4	9.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	3.5×10^{-2}	−7.7	1.1×10^{-11}	-2.6×10^{-3}	-1.0×10^{-10}	1.8×10^{-6}
96.6	8.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	9.1×10^{-12}	-2.6×10^{-3}	-1.7×10^{-12}	1.7×10^{-7}
96.6	8.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	2.0×10^{-11}	-2.6×10^{-3}	-1.1×10^{-12}	1.4×10^{-7}
96.6	8.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	2.7×10^{-11}	-2.6×10^{-3}	-6.7×10^{-13}	1.1×10^{-7}
96.6	8.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	3.1×10^{-11}	-2.6×10^{-3}	-4.2×10^{-13}	8.8×10^{-8}
96.6	8.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	3.4×10^{-11}	-2.6×10^{-3}	-2.7×10^{-13}	7.0×10^{-8}
96.6	8.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	3.6×10^{-11}	-2.6×10^{-3}	-1.7×10^{-13}	5.5×10^{-8}
96.6	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	3.7×10^{-11}	-2.6×10^{-3}	-1.1×10^{-13}	4.4×10^{-8}
96.6	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	3.8×10^{-11}	-2.6×10^{-3}	-6.7×10^{-14}	3.5×10^{-8}
96.6	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	3.8×10^{-11}	-2.6×10^{-3}	-4.4×10^{-14}	2.8×10^{-8}
96.6	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	3.8×10^{-11}	-2.6×10^{-3}	-3.3×10^{-14}	2.5×10^{-8}
96.6	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	3.8×10^{-11}	-2.6×10^{-3}	-4.2×10^{-14}	2.8×10^{-8}
96.6	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	3.9×10^{-11}	-2.6×10^{-3}	-1.1×10^{-13}	4.5×10^{-8}
96.6	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	3.9×10^{-11}	-2.6×10^{-3}	-4.1×10^{-13}	8.6×10^{-8}
96.6	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	3.8×10^{-11}	-2.6×10^{-3}	-1.6×10^{-12}	1.7×10^{-7}
96.6	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	3.8×10^{-11}	-2.6×10^{-3}	-6.4×10^{-12}	3.4×10^{-7}
96.6	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	3.4×10^{-11}	-2.6×10^{-3}	-2.5×10^{-11}	6.8×10^{-7}
96.6	9.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.7×10^{-2}	−7.8	2.1×10^{-11}	-2.6×10^{-3}	-1.0×10^{-10}	1.4×10^{-6}
96.8	8.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	1.9×10^{-2}	−7.9	4.3×10^{-12}	-2.6×10^{-3}	-2.6×10^{-12}	1.5×10^{-7}
96.8	8.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	1.9×10^{-2}	−7.9	2.2×10^{-11}	-2.6×10^{-3}	-1.7×10^{-12}	1.2×10^{-7}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\|Q_{PIT}\|$
96.8	8.4	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	3.3×10^{-11}	-2.6×10^{-3}	-1.0×10^{-12}	9.7×10^{-8}
96.8	8.5	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	4.0×10^{-11}	-2.6×10^{-3}	-6.6×10^{-13}	7.7×10^{-8}
96.8	8.6	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	4.4×10^{-11}	-2.6×10^{-3}	-4.2×10^{-13}	6.1×10^{-8}
96.8	8.7	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	4.7×10^{-11}	-2.6×10^{-3}	-2.6×10^{-13}	4.9×10^{-8}
96.8	8.8	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	4.9×10^{-11}	-2.6×10^{-3}	-1.7×10^{-13}	3.9×10^{-8}
96.8	8.9	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	5.0×10^{-11}	-2.6×10^{-3}	-1.0×10^{-13}	3.1×10^{-8}
96.8	9.0	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	5.1×10^{-11}	-2.6×10^{-3}	-6.7×10^{-14}	2.4×10^{-8}
96.8	9.1	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	5.1×10^{-11}	-2.6×10^{-3}	-4.3×10^{-14}	2.0×10^{-8}
96.8	9.2	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	5.1×10^{-11}	-2.6×10^{-3}	-3.3×10^{-14}	1.7×10^{-8}
96.8	9.3	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	5.2×10^{-11}	-2.6×10^{-3}	-4.2×10^{-14}	1.9×10^{-8}
96.8	9.4	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	5.2×10^{-11}	-2.6×10^{-3}	-1.1×10^{-13}	3.2×10^{-8}
96.8	9.5	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	5.2×10^{-11}	-2.6×10^{-3}	-4.1×10^{-13}	6.0×10^{-8}
96.8	9.6	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	5.1×10^{-11}	-2.6×10^{-3}	-1.6×10^{-12}	1.2×10^{-7}
96.8	9.7	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	5.1×10^{-11}	-2.6×10^{-3}	-6.3×10^{-12}	2.4×10^{-7}
96.8	9.8	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	4.7×10^{-11}	-2.6×10^{-3}	-2.5×10^{-11}	4.8×10^{-7}
96.8	9.9	${}_{2}^{-2}f$	${}_{5}^{5}g_{10}$	1.9×10^{-2}	−7.9	3.4×10^{-11}	-2.6×10^{-3}	-1.0×10^{-10}	9.5×10^{-7}
97.0	8.2	${}_{3}^{-3}f$	${}_{6}^{6}g_{9}$	-1.2×10^{-3}	4.3	2.1×10^{-11}	-3.6×10^{-4}	-7.8×10^{-12}	6.1×10^{-8}
97.0	8.3	${}_{3}^{-3}f$	${}_{6}^{6}g_{9}$	-1.2×10^{-3}	4.3	3.9×10^{-11}	-3.6×10^{-4}	-4.9×10^{-12}	4.9×10^{-8}
97.0	8.4	${}_{3}^{-3}f$	${}_{6}^{6}g_{9}$	-1.2×10^{-3}	4.3	5.0×10^{-11}	-3.6×10^{-4}	-3.1×10^{-12}	3.9×10^{-8}
97.0	8.5	${}_{3}^{-3}f$	${}_{6}^{6}g_{9}$	-1.2×10^{-3}	4.3	5.7×10^{-11}	-3.6×10^{-4}	-2.0×10^{-12}	3.1×10^{-8}
97.0	8.6	${}_{3}^{-3}f$	${}_{6}^{6}g_{9}$	-1.2×10^{-3}	4.3	6.2×10^{-11}	-3.6×10^{-4}	-1.2×10^{-12}	2.4×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
97.0	8.7	${}^{-3}_{3}f$	${}^{6}_{6}g_9$	-1.2×10^{-3}	4.3	6.4×10^{-11}	-3.6×10^{-4}	-7.8×10^{-13}	1.9×10^{-8}
97.0	8.8	${}^{-3}_{3}f$	${}^{6}_{6}g_9$	-1.2×10^{-3}	4.3	6.6×10^{-11}	-3.6×10^{-4}	-4.9×10^{-13}	1.5×10^{-8}
97.0	8.9	${}^{-3}_{3}f$	${}^{6}_{6}g_9$	-1.2×10^{-3}	4.3	6.7×10^{-11}	-3.6×10^{-4}	-3.1×10^{-13}	1.2×10^{-8}
97.0	9.0	${}^{-3}_{3}f$	${}^{6}_{6}g_9$	-1.2×10^{-3}	4.3	6.8×10^{-11}	-3.6×10^{-4}	-2.0×10^{-13}	9.7×10^{-9}
97.0	9.1	${}^{-3}_{3}f$	${}^{6}_{6}g_9$	-1.2×10^{-3}	4.3	6.9×10^{-11}	-3.6×10^{-4}	-1.3×10^{-13}	7.9×10^{-9}
97.0	9.2	${}^{-3}_{3}f$	${}^{6}_{6}g_9$	-1.2×10^{-3}	4.3	6.9×10^{-11}	-3.6×10^{-4}	-1.0×10^{-13}	6.9×10^{-9}
97.0	9.3	${}^{-3}_{3}f$	${}^{6}_{6}g_9$	-1.2×10^{-3}	4.3	6.9×10^{-11}	-3.6×10^{-4}	-1.4×10^{-13}	8.2×10^{-9}
97.0	9.4	${}^{-3}_{3}f$	${}^{6}_{6}g_9$	-1.2×10^{-3}	4.3	6.9×10^{-11}	-3.6×10^{-4}	-3.9×10^{-13}	1.4×10^{-8}
97.0	9.5	${}^{-3}_{3}f$	${}^{6}_{6}g_9$	-1.2×10^{-3}	4.3	6.9×10^{-11}	-3.6×10^{-4}	-1.4×10^{-12}	2.6×10^{-8}
97.0	9.6	${}^{-3}_{3}f$	${}^{6}_{6}g_9$	-1.2×10^{-3}	4.3	6.9×10^{-11}	-3.6×10^{-4}	-5.7×10^{-12}	5.2×10^{-8}
97.0	9.7	${}^{-3}_{3}f$	${}^{6}_{6}g_9$	-1.2×10^{-3}	4.3	6.8×10^{-11}	-3.6×10^{-4}	-2.3×10^{-11}	1.0×10^{-7}
97.0	9.8	${}^{-3}_{3}f$	${}^{6}_{6}g_9$	-1.2×10^{-3}	v4.3	6.5×10^{-11}	-3.6×10^{-4}	-9.0×10^{-11}	2.1×10^{-7}
97.0	9.9	${}^{-3}_{3}f$	${}^{6}_{6}g_9$	-1.2×10^{-3}	4.3	5.1×10^{-11}	-3.6×10^{-4}	-3.6×10^{-10}	4.1×10^{-7}
97.2	8.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	1.5×10^{-11}	-2.6×10^{-3}	-4.1×10^{-12}	3.3×10^{-8}
97.2	8.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	4.4×10^{-11}	-2.6×10^{-3}	-2.6×10^{-12}	2.7×10^{-8}
97.2	8.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	6.2×10^{-11}	-2.6×10^{-3}	-1.6×10^{-12}	2.1×10^{-8}
97.2	8.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	7.3×10^{-11}	-2.6×10^{-3}	-1.0×10^{-12}	1.7×10^{-8}
97.2	8.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	8.0×10^{-11}	-2.6×10^{-3}	-6.5×10^{-13}	1.3×10^{-8}
97.2	8.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	8.5×10^{-11}	-2.6×10^{-3}	-4.1×10^{-13}	1.1×10^{-8}
97.2	8.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	8.8×10^{-11}	-2.6×10^{-3}	-2.6×10^{-13}	8.4×10^{-9}
97.2	8.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	8.9×10^{-11}	-2.6×10^{-3}	-1.6×10^{-13}	6.7×10^{-9}
97.2	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	9.1×10^{-11}	-2.6×10^{-3}	-1.0×10^{-13}	5.3×10^{-9}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{\text{unit}}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{\text{PIT}}\rvert$
97.2	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	9.1×10^{-11}	-2.6×10^{-3}	-6.6×10^{-14}	4.2×10^{-9}
97.2	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	9.2×10^{-11}	-2.6×10^{-3}	-4.3×10^{-14}	3.4×10^{-9}
97.2	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	9.2×10^{-11}	-2.6×10^{-3}	-3.2×10^{-14}	3.0×10^{-9}
97.2	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	9.2×10^{-11}	-2.6×10^{-3}	-4.1×10^{-14}	3.3×10^{-9}
97.2	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	9.2×10^{-11}	-2.6×10^{-3}	-1.1×10^{-13}	5.4×10^{-9}
97.2	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	9.2×10^{-11}	-2.6×10^{-3}	-4.0×10^{-13}	1.0×10^{-8}
97.2	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	9.2×10^{-11}	-2.6×10^{-3}	-1.6×10^{-12}	2.1×10^{-8}
97.2	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	9.1×10^{-11}	-2.6×10^{-3}	-6.2×10^{-12}	4.1×10^{-8}
97.2	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	8.8×10^{-11}	-2.6×10^{-3}	-2.5×10^{-11}	8.2×10^{-8}
97.2	9.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	7.4×10^{-11}	-2.6×10^{-3}	-9.9×10^{-11}	1.6×10^{-7}
97.2	10.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	2.3×10^{-3}	−8.2	1.9×10^{-11}	-2.6×10^{-3}	-3.9×10^{-10}	3.3×10^{-7}
97.4	8.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	4.5×10^{-11}	-2.6×10^{-3}	-4.1×10^{-12}	6.8×10^{-8}
97.4	8.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	7.4×10^{-11}	-2.6×10^{-3}	-2.6×10^{-12}	5.4×10^{-8}
97.4	8.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	9.2×10^{-11}	-2.6×10^{-3}	-1.6×10^{-12}	4.3×10^{-8}
97.4	8.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.0×10^{-10}	-2.6×10^{-3}	-1.0×10^{-12}	3.4×10^{-8}
97.4	8.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.1×10^{-10}	-2.6×10^{-3}	-6.5×10^{-13}	2.7×10^{-8}
97.4	8.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.2×10^{-10}	-2.6×10^{-3}	-4.1×10^{-13}	2.1×10^{-8}
97.4	8.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.2×10^{-10}	-2.6×10^{-3}	-2.6×10^{-13}	1.7×10^{-8}
97.4	8.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.2×10^{-10}	-2.6×10^{-3}	-1.6×10^{-13}	1.4×10^{-8}
97.4	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.2×10^{-10}	-2.6×10^{-3}	-1.0×10^{-13}	1.1×10^{-8}
97.4	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.2×10^{-10}	-2.6×10^{-3}	-6.5×10^{-14}	8.6×10^{-9}
97.4	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.2×10^{-10}	-2.6×10^{-3}	-4.2×10^{-14}	6.9×10^{-9}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
97.4	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.2×10^{-10}	-2.6×10^{-3}	-3.2×10^{-14}	6.0×10^{-9}
97.4	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.2×10^{-10}	-2.6×10^{-3}	-4.1×10^{-14}	6.8×10^{-9}
97.4	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.2×10^{-10}	-2.6×10^{-3}	-1.1×10^{-13}	1.1×10^{-8}
97.4	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.2×10^{-10}	-2.6×10^{-3}	-4.0×10^{-13}	2.1×10^{-8}
97.4	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.2×10^{-10}	-2.6×10^{-3}	-1.6×10^{-12}	4.2×10^{-8}
97.4	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.2×10^{-10}	-2.6×10^{-3}	-6.2×10^{-12}	8.4×10^{-8}
97.4	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.2×10^{-10}	-2.6×10^{-3}	-2.5×10^{-11}	1.7×10^{-7}
97.4	9.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	1.1×10^{-10}	-2.6×10^{-3}	-9.8×10^{-11}	3.3×10^{-7}
97.4	10.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-6.8×10^{-3}	−8.3	4.9×10^{-11}	-2.6×10^{-3}	-3.9×10^{-10}	6.6×10^{-7}
97.6	8.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	3.9×10^{-11}	-2.6×10^{-3}	-6.4×10^{-12}	1.9×10^{-7}
97.6	8.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	8.6×10^{-11}	-2.6×10^{-3}	-4.0×10^{-12}	1.5×10^{-7}
97.6	8.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.1×10^{-10}	-2.6×10^{-3}	-2.6×10^{-12}	1.2×10^{-7}
97.6	8.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.3×10^{-10}	-2.6×10^{-3}	-1.6×10^{-12}	9.5×10^{-8}
97.6	8.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.5×10^{-10}	-2.6×10^{-3}	-1.0×10^{-12}	7.6×10^{-8}
97.6	8.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.5×10^{-10}	-2.6×10^{-3}	-6.4×10^{-13}	6.0×10^{-8}
97.6	8.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.6×10^{-10}	-2.6×10^{-3}	-4.0×10^{-13}	4.8×10^{-8}
97.6	8.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.6×10^{-10}	-2.6×10^{-3}	-2.6×10^{-13}	3.8×10^{-8}
97.6	8.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.6×10^{-10}	-2.6×10^{-3}	-1.6×10^{-13}	3.0×10^{-8}
97.6	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.6×10^{-10}	-2.6×10^{-3}	-1.0×10^{-13}	2.4×10^{-8}
97.6	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.6×10^{-10}	-2.6×10^{-3}	-6.5×10^{-14}	1.9×10^{-8}
97.6	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.6×10^{-10}	-2.6×10^{-3}	-4.2×10^{-14}	1.5×10^{-8}
97.6	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.6×10^{-10}	-2.6×10^{-3}	-3.2×10^{-14}	1.3×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
97.6	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.6×10^{-10}	-2.6×10^{-3}	-4.1×10^{-14}	1.5×10^{-8}
97.6	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.6×10^{-10}	-2.6×10^{-3}	-1.1×10^{-13}	2.5×10^{-8}
97.6	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.6×10^{-10}	-2.6×10^{-3}	-3.9×10^{-13}	4.7×10^{-8}
97.6	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.6×10^{-10}	-2.6×10^{-3}	-1.5×10^{-12}	9.3×10^{-8}
97.6	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.6×10^{-10}	-2.6×10^{-3}	-6.1×10^{-12}	1.9×10^{-7}
97.6	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.6×10^{-10}	-2.6×10^{-3}	-2.4×10^{-11}	3.7×10^{-7}
97.6	9.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	1.5×10^{-10}	-2.6×10^{-3}	-9.7×10^{-11}	7.4×10^{-7}
97.6	10.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-1.6×10^{-2}	−8.5	9.0×10^{-11}	-2.6×10^{-3}	-3.9×10^{-10}	1.5×10^{-6}
97.8	7.9	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	1.8×10^{-11}	-3.7×10^{-4}	-3.4×10^{-11}	2.8×10^{-7}
97.8	8.0	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	9.2×10^{-11}	-3.7×10^{-4}	-2.2×10^{-11}	2.2×10^{-7}
97.8	8.1	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	1.4×10^{-10}	-3.7×10^{-4}	-1.4×10^{-11}	1.8×10^{-7}
97.8	8.2	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	1.7×10^{-10}	-3.7×10^{-4}	-8.6×10^{-12}	1.4×10^{-7}
97.8	8.3	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	1.9×10^{-10}	-3.7×10^{-4}	-5.5×10^{-12}	1.1×10^{-7}
97.8	8.4	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	2.0×10^{-10}	-3.7×10^{-4}	-3.4×10^{-12}	8.8×10^{-8}
97.8	8.5	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	2.1×10^{-10}	-3.7×10^{-4}	-2.2×10^{-12}	7.0×10^{-8}
97.8	8.6	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	2.1×10^{-10}	-3.7×10^{-4}	-1.4×10^{-12}	5.5×10^{-8}
97.8	8.7	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	2.2×10^{-10}	-3.7×10^{-4}	-8.6×10^{-13}	4.4×10^{-8}
97.8	8.8	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	2.2×10^{-10}	-3.7×10^{-4}	-5.5×10^{-13}	3.5×10^{-8}
97.8	8.9	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	2.2×10^{-10}	-3.7×10^{-4}	-3.4×10^{-13}	2.8×10^{-8}
97.8	9.0	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	2.2×10^{-10}	-3.7×10^{-4}	-2.2×10^{-13}	2.2×10^{-8}
97.8	9.1	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	2.2×10^{-10}	-3.7×10^{-4}	-1.5×10^{-13}	1.8×10^{-8}
97.8	9.2	${}^{-3}_{3}f$	${}^{6}_{6}g_{8}$	2.9×10^{-3}	4.7	2.2×10^{-10}	-3.7×10^{-4}	-1.2×10^{-13}	1.6×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
97.8	9.3	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	2.9×10^{-3}	4.7	2.2×10^{-10}	-3.7×10^{-4}	-1.9×10^{-13}	2.0×10^{-8}
97.8	9.4	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	2.9×10^{-3}	4.7	2.2×10^{-10}	-3.7×10^{-4}	-5.6×10^{-13}	3.5×10^{-8}
97.8	9.5	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	2.9×10^{-3}	4.7	2.2×10^{-10}	-3.7×10^{-4}	-2.1×10^{-12}	6.9×10^{-8}
97.8	9.6	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	2.9×10^{-3}	4.7	2.2×10^{-10}	-3.7×10^{-4}	-8.4×10^{-12}	1.4×10^{-7}
97.8	9.7	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	2.9×10^{-3}	4.7	2.2×10^{-10}	-3.7×10^{-4}	-3.3×10^{-11}	2.7×10^{-7}
97.8	9.8	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	2.9×10^{-3}	4.7	2.2×10^{-10}	-3.7×10^{-4}	-1.3×10^{-10}	5.5×10^{-7}
97.8	9.9	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	2.9×10^{-3}	4.7	2.0×10^{-10}	-3.7×10^{-4}	-5.3×10^{-10}	1.1×10^{-6}
97.8	10.0	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	2.9×10^{-3}	4.7	1.4×10^{-10}	-3.7×10^{-4}	-2.1×10^{-9}	2.2×10^{-6}
98.0	7.9	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	-5.4×10^{-3}	4.8	8.8×10^{-11}	-3.7×10^{-4}	-3.4×10^{-11}	5.0×10^{-7}
98.0	8.0	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	-5.4×10^{-3}	4.8	1.6×10^{-10}	-3.7×10^{-4}	-2.2×10^{-11}	3.9×10^{-7}
98.0	8.1	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	-5.4×10^{-3}	4.8	2.1×10^{-10}	-3.7×10^{-4}	-1.4×10^{-11}	3.1×10^{-7}
98.0	8.2	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	-5.4×10^{-3}	4.8	2.4×10^{-10}	-3.7×10^{-4}	-8.6×10^{-12}	2.5×10^{-7}
98.0	8.3	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	-5.4×10^{-3}	4.8	2.6×10^{-10}	-3.7×10^{-4}	-5.4×10^{-12}	2.0×10^{-7}
98.0	8.4	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	-5.4×10^{-3}	4.8	2.7×10^{-10}	-3.7×10^{-4}	-3.4×10^{-12}	1.6×10^{-7}
98.0	8.5	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	-5.4×10^{-3}	4.8	2.8×10^{-10}	-3.7×10^{-4}	-2.2×10^{-12}	1.2×10^{-7}
98.0	8.6	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	-5.4×10^{-3}	4.8	2.9×10^{-10}	-3.7×10^{-4}	-1.4×10^{-12}	9.9×10^{-8}
98.0	8.7	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	-5.4×10^{-3}	4.8	2.9×10^{-10}	-3.7×10^{-4}	-8.6×10^{-13}	7.9×10^{-8}
98.0	8.8	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	-5.4×10^{-3}	4.8	2.9×10^{-10}	-3.7×10^{-4}	-5.4×10^{-13}	6.2×10^{-8}
98.0	8.9	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	-5.4×10^{-3}	4.8	2.9×10^{-10}	-3.7×10^{-4}	-3.4×10^{-13}	5.0×10^{-8}
98.0	9.0	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	-5.4×10^{-3}	4.8	2.9×10^{-10}	-3.7×10^{-4}	-2.2×10^{-13}	4.0×10^{-8}
98.0	9.1	${}^{-3}_{3}f$	${}^{6}_{6}g_8$	-5.4×10^{-3}	4.8	2.9×10^{-10}	-3.7×10^{-4}	-1.4×10^{-13}	3.2×10^{-8}
98.0	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-3.6×10^{-2}	-8.8	2.9×10^{-10}	-2.6×10^{-3}	-3.1×10^{-14}	2.9×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
98.0	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-3.6×10^{-2}	−8.8	2.9×10^{-10}	-2.6×10^{-3}	-4.0×10^{-14}	3.2×10^{-8}
98.0	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-3.6×10^{-2}	−8.8	2.9×10^{-10}	-2.6×10^{-3}	-1.1×10^{-13}	5.3×10^{-8}
98.0	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-3.6×10^{-2}	−8.8	2.9×10^{-10}	-2.6×10^{-3}	-3.9×10^{-13}	1.0×10^{-7}
98.0	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-3.6×10^{-2}	−8.8	2.9×10^{-10}	-2.6×10^{-3}	-1.5×10^{-12}	2.0×10^{-7}
98.0	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-3.6×10^{-2}	−8.8	2.9×10^{-10}	-2.6×10^{-3}	-6.0×10^{-12}	4.0×10^{-7}
98.0	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-3.6×10^{-2}	−8.8	2.9×10^{-10}	-2.6×10^{-3}	-2.4×10^{-11}	7.9×10^{-7}
98.0	9.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-3.6×10^{-2}	−8.8	2.7×10^{-10}	-2.6×10^{-3}	-9.6×10^{-11}	1.6×10^{-6}
98.0	10.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-3.6×10^{-2}	−8.8	2.2×10^{-10}	-2.6×10^{-3}	-3.8×10^{-10}	3.2×10^{-6}
98.2	7.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	6.1×10^{-11}	-2.6×10^{-3}	-1.6×10^{-11}	8.2×10^{-7}
98.2	7.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	1.8×10^{-10}	-2.6×10^{-3}	-9.9×10^{-12}	6.5×10^{-7}
98.2	8.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	2.6×10^{-10}	-2.6×10^{-3}	-6.3×10^{-12}	5.1×10^{-7}
98.2	8.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.1×10^{-10}	-2.6×10^{-3}	-4.0×10^{-12}	4.1×10^{-7}
98.2	8.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.4×10^{-10}	-2.6×10^{-3}	-2.5×10^{-12}	3.2×10^{-7}
98.2	8.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.6×10^{-10}	-2.6×10^{-3}	-1.6×10^{-12}	2.6×10^{-7}
98.2	8.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.7×10^{-10}	-2.6×10^{-3}	-9.9×10^{-13}	2.0×10^{-7}
98.2	8.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.8×10^{-10}	-2.6×10^{-3}	-6.3×10^{-13}	1.6×10^{-7}
98.2	8.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.9×10^{-10}	-2.6×10^{-3}	-4.0×10^{-13}	1.3×10^{-7}
98.2	8.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.9×10^{-10}	-2.6×10^{-3}	-2.5×10^{-13}	1.0×10^{-7}
98.2	8.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.9×10^{-10}	-2.6×10^{-3}	-1.6×10^{-13}	8.2×10^{-8}
98.2	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.9×10^{-10}	-2.6×10^{-3}	-9.9×10^{-14}	6.5×10^{-8}
98.2	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.9×10^{-10}	-2.6×10^{-3}	-6.3×10^{-14}	5.2×10^{-8}
98.2	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.9×10^{-10}	-2.6×10^{-3}	-4.1×10^{-14}	4.2×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
98.2	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.9×10^{-10}	-2.6×10^{-3}	-3.1×10^{-14}	3.6×10^{-8}
98.2	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.9×10^{-10}	-2.6×10^{-3}	-4.0×10^{-14}	4.1×10^{-8}
98.2	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.9×10^{-10}	-2.6×10^{-3}	-1.1×10^{-13}	6.7×10^{-8}
98.2	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.9×10^{-10}	-2.6×10^{-3}	-3.8×10^{-13}	1.3×10^{-7}
98.2	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.9×10^{-10}	-2.6×10^{-3}	-1.5×10^{-12}	2.5×10^{-7}
98.2	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.9×10^{-10}	-2.6×10^{-3}	-6.0×10^{-12}	5.0×10^{-7}
98.2	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.9×10^{-10}	-2.6×10^{-3}	-2.4×10^{-11}	1.0×10^{-6}
98.2	9.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.7×10^{-10}	-2.6×10^{-3}	-9.5×10^{-11}	2.0×10^{-6}
98.2	10.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	3.1×10^{-10}	-2.6×10^{-3}	-3.8×10^{-10}	4.0×10^{-6}
98.2	10.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{10}$	-4.7×10^{-2}	−9.0	7.9×10^{-11}	-2.6×10^{-3}	-1.5×10^{-9}	8.0×10^{-6}
98.4	7.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{9}$	4.7×10^{-2}	−9.5	1.9×10^{-10}	-2.6×10^{-3}	-2.0×10^{-11}	9.1×10^{-7}
98.4	7.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{9}$	4.7×10^{-2}	−9.5	3.1×10^{-10}	-2.6×10^{-3}	-1.3×10^{-11}	7.2×10^{-7}
98.4	8.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{9}$	4.7×10^{-2}	−9.5	3.9×10^{-10}	-2.6×10^{-3}	-8.0×10^{-12}	5.7×10^{-7}
98.4	8.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{9}$	4.7×10^{-2}	−9.5	4.4×10^{-10}	-2.6×10^{-3}	-5.0×10^{-12}	4.5×10^{-7}
98.4	8.2	${}^{-2}_{2}f$	${}^{5}_{5}g_{9}$	4.7×10^{-2}	−9.5	4.7×10^{-10}	-2.6×10^{-3}	-3.2×10^{-12}	3.6×10^{-7}
98.4	8.3	${}^{-2}_{2}f$	${}^{5}_{5}g_{9}$	4.7×10^{-2}	−9.5	4.9×10^{-10}	-2.6×10^{-3}	-2.0×10^{-12}	2.9×10^{-7}
98.4	8.4	${}^{-2}_{2}f$	${}^{5}_{5}g_{9}$	4.7×10^{-2}	−9.5	5.0×10^{-10}	-2.6×10^{-3}	-1.3×10^{-12}	2.3×10^{-7}
98.4	8.5	${}^{-2}_{2}f$	${}^{5}_{5}g_{9}$	4.7×10^{-2}	−9.5	5.1×10^{-10}	-2.6×10^{-3}	-8.0×10^{-13}	1.8×10^{-7}
98.4	8.6	${}^{-2}_{2}f$	${}^{5}_{5}g_{9}$	4.7×10^{-2}	−9.5	5.2×10^{-10}	-2.6×10^{-3}	-5.0×10^{-13}	1.4×10^{-7}
98.4	8.8	${}^{-2}_{2}f$	${}^{5}_{5}g_{9}$	4.7×10^{-2}	−9.5	5.2×10^{-10}	-2.6×10^{-3}	-2.0×10^{-13}	9.1×10^{-8}
98.4	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_{9}$	4.7×10^{-2}	−9.5	5.2×10^{-10}	-2.6×10^{-3}	-1.3×10^{-13}	7.2×10^{-8}
98.4	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_{9}$	4.7×10^{-2}	−9.5	5.2×10^{-10}	-2.6×10^{-3}	-8.1×10^{-14}	5.7×10^{-8}
98.4	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_{9}$	4.7×10^{-2}	−9.5	5.2×10^{-10}	-2.6×10^{-3}	-5.3×10^{-14}	4.7×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
98.4	9.2	$^{-2}_{2}f$	$^{5}_{5}g_9$	4.7×10^{-2}	−9.5	5.3×10^{-10}	-2.6×10^{-3}	-4.2×10^{-14}	4.1×10^{-8}
98.4	9.3	$^{-2}_{2}f$	$^{5}_{5}g_{10}$	-5.8×10^{-2}	−9.2	5.3×10^{-10}	-2.6×10^{-3}	-3.9×10^{-14}	4.9×10^{-8}
98.4	9.4	$^{-2}_{2}f$	$^{5}_{5}g_{10}$	-5.8×10^{-2}	−9.2	5.3×10^{-10}	-2.6×10^{-3}	-1.0×10^{-13}	8.1×10^{-8}
98.4	9.5	$^{-2}_{2}f$	$^{5}_{5}g_{10}$	-5.8×10^{-2}	−9.2	5.3×10^{-10}	-2.6×10^{-3}	-3.8×10^{-13}	1.5×10^{-7}
98.4	9.6	$^{-2}_{2}f$	$^{5}_{5}g_{10}$	-5.8×10^{-2}	−9.2	5.3×10^{-10}	-2.6×10^{-3}	-1.5×10^{-12}	3.1×10^{-7}
98.4	9.7	$^{-2}_{2}f$	$^{5}_{5}g_{10}$	-5.8×10^{-2}	−9.2	5.2×10^{-10}	-2.6×10^{-3}	-6.0×10^{-12}	6.1×10^{-7}
98.4	9.8	$^{-2}_{2}f$	$^{5}_{5}g_{10}$	-5.8×10^{-2}	−9.2	5.2×10^{-10}	-2.6×10^{-3}	-2.4×10^{-11}	1.2×10^{-6}
98.4	9.9	$^{-2}_{2}f$	$^{5}_{5}g_{10}$	-5.8×10^{-2}	−9.2	5.1×10^{-10}	-2.6×10^{-3}	-9.4×10^{-11}	2.4×10^{-6}
98.4	10.0	$^{-2}_{2}f$	$^{5}_{5}g_{10}$	-5.8×10^{-2}	−9.2	4.5×10^{-10}	-2.6×10^{-3}	-3.8×10^{-10}	4.8×10^{-6}
98.4	10.1	$^{-2}_{2}f$	$^{5}_{5}g_{10}$	-5.8×10^{-2}	−9.2	2.1×10^{-10}	-2.6×10^{-3}	-1.5×10^{-9}	9.7×10^{-6}
98.6	7.7	$^{-2}_{2}f$	$^{5}_{5}g_9$	3.5×10^{-2}	−9.7	1.6×10^{-10}	-2.6×10^{-3}	-3.2×10^{-11}	8.4×10^{-7}
98.6	7.8	$^{-2}_{2}f$	$^{5}_{5}g_9$	3.5×10^{-2}	−9.7	3.6×10^{-10}	-2.6×10^{-3}	-2.0×10^{-11}	6.7×10^{-7}
98.6	7.9	$^{-2}_{2}f$	$^{5}_{5}g_9$	3.5×10^{-2}	−9.7	4.9×10^{-10}	-2.6×10^{-3}	-1.3×10^{-11}	5.3×10^{-7}
98.6	8.0	$^{-2}_{2}f$	$^{5}_{5}g_9$	3.5×10^{-2}	−9.7	5.7×10^{-10}	-2.6×10^{-3}	-7.9×10^{-12}	4.2×10^{-7}
98.6	8.1	$^{-2}_{2}f$	$^{5}_{5}g_9$	3.5×10^{-2}	−9.7	6.2×10^{-10}	-2.6×10^{-3}	-5.0×10^{-12}	3.4×10^{-7}
98.6	8.2	$^{-2}_{2}f$	$^{5}_{5}g_9$	3.5×10^{-2}	−9.7	6.5×10^{-10}	-2.6×10^{-3}	-3.2×10^{-12}	2.7×10^{-7}
98.6	8.3	$^{-2}_{2}f$	$^{5}_{5}g_9$	3.5×10^{-2}	−9.7	6.7×10^{-10}	-2.6×10^{-3}	-2.0×10^{-12}	2.1×10^{-7}
98.6	8.4	$^{-2}_{2}f$	$^{5}_{5}g_9$	3.5×10^{-2}	−9.7	6.8×10^{-10}	-2.6×10^{-3}	-1.3×10^{-12}	1.7×10^{-7}
98.6	8.5	$^{-2}_{2}f$	$^{5}_{5}g_9$	3.5×10^{-2}	−9.7	6.9×10^{-10}	-2.6×10^{-3}	-7.9×10^{-13}	1.3×10^{-7}
98.6	8.6	$^{-2}_{2}f$	$^{5}_{5}g_9$	3.5×10^{-2}	−9.7	6.9×10^{-10}	-2.6×10^{-3}	-5.0×10^{-13}	1.1×10^{-7}
98.6	8.7	$^{-2}_{2}f$	$^{5}_{5}g_9$	3.5×10^{-2}	−9.7	7.0×10^{-10}	-2.6×10^{-3}	-3.2×10^{-13}	8.4×10^{-8}
98.6	8.8	$^{-2}_{2}f$	$^{5}_{5}g_9$	3.5×10^{-2}	−9.7	7.0×10^{-10}	-2.6×10^{-3}	-2.0×10^{-13}	6.7×10^{-8}
98.6	8.9	$^{-2}_{2}f$	$^{5}_{5}g_9$	3.5×10^{-2}	−9.7	7.0×10^{-10}	-2.6×10^{-3}	-1.3×10^{-13}	5.3×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
98.6	9.0	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	3.5×10^{-2}	−9.7	7.0×10^{-10}	-2.6×10^{-3}	-8.0×10^{-14}	4.2×10^{-8}
98.6	9.1	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	3.5×10^{-2}	−9.7	7.0×10^{-10}	-2.6×10^{-3}	-5.3×10^{-14}	3.4×10^{-8}
98.6	9.2	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	3.5×10^{-2}	−9.7	7.0×10^{-10}	-2.6×10^{-3}	-4.2×10^{-14}	3.1×10^{-8}
98.6	9.3	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	3.5×10^{-2}	−9.7	7.0×10^{-10}	-2.6×10^{-3}	-6.1×10^{-14}	3.7×10^{-8}
98.6	9.4	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	3.5×10^{-2}	−9.7	7.0×10^{-10}	-2.6×10^{-3}	-1.7×10^{-13}	6.3×10^{-8}
98.6	9.5	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	3.5×10^{-2}	−9.7	7.0×10^{-10}	-2.6×10^{-3}	-6.5×10^{-13}	1.2×10^{-7}
98.6	9.6	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	3.5×10^{-2}	−9.7	7.0×10^{-10}	-2.6×10^{-3}	-2.6×10^{-12}	2.4×10^{-7}
98.6	9.7	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	3.5×10^{-2}	−9.7	7.0×10^{-10}	-2.6×10^{-3}	-1.0×10^{-11}	4.8×10^{-7}
98.6	9.8	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	3.5×10^{-2}	−9.7	7.0×10^{-10}	-2.6×10^{-3}	-4.1×10^{-11}	9.6×10^{-7}
98.6	9.9	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	3.5×10^{-2}	−9.7	6.8×10^{-10}	-2.6×10^{-3}	-1.6×10^{-10}	1.9×10^{-6}
98.6	10.0	${}_{6}^{-6}g_{10}$	${}_{9}^{9}g_{10}$	-2.3×10^{-4}	−42.4	6.2×10^{-10}	-8.6×10^{-10}	-3.5×10^{-9}	3.2×10^{-6}
98.6	10.1	${}_{6}^{-6}g_{10}$	${}_{9}^{9}g_{10}$	-2.3×10^{-4}	−42.4	3.8×10^{-10}	-3.4×10^{-9}	-1.4×10^{-8}	3.2×10^{-6}
98.8	7.6	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	2.4×10^{-2}	−9.9	5.4×10^{-11}	-2.6×10^{-3}	-5.0×10^{-11}	6.9×10^{-7}
98.8	7.7	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	2.4×10^{-2}	−9.9	3.8×10^{-10}	-2.6×10^{-3}	-3.1×10^{-11}	5.5×10^{-7}
98.8	7.8	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	2.4×10^{-2}	−9.9	5.9×10^{-10}	-2.6×10^{-3}	-2.0×10^{-11}	4.4×10^{-7}
98.8	7.9	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	2.4×10^{-2}	−9.9	7.2×10^{-10}	-2.6×10^{-3}	-1.2×10^{-11}	3.5×10^{-7}
98.8	8.0	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	2.4×10^{-2}	−9.9	8.0×10^{-10}	-2.6×10^{-3}	-7.9×10^{-12}	2.8×10^{-7}
98.8	8.1	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	2.4×10^{-2}	−9.9	8.5×10^{-10}	-2.6×10^{-3}	-5.0×10^{-12}	2.2×10^{-7}
98.8	8.2	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	2.4×10^{-2}	−9.9	8.8×10^{-10}	-2.6×10^{-3}	-3.1×10^{-12}	1.7×10^{-7}
98.8	8.3	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	2.4×10^{-2}	−9.9	9.1×10^{-10}	-2.6×10^{-3}	-2.0×10^{-12}	1.4×10^{-7}
98.8	8.4	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	2.4×10^{-2}	−9.9	9.2×10^{-10}	-2.6×10^{-3}	-1.2×10^{-12}	1.1×10^{-7}
98.8	8.5	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	2.4×10^{-2}	−9.9	9.3×10^{-10}	-2.6×10^{-3}	-7.9×10^{-13}	8.7×10^{-8}
98.8	8.6	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	2.4×10^{-2}	−9.9	9.3×10^{-10}	-2.6×10^{-3}	-5.0×10^{-13}	6.9×10^{-8}
98.8	8.7	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	2.4×10^{-2}	−9.9	9.4×10^{-10}	-2.6×10^{-3}	-3.1×10^{-13}	5.5×10^{-8}
98.8	8.8	${}_{2}^{-2}f$	${}_{5}^{5}g_9$	2.4×10^{-2}	−9.9	9.4×10^{-10}	-2.6×10^{-3}	-2.0×10^{-13}	4.4×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
98.8	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	2.4×10^{-2}	−9.9	9.4×10^{-10}	-2.6×10^{-3}	-1.2×10^{-13}	3.5×10^{-8}
98.8	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	2.4×10^{-2}	−9.9	9.4×10^{-10}	-2.6×10^{-3}	-7.9×10^{-14}	2.8×10^{-8}
98.8	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	2.4×10^{-2}	−9.9	9.4×10^{-10}	-2.6×10^{-3}	-5.2×10^{-14}	2.2×10^{-8}
98.8	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	2.4×10^{-2}	−9.9	9.4×10^{-10}	-2.6×10^{-3}	-4.1×10^{-14}	2.0×10^{-8}
98.8	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	2.4×10^{-2}	−9.9	9.4×10^{-10}	-2.6×10^{-3}	-6.0×10^{-14}	2.4×10^{-8}
98.8	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	2.4×10^{-2}	−9.9	9.4×10^{-10}	-2.6×10^{-3}	-1.7×10^{-13}	4.1×10^{-8}
98.8	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	2.4×10^{-2}	−9.9	9.4×10^{-10}	-2.6×10^{-3}	-6.5×10^{-13}	7.9×10^{-8}
98.8	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	2.4×10^{-2}	−9.9	9.4×10^{-10}	-2.6×10^{-3}	-2.5×10^{-12}	1.6×10^{-7}
98.8	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	2.4×10^{-2}	−9.9	9.4×10^{-10}	-2.6×10^{-3}	-1.0×10^{-11}	3.1×10^{-7}
98.8	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	2.4×10^{-2}	−9.9	9.4×10^{-10}	-2.6×10^{-3}	-4.0×10^{-11}	6.2×10^{-7}
98.8	9.9	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	2.4×10^{-2}	−9.9	9.2×10^{-10}	-2.6×10^{-3}	-1.6×10^{-10}	1.2×10^{-6}
98.8	10.0	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	2.4×10^{-2}	−9.9	8.6×10^{-10}	-2.6×10^{-3}	-6.4×10^{-10}	2.5×10^{-6}
98.8	10.1	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	2.4×10^{-2}	−9.9	6.1×10^{-10}	-2.6×10^{-3}	-2.5×10^{-9}	5.0×10^{-6}
99.0	7.6	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	1.1×10^{-2}	−10.2	3.5×10^{-10}	-2.7×10^{-3}	-4.9×10^{-11}	3.3×10^{-7}
99.0	7.7	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	1.1×10^{-2}	−10.2	6.9×10^{-10}	-2.7×10^{-3}	-3.1×10^{-11}	2.6×10^{-7}
99.0	7.8	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	1.1×10^{-2}	−10.2	9.0×10^{-10}	-2.7×10^{-3}	-2.0×10^{-11}	2.1×10^{-7}
99.0	7.9	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.0×10^{-9}	-2.7×10^{-3}	-1.2×10^{-11}	1.6×10^{-7}
99.0	8.0	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.1×10^{-9}	-2.7×10^{-3}	-7.8×10^{-12}	1.3×10^{-7}
99.0	8.1	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.2×10^{-9}	-2.7×10^{-3}	-4.9×10^{-12}	1.0×10^{-7}
99.0	8.2	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.2×10^{-9}	-2.7×10^{-3}	-3.1×10^{-12}	8.2×10^{-8}
99.0	8.3	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.2×10^{-9}	-2.7×10^{-3}	-2.0×10^{-12}	6.5×10^{-8}
99.0	8.4	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.2×10^{-9}	-2.7×10^{-3}	-1.2×10^{-12}	5.2×10^{-8}
99.0	8.5	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.2×10^{-9}	-2.7×10^{-3}	-7.8×10^{-13}	4.1×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{\rm unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\|Q_{\rm PIT}\|$
99.0	8.6	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.3×10^{-9}	-2.7×10^{-3}	-4.9×10^{-13}	3.3×10^{-8}
99.0	8.7	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.3×10^{-9}	-2.7×10^{-3}	-3.1×10^{-13}	2.6×10^{-8}
99.0	8.8	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.3×10^{-9}	-2.7×10^{-3}	-2.0×10^{-13}	2.1×10^{-8}
99.0	8.9	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.3×10^{-9}	-2.7×10^{-3}	-1.2×10^{-13}	1.6×10^{-8}
99.0	9.0	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.3×10^{-9}	-2.7×10^{-3}	-7.9×10^{-14}	1.3×10^{-8}
99.0	9.1	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.3×10^{-9}	-2.7×10^{-3}	-5.2×10^{-14}	1.1×10^{-8}
99.0	9.2	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.3×10^{-9}	-2.7×10^{-3}	-4.1×10^{-14}	9.5×10^{-9}
99.0	9.3	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.3×10^{-9}	-2.7×10^{-3}	-6.0×10^{-14}	1.1×10^{-8}
99.0	9.4	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.3×10^{-9}	-2.7×10^{-3}	-1.7×10^{-13}	1.9×10^{-8}
99.0	9.5	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.3×10^{-9}	-2.7×10^{-3}	-6.4×10^{-13}	3.7×10^{-8}
99.0	9.6	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.3×10^{-9}	-2.7×10^{-3}	-2.5×10^{-12}	7.4×10^{-8}
99.0	9.7	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.3×10^{-9}	-2.7×10^{-3}	-1.0×10^{-11}	1.5×10^{-7}
99.0	9.8	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.3×10^{-9}	-2.7×10^{-3}	-4.0×10^{-11}	3.0×10^{-7}
99.0	9.9	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.2×10^{-9}	-2.7×10^{-3}	-1.6×10^{-10}	5.9×10^{-7}
99.0	10.0	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	1.2×10^{-9}	-2.7×10^{-3}	-6.3×10^{-10}	1.2×10^{-6}
99.0	10.1	$^{-2}_{2}f$	$^{5}_{5}g_9$	1.1×10^{-2}	−10.2	9.2×10^{-10}	-2.7×10^{-3}	-2.5×10^{-9}	2.3×10^{-6}
99.2	7.5	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.7×10^{-3}	−10.4	2.2×10^{-10}	-2.7×10^{-3}	-7.8×10^{-11}	1.1×10^{-7}
99.2	7.6	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.7×10^{-3}	−10.4	7.6×10^{-10}	-2.7×10^{-3}	-4.9×10^{-11}	8.8×10^{-8}
99.2	7.7	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.7×10^{-3}	−10.4	1.1×10^{-9}	-2.7×10^{-3}	-3.1×10^{-11}	7.0×10^{-8}
99.2	7.8	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.7×10^{-3}	−10.4	1.3×10^{-9}	-2.7×10^{-3}	-1.9×10^{-11}	5.6×10^{-8}
99.2	7.9	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.7×10^{-3}	−10.4	1.5×10^{-9}	-2.7×10^{-3}	-1.2×10^{-11}	4.4×10^{-8}
99.2	8.0	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.7×10^{-3}	−10.4	1.5×10^{-9}	-2.7×10^{-3}	-7.8×10^{-12}	3.5×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
99.2	8.1	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.6×10^{-9}	-2.7×10^{-3}	-4.9×10^{-12}	2.8×10^{-8}
99.2	8.2	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.6×10^{-9}	-2.7×10^{-3}	-3.1×10^{-12}	2.2×10^{-8}
99.2	8.3	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-1.9×10^{-12}	1.8×10^{-8}
99.2	8.4	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-1.2×10^{-12}	1.4×10^{-8}
99.2	8.5	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-7.8×10^{-13}	1.1×10^{-8}
99.2	8.6	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-4.9×10^{-13}	8.8×10^{-9}
99.2	8.7	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-3.1×10^{-13}	7.0×10^{-9}
99.2	8.8	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-1.9×10^{-13}	5.6×10^{-9}
99.2	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-1.2×10^{-13}	4.4×10^{-9}
99.2	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-7.8×10^{-14}	3.5×10^{-9}
99.2	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-5.1×10^{-14}	2.9×10^{-9}
99.2	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-4.1×10^{-14}	2.6×10^{-9}
99.2	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-5.9×10^{-14}	3.1×10^{-9}
99.2	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-1.7×10^{-13}	5.2×10^{-9}
99.2	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-6.4×10^{-13}	1.0×10^{-8}
99.2	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-2.5×10^{-12}	2.0×10^{-8}
99.2	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-1.0×10^{-11}	4.0×10^{-8}
99.2	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-4.0×10^{-11}	8.0×10^{-8}
99.2	9.9	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.7×10^{-9}	-2.7×10^{-3}	-1.6×10^{-10}	1.6×10^{-7}
99.2	10.0	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.6×10^{-9}	-2.7×10^{-3}	-6.3×10^{-10}	3.2×10^{-7}
99.2	10.1	${}^{-2}_{2}f$	${}^{5}_{5}g9$	-1.7×10^{-3}	−10.4	1.3×10^{-9}	-2.7×10^{-3}	-2.5×10^{-9}	6.3×10^{-7}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{\mathrm{unit}}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{\mathrm{PIT}}\rvert$
99.2	10.2	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.7×10^{-3}	-10.4	3.0×10^{-10}	-2.7×10^{-3}	-1.0×10^{-8}	1.3×10^{-6}
99.4	7.5	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	7.6×10^{-10}	-2.7×10^{-3}	-7.7×10^{-11}	5.3×10^{-7}
99.4	7.6	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	1.3×10^{-9}	-2.7×10^{-3}	-4.9×10^{-11}	4.2×10^{-7}
99.4	7.7	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	1.7×10^{-9}	-2.7×10^{-3}	-3.1×10^{-11}	3.3×10^{-7}
99.4	7.8	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	1.9×10^{-9}	-2.7×10^{-3}	-1.9×10^{-11}	2.6×10^{-7}
99.4	7.9	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.0×10^{-9}	-2.7×10^{-3}	-1.2×10^{-11}	2.1×10^{-7}
99.4	8.0	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.1×10^{-9}	-2.7×10^{-3}	-7.7×10^{-12}	1.7×10^{-7}
99.4	8.1	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.2×10^{-9}	-2.7×10^{-3}	-4.9×10^{-12}	1.3×10^{-7}
99.4	8.2	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.2×10^{-9}	-2.7×10^{-3}	-3.1×10^{-12}	1.1×10^{-7}
99.4	8.3	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.2×10^{-9}	-2.7×10^{-3}	-1.9×10^{-12}	8.4×10^{-8}
99.4	8.4	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.2×10^{-9}	-2.7×10^{-3}	-1.2×10^{-12}	6.6×10^{-8}
99.4	8.5	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.2×10^{-9}	-2.7×10^{-3}	-7.7×10^{-13}	5.3×10^{-8}
99.4	8.6	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.3×10^{-9}	-2.7×10^{-3}	-4.9×10^{-13}	4.2×10^{-8}
99.4	8.7	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.3×10^{-9}	-2.7×10^{-3}	-3.1×10^{-13}	3.3×10^{-8}
99.4	8.8	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.3×10^{-9}	-2.7×10^{-3}	-1.9×10^{-13}	2.6×10^{-8}
99.4	8.9	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.3×10^{-9}	-2.7×10^{-3}	-1.2×10^{-13}	2.1×10^{-8}
99.4	9.0	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.3×10^{-9}	-2.7×10^{-3}	-7.8×10^{-14}	1.7×10^{-8}
99.4	9.1	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.3×10^{-9}	-2.7×10^{-3}	-5.1×10^{-14}	1.4×10^{-8}
99.4	9.2	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.3×10^{-9}	-2.7×10^{-3}	-4.1×10^{-14}	1.2×10^{-8}
99.4	9.3	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.3×10^{-9}	-2.7×10^{-3}	-5.9×10^{-14}	1.5×10^{-8}
99.4	9.4	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.3×10^{-9}	-2.7×10^{-3}	-1.7×10^{-13}	2.5×10^{-8}
99.4	9.5	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	-10.7	2.3×10^{-9}	-2.7×10^{-3}	-6.3×10^{-13}	4.8×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
99.4	9.6	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	−10.7	2.3×10^{-9}	-2.7×10^{-3}	-2.5×10^{-12}	9.5×10^{-8}
99.4	9.7	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	−10.7	2.3×10^{-9}	-2.7×10^{-3}	-9.9×10^{-12}	1.9×10^{-7}
99.4	9.8	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	−10.7	2.3×10^{-9}	-2.7×10^{-3}	-3.9×10^{-11}	3.8×10^{-7}
99.4	9.9	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	−10.7	2.2×10^{-9}	-2.7×10^{-3}	-1.6×10^{-10}	7.5×10^{-7}
99.4	10.0	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	−10.7	2.2×10^{-9}	-2.7×10^{-3}	-6.2×10^{-10}	1.5×10^{-6}
99.4	10.1	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	−10.7	1.9×10^{-9}	-2.7×10^{-3}	-2.5×10^{-9}	3.0×10^{-6}
99.4	10.2	$^{-2}_{2}f$	$^{5}_{5}g_9$	-1.5×10^{-2}	−10.7	8.4×10^{-10}	-2.7×10^{-3}	-9.9×10^{-9}	6.0×10^{-6}
99.6	7.4	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	5.8×10^{-10}	-2.7×10^{-3}	-1.2×10^{-10}	1.2×10^{-6}
99.6	7.5	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	1.5×10^{-9}	-2.7×10^{-3}	-7.6×10^{-11}	9.8×10^{-7}
99.6	7.6	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	2.1×10^{-9}	-2.7×10^{-3}	-4.8×10^{-11}	7.8×10^{-7}
99.6	7.7	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	2.4×10^{-9}	-2.7×10^{-3}	-3.0×10^{-11}	6.2×10^{-7}
99.6	7.8	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	2.6×10^{-9}	-2.7×10^{-3}	-1.9×10^{-11}	4.9×10^{-7}
99.6	7.9	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	2.8×10^{-9}	-2.7×10^{-3}	-1.2×10^{-11}	3.9×10^{-7}
99.6	8.0	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	2.9×10^{-9}	-2.7×10^{-3}	-7.6×10^{-12}	3.1×10^{-7}
99.6	8.1	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	2.9×10^{-9}	-2.7×10^{-3}	-4.8×10^{-12}	2.5×10^{-7}
99.6	8.2	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-3.0×10^{-12}	1.9×10^{-7}
99.6	8.3	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-1.9×10^{-12}	1.5×10^{-7}
99.6	8.4	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-1.2×10^{-12}	1.2×10^{-7}
99.6	8.5	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-7.6×10^{-13}	9.8×10^{-8}
99.6	8.6	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-4.8×10^{-13}	7.8×10^{-8}
99.6	8.7	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-3.0×10^{-13}	6.2×10^{-8}
99.6	8.8	$^{-2}_{2}f$	$^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-1.9×10^{-13}	4.9×10^{-8}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
99.6	8.9	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-1.2×10^{-13}	3.9×10^{-8}
99.6	9.0	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-7.7×10^{-14}	3.1×10^{-8}
99.6	9.1	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-5.1×10^{-14}	2.5×10^{-8}
99.6	9.2	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-4.0×10^{-14}	2.2×10^{-8}
99.6	9.3	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-5.8×10^{-14}	2.7×10^{-8}
99.6	9.4	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-1.7×10^{-13}	4.6×10^{-8}
99.6	9.5	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-6.3×10^{-13}	8.8×10^{-8}
99.6	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-2.5×10^{-12}	1.8×10^{-7}
99.6	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-9.8×10^{-12}	3.5×10^{-7}
99.6	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-3.9×10^{-11}	7.0×10^{-7}
99.6	9.9	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	3.0×10^{-9}	-2.7×10^{-3}	-1.6×10^{-10}	1.4×10^{-6}
99.6	10.0	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	2.9×10^{-9}	-2.7×10^{-3}	-6.2×10^{-10}	2.8×10^{-6}
99.6	10.1	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	2.7×10^{-9}	-2.7×10^{-3}	-2.5×10^{-9}	5.5×10^{-6}
99.6	10.2	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-2.9×10^{-2}	−11.0	1.6×10^{-9}	-2.7×10^{-3}	-9.8×10^{-9}	1.1×10^{-5}
99.8	7.3	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-4.4×10^{-2}	−11.4	7.9×10^{-11}	-2.7×10^{-3}	-1.9×10^{-10}	2.3×10^{-6}
99.8	7.4	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-4.4×10^{-2}	−11.4	1.6×10^{-9}	-2.7×10^{-3}	-1.2×10^{-10}	1.8×10^{-6}
99.8	7.5	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-4.4×10^{-2}	−11.4	2.5×10^{-9}	-2.7×10^{-3}	-7.6×10^{-11}	1.4×10^{-6}
99.8	7.6	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-4.4×10^{-2}	−11.4	3.1×10^{-9}	-2.7×10^{-3}	-4.8×10^{-11}	1.1×10^{-6}
99.8	7.7	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-4.4×10^{-2}	−11.4	3.4×10^{-9}	-2.7×10^{-3}	-3.0×10^{-11}	9.0×10^{-7}
99.8	7.8	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-4.4×10^{-2}	−11.4	3.7×10^{-9}	-2.7×10^{-3}	-1.9×10^{-11}	7.1×10^{-7}
99.8	7.9	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-4.4×10^{-2}	−11.4	3.8×10^{-9}	-2.7×10^{-3}	-1.2×10^{-11}	5.7×10^{-7}
99.8	8.0	${}^{-2}_{2}f$	${}^{5}_{5}g_9$	-4.4×10^{-2}	−11.4	3.9×10^{-9}	-2.7×10^{-3}	-7.6×10^{-12}	4.5×10^{-7}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{unit}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{PIT}\rvert$
99.8	8.1	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.0×10^{-9}	-2.7×10^{-3}	-4.8×10^{-12}	3.6×10^{-7}
99.8	8.2	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.0×10^{-9}	-2.7×10^{-3}	-3.0×10^{-12}	2.8×10^{-7}
99.8	8.3	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.0×10^{-9}	-2.7×10^{-3}	-1.9×10^{-12}	2.3×10^{-7}
99.8	8.4	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-1.2×10^{-12}	1.8×10^{-7}
99.8	8.5	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-7.6×10^{-13}	1.4×10^{-7}
99.8	8.6	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-4.8×10^{-13}	1.1×10^{-7}
99.8	8.7	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-3.0×10^{-13}	9.0×10^{-8}
99.8	8.8	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-1.9×10^{-13}	7.1×10^{-8}
99.8	8.9	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-1.2×10^{-13}	5.7×10^{-8}
99.8	9.0	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-7.6×10^{-14}	4.5×10^{-8}
99.8	9.1	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-5.0×10^{-14}	3.7×10^{-8}
99.8	9.2	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-4.0×10^{-14}	3.3×10^{-8}
99.8	9.3	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-5.8×10^{-14}	3.9×10^{-8}
99.8	9.4	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-1.7×10^{-13}	6.7×10^{-8}
99.8	9.5	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-6.2×10^{-13}	1.3×10^{-7}
99.8	9.6	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-2.5×10^{-12}	2.6×10^{-7}
99.8	9.7	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-9.8×10^{-12}	5.1×10^{-7}
99.8	9.8	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-3.9×10^{-11}	1.0×10^{-6}
99.8	9.9	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.1×10^{-9}	-2.7×10^{-3}	-1.5×10^{-10}	2.0×10^{-6}
99.8	10.0	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	4.0×10^{-9}	-2.7×10^{-3}	-6.2×10^{-10}	4.0×10^{-6}
99.8	10.1	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	3.7×10^{-9}	-2.7×10^{-3}	-2.5×10^{-9}	8.1×10^{-6}
99.8	10.2	$^{-2}_{\ 2}f$	$^5_5g_9$	-4.4×10^{-2}	−11.4	2.6×10^{-9}	-2.7×10^{-3}	-9.8×10^{-9}	1.6×10^{-5}
100.0	7.3	$^{-4}_{\ 4}f$	$^7_7g_4$	-1.1×10^{-3}	6.9	1.4×10^{-9}	-5.9×10^{-5}	-2.1×10^{-9}	1.5×10^{-6}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{\mathrm{unit}}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{\mathrm{PIT}}\rvert$
100.0	7.4	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	2.9×10^{-9}	-5.9×10^{-5}	-1.3×10^{-9}	1.2×10^{-6}
100.0	7.5	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	3.8×10^{-9}	-5.9×10^{-5}	-8.4×10^{-10}	9.2×10^{-7}
100.0	7.6	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	4.5×10^{-9}	-5.9×10^{-5}	-5.3×10^{-10}	7.3×10^{-7}
100.0	7.7	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	4.8×10^{-9}	-5.9×10^{-5}	-3.3×10^{-10}	5.8×10^{-7}
100.0	7.8	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.1×10^{-9}	-5.9×10^{-5}	-2.1×10^{-10}	4.6×10^{-7}
100.0	7.9	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.2×10^{-9}	-5.9×10^{-5}	-1.3×10^{-10}	3.7×10^{-7}
100.0	8.0	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.3×10^{-9}	-5.9×10^{-5}	-8.4×10^{-11}	2.9×10^{-7}
100.0	8.1	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.4×10^{-9}	-5.9×10^{-5}	-5.3×10^{-11}	2.3×10^{-7}
100.0	8.2	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.4×10^{-9}	-5.9×10^{-5}	-3.3×10^{-11}	1.8×10^{-7}
100.0	8.3	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.4×10^{-9}	-5.9×10^{-5}	-2.1×10^{-11}	1.5×10^{-7}
100.0	8.4	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.5×10^{-9}	-5.9×10^{-5}	-1.3×10^{-11}	1.2×10^{-7}
100.0	8.5	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.5×10^{-9}	-5.9×10^{-5}	-8.4×10^{-12}	9.2×10^{-8}
100.0	8.6	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.5×10^{-9}	-5.9×10^{-5}	-5.3×10^{-12}	7.3×10^{-8}
100.0	8.7	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.5×10^{-9}	-5.9×10^{-5}	-3.3×10^{-12}	5.8×10^{-8}
100.0	8.8	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.5×10^{-9}	-5.9×10^{-5}	-2.1×10^{-12}	4.6×10^{-8}
100.0	8.9	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.5×10^{-9}	-5.9×10^{-5}	-1.3×10^{-12}	3.7×10^{-8}
100.0	9.0	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.5×10^{-9}	-5.9×10^{-5}	-8.6×10^{-13}	2.9×10^{-8}
100.0	9.1	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.5×10^{-9}	-5.9×10^{-5}	-6.2×10^{-13}	2.5×10^{-8}
100.0	9.2	${}_{4}^{-4}f$	${}_{7}^{7}g_4$	-1.1×10^{-3}	6.9	5.5×10^{-9}	-5.9×10^{-5}	-7.0×10^{-13}	2.7×10^{-8}
100.0	9.3	${}_{2}^{-2}f$	${}_{5}^{5}g_8$	3.3×10^{-2}	-12.2	5.5×10^{-9}	-2.7×10^{-3}	-8.6×10^{-14}	3.5×10^{-8}
100.0	9.4	${}_{2}^{-2}f$	${}_{5}^{5}g_8$	3.3×10^{-2}	-12.2	5.5×10^{-9}	-2.7×10^{-3}	-2.6×10^{-13}	6.1×10^{-8}
100.0	9.5	${}_{2}^{-2}f$	${}_{5}^{5}g_8$	3.3×10^{-2}	-12.2	5.5×10^{-9}	-2.7×10^{-3}	-1.0×10^{-12}	1.2×10^{-7}

(continued)

Table G.1 (continued)

Ω/Ω_K (%)	$\log T$ (K)	Daughters		$\Delta\tilde{\omega}$	$\mathcal{H}/E_{\text{unit}}$	$\tilde{\gamma}_\alpha$	$\tilde{\gamma}_\beta$	$\tilde{\gamma}_\gamma$	$\lvert Q_{\text{PIT}}\rvert$
100.0	9.6	${}^{-2}_{2}f$	${}^{5}_{5}g_8$	3.3×10^{-2}	−12.2	5.5×10^{-9}	-2.7×10^{-3}	-4.0×10^{-12}	2.4×10^{-7}
100.0	9.7	${}^{-2}_{2}f$	${}^{5}_{5}g_8$	3.3×10^{-2}	−12.2	5.5×10^{-9}	-2.7×10^{-3}	-1.6×10^{-11}	4.7×10^{-7}
100.0	9.8	${}^{-2}_{2}f$	${}^{5}_{5}g_8$	3.3×10^{-2}	−12.2	5.5×10^{-9}	-2.7×10^{-3}	-6.3×10^{-11}	9.4×10^{-7}
100.0	9.9	${}^{-2}_{2}f$	${}^{5}_{5}g_8$	3.3×10^{-2}	−12.2	5.5×10^{-9}	-2.7×10^{-3}	-2.5×10^{-10}	1.9×10^{-6}
100.0	10.0	${}^{-2}_{2}f$	${}^{5}_{5}g_8$	3.3×10^{-2}	−12.2	5.4×10^{-9}	-2.7×10^{-3}	-9.9×10^{-10}	3.7×10^{-6}
100.0	10.1	${}^{-2}_{2}f$	${}^{5}_{5}g_8$	3.3×10^{-2}	−12.2	5.1×10^{-9}	-2.7×10^{-3}	-3.9×10^{-9}	7.4×10^{-6}
100.0	10.2	${}^{-2}_{2}f$	${}^{5}_{5}g_8$	3.3×10^{-2}	-12.2	3.9×10^{-9}	-2.7×10^{-3}	-1.6×10^{-8}	1.5×10^{-5}