Appendix
Background Material

In this appendix we collect precise statements of some basic facts on positive definite kernels and positive definite functions on groups and semigroups.

A.1 Positive Definite Kernels

Let X be a set. Classically, reproducing kernels arise from Hilbert spaces $\mathcal{H}$ which are subspaces of the space $\mathbb{C}^X$ of complex-valued functions on X, for which the evaluations $f \mapsto f(x)$ are continuous, hence representable by elements $K_x \in \mathcal{H}$ by

$$f(x) = \langle K_x, f \rangle \quad \text{for} \quad f \in \mathcal{H}, x \in X.$$

Then

$$K : X \times X \to \mathbb{C}, \quad K(x, y) := K_y(x) = \langle K_x, K_y \rangle$$

is called the *reproducing kernel* of $\mathcal{H}$. As the kernel K determines $\mathcal{H}$ uniquely, we write $\mathcal{H}_K \subseteq \mathbb{C}^X$ for the Hilbert space determined by K and $\mathcal{H}_K^0 \subseteq \mathcal{H}_K$ for the subspace spanned by the functions $(K_x)_{x \in X}$. A kernel function $K : X \times X \to \mathbb{C}$ is the reproducing kernel of some Hilbert space if and only if it is *positive definite* in the sense that, for any finite collection $x_1, \ldots, x_n \in X$, the matrix $(K(x_j, x_k))_{1 \leq j,k \leq n}$ is positive semidefinite (cf. [Ar50], [Nel64, Chap. 1]). There is a natural generalization to Hilbert spaces $\mathcal{H}$ of functions with values in a Hilbert space $\mathcal{V}$, i.e., $\mathcal{H} \subseteq \mathcal{V}^X$. Then $K_x(f) = f(x)$ is a linear operator $K_x : \mathcal{H} \to \mathcal{V}$ and we obtain a kernel $K(x, y) := K_x K_y^* \in B(\mathcal{V})$ with values in the bounded operators on $\mathcal{V}$. However, there are also situations where one would like to deal with kernels whose values are unbounded operators, so that one has to generalize this context further. The notion of a positive definite kernel with values in the space $\mathrm{Bil}(V)$ of bilinear complex-valued forms on a real linear space V provides a natural context to deal with all relevant cases.

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Definition A.1.1 Let X be a set and V be a real vector space. We write $\mathrm{Bil}(V) = \mathrm{Bil}(V, \mathbb{C})$ for the space of complex-valued bilinear forms on V. We call a map $K: X \times X \to \mathrm{Bil}(V)$ *a positive definite kernel* if the associated scalar-valued kernel

$$K^\flat: (X \times V) \times (X \times V) \to \mathbb{C}, \quad K^\flat((x, v), (y, w)) := K(x, y)(v, w)$$

is positive definite.

The corresponding reproducing kernel Hilbert space $\mathcal{H}_{K^\flat} \subseteq \mathbb{C}^{X \times V}$ is generated by the elements $K^\flat_{x,v}$, $x \in X$, $v \in V$, with the inner product

$$\langle K^\flat_{x,v}, K^\flat_{y,w} \rangle = K(x, y)(v, w) =: K^\flat_{y,w}(x, v),$$

so that, for all $f \in \mathcal{H}_{K^\flat}$, we have

$$f(x, v) = \langle K^\flat_{x,v}, f \rangle. \tag{A.1}$$

We identify $\mathcal{H}_{K^\flat}$ with a subspace of the space $(V^*)^X$ of functions on X with values in the space V^* of complex-valued linear functionals on V by identifying $f \in \mathcal{H}_{K^\flat}$ with the function $f^*: X \to V^*$, $f^*(x) := f(x, \cdot)$. We call

$$\mathcal{H}_K := \{f^*: f \in \mathcal{H}_{K^\flat}\} \subseteq (V^*)^X$$

the (*vector-valued*) *reproducing kernel space associated to* K. The elements

$$K_{x,v} := (K^\flat_{x,v})^* \quad \text{with} \quad K_{x,v}(y) = K(y, x)(\cdot, v) \quad \text{for} \quad x, y \in X, v, w \in V,$$

then form a dense subspace of $\mathcal{H}_K$ with

$$\langle K_{x,v}, K_{y,w} \rangle = K(x, y)(v, w) \tag{A.2}$$

and

$$\langle K_{x,v}, f \rangle = f(x)(v) \quad \text{for} \quad f \in \mathcal{H}_K, x \in X, v \in V. \tag{A.3}$$

Remark A.1.2 Equation (A.2) shows that positive definiteness of K implies the existence of a Hilbert space $\mathcal{H}$ and a map $\gamma: X \to \mathrm{Hom}(V, \mathcal{H})$, $\gamma(x)(v) := K_{x,v}$ such that

$$K(x, y)(v, w) = \langle \gamma(x)(v), \gamma(y)(w) \rangle.$$

If, conversely, such a factorization exists, then the positive definiteness follows from

$$\sum_{j,k=1}^n \overline{c_j} c_k K(x_j, x_k)(v_j, v_k) = \sum_{j,k=1}^n \overline{c_j} c_k \langle \gamma(x_j)(v_j), \gamma(x_k)(v_k) \rangle = \Big\| \sum_{k=1}^n c_k \gamma(x_k)(v_k) \Big\|^2 \geq 0.$$

Example A.1.3 If V is a complex Hilbert space, then we write $\mathrm{Sesq}(V) \subseteq \mathrm{Bil}(V)$ for the linear subspace of *sesquilinear maps*, i.e., maps which are anti-linear in the first and complex linear in the second argument. If X is a set and $K: X \times X \to B(V)$ is an operator-valued kernel, then K is positive definite if and only if the corresponding kernel

$$\tilde{K}: (X \times V) \times (X \times V) \to \mathbb{C}, \quad \tilde{K}((x, v), (y, w)) := \langle v, K(x, y)w \rangle$$

is positive definite (Definition A.1.1). Then, for each $f \in \mathcal{H}_{\tilde{K}}$, the linear functionals $f^*(x): V \to \mathbb{C}$ are continuous, hence can be identified with elements of V. Accordingly, we consider $\mathcal{H}_K$ as a space of V-valued functions (see [Nel64, Chap. 1] for more details).

Example A.1.4 Let $\mathcal{A}$ be a C^*-algebra. A linear functional $\omega \in \mathcal{A}^*$ is called *positive* if the kernel $K_\omega(A, B) := \omega(A^*B)$ on $\mathcal{A} \times \mathcal{A}$ is positive definite. Then the corresponding Hilbert space $\mathcal{H}_\omega := \mathcal{H}_{K_\omega}$ can be realized in the space $\mathcal{A}^\sharp$ of anti-linear functionals on $\mathcal{A}$. It can be obtained from the GNS representation $(\pi_\omega, \mathcal{H}_\omega, \Omega)$ [BR02, Corollary 2.3.17] by

$$\Gamma: \mathcal{H}_\omega \to \mathcal{A}^\sharp, \quad \Gamma(\xi)(A) := \langle \pi(A)\Omega, \xi \rangle$$

because $\langle \pi(A)\Omega, \pi(B)\Omega \rangle = \omega(A^*B) = K_\omega(A, B)$. Note that $\mathcal{A}$ has a natural representation on $\mathcal{A}^\sharp$ by $(A.\beta)(B) := \beta(A^*B)$ and that Γ is equivariant with respect to this representation.[1]

If $X = G$ is a group and the kernel K is invariant under right translations, then it is of the form $K(g, h) = \varphi(gh^{-1})$ for a function $\varphi: G \to \mathrm{Bil}(V)$.

Definition A.1.5 Let G be a group and let V be a real vector space. A function $\varphi: G \to \mathrm{Bil}(V)$ is said to be *positive definite* if the $\mathrm{Bil}(V)$-valued kernel $K(g, h) := \varphi(gh^{-1})$ is positive definite.

Suppose, more generally, that $(S, *)$ is an *involutive semigroup*, i.e., a semigroup S, endowed with an involutive map $s \mapsto s^*$ satisfying $(st)^* = t^*s^*$ for $s, t \in S$. A function $\varphi: S \to \mathrm{Bil}(V)$ is called *positive definite* if the kernel $K(s, t) := \varphi(st^*)$ is positive definite.

The following proposition generalizes the GNS construction to form-valued positive definite functions on groups [NÓ15b, Proposition A.4].

Proposition A.1.6 (GNS-construction for groups) *Let V be a real vector space.*

(a) *Let $\varphi: G \to \mathrm{Bil}(V)$ be a positive definite function. Then $(U^\varphi_g f)(h) := f(hg)$ defines a unitary representation of G on the reproducing kernel Hilbert space $\mathcal{H}_\varphi := \mathcal{H}_K \subseteq (V^*)^G$ with kernel $K(g, h) = \varphi(gh^{-1})$ and the range of the map*

[1] This realization of the Hilbert space $\mathcal{H}_\omega$ has the advantage that we can view its elements as elements of the space $\mathcal{A}^\sharp$ (see [Nel64] for many applications of this perspective). Usually, $\mathcal{H}_\omega$ is obtained as the Hilbert completion of a quotient of $\mathcal{A}$ by a left ideal which leads to a much less concrete space.

$$j\colon V \to \mathcal{H}_\varphi, \quad j(v)(g)(w) := \varphi(g)(w, v), \qquad j(v) = K_{e,v},$$

is a cyclic subspace, i.e., $U^\varphi_G j(V)$ spans a dense subspace of $\mathcal{H}$. We then have

$$\varphi(g)(v, w) = \langle j(v), U^\varphi_g j(w)\rangle \quad \text{for} \quad g \in G, v, w, \in V. \tag{A.4}$$

(b) *If, conversely, $(U, \mathcal{H})$ is a unitary representation of G and $j\colon V \to \mathcal{H}$ a linear map, then*

$$\varphi\colon G \to \mathrm{Bil}(V), \quad \varphi(g)(v, w) := \langle j(v), U_g j(w)\rangle$$

is a Bil(V)*-valued positive definite function. If, in addition, $j(V)$ is cyclic, then $(U, \mathcal{H})$ is unitarily equivalent to $(U^\varphi, \mathcal{H}_\varphi)$.*

Proof (a) For the kernel $K(g, h) := \varphi(gh^{-1})$ and $v \in V$, the right invariance of the kernel K on G implies on $\mathcal{H}_\varphi$ the existence of well-defined unitary operators U_g with

$$U_g K_{h,v} = K_{hg^{-1},v} \quad \text{for} \quad g, h \in G, v \in V.$$

In fact, (A.2) shows that

$$\langle K_{h_1g^{-1},v_1}, K_{h_2g^{-1},v_2}\rangle = K(h_1g^{-1}, h_2g^{-1})(v_1, v_2) = K(h_1, h_2)(v_1, v_2) = \langle K_{h_1,v_1}, K_{h_2,v_2}\rangle.$$

For $f \in \mathcal{H}_\varphi$, we then have

$$(U_g f)(h)(v) = \langle K_{h,v}, U_g f\rangle = \langle U_{g^{-1}} K_{h,v}, f\rangle = \langle K_{hg,v}, f\rangle = f(hg)(v),$$

i.e., $(U_g f)(h) = f(hg)$. Further, $j(v) = K_{e,v}$ satisfies $U_g j(v) = K_{g^{-1},v}$, which shows that $U_G j(V)$ is total in $\mathcal{H}_\varphi$. Finally we note that

$$\langle j(v), U_g j(w)\rangle = \langle K_{e,v}, K_{g^{-1},w}\rangle = K(g)(v, w) = \varphi(g)(v, w).$$

(b) The positive definiteness of φ follows with Remark A.1.2 easily from the relation $\varphi(gh^{-1})(v, w) = \langle U_g^{-1} v, U_h^{-1} w\rangle$. Since $j(V)$ is cyclic, the map $\Gamma(\xi)(g)(v) := \langle U_g^{-1} j(v), \xi\rangle$ defines an injection $\mathcal{H} \hookrightarrow (V^*)^G$ whose range is the subspace $\mathcal{H}_\varphi$ and which is equivariant with respect to the right translation representation U^φ. □

Remark A.1.7 (a) If $\varphi\colon G \to \mathrm{Bil}(V)$ is a positive definite function, then (A.4) shows that,if $\tilde{V} := \overline{j(V)}$, which is the real Hilbert space defined by completing V with respect to the positive semidefinite form $\varphi(e)$, then

$$\tilde{\varphi}(g)(v, w) = \langle v, U_g w\rangle \tag{A.5}$$

defines a positive definite function

$$\tilde{\varphi}: G \to \mathrm{Bil}(\tilde{V}) \quad \text{with} \quad \tilde{\varphi}(g)(j(v), j(w)) = \varphi(g)(v, w) \quad \text{for} \quad v, w \in V.$$

Therefore it often suffices to consider $\mathrm{Bil}(V)$-valued positive definite functions for real Hilbert space V for which $\varphi(e)$ is a positive definite hermitian form on V whose real part is the scalar product on V. In terms of (A.4), this means that $j: V \to \mathscr{H}$ is an isometric embedding of the real Hilbert space V.

(b) If $\mathscr{V}$ is a real Hilbert space and j is continuous, then the adjoint operator $j^*: \mathscr{H} \to \mathscr{V}$ is well-defined and we obtain from (A.5) the $B(\mathscr{V})$-valued positive definite function $\varphi(g) := j^* U_g j$ which can be used to realize $\mathscr{H}$ in $\mathscr{V}^G$.

Example A.1.8 (Vector-valued GNS construction for semigroups) [Nel64, Sect. 3.1] Let $(U, \mathscr{H})$ be a representation of the unital involutive semigroup $(S, *, e)$, $\mathscr{V}$ be a Hilbert space and $j: \mathscr{V} \to \mathscr{H}$ be a linear map for which $U_S j(\mathscr{V})$ is total in $\mathscr{H}$. Then $\varphi(s) := j^* U_s j$ is a $B(\mathscr{V})$-valued positive definite function on S with $\varphi(e) = j^* j$ (which is $\mathbf{1}$ if and only if j is isometric) because we have the factorization

$$\varphi(st^*) = j^* U_{st^*} j = (j^* U_s)(j U_t)^*.$$

The map

$$\Phi: \mathscr{H} \to \mathscr{V}^S, \quad \Phi(v)(s) = j^* U_s v$$

is an S-equivariant realization of $\mathscr{H}$ as the reproducing kernel space $\mathscr{H}_\varphi \subseteq \mathscr{V}^S$, on which S acts by right translation, i.e., $(U_s^\varphi f)(t) = f(ts)$.

Conversely, let S be a unital involutive semigroup and $\varphi: S \to B(\mathscr{V})$ be a positive definite function. Write $\mathscr{H}_\varphi \subseteq \mathscr{V}^S$ for the corresponding reproducing kernel space with kernel $K(s,t) = \varphi(st^*)$ and $\mathscr{H}_\varphi^0$ for the dense subspace spanned by $K_{s,v} = \mathrm{ev}_s^* v$, $s \in S$, $v \in \mathscr{V}$. Then $(U_s^\varphi f)(t) := f(ts)$ defines a $*$-representation of S on $\mathscr{H}_\varphi^0$. We say that φ is *exponentially bounded* if all operators U_s^φ are bounded, so that we actually obtain a representation of S by bounded operators on $\mathscr{H}_\varphi$ (cf. [Nel64, Sect. 2.4]). Then $\mathrm{ev}_e \circ U_s^\varphi = \mathrm{ev}_s$ leads to

$$\varphi(s) = \mathrm{ev}_s \mathrm{ev}_e^* = \mathrm{ev}_e U_s^\varphi \mathrm{ev}_e^* \quad \text{and} \quad \varphi v = \mathrm{ev}_e^* v = K_{e,v}. \tag{A.6}$$

If $S = G$ is a group with $s^* = s^{-1}$, then φ is always exponentially bounded and the representation $(U^\varphi, \mathscr{H}_\varphi)$ is unitary.

Lemma A.1.9 *Let $(S, *, e)$ be a unital involutive semigroup and $\varphi: S \to B(\mathscr{V})$ be a positive definite function with $\varphi(e) = \mathbf{1}$. We write $(U^\varphi, \mathscr{H}_\varphi)$ for the representation on the corresponding reproducing kernel Hilbert space $\mathscr{H}_\varphi \subseteq \mathscr{V}^S$ by $(U^\varphi(s)f)(t) := f(ts)$. Then the inclusion $\iota: \mathscr{V} \to \mathscr{H}_\varphi$, $\iota(v)(s) := \varphi(s)v$, is surjective if and only if φ is multiplicative, i.e., a representation.*

Proof If φ is multiplicative, then $(U_s^\varphi \iota(v))(t) = \varphi(ts)v = \varphi(t)\varphi(s)v \in \iota(\mathscr{V})$. Therefore the S-cyclic subspace $\iota(\mathscr{V})$ is invariant, which implies that ι is surjective.

Suppose, conversely, that ι is surjective. Then each $f \in \mathcal{H}_\varphi$ satisfies $f(s) = \varphi(s)f(e)$. For $v \in \mathcal{V}$ and $t, s \in S$, this leads to

$$\varphi(st)v = (U_t^\varphi \iota(v))(s) = \varphi(s) \cdot (U_t^\varphi \iota(v))(e) = \varphi(s)\iota(v)(t) = \varphi(s)\varphi(t)v.$$

Therefore φ is multiplicative. □

A.2 Integral Representations

For a realization of unitary representations associated to positive definite functions in L^2-spaces, integral representations are of crucial importance. The following result is a straight-forward generalization of Bochner's Theorem for locally compact abelian groups. Here we write $\mathrm{Sesq}^+(V) \subseteq \mathrm{Sesq}(V)$ for the convex cone of positive semidefinite forms if V is a complex linear space.

Theorem A.2.1 *Let G be a locally compact abelian group. Then a function $\varphi\colon G \to \mathrm{Sesq}(V)$ for which all functions $\varphi^{v,w} := \varphi(\cdot)(v, w)$, $v, w \in V$, are continuous is positive definite if and only if there exists a (uniquely determined) finite $\mathrm{Sesq}^+(V)$-valued Borel measure μ on the locally compact group $\hat{G}$ such that $\hat{\mu}(g) := \int_{\hat{G}} \chi(g)\, d\mu(\chi) = \varphi(g)$ holds for every $g \in G$ pointwise on $V \times V$.*

Proof If $\varphi = \hat{\mu}$ holds for a finite $\mathrm{Sesq}^+(V)$-valued Borel measure μ on the locally compact group $\hat{G}$, then the kernel $\varphi(gh^{-1})(\xi, \eta) = \int_{\hat{G}} \chi(g)\overline{\chi(h)}\, d\mu^{\xi,\eta}(\chi)$ on $G \times V$ is positive definite because

$$\begin{aligned}\sum_{j,k=1}^n \varphi(g_j g_k^{-1})(\xi_j, \xi_k) &= \sum_{j,k=1}^n \int_{\hat{G}} \chi(g_j)\overline{\chi(g_k)}\, d\mu^{\xi_j,\xi_k}(\chi)\\ &= \sum_{j,k=1}^n \int_{\hat{G}} d\mu^{\overline{\chi(g_j)}\xi_j, \overline{\chi(g_k)}\xi_k}(\chi) = \int_{\hat{G}} d\mu^{\xi,\xi} \geq 0\end{aligned}$$

holds for $\xi := \sum_{j=1}^n \overline{\chi(g_j)}\xi_j$ and $\mu^{\xi,\eta}(\cdot) = \mu(\cdot)(\xi, \eta)$.

Suppose, conversely, that φ is positive definite. Then Bochner's Theorem for scalar-valued positive definite functions yields for every $v \in V$ a finite positive measure μ^v on $\hat{G}$ such that

$$\varphi^{v,v}(g) = \hat{\mu^v}(g) = \int_{\hat{G}} \chi(g)\, d\mu^v(\chi).$$

By polarization, we obtain for $v, w \in V$ complex measures $\mu^{v,w} := \frac{1}{4}\sum_{k=0}^3 i^{-k}\mu^{v+i^k w}$ on $\hat{G}$ with $\varphi^{v,w} = \hat{\mu^{v,w}}$. Then the collection $(\mu^{v,w})_{v,w\in V}$ of complex measures on $\hat{G}$ defines a $\mathrm{Sesq}^+(V)$-valued measure by $\mu(\cdot)(v, w) := \mu^{v,w}$ for $v, w \in V$, and this measure satisfies $\hat{\mu} = \varphi$. □

Remark A.2.2 Suppose that E is the spectral measure on the character group $\hat{G}$ for which the continuous unitary representation $(U, \mathscr{H})$ is represented by $U_g = \int_{\hat{G}} \chi(g)\, dE(\chi)$. Then, for $\xi \in \mathscr{H}$, the positive definite function $U^\xi(g) := \langle \xi, U_g \xi \rangle$ is the Fourier transform of the measure $E^{\xi,\xi} = \langle \xi, E(\cdot)\xi \rangle$. This establishes a close link between spectral measures and the representing measures in the preceding theorem.

The following theorem follows from [NÓ15b, Theorem B.3]:

Theorem A.2.3 (Laplace transforms and positive definite kernels) *Let E be a finite dimensional real vector space and $\mathscr{D} \subseteq E$ be a non-empty open convex subset. Let V be a Hilbert space and $\varphi \colon \mathscr{D} \to B(V)$ be such that*

(L1) *the kernel $K(x, y) = \varphi\big(\frac{x+y}{2}\big)$ is positive definite.*
(L2) *φ is weak operator continuous on every line segment in $\mathscr{D}$, i.e., all functions $t \mapsto \langle v, \varphi(x + th)v \rangle$, $v \in V$, are continuous on $\{t \in \mathbb{R} \colon x + th \in \mathscr{D}\}$.*

Then the following assertions hold:

(i) *There exists a unique* $\mathrm{Herm}^+(V)$*-valued Borel measure μ on the dual space E^* such that*

$$\varphi(x) = \mathscr{L}(\mu)(x) := \int_{E^*} e^{-\lambda(x)}\, d\mu(\lambda) \quad \text{for} \quad x \in \mathscr{D}.$$

(ii) *Let $T_{\mathscr{D}} = \mathscr{D} + iE \subseteq E_{\mathbb{C}}$ be the tube domain over $\mathscr{D}$. Then the map*

$$\mathscr{F} \colon L^2(E^*, \mu; V) \to \mathscr{O}(T_{\mathscr{D}}, V), \quad \langle \xi, \mathscr{F}(f)(z) \rangle := \langle e_{-\overline{z}/2}\xi, f \rangle$$

is unitary onto the reproducing kernel space $\mathscr{H}_\varphi := \mathscr{H}_K$ corresponding to the kernel associated to φ. It intertwines the unitary representation

$$(U_x f)(\alpha) := e^{i\alpha(x)} f(\alpha) \quad \text{on} \quad L^2(E^*, \mu) \quad \text{and}$$
$$(\tilde{U}_x f)(z) := f(z - 2ix) \quad \text{on} \quad \mathscr{H}_\varphi.$$

(iii) *φ extends to a unique holomorphic function $\hat{\varphi}$ on the tube domain $T_{\mathscr{D}}$ which is positive definite in the sense that the kernel $\hat{\varphi}\big(\frac{z+\overline{w}}{2}\big)$ is positive definite.*

Corollary A.2.4 *A continuous function $\varphi \colon \mathscr{D} \to \mathbb{C}$ on an open convex subset of a finite dimensional real vector space E is positive definite if and only if there exists a positive measure μ on E^* such that $\varphi = \mathscr{L}(\mu)|_{\mathscr{D}}$.*

The preceding theorem generalizes in an obvious way to $\mathrm{Sesq}(V)$-valued functions, where the corresponding measure μ has values in the cone $\mathrm{Sesq}^+(V)$. One can use the same arguments as in the proof of Bochner's Theorem (Theorem A.2.1).

The following lemma sharpens the "technical lemma" in [KL82, Appendix A]. We recall the notation $\mathscr{S}_\beta = \{z \in \mathbb{C} \colon 0 < \mathrm{Im} z < \beta\}$ for horizontal strips in $\mathbb{C}$.

Lemma A.2.5 *Let $U_t = e^{itH}$ be a unitary one-parameter group on $\mathscr{H}$, E the spectral measure of H, $\xi \in \mathscr{H}$, $E^\xi := \langle \xi, E(\cdot)\xi\rangle$, $\beta > 0$ and $\varphi(t) := \langle \xi, U_t\xi\rangle = \int_{\mathbb{R}} e^{it\lambda}\, dE^\xi(\lambda)$. Then the following are equivalent:*

(i) *There exists a continuous function ψ on $\overline{\mathscr{S}_\beta}$, holomorphic on $\mathscr{S}_\beta$, such that $\psi|_{\mathbb{R}} = \varphi$.*
(ii) $\mathscr{L}(E^\xi)(\beta) = \int_{\mathbb{R}} e^{-\beta\lambda}\, dE^\xi(\lambda) < \infty$.
(iii) $\xi \in \mathscr{D}(e^{-\beta H/2})$.

Proof That (i) implies (ii) follows from [Ri66, p. 311]. If, conversely, (ii) is satisfied, then $\psi(z) := \mathscr{L}(E^\xi)(-iz)$ is defined on $\overline{\mathscr{S}_\beta}$, holomorphic on $\mathscr{S}_\beta$ and $\psi|_{\mathbb{R}} = \varphi$. Finally, the equivalence of (ii) and (iii) follows from the definition of the unbounded operator $e^{-\beta H/2}$ in terms of the spectral measure E. □

Lemma A.2.6 (Criterion for the existence of $\mathscr{L}(\mu)(x)$) *Let $\mathscr{V}$ be a Hilbert space and μ be a finite* $\mathrm{Herm}^+(\mathscr{V})$*-valued Borel measure on $\mathbb{R}$, so that we can consider its Laplace transform $\mathscr{L}(\mu)$, taking values in* $\mathrm{Herm}(\mathscr{V})$*, whenever the integral*

$$\mathrm{tr}\left(\mathscr{L}(\mu)(x)S\right) = \int_{\mathbb{R}} e^{-\lambda x}\, d\mu^S(\lambda) \quad \text{for} \quad d\mu^S(\lambda) = \mathrm{tr}(S d\mu(\lambda)),$$

exists for every positive trace class operator S on $\mathscr{V}$. This is equivalent to the finiteness of the integrals $\mathscr{L}(\mu^v)(x)$ for every $v \in \mathscr{V}$, where $d\mu^v(\lambda) = \langle v, d\mu(\lambda)v\rangle$.

Proof For $x \in \mathbb{R}$, the existence of $\mathscr{L}(\mu)(x)$ implies the finiteness of the integrals $\mathscr{L}(\mu^v)(x)$ for $v \in \mathscr{V}$. Suppose, conversely, that all these integrals are finite. Then we obtain by polarization a hermitian form $\beta(v, w) := \int_{\mathbb{R}} e^{-\lambda x}\, \langle v, d\mu(\lambda)w\rangle$ on $\mathscr{V}$. We claim that β is continuous. As $\mathscr{V}$ is in particular a Fréchet space, it suffices to show that, for every $w \in \mathscr{V}$, the linear functional $\lambda(v) := \beta(w, v)$ is continuous [Ru73, Theorem 2.17].

The linear functionals $f_n(v) := \int_{-n}^{n} e^{-\lambda x}\, \langle w, d\mu(\lambda)v\rangle$ are continuous because μ is a bounded measure and the functions $e_x(\lambda) := e^{\lambda(x)}$ are bounded on bounded intervals. By the Monotone Convergence Theorem, combined with the Polarization Identity, $f_n \to f$ holds pointwise on $\mathscr{V}$, and this implies the continuity of f [Ru73, Theorem 2.8].

For a positive trace class operators $S = \sum_n \langle v_n, \cdot\rangle v_n$ with $\mathrm{tr}\, S = \sum_n \|v_n\|^2 < \infty$, we now obtain

$$\mathscr{L}(\mu^S)(x) = \sum_n \mathscr{L}(\mu^{v_n})(x) = \sum_n \beta(v_n, v_n) \leq \|\beta\| \sum_n \|v_n\|^2 < \infty. \qquad \square$$

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