Appendix A
Linear Operator Theory

Here we recall some concepts from linear operator theory, in particular compact operators and Fredholm operators. It is assumed that the reader is familiar with basics of linear functional analysis like norm, metric, completeness, Banach space, Hilbert space etc. Some theorems are proved here, some are only stated.

Definition A.1 Let E, F be vector spaces (linear spaces) over a field Δ, a linear operator from E to F is a mapping $A : E \to F$ such that

$$A(\lambda x + \mu y) = \lambda Ax + \mu Ay \quad \forall \lambda, \mu \in \Delta;\ x, y \in E\,.$$

In the following, E, F are linear normed spaces.

Theorem A.1 *Let A be a linear operator from E into F. Then A is bounded, if and only if A is continuous, written $A \in \mathscr{B}(E, F)$.*

Definition A.2 The operator norm $\|A\| = \|A\|_{E\to F}$ of $A \in \mathscr{B}(E, F)$ is defined by

$$\|A\| := \sup_{\substack{x\in E\\ \|x\|\le 1}} \|Ax\|_F = \sup_{x\in E\setminus\{0\}} \frac{\|Ax\|_F}{\|x\|_E}$$

Definition A.3 An operator A is said to be continuously invertible if A^{-1} exists and is continuous.

Theorem A.2 (Banach's Theorem) *Let $A \in \mathscr{B}(X, Y)$ and X, Y be Banach spaces. Assume that A is injective and surjective (onto). Then A is continuously invertible.*

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Let E' denote the dual space of E. Let M be an arbitrary nonvoid subset of E. Then the set

$$\{x' \in E' : \langle x', x\rangle = 0 \ \forall x \in M\}$$

is the annihilator of M, written $M^\perp$. Similarly, $N^\perp \subset E$ for $N \subset E'$.

Let $A \in \mathscr{B}(E, F)$. Then the adjoint A' is in $\mathscr{B}(F', E')$. Moreover immediately

$$\ker A = (\operatorname{im} A')^\perp,\ \ker A' = (\operatorname{im} A)^\perp. \tag{A.1}$$

Also $\operatorname{im} A \subset (\ker A')^\perp$ is obvious; however, equality holds only with closedness. More precisely, due to Banach:

Theorem A.3 (Closed Range Theorem) *Let $A \in \mathscr{B}(X, Y)$ and X, Y be Banach spaces. Then the following assertions are equivalent:*

(i) im A is closed in Y,
(ii) im A' is closed in X',
(iii) im $A = (\ker A')^\perp$,
(iv) im $A' = (\ker A)^\perp$.

A class of continuous linear operators that have closed range are Fredholm operators, see below Definition A.8.

Definition A.4 Let X, Y be Banach spaces. A sequence of linear operators $\{A_n : X \to Y\}$ is called strongly convergent (or pointwise convergent) to an operator $A : X \to Y$, written "$A_n \to A$", if for all $x \in X$, $\lim_{n\to\infty} \|A_n x - Ax\|_Y = 0$.

Definition A.5 Let X, Y be Banach spaces. A sequence of linear operators $\{A_n : X \to Y\}$ is called convergent in the norm, written "$A_n \Rightarrow A$", if $\lim\limits_{n\to\infty} \|A_n - A\| = 0$.

Theorem A.4 (Theorem of Banach-Steinhaus) *The sequence $\{A_n : X \to Y\}$ of continuous linear operators converges pointwise to a continuous linear operator $A : X \to Y$ if and only if*

(i) The sequence of operator norms $\|A_n\|$ is bounded.
(ii) The sequence $\{A_n x\}$ converges for all $x \in M$ where M is a dense subset of X.

Definition A.6 Let X, Y be Banach spaces and $A \in \mathscr{B}(X, Y)$. Then $A : X \to Y$ is said to be compact if the image of the closed unit ball $B_1^X = \{x \in X : \|x\| \le 1\}$ under A is relatively compact in Y.

Remark A.1 A set $K \subset Y$ is relatively compact in Y if for every $\varepsilon > 0$ there is a finite number of elements in K, say $y_1, \ldots, y_m \in K$, such that the ε-balls around y_j $(j = 1, \ldots, m)$ cover K, $K \subset \cup_{j=1}^m B_\varepsilon^Y(y_j)$. Thus the y_j are a finite ε-net in K. Furthermore A is compact if and only if, for every bounded sequence $\{x_n\}$ in X, the sequence $\{Ax_n\}$ has a convergent subsequence in Y.

Lemma A.1

(i) Let $A : X \longrightarrow Y$ *be compact and* $B : Y \longrightarrow Z$ *bounded. Then, the operator* $AB : X \longrightarrow Z$ *is compact, too.*
(ii) Let again $A : X \longrightarrow Y$ *be compact and let* $B_n \longrightarrow 0$ *be strongly convergent. Then,* $B_n A \longrightarrow 0$ *is strongly convergent, too.*

Proof (i) Exercise. (ii) Suppose the assertion does not hold. Then there exists a sequence $\{x_n\} \subset X$ satisfying $\|x_n\|_X = 1 \; \forall n$ and $\|B_n A x_n\|_Z \geq \alpha > 0 \; \forall n$. With A compact there further exists a subsequence $\{x'_n\}$ with $Ax'_n \longrightarrow y$ in Y. Thus,

$$0 < \alpha \leq \|B_n A x'_n\| \leq \|B_n y\| + \|B_n(Ax'_n - y)\|$$

But $\|B_n y\| \mapsto 0$ and $\|B_n(Ax'_n - y)\| \leq M\|Ax'_n - y\| \mapsto 0$ which is a contradiction. □

Definition A.7 Let X, Y be Banach spaces and $\{A_n\}$ a sequence of linear operators. Then $\{A_n : X \to Y\}$ is called collectively compact, if $K := \cup_{n\in\mathbb{N}} A_n(B_1^X(0))$ is relatively compact in Y.

Lemma A.2 *Assume that the sequence of the linear operators* $A_n : X \to Y$ *converges pointwise to* A*. Then there holds:*

(i) If $K \subset X$ *is relatively compact in* X*, then* $A_n \rightrightarrows A$ *uniformly in* K*.*
(ii) Let $B : W \to X$ *(with another Banach space* W*) be a compact linear operator and assume that* $B_n \rightrightarrows B$*. Then* $A_n B_n \rightrightarrows AB$*.*
(iii) Let $\{B_n : W \to X\}$ *be collectively compact and* $A = 0$*. Then* $A_n B_n \rightrightarrows 0$*.*

Proof

1. Assume that $\{x_1, \ldots, x_m\} \subset K$ is a ε-net for a given (but arbitrary) $\varepsilon > 0$, i.e. $K \subset \cup_{j=1}^m B_\varepsilon^X(x_j)$. Choose a $n_0(\varepsilon) \in \mathbb{N}$ such that for all $n > n_0(\varepsilon)$, $j = 1, \ldots, m$, $\|A_n x_j - A x_j\|_Y < \varepsilon$. Let $x \in K$ arbitrary. Then there exists a $j \in \{1, \ldots, m\}$ with $\|x - x_j\|_X \leq \varepsilon$ and thus:

$$\begin{aligned} \|A_n x - Ax\| &\leq \|(A_n - A)(x - x_j)\| + \|(A_n - A)x_j\| \\ &\leq \|(A_n - A)\| \; \|(x - x_j)\| + \varepsilon \\ &\leq (1 + \|A_n - A\|) \cdot \varepsilon \end{aligned}$$

   By the Theorem of Banach-Steinhaus (see A.4) we have $\|A_n - A\| \leq C \; \forall n \in \mathbb{N}$ and therefore $\|A_n x - Ax\| \leq (1 + C) \cdot \varepsilon \to 0$ uniformly for all $x \in K$.
2. Since B is compact, $\{Bw | w \in W, \|w\| \leq 1\} \subset X$ is relatively compact in X. Therefore by 1., for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|(A_n - A)Bw\| \leq \varepsilon$ for $n \geq n_0$ and $\|w\| \leq 1$. Moreover since $B_n \rightrightarrows B$, $\|(B_n - B)w\| \leq \varepsilon$ for $n \geq n_0$

and $\|w\| \leq 1$. Therefore for $\|w\| \leq 1$,

$$\begin{aligned}\|A_n B_n w - ABw\| &\leq \|A_n(B_n - B)w\| + \|(A_n - A)Bw\| \\ &\leq \|A_n\| \cdot \|(B_n - B)w\| + \varepsilon \\ &\leq (1 + C)\varepsilon \quad \text{for } \|w\| \leq 1.\end{aligned}$$

where $\|A_n\| \leq C$ by Banach-Steinhaus Theorem. Hence $A_n B_n \Rightarrow AB$.

3. From 1. we have that A_n converges uniformly to $0 = A$ on $K := \cup_n B_n(B_1^W(0))$, i.e. $\forall \varepsilon > 0 \;\; \exists n_0 : \|A_n x\| \leq \varepsilon \; \forall n \geq n_0 \; \forall x \in K$. Therefore $\|A_n B_n w\| \leq \varepsilon$ for $\forall n \geq n_0$, and $\|w\| \leq 1$ and so

$$\|A_n B_n\|_{W \to Y} := \sup_{\|w\| \leq 1} \|A_n B_n w\| \longrightarrow 0 \text{ for } n \to \infty.$$

□

Definition A.8 Let X, Y be Banach spaces and let $A \in \mathscr{B}(X, Y)$. Then A is called a Fredholm operator, written $A \in \mathscr{F}(X, Y)$, if A enjoys the following properties:

(I) The kernel ker A has finite dimension,
(II) im A is closed in Y,
(III) the range im A has finite codimension: codim im A = dim $(Y/\text{im } A) < \infty$.

The number

$$\text{ind }(A) := \text{dim ker } A - \text{codim im } A$$

is called the (Fredholm) index of A.

Remark A.2 By the closed range theorem, here Theorem A.3, the operator equation $Au = f$ with $A \in \mathscr{F}(X, Y)$ is solvable, if and only if, $f \in (\text{ker } A')^{\perp}$.

Theorem A.5 *Let $A \in \mathscr{F}(X, Y)$ and X, Y be Banach spaces. Then the adjoint A' is in $\mathscr{F}(X, Y)$, too. Moreover,*

$$\textit{dim ker } A' = \textit{codim im } A \textit{ and codim im } A' = \textit{dim ker } A\,,$$

hence

$$\textit{ind } A' = -\textit{ind } A\,. \tag{A.2}$$

Proof By A.1, ker $A' = (\text{im } A)^{\perp} \cong (Y/\text{im } A)' = (\text{coker } A)'$. This shows the first formula in the theorem. To prove the second formula, by the closedness of im A, we can apply Theorem A.3, hence im $A' = (\text{ker } A)^{\perp}$. This gives $(\text{ker } A)' \cong X'/(\text{ker } A)^{\perp} = X'/\text{im } A' = \text{coker } A'$, hence the second formula, (A.2), and also $A' \in \mathscr{F}(X, Y)$. □

Without proof we list from [364, 435] the following.

Theorem A.6 *Let $A \in \mathscr{B}(X, Y)$ and X, Y be Banach spaces. Then A is a Fredholm operator, if and only if, there exist $Q_1, Q_2 \in \mathscr{B}(Y, X)$ such that*

$$Q_1 A = I - C_1 \text{ in } X \text{ and } A Q_2 = I - C_2 \text{ in } Y \tag{A.3}$$

with compact operators C_1, C_2.

Let $A \in \mathscr{F}(X, Y)$, $B \in \mathscr{F}(Y, Z)$ and X, Y, Z Banach spaces. Then $B \circ A \in \mathscr{F}(X, Z)$ and

$$ind\,(B \circ A) = ind\, B + ind\, A\,. \tag{A.4}$$

For a Fredholm operator A and a compact operator C, the sum $A+C$ is a Fredholm operator and

$$ind\,(A + C) = ind\, A\,. \tag{A.5}$$

The set of Fredholm operators is an open subset in the space of bounded linear operators and the index is a continuous function.

To conclude Appendix A we recall from [259] Fredholm's alternative for a sesquilinear form a in a Hilbert space H under a Gårding inequality,

$$\Re\Big\{a(v, v) + (Cv, v)_H\Big\} \geq \alpha_0\, \|v\|_H^2\,, \quad v \in H \tag{A.6}$$

for a constant $\alpha_0 > 0$ and a compact operator C from H into H.

Theorem A.7 *Suppose the continuous sesquilinear form $a : H \times H \to \mathbb{C}$ satisfies Gårding inequality (A.6). Then for the variational equation*

$$\text{Find } u \in H \text{ such that } a(u, v) = \ell(v)\,, \forall v \in H \tag{A.7}$$

there holds the alternative:

Either

(A.7) has exactly one solution $u \in H$ for every given $\ell \in H^*$

or

The homogeneous problem,

$$\text{Find } u_0 \in H \text{ such that } a(u_0, v) = 0\,, \forall v \in H \tag{A.8}$$

and its adjoint problem,

$$\text{Find } v_0 \in H \text{ such that } a(u, v_0) = 0\,, \forall u \in H \tag{A.9}$$

have finite dimensional kernels of the same dimension $k > 0$. *The nonhomogeneous problem (A.7) and its adjoint,*

$$\textit{Find } w \in H \textit{ such that } \overline{a(v, w)} = \ell^*(v)\,, \forall v \in H$$

have solutions iff the orthogonality conditions

$$\ell(v_{0(j)}) = 0, \textit{ respectively, } \ell^*(u_{0(j)}) = 0 \textit{ for } j = 1, \ldots, k$$

hold where $\{u_{0(j)}\}_{j=1}^k$ *spans the eigenspace of (A.8) and* $\{v_{0(j)}\}_{j=1}^k$ *spans the eigenspace of (A.9), respectively.*

Rewrite the variational equation (A.7) as

$$a(u, v) = (jAu, v)_H = (f, v)_H = \ell(v)\,,$$

where $f \in H$ represents $\ell \in H^*$ due to the Riesz representation theorem. Then the maps $jA : H \to H$, $A : H \to H^*$ are linear and bounded. Moreover from Fredholm's alternative above, Theorem A.7 we derive the following

Remark A.3 Suppose the Gårding inequality (A.6). Then the linear operator A has closed range, moreover A is a Fredholm operator with index zero.

Appendix B
Pseudodifferential Operators

Here we recall some concepts from Fourier transform and the theory of pseudodifferential operators. These are especially used in Chap. 4. For further reading see e.g. [253, 376, 415].

For $1 \le p < \infty$ the space $L^p(E)$ consists of all measurable functions $f : E \to \mathbb{R}$ $(\mathbb{C})$ with norm

$$\|f\|_{L^p} := \left(\int_E |f(x)|^p \, dx \right)^{1/p} < \infty,$$

whereas $L^\infty(E)$ has norm $\|f\|_{L^\infty} := \text{ess sup}_{x \in E} |f(x)| < \infty$.

Definition B.1 Let $u \in L^1(\mathbb{R}^n)$. Then the Fourier transform of u is given by

$$\hat{u}(\xi) = (\mathscr{F}u)(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} u(x) \, dx \tag{B.1}$$

with its inverse

$$\tilde{u}(\xi) = \left(\tilde{\mathscr{F}}u\right)(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} u(x) \, dx. \tag{B.2}$$

Clearly $\hat{u} \in L^\infty(\mathbb{R}^n)$, $\| \hat{u} \|_{L^\infty} \le \| u \|_{L^1}$.

Definition B.2 (Rapidly Decreasing Functions) $\varphi \in \mathscr{S}(\mathbb{R}^n) \iff \varphi \in C^\infty(\mathbb{R}^n, \mathbb{C})$, $\sup_{x \in \mathbb{R}^n} | < x >^\alpha D^\beta \varphi(x) | < \infty \ \forall \alpha, \beta \in \mathbb{N}_0^n$, with $< x > := 1 + |x|$.

Proposition B.1 $u \in \mathscr{S}(\mathbb{R}^n) \implies \hat{u}, \ \tilde{u} \in \mathscr{S}(\mathbb{R}^n)$.

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Proof $u \in \mathscr{S}(\mathbb{R}^n) \Longrightarrow x^\alpha u \in \mathscr{S}(\mathbb{R}^n) \quad \forall \alpha \in \mathbb{N}_0^n; \quad D^\beta u \in \mathscr{S}(\mathbb{R}^n) \quad \forall \beta \in \mathbb{N}_0^n$. Therefore we have $D^\beta(x^\alpha u) \in \mathscr{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and consequently $\mathscr{F} D^\beta(x^\alpha u)$ is absolutely convergent. So we can interchange differentiation and integration in (B.1). With $\partial_j := \frac{\partial}{\partial x_j}$ and $D_j := \frac{1}{i}\partial_j$ we have

$$D_\xi^\alpha e^{-ix\xi} = (-i)^{|\alpha|}(-ix)^\alpha \, e^{-ix\xi} = (-1)^{|\alpha|} \, x^\alpha \, e^{-ix\xi}$$

and thus $\xi^\beta e^{-ix\xi} = (-1)^\beta \; D_x^\beta \, e^{-ix}$. It follows that

$$D_\xi^\alpha \hat{u}(\xi) = \int_{\mathbb{R}^n} (-x)^\alpha \, u(x) \, e^{-ix\xi} \, dx = \mathscr{F}\left((-x)^\alpha u\right)(\xi) \tag{B.3}$$

and

$$\begin{aligned} \xi^\beta \hat{u}(\xi) &= (-i)^\beta \int_{\mathbb{R}^n} u(x) \, D_x^\beta \, e^{-ix\xi} \, dx \\ &= \int_{\mathbb{R}^n} D_x^\beta \, u(x) \, e^{-ix\xi} \, dx = \mathscr{F}\left(D^\beta u\right)(\xi). \end{aligned} \tag{B.4}$$

Applying (B.3) and (B.4) yields

$$\xi^\beta D^\alpha \hat{u}(\xi) = \mathscr{F}\left(D^\beta (-x)^\alpha u\right)(\xi) \in L^\infty\left(\mathbb{R}^n\right) \quad \forall \alpha, \beta \in \mathbb{N}_0^n.$$

Hence $\hat{u} \in \mathscr{S}(\mathbb{R}^n)$. □

Proposition B.2 *The map* $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ *is an isomorphism with* $\mathscr{F}\widetilde{\mathscr{F}} = \widetilde{\mathscr{F}}\mathscr{F} = 1$. *Moreover, for* $u \in \mathscr{S}(\mathbb{R}^n)$ *there holds:*

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{u}(\xi) \, e^{ix\xi} \, d\xi. \tag{B.5}$$

Remark B.1 By Proposition B.1 we have $u = \widetilde{\mathscr{F}}(\mathscr{F}(u))$ for all $u \in \mathscr{S}(\mathbb{R}^n)$. Moreover, $\mathscr{F}$ and $\widetilde{\mathscr{F}}$ are related to each other via $(\mathscr{F}u)(x) = (2\pi)^n \left(\widetilde{\mathscr{F}}u\right)(-x)$.

It follows that

$$\mathscr{F}\mathscr{F}u(x) = (2\pi)^n \, u(-x). \tag{B.6}$$

Remark B.2 With equation (B.5) we write the differential operator

$p(x, D) = \sum_{|\alpha|=0}^{N} a_\alpha(x) \, D_x^\alpha$ as

$$p(x, D) \, u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} p(x, \xi) \, \hat{u}(\xi) \; d\xi.$$

Definition B.3 $p(x,\xi) = \sum_{|\alpha|=0}^{N} a_\alpha(x)\,\xi^\alpha$ is called the symbol of $p(x,D)$.

Definition B.4 A tempered distribution T on $\mathbb{R}^n$ is a continuous linear functional $T : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$. The linear space of tempered distributions is denoted by $\mathscr{S}'(\mathbb{R}^n)$.

Next we extend $\mathscr{F}$ to a map on tempered distributions:

Find $\mathscr{F}^+ : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ such that

$$< \mathscr{F} f, \varphi > = < f, \mathscr{F}^+ \varphi > \quad \forall f \in \mathscr{S}'\left(\mathbb{R}^n\right), \varphi \in \mathscr{S}\left(\mathbb{R}^n\right),$$

$< f, \varphi > := \int\limits_{\mathbb{R}^n} f(x)\varphi(x)\,dx \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^n)$ and $|< f, \varphi >| \le c$ bounded with $c \in \mathbb{R}$.

Then we define

$$\mathscr{F}\,|_{\mathscr{S}'(\mathbb{R}^n)} := (\mathscr{F}^+\,|_{\mathscr{S}(\mathbb{R}^n)})'.$$

If $f, \varphi \in \mathscr{S}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$, then by Fubini's theorem

$$\begin{aligned} < \mathscr{F} f, \varphi > &= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} f(x)\, e^{-ix\xi}\, dx\, \varphi(\xi)\, d\xi \\ &= \int\limits_{\mathbb{R}^n} f(x) \int\limits_{\mathbb{R}^n} e^{-ix\xi}\varphi(\xi)\, d\xi\, dx = < f, \mathscr{F}\varphi > . \end{aligned}$$

So we get that $\mathscr{F}^+ = \mathscr{F}$ on $\mathscr{S}(\mathbb{R}^n)$.

Definition B.5 If $f \in \mathscr{S}'(\mathbb{R}^n)$, then $\mathscr{F} f = \hat{f} \in \mathscr{S}'(\mathbb{R}^n)$ is defined by

$$< \mathscr{F} f, \varphi > = < f, \mathscr{F}\varphi > \quad \forall \varphi \in \mathscr{S}\left(\mathbb{R}^n\right). \tag{B.7}$$

(B.7) is equivalent to

$$(\mathscr{F} f, \varphi) = (2\pi)^n \left(f, \widetilde{\mathscr{F}}\varphi\right) \quad \forall \varphi \in \mathscr{S}\left(\mathbb{R}^n\right) \tag{B.8}$$

with $(f, \varphi) := < f, \bar{\varphi} >$.

Remark B.3 Also on $\mathscr{S}'(\mathbb{R}^n)$ there holds $\mathscr{F}\widetilde{\widetilde{\mathscr{F}}} = \widetilde{\widetilde{\mathscr{F}}}\mathscr{F} = 1$.
Furthermore, $\mathscr{F}\,|_{\mathscr{S}'(\mathbb{R}^n)}$ is continuous extension of $\mathscr{F}\,|_{\mathscr{S}(\mathbb{R}^n)}$.
For $u \in \mathscr{S}'(\mathbb{R}^n)$ there holds

$$D^\alpha u = \mathscr{F}^{-1}\left(\xi^\alpha \hat{u}\right), \tag{B.9}$$

$$(-x)^\beta\, u = \mathscr{F}^{-1}\left(D^\beta \hat{u}\right). \tag{B.10}$$

For all $u \in \mathscr{S}(\mathbb{R}^n)$ there holds Parseval's equality:

$$\| \hat{u} \|^2_{L^2(\mathbb{R}^n)} = \left(\mathscr{F}u, \underbrace{\mathscr{F}u}_{=:\varphi} \right) = (2\pi)^n \, (u, \widetilde{\mathscr{F}}\varphi) = (2\pi)^n \, \| u \|^2_{L^2(\mathbb{R}^n)} .$$

Thus $u \mapsto (2\pi)^{\frac{-n}{2}} \hat{u}$ is an *isometry* on $L^2(\mathbb{R}^n)$. There holds

$$\left((2\pi)^{\frac{-n}{2}} \mathscr{F} \right)^{-1} = (2\pi)^{\frac{n}{2}} \widetilde{\mathscr{F}} = \left((2\pi)^{\frac{-n}{2}} \mathscr{F} \right)^{*},$$

where the adjoint A^* of a given operator A is defined by

$$(Af, \varphi) = \left(f, A^*\varphi \right) \quad \forall \varphi \in \mathscr{S}\left(\mathbb{R}^n\right), \; f \in \mathscr{S}'\left(\mathbb{R}^n\right).$$

Definition B.6 For all u, v in $\mathscr{S}(\mathbb{R}^n)$ the convolution is defined by $(u * v)(x) := \int_{\mathbb{R}^n} u(x-y)\, v(y)\, dy$.

The Fourier transform satisfies:

$$\left(\hat{u} * \hat{v} \right)(\xi) = \int_{\mathbb{R}^n} \hat{u}(\xi - \eta)\, \hat{v}(\eta)\, d\eta = < \hat{u}(\xi - \cdot), \hat{v} >$$
$$= < \mathscr{F}\hat{u}(\xi - \cdot), v > = (2\pi)^n \, \widehat{u \cdot v}(\xi) .$$

The last equation follows from

$$\hat{u}(\xi - \eta) = \int_{\mathbb{R}^n} u(x)\, e^{-ix(\xi-\eta)}\, dx = (2\pi)^n \left(\widetilde{\mathscr{F}}_{x \mapsto \eta} e^{-ix\xi} u(x) \right)(\eta)$$

Thus

$$\left(\mathscr{F}_{\eta \mapsto x} \hat{u}(\xi - \eta) \right)(x) = (2\pi)^n \, e^{-ix\xi} u(x)$$

and

$$< \mathscr{F}\hat{u}(\xi - \cdot), v > = (2\pi)^n \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\, v(x)\, dx = (2\pi)^n \, \widehat{u \cdot v}(\xi) .$$

The following formulae are valid for $u, v \in \mathscr{S}(\mathbb{R}^n)$:

$$\hat{u} * \hat{v} = (2\pi)^n \, \widehat{u \cdot v} \tag{B.11}$$
$$\widehat{u * v} = \hat{u} \cdot \hat{v}. \tag{B.12}$$

We prove (B.12): $\widehat{\tilde{u} \cdot \tilde{v}} = \hat{\tilde{u}} * \hat{\tilde{v}}\, (2\pi)^{-n} = u * v\, (2\pi)^{-n}$.

Hence by Remark B.1

$$\widehat{u * v}\,(x) = (2\pi)^n \left(\mathscr{F}\widehat{\tilde{u}\cdot\tilde{v}}\right)(x) = (2\pi)^{2n}\,(\tilde{u}\cdot\tilde{v})\,(-x)$$
$$= (2\pi)^{2n}\,\tilde{u}\,(-x)\,\tilde{v}\,(-x) = (2\pi)^{2n}\,(2\pi)^{-n}\,\hat{u}\,(x)\,(2\pi)^{-n}\,\hat{v}\,(x) = \left(\hat{u}\cdot\hat{v}\right)(x)$$

which completes the proof. □

Next we give a brief introduction in pseudodifferential operators and symbol-classes. We define for $s \in \mathbb{R}$

$$||\phi||^2_{H^s(\mathbb{R}^n)} := (2\pi)^{-n}\int_{\mathbb{R}^n}(1+|\xi|^2)^s|\widehat{\phi}(\xi)|^2\,d\xi \quad \phi \in \mathscr{S}(\mathbb{R}^n) \tag{B.13}$$

In the following we also use the abbreviation $< \xi >^{2s} := (1+|\xi|^2)^s$.

We remark

$$||\phi||^2_{H^0(\mathbb{R}^n)} \equiv ||\phi||^2_{L^2(\mathbb{R}^n)}$$

and introduce the Sobolev space

$$H^s(\mathbb{R}^n) := \{\varphi \in \mathscr{S}'(\mathbb{R}^n) | \widehat{\varphi} \in L^2_{loc}(\mathbb{R}^n), ||\varphi||_{H^s(\mathbb{R}^n)} < \infty\}$$
$$= \text{completion of } \mathscr{S}'(\mathbb{R}^n) \text{ with } \;||\cdot||_{H^s(\mathbb{R}^n)}$$

This gives rise to the Bessel potential operator $\Lambda^s : H^t \to H^{t-s} \qquad \forall t, s \in \mathbb{R}$.

The differentiation $D^\alpha : H^s \longmapsto H^{s-|\alpha|}$ is continuous and for all $0 \le s \le t$,

$$\mathscr{S} \subset H^t \subset H^s \subset H^0 = L^2 \subset H^{-s} \subset \mathscr{S}'\,.$$

Now we introduce symbol classes. Let $p(x, D) = \sum_{|\alpha|\le k} a_\alpha(x) D^\alpha$. Then we have

$$p(x, D)u(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\xi}\,p(x,\xi)\widehat{u}(\xi)d\xi\;, \forall u \in \mathscr{S}(\mathbb{R}^n)\,. \tag{B.14}$$

By the property of the Fourier transform,

$$\mathscr{F}^{-1}\widehat{D^\alpha u} = \mathscr{F}^{-1}(\xi^\alpha \widehat{u})\,,$$

hence

$$p(x, D)u(x) = \sum a_\alpha(x)\mathscr{F}^{-1}\widehat{D^\alpha u} = \sum a_\alpha(x)\mathscr{F}^{-1}(\xi^\alpha\widehat{u}) = \mathscr{F}^{-1}p(x,\xi)\widehat{u}\,.$$

Example B.1

(i) $p(\xi) =< \xi >^s$ $(= (1 + |\xi|^2)^{\frac{s}{2}})$. This gives $p(D) = (1 - \Delta)^s$, since $-\widehat{\Delta u} = -\widehat{\frac{d^2}{dx^2}u} = -(i\,\xi)^2\widehat{u}(\xi) = \xi^2\widehat{u}(\xi)$.

(ii) $p(\xi) = e^{ia\cdot\xi}$ with $a \in \mathbb{R}$ fixed gives $p(D)u(x) = u(x + a)$.

(iii) For $n = 1$, define using the Poisson kernel,

$$K_t f(x) := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t f(y)}{t^2 + (x - y)^2}\, dy\,.$$

Then $K_t = p(x, D)$ with $p(x, \xi) = e^{-t|\xi|}$.

Next we write (B.14) for some special functions $p(x, \xi)$.

$p(x, \xi) = p(x)$ gives a multiplication operator that maps $C_0^\infty \to C^\infty$

$p(x, \xi) = p(\xi)$ gives all convolution operators $\mathscr{S} \to \mathscr{S}'$, provided $p \in \mathscr{S}'$, (respectively $L^2 \to L^2 \Leftrightarrow p \in L^\infty$).

In the general case, formal calculus gives

$$\begin{aligned} p(x, D)u(x) &= (2\pi)^{-n}\int e^{ix\cdot\xi} p(x, \xi)\Big(\int e^{-iy\cdot\xi} u(y)\, dy\Big)\, d\xi \\ &= (2\pi)^{-n}\iint e^{-i(x-y)\cdot\xi} p(x, \xi) u(y)\, dy\, d\xi \\ &= \int \check{p}(x, x - y) u(y)\, dy\,, \end{aligned}$$

where $\check{p}(x, z) = (2\pi)^{-n}\int e^{iz\cdot\xi} p(x, \xi)\, d\xi$ or equivalently,

$$p(x, \xi) = \mathscr{F}_{z\to\xi}(\check{p}(x, z))(\xi) \tag{B.15}$$

Hence $p(x, D)$ has a kernel $k(x, y) = \check{p}(x, x - y)$ or equivalently, $\check{p}(x, z) = k(x, x - z)$, and thus

$$\begin{aligned} p(x, \xi) &= \int e^{-iz\cdot\xi} k(x, x - z)\, dz \\ &= \mathscr{F}k(x, x - \cdot)(\xi) \quad \text{if } k(x, x - \cdot) \in \mathscr{S}'\,. \end{aligned}$$

Let us introduce the following notions.

(i) Let Ω open in $\mathbb{R}^n$, fix $m, \rho, \delta \in \mathbb{R}$ with $\rho \le 1$, $\delta \ge 0$. Then define

$$\begin{aligned} S^m_{\rho,\delta}(\Omega) :=&\{p \in C^\infty(\Omega \times \mathbb{R}^n) | \forall K \subset\subset \Omega\ \forall \alpha, \beta \in \mathbb{N}_0^n\ \exists\, C_{K,\alpha,\beta} : \\ & |D_x^\alpha D_\xi^\beta p(x, \xi)| \le C_{K,\alpha,\beta} < \xi >^{n-\rho|\beta|+\delta|\alpha|}, \forall x \in K, \xi \in \mathbb{R}^n\} \end{aligned} \tag{B.16}$$

In the following we often have $\rho = 1, \delta = 0$.

(ii) Classical symbols:
$p \in S^m(\Omega) :\Leftrightarrow p \in S^m_{1,0}$ and
$\exists$ sequence $(p_{m-j})_{j\in\mathbb{N}_0} \subset C^\infty(\Omega \times \mathbb{R}^n)$ with

$$p_{m-j}(x, r\xi) = r^{m-j} p_{m-j}(x, \xi)\,, \forall\, |\xi| \geq 1, r \geq 1$$

(positively homogenous of degree $m - j$ for $|\xi| \geq 1, r \geq 1$) that decays faster than any power,

$$p - \sum_{j=0}^{N} p_{m-j} \in S^{m-N-1}_{1,0}(\Omega) \qquad (\Leftrightarrow\ p \sim \sum_{j\geq 0} p_{m-j})$$

(iii)

$$S^{-\infty} := \bigcap_m S^m = \bigcap_m S^m_{\rho,\delta} \quad \text{(independent of } \rho, \delta)$$

Note that when $v(x) \longmapsto \widehat{v}(\xi)$, the asymptotic behaviour of v for small x corresponds to the asymptotic behaviour of $\widehat{v}$ for large ξ.

Remember

$$(v * u)(x) = \int v(x - y)u(y)\,dy \qquad \widehat{v * u}(\xi) = \widehat{v}(\xi)\widehat{u}(\xi)$$

and consider the following

Example B.2

$$\Delta u = 0 \text{ in } \Omega = \mathbb{R}^+_2\,, u = g \text{ on } \partial\Omega = \mathbb{R}_1$$

Then

$$u(x) = -\frac{1}{2\pi}\int_{\mathbb{R}} \ln|x - y|\phi(y)\,dy$$

satisfies $\Delta u = 0$ in Ω. We have the convolution $u(x) = \int v(x - y)\,\phi(y)\,dy$ with the simple kernel $v(x) = \ln|x|$. Hence $\widehat{g}(\xi) = \widehat{v}(\xi)\widehat{\phi}(\xi)$ and thus

$$\phi(x) = \mathscr{F}^{-1}_{\xi\to x}\widehat{\phi}(\xi) = \mathscr{F}^{-1}_{\xi\to x}\frac{\widehat{g}}{\widehat{v}}(\xi)\,.$$

More generally let

$$u = A\phi\,,\ A\phi(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}a(x, \xi)\widehat{\phi}(\xi)\,d\xi$$

Here the singularity of the kernel $k(x, x-y)$ of A when $x \to y$ is determined by the behaviour of $a(x,\xi)$ for $|\xi| \to \infty$.

Example B.3

(i) Let $p(x,\xi) = \sum_{|\alpha|\le k} a_\alpha(x)\xi^\alpha$, then $p(x,\xi) \in S^k$. Here we have $p_{k-j}(x,\xi) = \sum_{|\alpha|=k-j} a_\alpha(x)\xi^\alpha$. Note that we can write $D_\xi^\beta \xi^\alpha = C_{\alpha\ \beta}\xi^{\alpha-\beta}$, $\forall\, \beta \le \alpha$ with some constants $C_{\alpha\ \beta}$.

(ii) Let $q(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be homogeneous of degree m,

$$q(r\xi) = r^m q(\xi)\,, \quad \forall r > 0, \xi \neq 0\,.$$

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ in a neighborhood of 0 and $\operatorname{supp}\chi \subset\subset B_1(0)$. Then put $p(x,\xi) = p(\xi) := (1-\chi(\xi))q(\xi)$. Hence there holds $p \in S^m_{1,0}(\mathbb{R}^n)$ (even $\in S^m(\mathbb{R}^n)$). Note that we can modify symbols for small $|\xi|$, since we are interested in the behaviour for large $|\xi|$.

(iii) Let $p(\xi) = <\xi>^{-2} = (1+|\xi|^2)^{-1}$. Then $p \in S^{-2} \subset S^{-2}_{1,0}$.
Indeed, $p \in C^\infty$, $D^\alpha p(\xi) = (1+|\xi|^2)^{-1-|\alpha|}\cdot h^{|\alpha|}(\xi)$ with an appropriate polynomial h. Hence $|D^\alpha p(\xi)| \le C <\xi>^{-2-|\alpha|}$ and $p \in S^{-2}_{1,0}$ follows.
Moreover, we use the asymptotic expansion

$$\frac{1}{1+|\xi|^2} = |\xi|^{-2}\frac{1}{1+|\xi|^{-2}} = -\sum_{k=1}^{\infty}(-1)^k|\xi|^{-2k}\ (|\xi|>1)$$

and put

$$p_{-2k}(\xi) := (1-\chi(\xi))(-1)^{k+1}|\xi|^{-2k}$$

Then as seen above in (2.) $p_{-2k} \in S^{-2k}_{1,0}$ and hence

$$p(\xi) - \sum_{k=1}^{N} p_{-2k}(\xi) \in S^{-2N-2}_{1,0} \subset S^{-2N-1}_{1,0} \quad (N \ge 1)$$

(iv) Let $p(\xi) = <\xi>^s$. Then $p \in S^s \subset S^s_{1,0}$, $\forall s \in \mathbb{R}$

Lemma B.1 $D^\alpha(\frac{1}{u}) = \frac{1}{u}\ \sum_{k\le|\alpha|} C_{\alpha_1,\ldots,\alpha_k}\frac{D^{\alpha_1}u}{u}\cdot\ldots\cdot\frac{D^{\alpha_k}u}{u}$

Theorem B.1 *Let $p \in S^m_{1,0}(\Omega)$ and*

$$|\frac{1}{p(x,\xi)}| \le c <\xi>^{-m} \quad \textit{for } |\xi| \ge 1 \qquad (\Longleftrightarrow \textit{elliptic})$$

Then $\frac{1-\chi}{p} \in S^{-m}_{1,0}(\Omega)$.

Proof We have

$$|\frac{D_x^{\alpha_\varkappa} D_\xi^{\beta_\lambda} p(x,\xi)}{p(x,\xi)}| \leq C < \xi >^{m-|\beta_\lambda|} < \xi >^{-m} = C < \xi >^{-|\beta_\lambda|}$$

and hence by the lemma above

$$|D_x^\alpha D_\xi^\beta \frac{1}{p(x,\xi)}| \leq |\tfrac{1}{p(x,\xi)}| \sum_{\substack{\sum \alpha_\kappa = \alpha \\ \sum \beta_\lambda = \beta}} C_{\alpha\beta} \prod |\tfrac{D_x^{\alpha_\varkappa} D_\xi^{\beta_\lambda} p}{p}|$$
$$\leq C < \xi >^{-m} < \xi >^{-|\beta|} \; .$$

□

Theorem B.2 *Let* $p \in S^m_{1,0}(\Omega), q \in S^{m'}_{1,0}(\Omega)$. *Then*

$$D_x^\alpha D_\xi^\beta p \;\in\; S_{1,0}^{m-|\beta|}(\Omega) \quad \textit{and } p \cdot q \in S_{1,0}^{m+m'}(\Omega)\,.$$

Proof Use Leibniz rule

$$D_x^\alpha D_\xi^\beta (p \cdot q)(x,\xi) = \sum_{\substack{\alpha'+\alpha''=\alpha \\ \beta'+\beta''=\beta}} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} D_x^{\alpha'} D_\xi^{\beta'} p(x,\xi)\, D_x^{\alpha''} D_\xi^{\beta''} q(x,\xi)$$

and estimate the partial derivatives of p by $< \xi >^{m-|\beta'|}$, respectively the partial derivatives of q by $< \xi >^{m'-|\beta''|}$ modulo some positive constant factor, what leads to the upper bound $< \xi >^{m+m'-|\beta|}$ modulo a positive constant.

□

We define a pseudodifferential operator of class $S^m_{1,0}$

$$p(x,D) \in OP\; S^m_{1,0}(\Omega) \quad :\Leftrightarrow \quad p(x,\xi) \in \; S^m_{1,0}(\Omega)$$

with

$$p(x,D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\widehat{u}(\xi)\, d\xi\,, \qquad \forall u \in C_0^\infty(\mathbb{R}^n) \tag{B.17}$$

Theorem B.3 *Let* $m \in \mathbb{R}$, , $p(x,D) \in OP\; S^m_{1,0}(\Omega)$. *There holds*

$$p(x,D) : C_0^\infty(\Omega) \longmapsto C^\infty(\Omega) \quad \textit{continuous, linear.}$$

Proof Now $u \in C_0^\infty(\mathbb{R}^n)$ implies $\widehat{u} \in \mathscr{S}(\mathbb{R}^n)$. Hence $p(x,\xi)\widehat{u}(\xi)$ still decays fast and the integral converges absolutely. Hence interchanging differentiation and integration yields with (B.17)

$$|D_x^\alpha(p(x,D)u(x))| \le c \sum_{\alpha'+\alpha''=\alpha} \int <\xi>^m <\xi>^{|\alpha''|} |\widehat{u}(\xi)|\, d\xi < \infty \quad \forall \alpha$$

Thus $p(x, D)u \in C^\infty$. □

Exercise: Show the mapping $p(x, D) \in OP\ S^m_{1,0} : H^s(\mathbb{R}^m) \longmapsto H^{s-m}(\mathbb{R}^m)$ is continuous, that is, there exists $C > 0$ such that $\|p(x, D)u\|_{H^{s-m}} \le C\,\|u\|_{H^s}$.

Next we consider the relation between strong ellipticity of a pseudodifferential operator and Gårding's inequality. As shown in Sect. 4.2 with the example of the single layer operator, considering integral operators as pseudodifferential operators allows to deduced the mapping properties of boundary integral operators by examining the symbols of the pseudodifferential operators. On the other hand, Garding's inequality for integal equations is the key property to guarantee convergence of Galerkin's method, see Theorem 6.1, Theorem 6.11. Now, Garding's inequality follows from the definition of uniform strong ellipticity of pseudodifferential operators, see Theorem 6.2.7 in [259].

Definition B.7 A system of pseudodifferential operators $A_{jk} \in OP\ S^{s_j+t_k}_{1,0}(\Omega)$ is called uniformly strongly elliptic if for the principal part matrix $a^0(x;\xi) = ((a^{jk0}_{s_j+t_k}(x;\xi)))_{p\times p}$ there exist a C^∞ -matrix valued function $\Theta(x) = ((\Theta_{jk}(x)))_{p\times p}$ and a constant $\gamma_0 > 0$ such that

$$\Re \zeta^T \Theta(x) a^0(x,\xi)\overline{\zeta} \ge \gamma_0 |\zeta|^2$$

for all $x \in \Omega$, $\zeta \in \mathbb{C}^p$ and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$.

A uniformly strongly elliptic system of pseudodifferential operators satisfies a Gårding inequality, see [259, Theorem 6.2.7.] . In the following we present and prove the corresponding result for a single pseudodifferential operator:

Theorem B.4 (Gårding Inequality) *Let $p(x,\xi) \in OP\ S^m_{1,0}(\Omega)$ be strongly elliptic , i.e. $\forall K \subset\subset \Omega$ let there exist positive constants C_K, R_K such that there holds*

$$\Re p(x,\xi) \ge C_K <\xi>^m \ \forall x \in K, |\xi| \ge R_K$$

Then $\forall K \subset\subset \Omega$ and $\forall s \in \mathbb{R}$ there exist constants γ_K, $C_{K,s}$ such that

$$\Re(p(x,D)u, u) \ge \gamma_K ||u||^2_{H^{m/2}(\Omega)} - C_{K,s}||u||^s_{H^s(\Omega)}$$

$\forall u \in C_0^\infty(\Omega)$.

Lemma B.2 *Let $p \in S^0_{1,0}(\Omega)$, $\Re p(x,\xi) \geq C > 0\ \forall x, \xi$ ($|\xi|$ sufficiently large) then there exist $B \in OP\ S^m_{1,0}(\Omega)$, $K \in OP\ S^{-\infty}(\Omega)$ such that*

$$\Re p(x, D) = B^* B + K.$$

Proof Setting $q(x, D) := \Lambda^{-\frac{m}{2}} p \Lambda^{-\frac{m}{2}}$ we have for $u \in C_0^\infty$ $(pu, u) = (q\Lambda^{\frac{m}{2}} u, \Lambda^{\frac{m}{2}} u)$ and $||u||^2_{H^{1/2}} \sim ||\Lambda^{\frac{m}{2}} u||^2_{L^2}$. Therefore it suffices to show for $m = 0$. Now we use the above lemma for $p_0(x, \xi) := \Re p(x, \xi) - c'$, with 0<c'<c. Hence $p_0(x, \xi) \geq c - c' > 0$. Then there exists $b \in OP\ S^0_{1,0}$ such that $\Re p_0(x, D) - B^* B =: S \in OP\ S^\infty$ such that

$$\Re(p(x, D)u, u) - c'||u||^2_{L^2} = ||Bu||^2_{L^2} + \Re(Su, u)$$

and this yields

$$\Re(p(x, D)u, u) \geq c''||u||^2_{L^2} + \Re(Su, u).$$

□

As a consequence, any strongly elliptic pseudodifferential operator defines a Fredholm operator of index zero since for the corresponding bilinear form one may apply the classical Fredholm alternative (A.3).

Example B.4 Writing the single layer potential as

$$V\psi(x) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{ix'\xi'} \frac{\widehat{u}(\xi')}{|\xi'|} d\xi'$$

gives

$$\int_\Gamma (V\psi(x')\psi(x')dx' = \Re(\widehat{V\psi}(\xi'), \widehat{\psi}(\xi')) = \Re(2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} \frac{1}{|\xi'|} \widehat{\psi}(\xi') \overline{\widehat{\psi}}(\xi') d\xi'$$

$$\geq \gamma ||\psi||^2_{H^{-1/2}(\Gamma)} - \text{ compact perturbation .}$$

Example B.5 The single layer potential in linear elasticity with fundamental solution

$$E(x, y) = 1/|x - y| I + \kappa(x - y)(x - y)^T$$

where $\kappa = \frac{\lambda+\mu}{\lambda+3\mu}$ has principal symbol

$$\sigma_0(V)(\xi) = \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)} \frac{1}{|\xi|^3} \begin{pmatrix} |\xi|^2 + \kappa\xi_2^2 & -\kappa\xi_1\xi_2 & 0 \\ -\kappa\xi_1\xi_2 & |\xi|^2 + \kappa\xi_1^2 & 0 \\ 0 & 0 & |\xi|^2 \end{pmatrix}$$

The corresponding hypersingular operator has principal symbol

$$\sigma_0(W)(\xi) = \frac{-\mu^2}{|\xi|} \begin{pmatrix} |\xi|^2 + \epsilon\xi_1^2 & \epsilon\xi_1\xi_2 & 0 \\ \epsilon\xi_1\xi_2 & |\xi|^2 + \epsilon\xi_2^2 & 0 \\ 0 & 0 & (1+\epsilon)|\xi|^2 \end{pmatrix}$$

with $-1/2 < \epsilon := \lambda(\lambda + 2\mu)^{-1} < 1$ see [130, 396].

Appendix C
Convex and Nonsmooth Analysis, Variational Inequalities

C.1 Convex Optimization, Lagrange Multipliers

By this section of Appendix C we invite the reader to get acquainted with some fundamental concepts, methods, and results of convex optimization that are necessary for the proper understanding of the mathematical and numerical treatment of inequality constrained problems that occur in the Signorini boundary value problem and in further nonsmooth boundary value problems, see Chap. 5, and in contact problems, see Chap. 11 and also Sect. 12.5

Based on the monograph [55] of Blum and Oettli, we start with convex quadratic optimization in finite dimensions. Already at this level we encounter different formulations, namely a "primal" and a "mixed" formulation with signed Lagrange multipliers that are associated to inequality constraints. In fact, the existence of such Lagrange multipliers can be derived from the celebrated duality theory of linear optimization ("linear programming") without any further assumptions. Moreover, a solution in convex quadratic optimization is characterized by a "linear complementarity problem" and by a variational inequality (VI) of a special structure.

Then we proceed to convex variational problems in Hilbert space. As a straightforward extension of the finite dimensional case, we characterize solutions by variational inequalities with symmetric bilinear forms. Also guided by the finite dimensional case, we readily introduce the Lagrange function for convex cone constraints. However, the existence of Lagrange multipliers is more involved than in the finite dimensional case. First we construct the Lagrange multiplier in the space dual to the solution space of the primal variable, which is a Sobolev space of negative order in application to contact problems. Then in the subsequent subsection we follow [219] and present mixed formulations with Lagrange multipliers that live in the Hilbert space of constraints, which is the more regular L^2 function space on the contact boundary part in the application to unilateral contact problems. To this end we provide an extension of the famous Brezzi splitting theorem that originally covers sad-

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dle point problems with equality constraints, only, to a class of nonsmooth inequality constrained variational problems. Under the celebrated Babuška-Brezzi condition we obtain independent Lagrange multipliers in the ordering cone of the inequality constraints and in the subdifferential of the convex nonsmooth sublinear functional.

C.1.1 Convex Quadratic Optimization in Finite Dimensions

For given data $b \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, $C \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{n\times n}$, where A is a symmetric positive semidefinite matrix, shortly $A = A^T \geq 0$, we consider the convex finite dimensional quadratic optimization problem with linear inequality constraints ("quadratic program")

$$(\mathrm{QP}) \begin{cases} \text{minimize } f(x) = \frac{1}{2}x^T Ax - b^T x \\ \text{subject to } x \geq 0,\ Cx \leq d\,. \end{cases}$$

Put in another way, among all feasible solutions x to (QP), that is, $x \in \mathbb{R}^n_+$, shortly $x \geq 0$, that satisfy the constraints $(Cx)_j \leq d_j \ \ (\forall j = 1, \ldots, m)$ we are looking for that feasible $\hat{x}$ that minimizes the objective function f.

The symmetry requirement $A = A^T$ is not essential, since we can replace the matrix A by its symmetric part $\dfrac{1}{2}(A + A^T)$ in the objective function f. In the formal discussion to follow, considering only signed variables x_i for $i = 1, \ldots, n$ does not lead to a loss of generality either, since for a free variable x_i we can use its decomposition $x_i = x_i^+ - x_i^-$, $x_i^+ \geq 0$, $x_i^- \geq 0$. Also an equality constraint $c_j^T x = d_j$ can be rewritten as

$$\begin{Bmatrix} c_j^T x \leq & d_j \\ -c_j^T x \leq & -d_j \end{Bmatrix}.$$

Of course, these two latter trivial reformulations are not appropriate in numerical computation, but are convenient here to reduce the discussion of constrained optimization problems to the standard form (QP) given above.

Now we take (QP) as primal optimization problem ("primal program") and proceed to its mixed formulation via the Lagrange function

$$L(x, y) = f(x) + y^T(Cx - d)\,.$$

In view of the sign conditions and the inequality constraints, the Lagrange function is considered only for $x \geq 0,\ y \geq 0$, since we have for any $x \geq 0$,

$$\sup_{y\geq 0} L(x, y) = \begin{cases} f(x) & \text{if } x \text{ is feasible;} \\ +\infty & \text{otherwise.} \end{cases}$$

This gives

$$\inf_{x\geq 0}\sup_{y\geq 0} L(x,y) = \inf(\text{QP}) ,$$

where inf (QP) denotes the optimal value of (QP). Therefore in the sense of convex duality theory, the dual optimization problem ("dual quadratic program") to (QP) reads

$$(\text{DQP}) \begin{cases} \text{maximize } \inf_{x\geq 0} L(x,y) \\ \text{subject to } y \geq 0 . \end{cases}$$

Obviously, $\inf\sup L \geq \sup\inf L$ is trivial. But in finite dimensions, without further assumptions, we have even the "duality equality" $\inf(\text{QP}) = \sup(\text{DQP})$; moreover, the dual problem attains an optimal solution, what is nothing else than a Lagrange multiplier to the inequality constrained optimization problem (QP):

Theorem C.1 *If (QP) has an optimal solution $\hat{x}$, then there exists a Lagrange multiplier $\hat{y} \geq 0$ such that $(\hat{x}, \hat{y})$ is a saddle point of L on $\mathbb{R}^n_+ \times \mathbb{R}^m_+$, that is, we have*

$$\textit{(SP)} \quad L(\hat{x}, y) \leq L(\hat{x}, \hat{y}) \leq L(x, \hat{y}); \quad \forall x \in \mathbb{R}^n_+, y \in \mathbb{R}^m_+ .$$

Proof To prove (SP) it is enough to establish the Karush–Kuhn–Tucker conditions, which read for the linear constraints in (QP) here

$$(\text{KKT}) \quad f(\hat{x}) \leq f(x) + \hat{y}^T(Cx - d) \ \ \forall x \in \mathbb{R}^n_+ .$$

Indeed, in view of feasibility, $C\hat{x} - d \leq 0$, (KKT) implies the equality

$$(*) \quad \hat{y}^T(C\hat{x} - d) = 0 .$$

Hence, the right hand side of (SP) follows from (KKT) directly, whereas the left hand side of (SP) is equivalent to (*) and the feasibility of $\hat{x}$.

Therefore it remains to show the existence of $\hat{y} \in \mathbb{R}^m_+$ that satisfies (KKT). Here we rely on the duality theorem of finite dimensional linear optimization ("linear programming") and first show the following

Proposition C.1 *Let $\hat{x}$ be an optimal solution to (QP). Then $\hat{x}$ is an optimal solution to the linear program*

$$(LP) \begin{cases} \textit{minimize } (A\hat{x} - b)^T x =: c^T x \\ \textit{subject } x \geq 0, Cx \leq d . \end{cases}$$

Proof of the Proposition Since the constraints of (QP) and (LP) are the same, it is enough to give the following contradiction argument. Suppose there exists $\tilde{x} \in \mathbb{R}^n_+$

such that $c^T\tilde{x} < c^T\hat{x}$ and $C\tilde{x} \leq d$. Then consider $x_t = \hat{x} + t(\tilde{x} - \hat{x})$, where $0 < t < 1$; x_t is feasible for (QP). By $\nabla f(\hat{x})^T(\tilde{x} - \hat{x}) = c^T(\tilde{x} - \hat{x}) < 0$, for small enough $t > 0$, we arrive at $f(x_t) < f(\hat{x})$ contradicting the optimality of $\hat{x}$. □

Proof of the theorem continued. The dual linear optimization problem ("dual program") to (LP) reads

$$(DLP) \begin{cases} \text{maximize } -d^T y \\ \text{subject } y \geq 0,\, C^T y + c \geq 0\,; \end{cases}$$

this can be seen by means of the associated Lagrange function

$$l(x, y) = c^T x + y^T(Cx - d) = -d^T y + x^T(c + C^T y)$$

on $\mathbb{R}^n_+ \times \mathbb{R}^m_+$ and by the relation

$$\inf_{x \geq 0} l(x, y) = \begin{cases} -d^T y & \text{if } C^T y + c \geq 0\,; \\ -\infty & \text{otherwise.} \end{cases}$$

In virtue of the duality theorem of linear programming, see [55, 140], there exists $\hat{y} \in \mathbb{R}^m_+$ such that

$$\text{(i) } C^T\hat{y} \geq -c\,,$$
$$\text{(ii) } c^T\hat{x} = -d^T\hat{y}\,.$$

Then multiplying (i) by arbitrary $x \geq 0$ gives

$$x^T C^T \hat{y} \geq -x^T A\hat{x} + b^T x\,,$$

hence by (ii)

$$(Cx - d)^T\hat{y} \geq \hat{x}^T A(\hat{x} - x) + b^T x - b^T\hat{x}\,.$$

Thus we obtain

$$\begin{aligned} f(x) + (Cx - d)^T\hat{y} &\geq \hat{x}^T A(\hat{x} - x) + \tfrac{1}{2}x^T Ax - b^T\hat{x} \\ &= \tfrac{1}{2}\hat{x}^T A\hat{x} + \tfrac{1}{2}[(\hat{x} - x)^T A(\hat{x} - x)] - b^T\hat{x}\,, \end{aligned}$$

and since A is positive semidefinite,

$$f(x) + (Cx - d)^T\hat{y} \geq \frac{1}{2}\hat{x}^T A\hat{x} - b^T\hat{x}\,,$$

what is the claimed (KKT) inequality. □

We remark that the saddle point inequalities (SP) are clearly also sufficient for the optimality of $\hat{x}$.

We can characterize the optimality of $\hat{x}$ in another way using slack variables. Define the primal slack variable

$$v = d - Cx \in \mathbb{R}^m ,$$

then feasibility is equivalent to $v \geq 0$ and (*) reads $\hat{v}^T \hat{y} = 0$ with $\hat{v} = d - C\hat{x}$. Likewise define the dual slack variable

$$u = c + C^T y = A\hat{x} - b + C^T y \in \mathbb{R}^n .$$

Then for $y \geq 0$, feasibility in (DLP) is equivalent to $u \geq 0$ and with $\hat{u} = c + C^T \hat{y} = A\hat{x} - b + C^T \hat{y}$, we conclude from (ii) and (*) that

$$(**) \quad \hat{x}^T \hat{u} = 0 .$$

Since $\hat{v}_j \geq 0, \hat{y}_i \geq 0, \hat{u}_i \geq 0, \hat{x}_i \geq 0$, (*) means $\hat{v}_j = 0$ or $\hat{y}_j = 0$ and (**) means $\hat{u}_i = 0$ or $\hat{x}_i = 0$. In this sense $\hat{v}$ and $\hat{y}$, respectively $\hat{u}$ and $\hat{x}$ are "complementary variables".

Altogether we obtain the following

Corollary C.1 *$\hat{x}$ is an optimal solution to (QP), if and only if $(\hat{x}, \hat{y}, \hat{u}, \hat{v}) \in \mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$ satisfies*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & C^T \\ -C & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -b \\ d \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \geq 0 \qquad \begin{pmatrix} x \\ y \end{pmatrix} \geq 0 \qquad \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} = 0 .$$

The above system of linear equations and sign inequalities can be considered as a "mixed formulation" of the convex quadratic optimization problem (QP). It leads to the

Definition Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be given. Then the "complementarity problem" consists in finding $\hat{z} \in \mathbb{R}^N_+$ such that $F(\hat{z}) \in \mathbb{R}^N_+$ and $\hat{z}^T F(\hat{z}) = 0$ hold. We have a "linear complementarity problem" (LCP), if $F(z) = Bz - a$ is affine-linear for some $B \in \mathbb{R}^{N \times N}$, $a \in \mathbb{R}^N$.

Thus the solution of (QP) can be characterized as the solution of a special (LCP), where the matrix B has the special saddle point structure

$$B = \begin{pmatrix} A & C^T \\ -C & 0 \end{pmatrix} .$$

Furthermore a solution $\hat{z}$ to the complementarity problem can be characterized by the following "variational inequality":

$$\hat{z} \in \mathbb{R}^N_+ , \quad F(\hat{z})^T(z - \hat{z}) \geq 0 \quad \forall z \in \mathbb{R}^N_+ .$$

Indeed, the direct implication being obvious, only the reverse implication needs an argument; for that choose $z = \frac{1}{2}\hat{z}$ and $z = 2\hat{z}$. In the case of a linear complementarity problem, the variational inequality reads

$$\hat{z} \in \mathbb{R}^N_+ , \quad (B\hat{z})^T(z - \hat{z}) \geq a^T(z - \hat{z}) \quad \forall z \in \mathbb{R}^N_+ .$$

Remark Also in the case of a linear complementarity problem, the solution generally depends *nonlinear* on the data, e.g. on the datum a!

An unessential extension of the problem is obtained by a simple translation: Let $c \in \mathbb{R}^N$ be given; find $z \in \mathbb{R}^N$, such that $z \geq c$, $F(z) \geq 0$, $(z - c)^T F(z) = 0$.

An essential extension of the problem is obtained as follows. Instead of R^N_+, consider an arbitrary convex cone K (that is, $K + K \subseteq K$, $\mathbb{R}_+ K \subseteq K$) in $\mathbb{R}^N$, not necessarily polyhedric, define the positive polar cone $K^+ = \{u \in \mathbb{R}^N | u^T x \geq 0, \forall x \in K\}$. Then the complementarity problem consists in finding $\hat{z} \in \mathbb{R}^N$ such that $\hat{z} \in K$, $F(\hat{z}) \in K^+$, $\hat{z}^T F(\hat{z}) = 0$. Again, this can be characterized by a variational inequality. In the case of a linear complementarity problem with F affine-linear as above, this variational inequality reads

$$\hat{z} \in K , \quad (B\hat{z})^T(z - \hat{z}) \geq a^T(z - \hat{z}) \quad \forall z \in K .$$

For more information on linear complementarity problems and variational inequalities in finite dimensions we refer to the monographs of Cottle, Pang, and Stone [139] and of Facchinei and Pang [168, 169], respectively.

C.1.2 Convex Quadratic Optimization in Hilbert Spaces

Let V be a real Hilbert space (may be also a reflexive Banach space) and Z another real Hilbert space with its dual Z'. Let $A \in \mathscr{L}(V, V')$ with $A = A'$, $A \geq 0$ (i.e. $\langle Av, v\rangle \geq 0, \forall v \in V$), further $B \in \mathscr{L}(V, Z')$ and $f \in V'$, $g \in Z'$ fixed elements. We also need the adjoint $B' \in \mathscr{L}(Z, V')$. Moreover let an order $\leq$ be defined in Z via a convex closed cone $P \subset Z$ via $z \geq 0$ iff $z \in P$. Also $\zeta \in Z' \leq 0$ iff ζ lies in the negative dual cone $P^- = \{\zeta \in Z' : \zeta(p) \leq 0, \forall p \in P\}$. With these given data, similar to (QP) in C.1.1, we consider the convex quadratic optimization problem

$$(CP) \begin{cases} \text{minimize } f(v) = \frac{1}{2}\langle Av, v\rangle - \langle f, v\rangle \\ \text{subject to } Bv \leq g . \end{cases}$$

This gives rise to the bilinear form $a(u, v) := \langle Au, v\rangle$ and the convex closed sets,

$$K(g) := \{v \in V \mid Bv \leq g\}$$

which is a translate of the convex closed cone (with vertex at zero)

$$K := \{v \in V \mid Bv \leq 0\}.$$

As in C.1.1 a solution u of (CP) is characterized by a variational inequality, here

$$u \in K(g),\ a(u, v - u) \geq \langle f, v - u\rangle\,, \forall v \in K(g)\,. \tag{C.1}$$

Analogously to C.1.1 , we introduce the Lagrangian

$$L(v, p) := f(v) + \langle p, Bv - g\rangle_{Z\times Z'} = f(v) + \langle B'p, v\rangle_{V'\times V} - g(p), \quad v \in V, p \in P\,,$$

to arrive at saddle points and to mixed formulations.

We can drop the requirement that $A = A'$ and now start from the primal VI (C.1). However, the existence of Lagrange multipliers in the cone P in the infinite dimensional space Z is more involved than in finite dimensions. As the recent paper [219] shows, this can be accomplished by an extension of the Brezzi splitting theorem under the Babuška-Brezzi condition. We postpone a sketch of this approach to Lagrange multipliers to the next subsection.

Before that we describe here first an easier approach under the assumption that there exists a preimage w of g under B, thus $Bw = g$. This allows to to work with the duality on $V \times V'$ and to obtain the following characterization via multipliers in the negative dual cone K^- to K.

Proposition C.2 *$u \in K(g)$ solves the VI (C.1), iff there exists $\lambda \in V'$ such that $(u, \lambda) \in K(g) \times K^-$ solves the mixed system*

$$(MP) \quad \begin{cases} a(u, v) + \langle \lambda, v\rangle = \langle f, v\rangle \\ \langle \kappa - \lambda, u - w\rangle \leq 0\,, \end{cases}$$

for all $v \in V, \kappa \in K^-$. Then there holds the complementarity condition

$$(*) \quad \langle \lambda, u - w\rangle = 0\,.$$

Proof Let $u \in K(g)$ solve the VI (C.1). Define $\lambda \in V'$ by $\lambda(v) = f(v) - a(u, v)$. Then $(MP)_1$ holds. Further, for any $v \in K$, $\tilde{v} := v + u$ lies in $K(g)$ and hence

$$-\lambda(v) = a(u, \tilde{v} - u) - f(\tilde{v} - u) \geq 0\,.$$

Thus $\lambda \in K^-$. Since $w \in K(g)$, $u - w \in K$,

$$\langle \kappa - \lambda, u - w \rangle = \langle \kappa, u - w \rangle + [a(u, u - w) - f(u - w)] \le 0$$

for any $\kappa \in K^-$ and therefore (MP) holds.

The complementarity condition $(*)$ follows from $(MP)_2$ by the choice $\mu = 2\lambda$, $\mu = 0$.

Vice versa, let $v \in K(g)$, hence $v - w \in K$. This implies by the complementarity condition $(*)$

$$\langle \lambda, v - u \rangle = \langle \lambda, v - w \rangle - \langle \lambda, u - w \rangle \le 0 \,.$$

Hence we arrive at $a(u, v - u) = (f - \lambda)(v - u) \ge f(v - u)\,.$ □

From the proof above, it follows that $u \in K(g)$ solves the VI (C.1), iff there exists λ such that $[u, \lambda] \in K(g) \times K^-$ solves $(MP)_1$ and $(*)$ holds. Therefore the above mixed form does not depend on the chosen preimage w. Indeed, let $Bw_i = g$ $(i = 1, 2)$. Then $u \pm (w_1 - w_2) \in K(g)$ and thus by the VI (C.1), $\lambda(w_1 - w_2) = 0$.

The mixed formulation above applies to unilateral contact problems with Signorini condition on some boundary part Γ_c in appropriate function spaces, where for a boundary variable u the linear map $u \mapsto Bu$ is the restriction to the boundary part Γ_c; see Sect. 11.4.1.

With friction problems we encounter nonsmooth optimization problems of the form

$$(NOP) \quad \text{minimize } f(v) = \frac{1}{2}\langle Av, v \rangle - \langle f, v \rangle + \varphi(v),\ \ v \in V\,,$$

where φ is convex, even positively homogeneous, hence sublinear on V, but not differentiable in the classic sense. A prominent example is

$$\varphi(v) = \int_{\Gamma_c} g|v|\, ds \quad (g \in L^\infty(\Gamma_c), g > 0)\,.$$

An optimal solution of (NOP) is characterized as solution to the so-called variational inequality of the second kind:

$$u \in V,\ \langle Au, v - u \rangle + \varphi(v) - \varphi(u) \ge f(v - u),\ \forall v \in V\,.$$

Here one can obtain by (L^1, L^∞) duality and density the useful duality formula

$$\varphi(v) = \int_{\Gamma_c} g|v|\, ds = \sup\{\int_{\Gamma_c} gv\mu\, ds \mid \mu \in C(\Gamma), |\mu| \le 1\}\,.$$

C.1.3 Lagrange Multipliers for Some Inequality Constrained Variational Inequalities

In this subsection we deal with a canonical class of inequality constrained variational inequalities of the second kind, where the sum of a bilinear form and a sublinear functional and further a linear functional as right hand side occur and where the constraints are defined by linear inequalties with respect to a closed convex ordering cone. More precisely, let V, Z be real reflexive Banach spaces with (topological) dual spaces V', Z'. Let $P \subset Z$ be a closed convex cone with vertex at zero. Let $A \in \mathscr{L}(V, V')$, $B \in \mathscr{L}(V, Z')$ be continuous linear operators that give rise to the continuous bilinear forms $a : V \times V \to \mathbb{R}$, $b : V \times Z \to \mathbb{R}$ via $a(v, w) = \langle Av, w\rangle_{V' \times V}$, $b(v, z) = \langle Bv, z\rangle_{Z' \times Z}$. We use the null space $W := \ker B$ of B and its polar W° contained in V'. Further let $\varphi : V \to \mathbb{R}$ be sublinear, thus there holds the representation formula

$$\varphi(v) = \max_{\sigma \in S} \langle \sigma, v\rangle, \quad \forall v \in V, \tag{C.2}$$

where $S \subset V'$ is weak* compact and coincides with the convex subdifferential $\partial\varphi(0) = \{\xi \in V' | \langle \xi, \cdot\rangle \le \varphi\}$. In other words, φ is the support function [252] of S. Finally let $f \in V'$, $g \in Z'$ be fixed. Then introduce the feasible set

$$K(g) = \{v \in V \ : \ b(v, p) \le \langle g, p\rangle, \ \forall p \in P\}$$

and pose the variational inequality in its primal form: Find $u \in V$ that satisfies

$$(VI) \quad u \in K(g), \ a(u, v - u) + \varphi(v) - \varphi(u) \ge \langle f, v - u\rangle, \ \forall v \in K(g)\,.$$

Our goal in this subsection is to arrive at the following mixed form with Lagrange multipliers $q \in P$ and $\tau \in S$:

$$(MF) \begin{cases} (MF-1) \ a(u, v) + b(v, q) + \langle \tau, v\rangle = \langle f, v\rangle\,, & \forall v \in V\,, \\ (MF-2) \ b(u, p - q) + \langle u, \sigma - \tau\rangle \le \langle g, p - q\rangle\,, & \forall [p, \sigma] \in P \times S\,. \end{cases}$$

To achieve this goal we use the famous Brezzi lemma which characterizes that B', the adjoint operator of B, is isomorph, i.e. is bijective with continuous inverse, by the celebrated Babuška-Brezzi condition (BB). More precisely, there holds

Lemma C.1 *The following assertions are equivalent.*

(i) There exists a number $\beta > 0$ *such that*

$$(BB) \qquad \sup_{v \in V, v \ne 0} \frac{b(v, z)}{\|v\|_V} \ge \beta\, \|z\|_Z\,, \qquad \forall z \in Z\,,$$

(ii) $B' : Z \to W^\circ$ *is isomorph with*

$$\|B'z\|_{V'} \geq \beta \, \|z\|_Z \,, \qquad \forall z \in Z \tag{C.3}$$

for some $\beta > 0$.

For the proof of the Brezzi lemma we can e.g. refer to [60, Theorem 3.6, Lemma 4.2].

Now we focus to the homogeneous case, where $g = 0$ with feasible set $K =: K(0)$, since the proof of this case is simpler and nearer to the linear functional analytical proof of the classic case of equality constrained variational problems than the proof for general g.

Theorem C.2 *The two problems* (VI) *and* (MF) *are related as follows. If* $[u, q, \lambda] \in V \times Z \times V'$ *solves* (MF) *(with* $g = 0$*), then* u *lies in* K *and solves* (VI)*. Vice versa, let* $u \in K$ *solve* (VI)*, then there exist* $q \in P$ *and* $\tau \in S$ *such that* $[u, q, \tau]$ *solves* (MF) *(with* $g = 0$*), provided* (BB) *holds for some* $\beta > 0$.

Proof We give a sketch of the proof divided in several steps.

I. Since P is a cone, we can choose $p = 1/2\, q$ and $p = 2q$ in $(MF-2)$. Moreover we use (C.2). Thus we first observe that $(MF-2)$ with $g = 0$ splits equivalently into the statements

$$(MF-3) \qquad \begin{cases} b(u, p) \leq 0 \,, \forall p \in P \,, \\ b(u, q) = 0 \,, \\ \varphi(u) = \langle \tau, u \rangle \,. \end{cases}$$

II. Let $[u, q, \tau] \in V \times Z \times V'$ solve (MF) with $g = 0$. Then from $(MF-3)_1$ it is immediate that $u \in K$.

To show that u solves (VI), let $v \in K$ be arbitrary. Then $b(v, q) \leq 0$ and from $(MF-3)_2$, $b(v-u, q) \leq 0$. Hence from $(MF-3)_3$, (C.2), and $(MF-1)$,

$$\begin{aligned} & a(u, v-u) + \varphi(v) - \varphi(u) \\ & \geq a(u, v-u) + \langle \tau, v-u \rangle \\ & = -b(v-u, q) + \langle f, v-u \rangle \\ & \geq \langle f, v-u \rangle \,. \end{aligned}$$

III. The proof of the second part of the theorem runs in 5 steps.

1. Let $u \in K$ solve (VI). Since K is a cone, we can choose $v = 2u$ and $v = 1/2\, u$. This gives

$$a(u, u) + \varphi(u) = \langle f, u \rangle \,, \tag{C.4}$$

hence by addition,

$$a(u, v) + \varphi(v) \geq \langle f, v \rangle , \forall v \in K \, . \tag{C.5}$$

Note that (C.4) and (C.5) are equivalent to (VI).

2. By (C.2), (C.5) means: $\forall v \in K \, \exists \sigma \in S$ such that $a(u, v) + \langle \sigma, v \rangle \geq \langle f, v \rangle$. Since S is convex and weak* compact, it can be shown that there exists some $\tau \in S$ such that

$$a(u, v) + \langle \tau, v \rangle \geq \langle f, v \rangle , \forall v \in K \, . \tag{C.6}$$

3. By construction, $W = \ker B \subset K$. Hence (C.6) implies

$$a(u, w) + \langle \tau, w \rangle = \langle f, w \rangle , \forall w \in W \, ,$$

or $f - Au - \tau \in W^\circ$. In virtue of the (BB) condition, Lemma C.1 applies and entails the existence of $q \in Z$ such that $B'q = f - Au - \tau$ or

$$a(u, v) + b(v, q) + \langle \tau, v \rangle = \langle f, v \rangle , \forall v \in V \, .$$

Thus we obtain $(MF - 1)$.

4. We claim that $q \in P$. Indeed, (C.6) gives by definition of q,

$$\langle B'q, v \rangle = \langle Bv, q \rangle \leq 0 \, , \forall v \in K \, .$$

This means $Bv \in P^- \Rightarrow Bv \in Q^-$, where $P^- = \{\zeta \in Z' | \langle \zeta, p \rangle \leq 0, \ \forall p \in P\}$ is the negative dual cone to P and $Q := \mathbb{R}_+ q \subset Z$. In virtue of the (BB) condition, Lemma C.1 applies and hence $B : (W^\circ)' \to Z'$ is isomorph, in particular is onto. Therefore the implication above gives $P^- \subset Q^-$, what results by the bipolar theorem in $P^{--} = P \supset Q^{--} = Q$. This proves the claim.

5. To prove $(MF - 2)$, we show $(MF - 3)$. By feasibility of $u \in K$, $(MF - 3_1)$ is obvious. Since $\tau \in S = \partial \varphi(0)$, $\varphi(u) \geq \langle \tau, u \rangle$. From (C.4) and (C.6), we get

$$\langle f, v \rangle = a(u, u) + \varphi(u) \geq a(u, u) + \langle \tau, u \rangle \geq \langle f, v \rangle \, ,$$

hence $(MF - 3)_3$, and also by definition of q, $\langle B'q, u \rangle = b(u, q) = 0$, thus finally $(MF - 3)_2$. □

For a more detailed proof and for the proof of the general case of arbitrary g, moreover for further references see [219].

Here let us first consider the special case $\varphi = 0, S = \{0\}$. Our aim is to derive from the present mixed form (MF) the mixed form (MP) of the previous subsection.

The present mixed form becomes then with some preimage $w = B^{-1}g$

$$(MF)_0 \begin{cases} a(u,v) + \langle B'q, v\rangle = \langle f, v\rangle\,, & \forall v \in V\,,\\ \langle u, B'p - B'q\rangle \le \langle w, B'p - B'q\rangle\,, & \forall p \in P\,, \end{cases}$$

where the multiplier q exists in P. Note that $\lambda := B'q \in K^-$, the latter inequality $(MF)_{0-2}$ extends to the closure of $B'P$, what coincides with $[B^{-1}(P^-)]^- = K^-$. Hence we arrive at the mixed form (MP).

To conclude this subsection, we want to bring the present mixed form (MF) in relation to the mixed form used in BEM solution of frictional unilateral contact problems in [33, 37], see Sect. 11.4.1. To this end, we proceed as in the special case above and obtain from (MF) with again $Bw = g$ the pair $[\lambda, \tau] \in K^- \times S$ that together with $u \in V$ solves the mixed system

$$\begin{cases} a(u,v) + \langle \lambda, v\rangle + \langle \tau, v\rangle = \langle f, v\rangle, & \forall v \in V\,,\\ \langle \kappa - \lambda, u\rangle + \langle \sigma - \tau, u\rangle \le \langle \kappa - \lambda, w\rangle, & \forall [\kappa, \sigma] \in K^- \times S\,. \end{cases}$$

Note that $K^- + S$ is convex and closed in V'. Thus using the indicator function χ_K of K ($\chi_K(v) = 0$ iff $v \in K$, $= +\infty$ elsewhere),

$$\begin{aligned} K^- + S = \partial\chi_K(0) + \partial\varphi(0) &= \partial(\chi_K + \varphi)(0)\\ &= \{\mu | \langle \mu, \cdot\rangle \le \chi_K + \varphi\}\\ &= \{\mu | \langle \mu, v\rangle \le \varphi(v), \forall v \in V \text{ with } Bv \in P^-\} =: M \end{aligned}$$

what is the analog to the set of multipliers in [33, 37].

On the other hand, for any $\mu \in M$ - in the case of a general reflexive Banach space V in virtue of Troyanski's renorming theorem an equivalent norm can be introduced so that V and V' are locally uniformly convex, and thus also strictly convex - the constrained best approximation problem

$$\begin{aligned} &\text{minimize } \|\kappa\|^2 + \|\sigma\|^2,\ \kappa \in K^-, \sigma \in S\\ &\text{subject to } \kappa + \sigma = \mu \end{aligned}$$

admits unique solutions $\mu_- \in K^-, \mu_S \in S$ with $\mu = \mu_- + \mu_S$.

Therefore we arrive at the multiplier $\nu := \lambda + \tau \in M$ that together with $u \in V$ solves the somewhat condensed mixed system

$$(MF)_c \begin{cases} a(u,v) + \langle \nu, v\rangle = \langle f, v\rangle\,, & \forall v \in V\,,\\ \langle \mu - \nu, u\rangle \le \langle \mu_- - \nu_-, w\rangle, & \forall \mu \in M\,, \end{cases}$$

what corresponds to the mixed form in [33, 37].

C.2 Nonsmooth Analysis

With nonmonotone contact problems we encounter locally Lipschitz functions that are not necessarily convex or smooth in the sense of classical differentiability. Therefore in this section we draw some basics from Clarke's monograph [109] on nonsmooth analysis. We collect some fundamental concepts of the Clarke generalized differential calculus, in particular introduce his generalized directional derivative along with its basic properties. Following [332, 335] we also provide regularization techniques of nondifferentiable optimization to smooth locally Lipschitz functions that are minima or maxima of smooth functions. These regularization techniques are needed in addition for the numerical treatment of nonmonotone contact problems, see Sect. 11.5.

C.2.1 Nonsmooth Analysis of Locally Lipschitz Functions

Throughout this subsection, let X denote a (real) Banach space. Let $f : X \to \mathbb{R}$ be Lipschitz of rank K near a given point $x \in X$; that is, for some $\varepsilon > 0$, we have

$$|f(y) - f(z)| \leq K \, \|y - z\|; \; \forall y, z \in B(x, \varepsilon)\,.$$

Definition C.1

$$f^0(x; v) := \limsup\{\frac{f(y + tv) - f(y)}{t} \mid y \in X, y \to x; t > 0, t \to 0\}$$

is called the **generalized directional derivative** of f in the direction v.

Note that this definition does not presuppose the existence of a limit and that it differs from the common definition of the directional derivative (or Gâteaux derivative, which is continuous in v) in that the base point y in the difference quotient varies. Also note that in general $f^0(x; \cdot)$ is not linear. The utility of this definition is seen from the properties listed below.

Proposition C.3 *Let f be Lipschitz of rank K near x. Then:*

(i) The function $v \mapsto f^0(x; v) \in \mathbb{R}$ is sublinear, hence convex, and satisfies $|f^0(x;v)| \leq K \, \|v\|$ for all $v \in X$;
(ii) The function $(z, w) \mapsto f^0(z; w)$ is upper semicontinuous at (x, v); the function $w \mapsto f^0(x; w)$ is Lipschitz of rank K on X;
(iii) There holds $f^0(x; -v) = (-f)^0(x; v)$ for $v \in X$.

Definition C.2 The **generalized gradient** of the function f at x, denoted by (simply) $\partial f(x)$, is the unique nonempty weak* compact convex subset of the dual space X', whose support function is $f^0(x; .)$.

Thus

$$\xi \in \partial f(x) \Leftrightarrow f^0(x, v) \geq \langle \xi, v \rangle, \ \forall v \in X ,$$
$$f^0(x; v) = \max\{\langle \xi, v \rangle \ : \ \xi \in \partial f(x)\}, \ \forall v \in X .$$

A function $f : X \to \mathbb{R}$ which is continously differentiable near a point x is locally Lipschitz near x by the mean value theorem. Also a function $f : X \to \mathbb{R}$ which is convex and lower semicontinuous is locally Lipschitz on all of X. In either case, ∂f reduces to the familiar concept of the derivative, respectively of that of the subdifferential of convex analysis:

Theorem C.3 *If $f : X \to \mathbb{R}$ is continously differentiable near x, then $\partial f(x) = \{f'(x)\}$. If $f : X \to \mathbb{R}$ is convex and lower semicontinuous on X, then for any $x \in X$,*

$$\partial f(x) = \{\xi \in X^* \ : \ \langle \xi, y - x \rangle \leq f(y) - f(x), \ \forall y \in X\} .$$

On the other hand, let f be Lipschitz near x and suppose that $\partial f(x)$ is a singleton $\{\xi\}$, then f is Gâteaux differentiable with $f'(x) = \xi$.

Definition C.3 Let $f : X \to \mathbb{R}$ be locally Lipschitz near x. Then f is called **regular** at x, if $f^0(x; v)$ coincides with the classical directional derivative $f'(x, v)$ for all $v \in X$.

There is a calculus of generalized gradiens including a sum rule, mean value theorem, and chain rule; see [109] for details. Here we only provide an important formula of nonsmooth analysis ('Danskin's formula', see [109, (2.3.12)]) that chararacterizes the generalized directional derivative of max functions.

Let I be a finite index set and let $\{f_i : i \in I\}$ be a finite collection of functions that are Lipschitz near x. Then the function f defined by

$$f(x) := \max_{i \in I} f_i(x)$$

is Lipschitz near x as well. Let $I(x) := \{i \in I : f_i(x) = f(x)\}$ and "co" denote the convex hull.

Theorem C.4 *There holds*

$$\partial f(x) \subset \ co\,\{\partial f_i(x) : i \in I(x)\}.$$

If f_i is regular at x for each $i \in I(x)$, then equality holds and f is regular at x.

C.2.2 Regularization of Nonsmooth Functions

In this subsection we follow [332, 335] and present a unified approach to regularization of nonsmooth functions with focus to locally Lipschitz functions that are minima or maxima of smooth functions.

According to Bertsekas [49] the maximum function $f : \mathbb{R}^n \to \mathbb{R}$,

$$f(x) = \max\{g_1(x), g_2(x), \ldots, g_m(x)\} \tag{C.7}$$

of m continuously differentiable functions g_i can be expressed by means of the plus function $\mathrm{p}(x) = x^+ = \max(x, 0)$, $x \in \mathbb{R}$ as

$$f(x) = g_1(x) + \mathrm{p}\left[g_2(x) - g_1(x) + \ldots + \mathrm{p}\left[g_m(x) - g_{m-1}(x)\right]\right]. \tag{C.8}$$

Replacing now the plus function by an approximation $P(\varepsilon, \dot{})$, the smoothing function $S : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}$ is given by

$$S(x, \varepsilon) := g_1(x) + P\left[\varepsilon, g_2(x) - g_1(x) + \ldots + P\left[\varepsilon, g_m(x) - g_{m-1}(x)\right]\right] \tag{C.9}$$

as suggested by Chen et al. in [100]. The advantage of this procedure lies in the use of one single regularization parameter ε to smooth an eventually larger number of kinks. Here, $P : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}$ is the smoothing function via convolution for the plus function p defined by

$$P(\varepsilon, t) = \int_{-\infty}^{\frac{t}{\varepsilon}} (t - \varepsilon s)\rho(s)\, ds.$$

We restrict $\rho : \mathbb{R} \to \mathbb{R}_+$ to be a density function of finite absolute mean; that is

$$k := \int_{\mathbb{R}} |s|\rho(s)\, ds < \infty.$$

The major properties of S, see [346], that follow from the properties of the function P, see [169, section 11.8.2], are collected in the following lemma.

Lemma C.2

(i) For any $\varepsilon > 0$ and for all $x \in \mathbb{R}^n$,

$$|S(x, \varepsilon) - f(x)| \le (m - 1)k\varepsilon. \tag{C.10}$$

(ii) The function S is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}_{++}$ *and for any* $x \in \mathbb{R}^n$ *and* $\varepsilon > 0$ *there exist* $\Lambda_i \geq 0$ *such that* $\sum_{i=1}^{m} \Lambda_i = 1$ *and*

$$\nabla_x S(x, \varepsilon) = \sum_{i=1}^{m} \Lambda_i \nabla g_i(x). \tag{C.11}$$

Moreover,

$$co\{\xi \in \mathbb{R}^n \, : \, \xi = \lim_{k\to\infty} \nabla_x S(x_k, \varepsilon_k), \; x_k \to x, \; \varepsilon_k \to 0^+\} \subseteq \partial f(x), \tag{C.12}$$

where "co" denotes the convex hull and $\partial f(x)$ *is the Clarke subdifferential.*

We recall that the Clarke subdifferential of a locally Lipschitz function f at a point $x \in \mathbb{R}^n$ can be characterized by

$$\partial f(x) = co\,\{\xi \in \mathbb{R}^n \, : \, \xi = \lim_{k\to\infty} \nabla f(x_k), \; x_k \to x, \; f \text{ is differentiable at } x_k\},$$

since in finite dimensional case, according to Rademacher's theorem, f is differentiable almost everywhere.

The maximum function given by (C.7) is clearly locally Lipschitz continuous and by Theorem C.4, the Clarke subdifferential can be written as

$$\partial f(x) = co\{\nabla g_i(x) \, : \, i \in I(x)\}$$

with

$$I(x) := \{i \, : \, f(x) = g_i(x)\}.$$

In particular, if $x \in \mathbb{R}^n$ is a point such that $f(x) = g_i(x)$ then $\partial f(x) = \{\nabla g_i(x)\}$. For such a point $x \in \mathbb{R}^n$ we show later on that

$$\lim_{z\to x, \varepsilon\to 0^+} \nabla_x S(z, \varepsilon) = \nabla g_i(x).$$

Note that the set on the left-hand side in (C.12) goes back to [353]. In [99], this set is denoted by $G_S(x)$ and is called there the subdifferential associated with the smoothing function. The inclusion (C.12) shows in fact that $G_S(x) \subseteq \partial f(x)$. Moreover, according to the part (b) of Corollary 8.47 in [353], $\partial f(x) \subseteq G_S(x)$. Thus, $\partial f(x) = G_S(x)$.

Remark C.1 Note also that S is a smoothing approximation of f in the sense that

$$\lim_{z\to x, \varepsilon\to 0^+} S(z, \varepsilon) = f(x) \quad \forall x \in \mathbb{R}^n. \tag{C.13}$$

This is immediate from (C.10).

Remark C.2 The regularization procedure (C.9) can be also applied to a minimum function by

$$\min\{g_1(x), g_2(x), \ldots, g_m(x)\} = -\max\{-g_1(x), -g_2(x), \ldots, -g_m(x)\} \approx -S(x, \varepsilon).$$

Denote now

$$S_i = g_i - g_{i-1} + P\,[\varepsilon, g_{i+1} - g_i + P\,[\varepsilon, g_{i+2} - g_{i+1} + \ldots + P\,[\varepsilon, g_m - g_{m-1}]]]\,.$$

This function should approximate

$$g_i - g_{i-1} + \mathrm{p}\left[g_{i+1} - g_i + \mathrm{p}\left[g_{i+2} - g_{i+1} + \ldots + \mathrm{p}\left[g_m - g_{m-1}\right]\right]\right]$$
$$\overset{(C.8)}{=} \max\{g_i - g_{i-1}, g_{i+1} - g_{i-1}, \ldots, g_m - g_{i-1}\} =: T_{i-1}.$$

Lemma C.3 *It holds*

$$\lim_{z \to x, \varepsilon \to 0^+} P(\varepsilon, S_i(z, \varepsilon)) = \mathrm{p}(T_{i-1}(x)). \tag{C.14}$$

Proof First, for any $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that

$$|P(\varepsilon, z) - \mathrm{p}(T_{i-1}(x))| < \varepsilon_0 \tag{C.15}$$

for any $z \in B_{\delta_0}(T_{i-1}(x))$ and $\varepsilon \in (0, \delta_0)$. Next, since S_i is a smoothing approximation of T_{i-1} in the sense of (C.13), there exists $\bar{\delta}_0 > 0$ such that

$$|S_i(z, \varepsilon) - T_{i-1}(x)| < \delta_0 \tag{C.16}$$

for any $z \in B_{\bar{\delta}_0}(x)$ and $\varepsilon \in (0, \bar{\delta}_0)$. Combining (C.15) and (C.16), it follows that

$$|P(\varepsilon, S_i(z, \varepsilon)) - \mathrm{p}(T_{i-1}(x))| < \varepsilon_0$$

holds for any $\varepsilon < \min\{\delta_0, \bar{\delta}_0\}$ and any $z \in B_{\bar{\delta}_0}(x)$. Thus, the assertion of the lemma is proved. □

Since the nonsmooth functions that occur in the nonmonotone contact problems can be reformulated by using the plus function, all our regularizations are based in fact on a class of smoothing approximations for the plus function. Some examples from [168] and the references therein are in order:

$$P_1(\varepsilon, t) = \int_{-\infty}^{\frac{t}{\varepsilon}} (t - \varepsilon s)\, \rho_1(s)\, ds = t + \varepsilon\, ln(1 + e^{-\frac{t}{\varepsilon}}) = \varepsilon\, ln(1 + e^{\frac{t}{\varepsilon}}), \tag{C.17}$$

where $\rho_1(s) = \dfrac{e^{-s}}{(1 + e^{-s})^2}$

$$P_2(\varepsilon, t) = \int_{-\infty}^{\frac{t}{\varepsilon}} (t - \varepsilon s)\, \rho_2(s)\, ds = \frac{\sqrt{t^2 + 4\varepsilon^2} + t}{2}, \tag{C.18}$$

where $\rho_2(s) = \dfrac{2}{(s^2 + 4)^{3/2}}$

$$P_3(\varepsilon, t) = \int_{-\infty}^{\frac{t}{\varepsilon}} (t - \varepsilon s)\, \rho_3(s)\, ds = \begin{cases} 0 & \text{if } t < -\frac{\varepsilon}{2} \\ \frac{1}{2\varepsilon}(t + \frac{\varepsilon}{2})^2 & \text{if } -\frac{\varepsilon}{2} \le t \le \frac{\varepsilon}{2} \\ t & \text{if } t > \frac{\varepsilon}{2}, \end{cases} \tag{C.19}$$

where $\rho_3(s) = \begin{cases} 1 \text{ if } -\frac{1}{2} \le s \le \frac{1}{2} \\ 0 \text{ otherwise.} \end{cases}$

$$P_4(\varepsilon, t) = \int_{-\infty}^{\frac{t}{\varepsilon}} (t - \varepsilon s)\, \rho_4(s)\, ds = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t^2}{2\varepsilon} & \text{if } 0 \le t \le \varepsilon \\ t - \frac{\varepsilon}{2} & \text{if } t > \varepsilon, \end{cases} \tag{C.20}$$

where $\rho_4(s) = \begin{cases} 1 \text{ if } 0 \le s \le 1 \\ 0 \text{ otherwise.} \end{cases}$

In the following we need

$$A_i = \{x \in \mathbb{R}^n \ :\ g_i(x) > g_j(x),\ \forall j = 1, \ldots, m,\ \ j \neq i\} \quad \text{for all } i = 1, \ldots, m$$

and compute

$$P_t(\varepsilon, t) = \int_{-\infty}^{\frac{t}{\varepsilon}} \rho(s)\, ds. \tag{C.21}$$

Lemma C.4 *The following properties hold:*

a) If $x \in A_i$, $i = 1, 2, \ldots, m-1$, then

$$\lim_{z \to x, \varepsilon \to 0^+} P_t(\varepsilon, S_{i+1}(z, \varepsilon)) = 0. \tag{C.22}$$

b) if $x \in A_i$, $i = 2, 3, \ldots, m$, then

$$\lim_{z \to x, \varepsilon \to 0^+} P_t(\varepsilon, S_j(z, \varepsilon)) = 1 \quad \textit{for all } j = 2, 3, \ldots, i. \tag{C.23}$$

Proof

a) Let $i \in \{1, 2, \ldots, m-1\}$ and $x \in A_i$, i.e., $g_i(x) > g_j(x)$ for all $j = 1, \ldots, m$, $j \neq i$, and S_{i+1} be a smoothing approximation of T_i defined as above by

$$T_i = \max\{g_{i+1} - g_i, g_{i+2} - g_i, \ldots, g_m - g_i\}.$$

Clearly, $T_i(x) < 0$. Since by (C.16) in the proof of Lemma C.3

$$S_{i+1}(z,\varepsilon) \to T_i(x) \quad \text{as } z \to x \text{ and } \varepsilon \to 0^+ \tag{C.24}$$

and due to $T_i(x) < 0$, it follows from (C.21) that

$$P_t(\varepsilon, S_{i+1}(z,\varepsilon)) = \int_{-\infty}^{\frac{S_{i+1}(z,\varepsilon)}{\varepsilon}} \rho(s)\,ds \to 0 \quad \text{as } z \to x, \varepsilon \to 0^+$$

and (C.22) is verified.

b) Let now $x \in A_i$, $i \in \{2, 3, \ldots, m\}$. We first prove the statement of the lemma for $j = i$. By the representation

$$S_i(z,\varepsilon) = g_i(z) - g_{i-1}(z) + P(\varepsilon, S_{i+1}(z,\varepsilon))$$

and using (C.14) from Lemma C.3, it follows that

$$S_i(z,\varepsilon) \to g_i(x) - g_{i-1}(x) \quad \text{as } z \to x \text{ and } \varepsilon \to 0^+. \tag{C.25}$$

Hence, since $g_i(x) - g_{i-1}(x) > 0$, we have

$$P_t(\varepsilon, S_i(z,\varepsilon)) = \int_{-\infty}^{\frac{S_i(z,\varepsilon)}{\varepsilon}} \rho(s)\,ds \to 1 \quad \text{as } z \to x,\ \varepsilon \to 0^+ \tag{C.26}$$

and therefore (C.23) is verified for $j = i$. Thus, we completely proved the lemma in the case $m = 2$. The remaining case can be based on an induction argument, see [335]. □

Now we are ready to show that the gradient of the given function g_i on A_i can be approximated by the gradients of the smoothing function.

Theorem C.5 *For any $x \in A_i$, $i = 1, 2, \ldots, m$,*

$$\lim_{z \to x, \varepsilon \to 0^+} \nabla_x S(z,\varepsilon) = \nabla g_i(x).$$

Proof From (C.9), by direct differentiation with respect to x, it follows that

$$\begin{aligned}
\nabla_x S(z,\varepsilon) &= \Big(1 - P_t(\varepsilon, S_2(z,\varepsilon))\Big)\nabla g_1(z) \\
&\quad + \sum_{i=2}^{m-1} \Big(1 - P_t(\varepsilon, S_{i+1}(z,\varepsilon))\Big) \prod_{j=2}^{i} P_t(\varepsilon, S_j(z,\varepsilon)) \nabla g_i(z) \\
&\quad + \prod_{i=2}^{m} P_t(\varepsilon, S_i(z,\varepsilon)) \nabla g_m(z).
\end{aligned}$$

We shall distinguish the following three cases.

1) First, we take $x \in A_1$. From Lemma C.4 a)

$$\lim_{z\to x, \varepsilon\to 0^+} P_t(\varepsilon, S_2(z,\varepsilon)) = 0$$

and, consequently, the following relations hold as $z \to x$ and $\varepsilon \to 0^+$:

$$\Lambda_1 := 1 - P_t(\varepsilon, S_2(z,\varepsilon)) \to 1,$$

$$\Lambda_i := \Big(1 - P_t(\varepsilon, S_{i+1}(z,\varepsilon))\Big) \prod_{j=2}^{i} P_t(\varepsilon, S_j(z,\varepsilon)) \to 0, \quad i=2,\dots,m-1 \quad (m\geq 3)$$

and

$$\Lambda_m := \prod_{j=2}^{m} P_t(\varepsilon, S_j(z,\varepsilon)) \to 0.$$

Hence, if $x \in A_1$ then $\lim_{z\to x, \varepsilon\to 0^+} \nabla_x S(z,\varepsilon) = \nabla g_1(x)$.

2) Let now $x \in A_i$ for some $i \in \{2, 3, \dots, m-1\}$, $m \geq 3$. By (C.22) and (C.23), it follows immediately that

$$\Lambda_i \to 1 \quad \text{as } z \to x,\ \varepsilon \to 0^+.$$

Further, we shall show that for any k, $k \in \{1, 2, \dots, m\}$, $k \neq i$, it holds for any $z \to x$ and $\varepsilon \to 0^+$ that $\Lambda_k \to 0$.
Indeed, the relation (C.23) implies

$$\lim_{z\to x, \varepsilon\to 0^+} P_t(\varepsilon, S_{k+1}(z,\varepsilon)) = 1 \quad \forall k = 1, \dots, i-1.$$

Therefore,

$$\Lambda_1 = 1 - P_t(\varepsilon, S_2(z,\varepsilon)) \to 0$$

and

$$\Lambda_k = (1 - P_t(\varepsilon, S_{k+1}(z,\varepsilon))) \prod_{j=2}^{k} P_t(\varepsilon, S_j(z,\varepsilon)) \to 0 \quad \forall k = 2, \dots, i-1 \tag{C.27}$$

as $z \to x$ and $\varepsilon \to 0^+$. Altogether, $\Lambda_k \to 0$ for all $k = 1, \dots, i-1$.
Let now $k \in \{i+1, i+2, \dots, m-1\}$. According to (C.22), the $(i+1)-$ multiplier $P_t(\varepsilon, S_{i+1}(z,\varepsilon))$ in (C.27) goes to zero and consequently, $\Lambda_k \to 0$.

Further, since $(i+1) \in \{3, 4, \ldots, m\}$ and $P_t(\varepsilon, S_{i+1}(z, \varepsilon))$ goes to zero, it follows that

$$\Lambda_m = \prod_{j=2}^{m} P_t(\varepsilon, S_j(z, \varepsilon)) \to 0 \quad \text{as } z \to x,\ \varepsilon \to 0^+.$$

In this way, we have proved that $\Lambda_k \to 0$ for every $k = 1, \ldots, m$, $k \neq i$, and therefore, if $x \in A_i$ then $\lim\limits_{z \to x, \varepsilon \to 0^+} \nabla_x S(z, \varepsilon) \to \nabla g_i(x)$.

3) Finally, let $x \in A_m$. From Lemma C.4 b),

$$\lim_{z \to x, \varepsilon \to 0^+} P_t(\varepsilon, S_i(z, \varepsilon)) = 1 \quad i = 2, \ldots, m.$$

Hence,

$$\Lambda_1 = 1 - P_t(\varepsilon, S_2(z, \varepsilon)) \to 0 \quad \text{and} \quad \Lambda_m \to 1.$$

Clearly, we can also write

$$P_t(\varepsilon, S_{i+1}(z, \varepsilon)) \to 1 \quad \forall i = 2, \ldots, m-1$$

and consequently,

$$\Lambda_i \to 0 \quad \text{for all } i = 2, \ldots, m-1.$$

Therefore, we have proved that if $x \in A_m$ then $\lim\limits_{z \to x, \varepsilon \to 0^+} \nabla_x S(z, \varepsilon) \to \nabla g_m(x)$.

Collecting all cases, the proof of the theorem is complete. □

Remark C.3 Note that if $x \in \mathbb{R}^n$ is a point such that $g_i(x) = g_j(x)$ for some i and j, $i \neq j$, then for any sequences $\{x_k\} \subset \mathbb{R}^n$, $\{\varepsilon_k\} \subset \mathbb{R}_{++}$ such that $x_k \to x$ and $\varepsilon_k \to 0^+$ we have

$$\lim_{k \to \infty} \nabla_x S(x_k, \varepsilon_k) \in \partial f(x).$$

C.3 Existence and Approximation Results for Variational Inequalities

C.3.1 Existence Results for Linear VIs

Let $(V, \|.\|, \langle ., . \rangle)$ be a real Hilbert space. Let $\lambda \in V^*$ be a continuous linear form, $K \subset V$ a nonvoid closed, convex set, and $\beta : V \times V \to \mathbb{R}$ be a continuous bilinear form, not necessarily symmetric. With these data given we consider the subsequent variational inequality (P): Find $\hat{u} \in K$ such that

$$\beta(\hat{u}, v - \hat{u}) \geq \lambda(v - \hat{u}) \quad \forall v \in K .$$

We require that β is positive semidefinite, i.e. $\beta(v, v) \geq 0$ for all $v \in V$. Hence the closed set

$$\mathscr{N} := \{u \in V : \beta(u, u) = 0\}$$

is a (generally nontrivial) subspace, as it is seen as follows. Clearly, $\mathbb{R}\mathscr{N} \subseteq \mathscr{N}$. The symmetric bilinear form

$$\beta^{\text{symm}}(u, v) := \frac{1}{2}\{\beta(u, v) + \beta(v, u)\}$$

satisfies the Schwarz inequality. Therefore for any $u, v \in \mathscr{N}$

$$0 \leq \beta(u + v, u + v) = 2\,\beta^{\text{symm}}(u, v) \leq 0 ,$$

hence $u + v \in \mathscr{N}$. This also shows that

$$\mathscr{N} = \{u \in V : \beta^{\text{symm}}(u, .) \equiv 0\} .$$

Although the solution of (P) generally depends nonlinearly on the datum λ, the solution set of (P) is convex. This is an easy consequence of the following useful characterization.

Lemma C.5 *Let $\hat{u} \in K$. Then $\hat{u}$ solves (P), if and only if*

$$\beta(v, \hat{u} - v) \leq \lambda(\hat{u} - v) \quad \forall v \in K .$$

Proof To show the "$\leq$" inequality, use positive semidefiniteness of β and obtain

$$\beta(v, \hat{u} - v) \leq -\beta(\hat{u}, v - \hat{u}) \leq -\lambda(v - \hat{u}) \quad \forall v \in K .$$

To show conversely (P), for any $v \in K$ take $w_t := \hat{u} + t(v - \hat{u})$, $t \in (0, 1)$. Then $w_t \in K$ and

$$\beta(w_t, \hat{u} - w_t) \leq \lambda(\hat{u} - w_t) ,$$

hence

$$\beta(w_t, v - \hat{u}) \geq \lambda(v - \hat{u}) .$$

Letting $t \to 0$, the inequality of (P) follows. □

To obtain existence results one needs further assumptions. In accordance to the Signorini problem in Sect. 5.1 , we assume that β is semicoercive in the sense that β should satisfy a Gårding inequality:

$$(G) \quad \beta(v, v) + \langle Cv, v\rangle \ \geq \ c \ \|v\|^2 \quad \text{for all } v \in V$$

with some real number $c > 0$ and a compact linear operator $C : V \to V^*$. If (G) holds with $C = 0$, then β is usually termed coercive or elliptic. In the coercive case, the Lions - Stampacchia theorem that extends the Lax - Milgram lemma guarantees unique solvability of (P) for each $\lambda \in V^*$:

Theorem C.6 (Lions - Stampacchia Theorem) *Let $\beta : V \times V \to \mathbb{R}$ be a continuous elliptic bilinear form on the Hilbert space V. Moreover, let $K \neq \emptyset$, convex, closed $\subset V$, $\lambda \in V^*$. Then the variational inequality (P) has a unique solution $\hat{u}$. Moreover, the mapping $\lambda \mapsto \hat{u}$ is Lipschitz continuous.*

Proof We give a sketch of the proof divided in three steps.

1. Let u_i be solutions to the data λ_i. Then choose $v = u_2$, respectively $v = u_1$ in (P), sum up and obtain $\beta(u_1 - u_2, u_1 - u_2) \leq (\lambda_1 - \lambda_2)(u_1 - u_2)$. Since β is elliptic, $c \|u_1 - u_2\|^2 \leq \|\lambda_1 - \lambda_2\|_{V^*} \|u_1 - u_2\|$, what shows Lipschitz continuity and uniqueness.
2. Existence in the case of symmetric β
   Method: Minimize "energy" $J(v) = \dfrac{1}{2}\beta(v, v) - \lambda(v)$, since minimization problem on K is equivalent to (P) in the symmetric case.
   Consider minimizing sequence $\{u_n\}$; this is a Cauchy sequence, what can be seen by the parallelogram rule. Then $u_n \to \hat{u} \in K$, $J(u_n) \to J(\hat{u})$, since J is continuous.
3. Existence in the general case.
   Let in addition σ a symmetric elliptic bilinear form, e. g. $\sigma(v, w) = \langle v, w\rangle$ or $\sigma(v, w) = \beta^{\text{symm}}(v, w) = \frac{1}{2}\Big(\beta(v, w) + \beta(w, v)\Big)$.
   For fixed $u \in K$, $\rho > 0$ $\exists^1 w \in K$ (according to the symmetric case above) such that

   $$\sigma(w, v - w) \geq \sigma(u, v - w) - \rho[\beta(u, v - w) - \lambda(v - w)] \ \forall v \in K\,.$$

   Hence $u \mapsto w = S(u)$ gives a mapping S: $K \to K$. Clearly $\hat{u}$ solves (P), if and only if $\hat{u} = S(\hat{u})$. Choose now $\rho > 0$ sufficiently small, such that S is a contraction that gives the fixed point $\hat{u}$.
   For details see e.g. the monograph of Kinderlehrer and Stampacchia [267, theorem II.2.19]. □

Coercivity is also necessary for well-posedness; this is clarified in the following

Proposition C.4 *Let $A : H \to H$ be a linear continuous operator. Suppose that the bilinear form*

$$\alpha(x, y) = < Ax, y >$$

is symmetric and positive semidefinite. If A is bijective, then α is coercive.

Proof By Banach's inverse mapping theorem, A^{-1} is continuous. Then for any fixed $x \in H$ with $\|x\| = 1$ we have

$$\|A^{-1}\| \sup_{\|y\|=1} |\langle Ax, y\rangle| \geq \left\| A^{-1} \frac{Ax}{\|Ax\|} \right\| \left| \left\langle Ax, \frac{Ax}{\|Ax\|} \right\rangle \right| = \frac{\|x\|}{\|Ax\|} \cdot \frac{\|Ax\|^2}{\|Ax\|} = 1$$

and hence

$$\inf_{||x||=1} \sup_{||y||=1} |\langle Ax, y\rangle| \;\geq\; \frac{1}{||A^{-1}||}.$$

Thus by the Cauchy-Schwarz inequality applied to the positive semidefinite and symmetric bilinear form a,

$$\begin{aligned} \alpha(x, x) &\geq \sup_{y \neq 0} \frac{|\alpha(x, y)|}{\alpha(y, y)} \\ &\geq \frac{1}{\|A\|} \, [\, \sup_{y \neq 0} \frac{|\alpha(x, y)|}{\|y\|} \,]^2 \\ &\geq \frac{1}{\|A\| \, \|A^{-1}\|^2} \|x\|^2 \end{aligned}$$

what proves the asssertion. □

Thus for the general semicoercive bilinear form β under study, we need extra conditions for the specific $\lambda \in V^*$ to yield existence of solutions to (P). Referring to [28, 201] a sufficient condition for solvability is the recession condition $\mathscr{C} = -\mathscr{C}$, where the convex cone $\mathscr{C}$ is given by

$$\mathscr{C} := \{w \in \text{ac}\, K \cap \mathscr{N} : \; \beta(v, w) \leq \lambda(w) \, \forall v \in K\}$$

and with some fixed $k_0 \in K$

$$\text{ac}\, K := \bigcap_{t>0} t(K - k_0)$$

denotes the *asymptotic cone* or the *recession cone* of K. A stronger condition is that there exists some $v_0 \in K$ such that

$$\lambda(w) \ < \ \beta(v_0, w) \quad \forall w \in \text{ac}\, K \cap \mathscr{N} \setminus \{0\}\,,$$

since this latter condition obviously implies that $\mathscr{C} = \{0\}$. In the case $0 \in K$, this latter condition simplifies to

$$\lambda(w) \ < \ 0 \quad \forall w \in \text{ac}\, K \cap \mathscr{N} \setminus \{0\}\,,$$

which can already be found with Fichera [179] and Stampacchia [388].

In the Signorini problem discussed in Sect. 5.1, see (5.6), K is already a convex cone (with vertex at zero) and the set $K \cap \mathscr{N}$ coincides with the set of constant functions that are nonpositive on Γ_S, thus nonpositive throughout $\mathbb{R}^d$. Therefore the recession condition of Fichera–Stampacchia is here simply

$$\ell(\underline{1}) = \int_{\Gamma_N} g\, ds + \int_{\Gamma_S} h\, ds > 0\,. \tag{C.28}$$

This latter condition also guarantees the uniqueness of the solution of the Neumann–Signorini problem, where Γ_D may be empty.

C.3.2 Approximation of Linear VIs

In this subsection we present an approximation result, which is based on [213], for linear variational inequalities in Hilbert space. So we have the same setting as in the previous section and consider the problem (P), but now for simplicity K is assumed to be a nonvoid closed, convex cone (with vertex at zero).

To describe the approximation of our variational problem (P) we suppose that we are given a positive parameter h converging to 0 and a family $\{V^h\}_{h>0}$ of closed finite dimensional subspaces contained in V. In addition we have a family $\{K^h\}_{h>0}$ of closed convex nonempty cones of V^h. These sets K^h should approximate the given set K. However, piecewise polynomial interpolation - except piecewise linear interpolation - does not preserve order, thus generally K^h cannot assumed to be contained in K. To cope with this difficulty of nonconforming approximation we follow the discretization theory of Glowinski [199, Chapter 1], which refines the set convergence notion due to Mosco [309] (see also [6] for definition and further study) and independently to Stummel, see [414] and introduce the following two hypotheses (H1) and (H2):

(H1) If for some sequence $\{h_j\}_{j\in\mathbb{N}}$ with $h_j \to 0$, $v^{h_j} \in K^{h_j}$ $(j \in \mathbb{N})$ and v^{h_j} converges weakly to $v \in V$ $(j \to \infty)$, then $v \in K$.

(H2) There exist a subset $M \subset V$ such that $\overline{M} = K$ and mappings $r^h : M \to V^h$ with the property that, for each $v \in M$, $r^h v \to v$ $(h \to 0)$ and $r^h v \in K^h$ for all $h \leq h_0(v)$ for some $h_0(v) > 0$.

Thus we approximate the problem (P) by the following variational inequality (P^h): Find $u^h \in K^h$ such that

$$\beta(u^h, v^h - u^h) \geq \lambda(v^h - u^h) \quad \forall v^h \in K^h \,.$$

By the existence theory in the infinite dimensional case, also solutions u^h to these finite dimensional problems exist.

Note that in most computations, however, it will be necessary to replace also β and λ by some approximations β^h and λ^h , defined by a numerical integration rule which is used in the finite element, respectively boundary element discretization. Since there is nothing new compared to the case of linear elliptic boundary value problems and variational equalities, we do not discuss this aspect here.

Now we can state and prove our basic convergence result.

Theorem C.7 *Let β , λ, K, and $\{K^h\}_h$ satisfy the conditions (G), (H1) and (H2). If the solution $\hat{u}$ of (P) is unique, then* $\lim_{h\to 0} \|u^h - \hat{u}\| = 0$ *holds.*

Proof We divide the proof in five parts. We first show a priori estimates for $\{u^h\}_h$, before we can establish the convergence results.

1) $|\,.\,|$ *– estimate for* $\{u^h\}$.
Fix $w_0 \in M$, let $w^h := r^h w_0 \in K^h$ for $0 < h = h_0 := h_0(w_0)$. Then we have $\lim \|w^h - w_0\| = 0$, and with u^h, a solution of (P^h)

$$|u^h|^2 = \beta(u^h, u^h) \leq c_0 + c_1\|u^h\| + \lambda(u^h) \tag{C.29}$$

$$\leq c_0 + c_2\|u^h\| \,. \tag{C.30}$$

Here and in the following $c_0, c_1, c_2, \ldots$ are generic positive constants. Moreover, by positive semidefiniteness,

$$\beta(w^h, u^h) - \lambda(u^h) \leq \beta(w^h, w^h) - \lambda(w^h) \leq c_3 \,. \tag{C.31}$$

2) *Norm-boundedness of* $\{u^h\}$.
Here we modify a contradiction argument, which in the existence theory of semicoercive variational inequalities goes back to Fichera [179] and Stampacchia [388]. We assume there exists a subsequence $\{u_\ell\}_{\ell\in\mathbb{N}} := \{u^{h_\ell}\}$ such that $\|u_\ell\| \to +\infty\,(\ell \to \infty)$. With $y_\ell := \|u_\ell\|^{-1}u_\ell$ in the Hilbert space V, we can extract a subsequence, again denoted by $\{y_\ell\}$, that converges weakly to some $y \in V$. In virtue of (C.30) , we get

$$|y_\ell|^2\ \|u_\ell\| \leq c_4 \,.$$

Thus we have $|y_\ell| \to 0$. [Assume not. Then for a subsequence $|y_{\ell_k}| \geq c_5 > 0$ and hence

$$|y_{\ell_k}| \leq \frac{c_4}{c_5 \|u_{\ell_k}\|} ,$$

what by $\|u_{\ell_k}\| \to +\infty$ leads to a contradiction.]
Since $| \, . \, |$ is continuous and sublinear, hence weakly sequentially lower semicontinuous, we obtain $y \in \mathscr{N}$. Since $\{u_\ell\}$ belongs to the cone K^{h_ℓ}, (H1) implies that $y \in K$, too.
We claim that $y = 0$. From (C.31) we obtain

$$\beta(w_\ell, y_\ell) - \lambda(y_\ell) \leq \frac{c_3}{\|u_\ell\|} ,$$

hence

$$\beta(w_0, y) \leq \lambda(y) \quad \forall w_0 \in M , \tag{C.32}$$

which extends to $\overline{M} = K$ by continuity. Moreover, for the solution $\hat{u}$ we have by the characterization lemma C.5

$$\beta(u, \hat{u}) - \beta(u, u) \leq -\lambda(u - \hat{u}) \quad \forall u \in K . \tag{C.33}$$

From (C.32) and (C.33) it follows for any $t > 0$

$$\beta(u, \hat{u} + tu) - \beta(u, u) \leq \lambda(\hat{u} + tu) - \lambda(u) \quad \forall u \in K .$$

Hence by the characterization lemma C.5, $\hat{u} + ty$ solves (P), and by uniqueness, $y = 0$ follows.

Now we use (G). By compactness of C, for some subsequence $\lim_{k\to\infty} \|Cy_{\ell_k}\| = 0$ and

$$c\|y_{\ell_k}\|^2 \leq \beta(y_{\ell_k}, y_{\ell_k}) + \langle Cy_{\ell_k}, y_{\ell_k} \rangle ,$$

hence $y_{\ell_k} \to 0$. However, $\|y_{\ell_k}\| = 1$, and a contradiction is reached proving the boundedness of $\{u^h\}$.

3) *Any weak limit point u^* of $\{u^h\}$ solves (P) .*
By the preceding step, there exists a subsequence, again denoted by $\{u_\ell\}$ such that $u_\ell \rightharpoonup u^*$. By (H1) , u^* belongs to K. To show that u^* solves (P), take $v \in M$ arbitrarily. Then $v_\ell := r^{h_\ell} v$ converges strongly to v , and for $h_\ell \leq h_0(v)$ we have

$$\beta(u_\ell, v_\ell - u_\ell) \geq \lambda(v_\ell - u_\ell) .$$

Since β is positive semidefinite,

$$\beta(v_\ell, u_\ell - v_\ell) \leq \lambda(u_\ell - v_\ell) .$$

Hence in the limit

$$\beta(v, u^* - v) \leq \lambda(u^* - v) \,.$$

This inequality extends by continuity to $\overline{M} = K$. Finally by the characterization lemma C.5, we conclude for any $v \in K$

$$\beta(u^*, v - u^*) \geq \lambda(v - u^*) \,.$$

4) *Convergence with respect to* $| \,.\, |$.
Here we use an argument due to Glowinski [199, Chapter 1]. Since the solution $\hat{u}$ of (P) is unique, the entire family $\{u^h\}$ converges weakly to $\hat{u}$. Now take $v \in M$ arbitrarily. Then $v^h := r^h v$ converges strongly to v , and for $h_\ell \leq h_0(v)$ we have

$$\begin{aligned}\beta(u^h - \hat{u}, u^h - \hat{u}) &= \beta(u^h, v^h - \hat{u}) - \beta(u^h, v^h - u^h) - \beta(\hat{u}, u^h - \hat{u}) \\ &\leq c_6 \, \|v^h - \hat{u}\| + \lambda(u^h - v^h) - \beta(\hat{u}, u^h - \hat{u}) \,.\end{aligned}$$

Hence in the limit, for any $v \in M$,

$$0 \leq \limsup_{h \to 0} |u^h - \hat{u}|^2 \ \leq c_6 \, \|v - \hat{u}\| + \lambda(\hat{u} - v) \,.$$

The obtained inequality extends to K by density and continuity. Finally, the choice $v = \hat{u}$ leads to the desired $|.|$ – convergence.

5) *Convergence with respect to* $\|.\|$.
Assume there exists a sequence $\{u_\ell\}$ such that u_ℓ is a solution to (P^{h_ℓ}) and $\|u_\ell - \hat{u}\| \ \geq \ \delta \ > \ 0$. By part (2), $\|u_\ell - \hat{u}\|$ is bounded and therefore we can extract a subsequence, again denoted by $\{u_\ell\}$ such that $u_\ell - \hat{u}$ converges weakly to some $w \in V$. By part (3), $\hat{u} + w$ solves (P), hence by uniqueness $w = 0_V$. Now we again use (G). By compactness of C, we can extract a subsequence, again denoted by $\{u_\ell\}$ such that $C\, u_\ell$ converges strongly to $C\, \hat{u}$ and moreover by part (4), $|u_\ell - \hat{u}| \to 0$ $(\ell \to \infty)$. Therefore by (G), $\|u_\ell - \hat{u}\| \to 0$ $(\ell \to \infty)$, and a contradiction is reached. □

For the more general approximation of general convex closed sets (instead of cones) we refer to [213].

C.3.3 Pseudomonotone VIs—Existence Result

The Lions-Stampacchia theorem was substantially extended by Brézis to a very large class of (non-linear) operators, called *pseudomonotone* operators in [64, Theorem 24], see also [438, section 27.2]. With the symbol $\rightharpoonup$ denoting weak convergence on V, $T \ : \ V \to V^*$ is called *pseudomonotone*, if it is bounded and if for any sequence $\{u_n\}_{n\in\mathbb{N}}$ in V,

$$u_n \rightharpoonup u \;\text{ and }\; \liminf_{n\to\infty} \langle T(u_n), u - u_n\rangle \ \geq 0,$$

imply

$$\langle T(u), v - u\rangle \ \geq \ \limsup_{n\to\infty} \ \langle T(u_n),\ v - u_n\rangle,\ \forall v \in V.$$

Such a pseudomonotone operator $T : K \subset V \to V^*$ as defined above gives rise to the bifunction $\psi : K \times K \to \mathbb{R}$ via $\psi(u, v) := \langle T(u), v - u\rangle$. Then ψ is pseudomonotone (PM) in the sense that for any sequence $\{u_n\}$ in K,

$$u_n \rightharpoonup u \quad \text{and} \quad \liminf_{n\to\infty} \psi(u_n, u) \geq 0$$

imply that for any $v \in K$ there holds

$$\psi(u, v) \geq \limsup_{n\to\infty} \ \psi(u_n, v)\,.$$

A simple example of a pseudomonotone bifunction (not represented by an operator) is $\psi(u, v) = g(v) - g(u)$, where g is a weakly lower semicontinuous function.

Let T be weakly continuous on subsets $\mathrm{F} \cap K$ of K, where F is a finite dimensional subspace of V. Then the function $\psi(\cdot, v)$ becomes upper semicontinuous on each finite dimensional part $\mathrm{F} \cap K$ of K. Here, we assume only that $\psi(u, u) \geq 0$ and $\psi(u, \cdot)$ is convex for any $u \in K$; thus we do not require that $\psi(u, \cdot)$ is linear-affine. This is a suitable extension for the treatment of hemivariational inequalities to follow. In this setting we have the following existence result from [212, Theorem 3].

Theorem C.8 *Let K be a closed convex nonvoid subset of a reflexive Banach space V. Let the bifunction $\psi : K \times K \to \mathbb{R}$ be pseudomonotone with $\psi(\cdot, v)$ upper semicontinuous on each finite dimensional part of K, $\psi(u, u) \geq 0$ and $\psi(u, \cdot)$ convex for any $u \in K$. Suppose that for some $u_0 \in K$, ψ satisfies the coercivity condition*

$$(CC) \quad \frac{\psi(u, u_0)}{\|u - u_0\|} \to -\infty \quad \text{as } u \in K,\ \|u\| \to \infty\,.$$

Then for any $f \in V^*$ the variational inequality $VI(\psi, f, K)$ admits a solution, i.e. there exists $u \in K$ such that

$$\psi(u, v) \geq \langle f, v - u\rangle,\ \forall v \in K\,. \tag{C.34}$$

C.3.4 Mosco Convergence, Approximation of Pseudomonotone VIs

In this subsection we present an approximation procedure for pseudomonotone variational inequalities, where the given data (ψ, f, K) of the variational inequality are approximated by bifunctions ψ_t, linear continuous functionals f_t and closed convex sets K_t, respectively, indexed by a directed set T. While K is contained in a general reflexive Banach space V, K_t is a subset of a subspace V_t of V. For the approximation of K by K_t we employ Mosco convergence, since we do not assume that K_t is a subset of K. We provide a general approximation result, which with finite-dimensional subspaces V_t of V can be considered as an abstract convergence result for the Galerkin method for the solution of $VI(\psi, f, K)$. Our approximation result includes also the existence of solutions to the approximate $VI(\psi_t, f_t, K_t)$ under an appropriate coerciveness condition.

We assume the following hypotheses:

(H1) If $\{v_{t'}\}_{t'\in T'}$ weakly converges to v in V, $v_{t'} \in K_{t'}$ $(t' \in T')$ for a subnet $\{K_{t'}\}_{t'\in T'}$ of the net $\{K_t\}_{t\in T}$, then $v \in K$.
(H2) For any $v \in K$ and any $t \in T$ there exists $v_t \in K_t$ such that $v_t \to v$ (strongly) in V.
(H3) ψ_t is pseudomonotone for any $t \in T$.
(H4) $f_t \to f$ in V^*.
(H5) For any nets $\{u_t\}$ and $\{v_t\}$ such that $u_t \in K_t$, $v_t \in K_t$, $u_t \rightharpoonup u$, and $v_t \to v$ in V it follows that

$$\liminf_{t\in T} \; \psi_t(u_t, v_t) \le \psi(u, v)\,.$$

(H6) The family $\{-\psi_t\}$ is uniformly bounded from below in the sense that there exist constants $c > 0$, $d, d_0 \in \mathbb{R}$ and $\alpha > 1$ (independent of $t \in T$) such that for some $w_t \in K_t$ with $w_t \to w$ there holds

$$-\psi_t(u_t, w_t) \ge c\|u_t\|_V^{\alpha} + d\|u_t\|_V + d_0, \quad \forall u_t \in K_t,\ \forall t \in T\,.$$

Remark C.4 The hypotheses (H1) and (H2) describe the Mosco convergence [6] of the family $\{K_t\}$ to K.

Remark C.5 Without loss of generality we can assume that $0 \in K \cap \{\cap_{t\in T} K_t\}$. Indeed, since K is nonvoid, by (H2) for any $w \in K$ there exist $w_t \in K_t$ such that $w_t \to w$. Then we consider the transformations $v \in K \mapsto v - w \in \tilde{K} := K - w$; $v_t \in K_t \mapsto v_t - w_t \in \tilde{K}_t := K_t - w_t$. Thus, $\tilde{K}_t$ Mosco converges to $\tilde{K}$ and the hypotheses (H3), (H5), (H6) hold for the transformed bifunctions $\tilde{\psi}$, $\tilde{\psi}_t$ as well.

Under these hypotheses we have the following basic convergence result.

Theorem C.9 (General Approximation Result) *Under conditions* (H1)–[(H6), *there exist solutions u_t to the approximate problem $VI(\psi_t, f_t, K_t)$ and the family $\{u_t\}$ is uniformly bounded in V. Moreover, there exists a subnet of $\{u_t\}$ that* *converges weakly in V to a solution of the problem $VI(\psi, f, K)$. Furthermore, any weak accumulation point of $\{u_t\}$ is a solution to the problem $VI(\psi, f, K)$.*

Proof Using (H3) and (H6), the existence of a solution u_t to $VI(\psi_t, f_t, K_t)$ follows from Theorem C.8. Inserting $v_t = 0$ in $VI(\psi_t, f_t, K_t)$ and using (H6) and (H4) we obtain

$$c\|u_t\|_V^\alpha + d\|u_t\|_V + d_0 \leq -\psi_t(u_t, 0) \leq \|f_t\|_{V^*}\|u_t\| \leq C\,\|f\|_{V^*}\,\|u_t\|_V,$$

what proves the norm boundedness of $\{u_t\}$. So we can extract a subnet of $\{u_t\}$ denoted by $\{u_{t'}\}_{t'\in T'}$ such that $u_{t'}$ converges weakly to u in V. By (H1), $u \in K$. Now, take an arbitrary $v \in K$. By (H2), there exists a net $\{v_t\}$ such that $v_t \in K_t$ and $v_t \to v$ in V. By (H4) and (H5), we get from $VI(\psi_t, f_t, K_t)$ that for any $v \in K$

$$\psi(u, v) \geq \liminf_{t'\in T'} \psi_{t'}(u_{t'}, v_{t'}) \geq \lim_{t'\in T'} \langle f_{t'}, v_{t'} - u_{t'}\rangle = \langle f, v - u\rangle$$

and consequently u is a solution to $VI(\psi, f, K)$. At the same time we have proved that any weak accumulation point of $\{u_t\}$ is a solution to $VI(\psi, f, K)$. This should be understood in the sense that every weak limit of any subnet of $\{u_t\}$ is a solution to $VI(\psi, f, K)$. □

Remark C.6 Without the coercivity hypothesis (H6) we get a stability result in the sense of Painlevé-Kuratowski set convergence that guarantees the inclusion

$$\limsup_{t\in T}\ \mathscr{S}(\psi_t, f_t, K_t) \subset \mathscr{S}(\psi, f, K)\,.$$

Here, the set $\mathscr{S}(\psi, f, K)$, depending on ψ, f and K, consists of all functions $u \in K$ satisfying the variational inequality $VI(\psi; f; K)$.

C.3.5 A Hemivariational Inequality as a Pseudomonotone VI

Let V be the classical Sobolev space $H^1(\Omega; \mathbb{R}^d)$, where $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ is a bounded domain with Lipschitz boundary $\partial\Omega$, and let $K \subseteq V$ be a nonempty closed, convex set specified later. Further, let the boundary $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_c \cup \bar{\Gamma}_F$ be composed of three mutually disjoint parts: a Dirichlet boundary Γ_D, a contact boundary Γ_c and a part Γ_F, where given external forces are applied. We also assume that the measure of Γ_D and Γ_c is strictly positive.

With γ we denote the trace operator from V into $L^2(\Gamma_c; \mathbb{R}^d)$, which is a linear continuous mapping. Hence, there exists a constant c_0 depending on Ω and Γ_c such that

$$\|\gamma \mathbf{v}\|_{L^2(\Gamma_c;\mathbb{R}^d)} \leq c_0\|\mathbf{v}\|_V, \quad \forall \mathbf{v} \in V. \tag{C.35}$$

Moreover, by the trace theorem [277, Theorem 6.10.5], γ is compact.

We introduce the linear elastic operator $A : V \to V^*$,

$$\langle A\mathbf{u}, \mathbf{v}\rangle = \int_{\Omega} \varepsilon(\mathbf{u}) : \sigma(\mathbf{v})\, dx, \tag{C.36}$$

where $\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ is the linearized strain tensor and $\sigma(\mathbf{v}) = C : \varepsilon(\mathbf{v})$ is the stress tensor. Here, C is the elasticity tensor with symmetric positive L^∞ coefficients. Hence, the linear elastic operator $A : V \to V^*$ is continuous, symmetric and due to the Korn's inequality coercive, i.e. there exists a constant $c_K > 0$ such that

$$\langle A\mathbf{v}, \mathbf{v}\rangle \geq c_K \, \|\mathbf{v}\|_V^2, \quad \forall \mathbf{v} \in V\,. \tag{C.37}$$

We define the linear form $f : V \to \mathbb{R}$ by

$$\langle f, \mathbf{v}\rangle = \int_{\Omega} \mathbf{f}_0^T \mathbf{v}\, dx + \int_{\Gamma_F} \mathbf{f}_1^T \mathbf{v}\, ds,$$

where $\mathbf{f}_0 \in L^2(\Omega; \mathbb{R}^d)$ are the prescribed body forces and $\mathbf{f}_1 \in L^2(\Gamma_F; \mathbb{R}^d)$ are the prescribed surface tractions on Γ_F.

In what follows we consider a function $j : \Gamma_c \times \mathbb{R}^d \to \mathbb{R}$ such that $j(\cdot, \xi) : \Gamma_c \to \mathbb{R}$ is measurable on Γ_c for all $\xi \in \mathbb{R}^d$ and $j(s, \cdot) : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz on $\mathbb{R}^d$ for almost all (a.a.) $s \in \Gamma_c$. Moreover, $j^0(s, \cdot\,;\,\cdot)$ stands for the generalized Clarke directional derivative [109] of $j(s, \cdot)$, as used in Sect. 5.3 and analyzed in Sect. C.2.1 above. With this data we consider the following hemivariational inequality: Find $u \in K$ such that

$$\langle A\mathbf{u}, \mathbf{v}-\mathbf{u}\rangle + \int_{\Gamma_c} j^0(s, \gamma\mathbf{u}(s); \gamma\mathbf{v}(s) - \gamma\mathbf{u}(s))\, ds \geq \langle f, \mathbf{v}-\mathbf{u}\rangle, \quad \forall \mathbf{v} \in K. \tag{C.38}$$

We denote by $\partial j(s, \xi) := \partial j(s, \cdot)(\xi)$ the Clarke generalized subdifferential of $j(s, \cdot)$ at the point ξ. We assume that there exist positive constants c_1 and c_2 such that for a.a. $s \in \Gamma_c$, all $\xi \in \mathbb{R}^d$ and for all $\eta \in \partial j(s, \xi)$ the following inequalities hold

(i) $|\eta| \leq c_1(1 + |\xi|)$;
(ii) $\eta^T \xi \geq -c_2|\xi|$.

This growth condition assures that the integral in (C.38) is well defined. Indeed, it follows from (i) and (ii) that for a.a. $s \in \Gamma_c$

$$\left|j^0(s, \xi; \varsigma)\right| = \left|\max_{\eta \in \partial j(s,\xi)} \eta^T \varsigma\right| \leq \max_{\eta \in \partial j(s,\xi)} |\eta|\,|\varsigma| \leq c_1(1 + |\xi|)|\varsigma|, \quad \forall\, \xi, \varsigma \in \mathbb{R}^d \tag{C.39}$$

and

$$j^0(s,\xi;-\xi) = \max_{\eta\in\partial j(s,\xi)} \eta^T(-\xi) \le c_2|\xi|, \quad \forall \xi \in \mathbb{R}^d. \tag{C.40}$$

The existence of a solution u to problem (P) can be derived from Theorem C.8. To this end we define the functional $\varphi : V \times V \to \mathbb{R}$ by

$$\varphi(\mathbf{u},\mathbf{v}) = \int_{\Gamma_c} j^0(s,\gamma\mathbf{u}(s);\gamma\mathbf{v}(s)-\gamma\mathbf{u}(s))\,ds, \quad \forall \mathbf{u},\mathbf{v}\in V. \tag{C.41}$$

Lemma C.6 ([220]) *The functional φ is pseudomonotone and satisfies*

$$\varphi(\mathbf{u},0) \le c_3\|\mathbf{u}\|_V, \quad \forall \mathbf{u}\in V \tag{C.42}$$

for some positive constant c_3.

Proof Let $\{\mathbf{u}_m\}$ be a sequence in V such that $\mathbf{u}_m \rightharpoonup \mathbf{u}$ in V as $m\to\infty$. Since γ is compact, it follows for a subsequence of $\{\gamma\mathbf{u}_m\}$, which we denote again by $\{\gamma\mathbf{u}_m\}$, that

$$\gamma\mathbf{u}_m \to \gamma\mathbf{u} \text{ in } L^2(\Gamma_c;\mathbb{R}^d) \text{ as } m\to\infty. \tag{C.43}$$

Now, we fix $\mathbf{v}\in V$ and show that

$$\limsup_{m\to\infty} \varphi(\mathbf{u}_m,\mathbf{v}) \le \varphi(\mathbf{u},\mathbf{v}). \tag{C.44}$$

We first observe that by (C.43) there exists a subsequence of $\{\gamma\mathbf{u}_m\}$, which we denote again by $\{\gamma\mathbf{u}_m\}$, such that

$$\gamma\mathbf{u}_m(s) \to \gamma\mathbf{u}(s) \quad \text{for a.a. } s\in\Gamma_c \tag{C.45}$$

and

$$|\gamma\mathbf{u}_m(s)| \le \kappa_0(s) \quad \text{for some nonnegative function } \kappa_0 \in L^2(\Gamma_c). \tag{C.46}$$

Using (C.39) and (C.46), it follows that

$$\begin{aligned} j^0(s,\gamma\mathbf{u}_m(s);\gamma\mathbf{v}(s)-\gamma\mathbf{u}_m(s)) &\le c_1(1+|\gamma\mathbf{u}_m(s)|)|\gamma\mathbf{v}(s)-\gamma\mathbf{u}_m(s)| \\ &\le c_1(1+\kappa_0(s))\big(|\gamma\mathbf{v}(s)|+\kappa_0(s)\big) \in L^1(\Gamma_c). \end{aligned}$$

From (C.45) and the upper semicontinuity of $j^0(s; \cdot, \cdot)$, we conclude by applying the Fatou lemma that

$$\begin{aligned}\limsup_{m\to\infty} \varphi(\mathbf{u}_m, \mathbf{v}) &= \limsup_{m\to\infty} \int_{\Gamma_c} j^0(s, \gamma\mathbf{u}_m(s); \gamma\mathbf{v}(s) - \gamma\mathbf{u}_m(s))\, ds \\ &\le \int_{\Gamma_c} \limsup_{m\to\infty} j^0(s, \gamma\mathbf{u}_m(s); \gamma\mathbf{v}(s) - \gamma\mathbf{u}_m(s))\, ds \\ &\le \int_{\Gamma_c} j^0(s, \gamma\mathbf{u}(s); \gamma\mathbf{v}(s) - \gamma\mathbf{u}(s))\, ds = \varphi(\mathbf{u}, \mathbf{v}) \quad \text{(C.47)}\end{aligned}$$

and thus, (C.44) is shown. Hence, the functional φ is pseudomonotone.

Furthermore, by (C.40) for any $\mathbf{u} \in V$ we can estimate

$$\begin{aligned}\varphi(\mathbf{u}, 0) &= \int_{\Gamma_c} j^0(s, \gamma\mathbf{u}(s); -\gamma\mathbf{u}(s))\, ds \le c_2 \int_{\Gamma_c} |\gamma\mathbf{u}(s)|\, ds \\ &\le c_2((\text{meas}\,(\Gamma_c))^{1/2} \|\gamma\mathbf{u}\|_{L^2(\Gamma_c;\mathbb{R}^d)} \overset{(C.35)}{\le} c_2((\text{meas}\,(\Gamma_c))^{1/2} c_0 \|\mathbf{u}\|_V,\end{aligned}$$

which implies (C.42). The proof of the lemma is thus complete. □

Since summation preserves pseudomonotonicity, see [216], the bifunction

$$\psi(\mathbf{u}, \mathbf{v}) := \langle A\mathbf{u}, \mathbf{v} - \mathbf{u}\rangle + \varphi(\mathbf{u}, \mathbf{v})$$

is pseudomonotone and satisfies the assumptions of Theorem C.8; in particular the coercivity condition (CC) holds, since by (C.37) and (C.42),

$$-\psi(\mathbf{u}, 0) \ge c_K \|\mathbf{u}\|_V^2 - c_3 \|\mathbf{u}\|_V\,.$$

We point out that uniqueness of the solution u to the hemivariational inequality (C.38) can be ensured for a large enough Korn constant c_K, see [332] for a proof of such an uniqueness result. We also refer to [326] for a similar uniqueness result to related nonconvex nonsmooth optimization problems.

Appendix D
Some Implementations for BEM

D.1 Symm's Equation on an Interval

$$Vu(x) := -\frac{1}{\pi} \int_{\Gamma} \ln|x-y| u(y) ds_y = f(x) \quad \text{for } x \in \Gamma = (-1, 1). \tag{D.1}$$

On a uniform mesh with meshsize h on Γ, $x_j = -1 + jh$, $\quad h = \frac{2}{n}$, $\quad j = 0, \ldots, n$, we take the space $\overline{V}_h$ of piecewise constant functions and perform the h-version of the Galerkin scheme for (D.1):

Find $u_h \in \overline{V}_h$, such that

$$a(u_h, v_h) := \langle Vu_h, v_h \rangle = \langle f, v_h \rangle \quad \forall v_h \in \overline{V}_h. \tag{D.2}$$

With the auxiliary function $F(x) = x^2 \ln|x|$ the Galerkin element a_{ij} becomes:

$$\begin{aligned} -\frac{1}{\pi} \int_{x_i}^{x_{i+1}} \int_{x_j}^{x_{j+1}} \ln|x-y| \, dy \, dx = \frac{1}{2\pi} \Big(& F(x_{i+1} - x_{j+1}) - F(x_{i+1} - x_j) \\ & - F(x_i - x_{j+1}) + F(x_i - x_j) \\ & + 3(x_{i+1} - x_i)(x_{j+1} - x_j) \Big). \end{aligned} \tag{D.3}$$

Exercise: Write a program,which implements the Galerkin scheme (D.2).

(i) Compute (D.2) for $f = 1$ and $f = x$. Note $u = \frac{1}{\ln 2} \frac{1}{\sqrt{1-x^2}}$ for $f = 1$.

(ii) Plot the solution for $n = 4$, $n = 8$ and $n = 16$.

(iii) Write for the program *unilap2.f90* a subroutine, which computes the energy norm of the solution of the Galerkin equations. Compute for $f = 1$ the error

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in the energy norm:

$$\|u - u_h\|_V = \sqrt{\|u\|_V^2 - \|u_h\|_V^2} = \sqrt{\frac{\pi}{\ln 2} - \|u_h\|_V^2}.$$

Compute for different n the errors in the energy norm and plot them in a double logarithmic scale.

D.2 The Dirichlet Problem in 2D

Now we consider the integral equation

$$\begin{aligned} Vu(x) &= (I + K)g(x), \qquad \text{for } x \in \Gamma \\ Vu(x) &:= -\frac{1}{\pi}\int_\Gamma \ln|x - y| u(y)\, ds_y \\ Ku(x) &:= -\frac{1}{\pi}\int_\Gamma \frac{\partial}{\partial n_y} \ln|x - y| u(y)\, ds_y = -\frac{1}{\pi}\int_\Gamma \frac{n_y(y - x)}{|x - y|^2} u(y)\, ds_y. \end{aligned} \tag{D.4}$$

As geometry we take the L-shape Γ with vertices $(0,0)$, $(0,0.5)$, $(-0.5,0.5)$, $(-0.5,-0.5)$, $(0.5,-0.5)$, $(0.5,0)$. We use a uniform mesh with length h and define there the space $\overline{V}_h$ of piecewise constant functions. Then the h-Version of the Galerkin BEM for the equation (D.4) reads: Find $u_h \in \overline{V}_h$, such that

$$a(u_h, v_h) := \langle Vu_h, v_h\rangle = \langle (I + K)g, v_h\rangle \quad \forall v_h \in \overline{V}_h.$$

Write a program, which implements this method. Use routines of `maiprogs`.

To describe the mesh use the following data structure

```
  integer:: ng
integer,parameter :: ngmax=2048
real(kind=dp):: rx(0:1,0:ngmax-1)
real(kind=dp):: rh(0:1,0:ngmax-1)
real(kind=dp):: rn(0:1,0:ngmax-1)
```

Here `ngmax` denotes the maximal number of elements and `ng` denotes the actual amount. `rx(0,i)` and `rx(1,i)` are the x- and y- components of a vertex of the element with the number `i`. `rh(.,i)` points from a vertex to the next vertex and `rn(.,i)` is the direction of the exterior normal of this element.

(i) Create a mesh generator, which creates for an arbitrary number of elements a uniform mesh.
(ii) Compute the Galerkin matrix for this data structure. Use the routine lapintegmd of liblap2.f90.

(iii) Compute the right hand side with Gaussian quadrature using lapid und lapkspot.
(iv) Test the last routine with $g \equiv 1$. (There holds $(I + K)1 \equiv 0$, why?)
(v) Solve the linear system with Gauss elimination.

The above mentioned subroutines can be downloaded from the home page of M. Maischak, http://people.brunel.ac.uk/~mastmmm/. Use

$$\underline{\langle Kg, v_h\rangle} = -\frac{1}{\pi}\int_\Gamma v_h(x)\left[\int_\Gamma g(y)\frac{\mathbf{n}_y\cdot(\mathbf{y}-\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2}ds_y\right]ds_x$$

$$\overset{\text{Fubini}}{=} -\frac{1}{\pi}\int_\Gamma g(y)\left[\int_\Gamma \frac{\mathbf{n}_y\cdot(\mathbf{y}-\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2}v_h(x)ds_x\right]ds_y = \underline{\langle g, K'v_h\rangle}.$$

Let $x \in \Gamma_i$, $y \in \Gamma_j$. If Γ_i and Γ_j are on the same edge then $\mathbf{n}_x\cdot(\mathbf{y}-\mathbf{x}) = 0$ and $\langle g, K'v_h\rangle = 0$. Otherwise we compute with the parametrisation of the vector $\mathbf{y}-\mathbf{x}$

$$\frac{1}{\pi}\int_{\Gamma_i}\frac{\mathbf{n}\cdot(\mathbf{y}-\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2}ds_y = \frac{1}{\pi}\int_{-1}^{1}\frac{\mathbf{n}\cdot(\mathbf{a}t+\mathbf{b})}{|\mathbf{a}t+\mathbf{b}|^2}dt$$

$$= \frac{1}{\pi}\left(\mathbf{n}\cdot\mathbf{a}\int_{-1}^{1}\frac{t}{\mathbf{a}^2t^2+2\mathbf{a}\cdot\mathbf{b}t+\mathbf{b}^2}dt + \mathbf{n}\cdot\mathbf{b}\int_{-1}^{1}\frac{1}{\mathbf{a}^2t^2+2\mathbf{a}\cdot\mathbf{b}t+\mathbf{b}^2}dt\right).$$

For the determination of these integrals, let $g_k^n(\alpha,\beta,\gamma) := \int_{-1}^{1} t^k(\alpha t^2+\beta t+\gamma)^n dt$ with $\alpha = \mathbf{a}^2$, $\beta = 2\mathbf{a}\cdot\mathbf{b}$, $\gamma = \mathbf{b}^2$. Now with $\triangle = 4\alpha\gamma - \beta^2$:

$$g_0^{-1} = \int_{-1}^{1}\frac{1}{\alpha t^2+\beta t+\gamma}dt = \frac{2}{\sqrt{\triangle}}\left(\arctan\frac{2\alpha+\beta}{\sqrt{\triangle}} - \arctan\frac{-2\alpha+\beta}{\sqrt{\triangle}}\right)$$

and

$$g_1^{-1} = \int_{-1}^{1}\frac{t}{\alpha t^2+\beta t+\gamma}dt = \frac{2\alpha t+\beta}{\triangle(\alpha t^2+\beta t+\gamma)} + \frac{2\alpha}{\triangle}\int_{-1}^{1}\frac{dt}{\alpha t^2+\beta t+\gamma}.$$

D.3 Symm's Equation on a Surface Piece

We can consider on a plane surface piece Γ

$$\frac{1}{4\pi}\int_{\Gamma}\frac{\psi(y)}{|x-y|}\,d\sigma_y \;=\; f(x)\,,\quad x\in\Gamma\,. \tag{D.5}$$

Decomposing Γ into rectangles $R_i\in\mathcal{T}_h$, we can choose our basis functions $\varphi_j(x)$ to be 1 only on one element,

$$\psi_h(x) \;=\; \sum_{i=1}^{N}\psi_j\varphi_j(x)\,,\qquad \varphi_j(x) \;=\; \begin{cases}1\,, & x\in R_j\,,\\ 0\,, & \text{else.}\end{cases}$$

We get the Galerkin scheme: Find $\psi_h\in\tilde{H}^{-\frac{1}{2}}(\Gamma_1)$ such that, for $k=1,\ldots,N$,

$$\begin{aligned}\int_{\Gamma}\frac{1}{4\pi}\int_{\Gamma}\frac{\psi_h(y)}{|x-y|}\,d\sigma_y\varphi_k(x)\,d\sigma_x &= \sum_{j=1}^{N}\psi_j\underbrace{\frac{1}{4\pi}\int_{\Gamma}\int_{\Gamma}\frac{\varphi_j(y)\varphi_k(x)}{|y-x|}\,d\sigma_y\,d\sigma_x}_{=:I}\\ &= \int_{\Gamma}f(x)\varphi_k(x)\,d\sigma_x\end{aligned} \tag{D.6}$$

By definition of φ_j and φ_k, I can be written as

$$I=\frac{1}{4\pi}\int_{R_k}\int_{R_j}\frac{1}{|x-y|}d\sigma_y d\sigma_x=\frac{1}{4\pi}\int_{R_k}\int_{R_j}\frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}}d\sigma_y d\sigma_x\,.$$

The inner integral can be calculated by transforming R_j to the reference square $\tilde{R}=[0,1]^2$:

$$\begin{aligned}R_j &:= \left\{(\xi,\eta):\; x_j\le\xi\le x_j+h_x\,,\; y_j\le\eta\le y_j+h_y\right\}\\ \xi &= x_j+h_x u\,,\quad 0\le u\le 1\\ \eta &= y_j+h_y v\,,\quad 0\le v\le 1\,.\end{aligned}$$

With fixed points $x=(a,b)$ and $y=(\xi,\eta)$, we get for the inner integral

$$\begin{aligned}&\int_{R_j}\frac{1}{|x-y|}\,d\sigma_y \;=\\ &=\int_{R_j}\frac{d\xi\,d\eta}{\sqrt{(\xi-a)^2+(\eta-b)^2}} \;=\; \int_{\tilde{R}}\frac{h_x\,du\,h_y\,dv}{\left((x_j+h_x u-a)^2\;+\;(y_j+h_y v-b)^2\right)^{\frac{1}{2}}}\end{aligned}$$

$$
\begin{aligned}
=&\alpha\left(\sinh^{-1}\left(\frac{h_y-\beta}{|\alpha|}\right)+\sinh^{-1}\left(\frac{\beta}{|\alpha|}\right)\right)+\beta\left(\sinh^{-1}\left(\frac{h_x-\alpha}{|\beta|}\right)+\sinh^{-1}\left(\frac{\alpha}{|\beta|}\right)\right)\\
&+(h_x-\alpha)\left(\sinh^{-1}\left(\frac{h_y-\beta}{|h_x-\alpha|}\right)+\sinh^{-1}\left(\frac{\beta}{|h_x-\alpha|}\right)\right)\\
&+(h_y-\beta)\left(\sinh^{-1}\left(\frac{h_x-\alpha}{|h_y-\beta|}\right)+\sinh^{-1}\left(\frac{\alpha}{|h_y-\beta|}\right)\right),\ \alpha=a-x_j,\ \beta=b-y_j.
\end{aligned}
$$

The outer integral can be approximated, e.g. by a 4-point quadrature formula that is exact for polynomials of degree ≤ 2: Let the quadrature nodes $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$, $\hat{x}_4$ be given by $\hat{x}_i = (0.5 \pm \frac{\sqrt{3}}{6}, 0.5 \pm \frac{\sqrt{3}}{6}), i = 1,2,3,4$ on $\tilde{R}$, then on the reference square, for the polynomial P there holds

$$
\int_{\tilde{R}} P(u,v)\,du\,dv = \frac{1}{4}\sum_{i=1}^{4} P(\hat{x}_i)\;.
$$

Note that the outer integration can also be performed analytically. This is implemented in the software package *maiprogs*, see [291].

For decomposing Γ we consider 4 different methods, firstly a uniform mesh with axis-parallel rectangles, secondly a graded mesh described by a tensor product mesh based on a 1-d graded mesh with grading constant β. And finally two adaptive strategies based on a two-level error estimator, one where we split each appropriate element into four equal sized elements and another one where we split the element horizontally or vertically into two parts or into four equal parts depending on the composition of the local error indicator (see Fig. D.1).

For the hierarchical error estimator we decompose every brick function ϕ_i^h associated with the element i and element size h into a set of three jump functions $\beta_{i,j}$ by uniformly refining the element i into four equal sized sub elements. Then there holds for $1 \leq j \leq 3$

$$
\beta_{i,j} = \sum_{l=1}^{4} c_{l,j}\phi_{i,l}^{h/2} = \phi_i^h + \sum_{l=2}^{4} \tilde{c}_{l,j}\phi_{i,l}^{h/2},
$$

where $\phi_{i,l}^{h/2}$ is the brick function on the sub element l to the element i. Further we define with the energynorm $\|\cdot\|_V$

$$
\begin{aligned}
\vartheta_{i,j} :=& \frac{\left|\langle V\psi_N - f, \beta_{i,j}\rangle\right|}{\|\beta_{i,j}\|_V} = \frac{\left|\left\langle V\psi_N - f, \phi_i^h + \sum_{l=2}^{4}\tilde{c}_{l,j}\phi_{i,l}^{h/2}\right\rangle\right|}{\|\beta_{i,j}\|_V}\\
=& \frac{\left|\left\langle V\psi_N - f, \sum_{l=2}^{4}\tilde{c}_{l,j}\phi_{i,l}^{h/2}\right\rangle\right|}{\|\beta_{i,j}\|_V}
\end{aligned}
$$

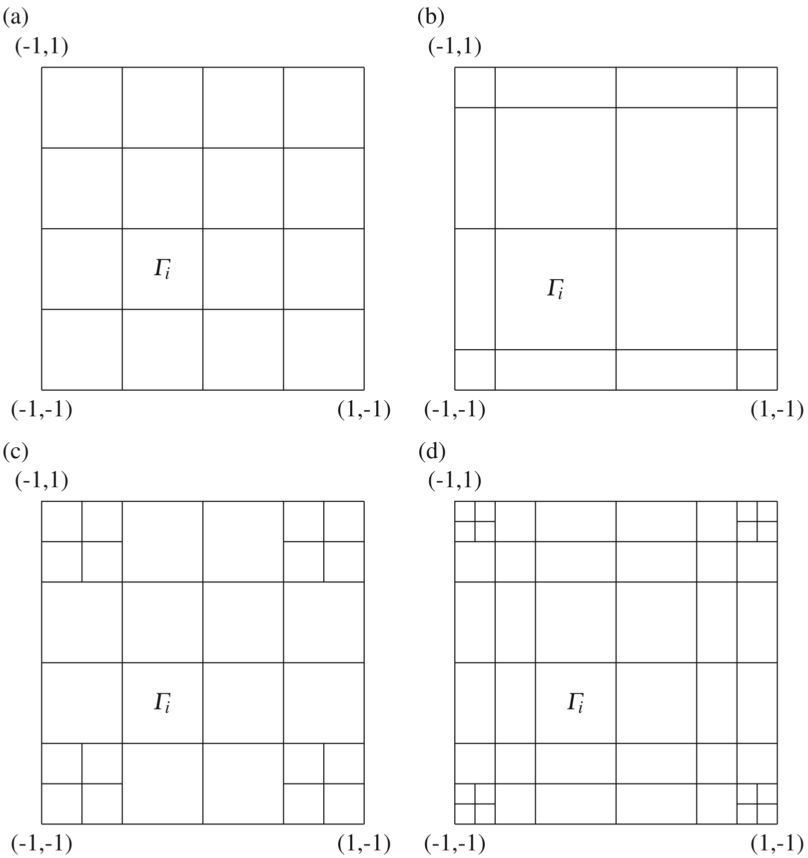


Fig. D.1 Different boundary decomposition techniques . (**a**) Uniform mesh decomposition. (**b**) Graded mesh decomposition. (**c**) Adaptive mesh decomposition strategies 1. (**d**) Adaptive mesh decomposition strategies 2

$$= \frac{\left|\left\langle V \sum_{i=1}^{N} \alpha_i \phi_i^h(x), \sum_{l=2}^{4} \tilde{c}_{l,j} \phi_{i,l}^{h/2} \right\rangle - \left\langle f, \sum_{l=2}^{4} \tilde{c}_{l,j} \phi_{i,l}^{h/2} \right\rangle\right|}{\left\| \beta_{i,j} \right\|_V}$$

Note that $\vartheta_{i,j}$ can be implemented efficiently when using the linearity of the scalar product and the reuse of old values. When making use of the Galerkin orthogonality as above we are able to reduce the computation time for $\vartheta_{i,j}$ by $\frac{1}{4}$. The local error indicator is now defined by

$$\vartheta_i := \sqrt{\vartheta_{i,1}^2 + \vartheta_{i,2}^2 + \vartheta_{i,3}^2}$$

For the adaptive strategy 2 we save in an additional vector if there holds $\vartheta_{i,2} \geq 1.5\vartheta_{i,1}$, $\vartheta_{i,1} \geq 1.5\vartheta_{i,2}$ or neither. If the element has been marked for refinement and the first condition is true split the element vertically into 2 equal sized rectangles, if the second condition is true then split horizontally into 2 equal sized rectangles and else into 4 equal sized rectangles. If the saturation assumption holds one can prove the efficiency and reliability of the error indicator $\eta = \|\vartheta\|_2$.

The numerical experiments were carried out by Lothar Banz on the Laptop Fujitsu Siemens Amilo M1439G with MatLab R2007. For solving the discrete linear system a CG algorithm is applied.

As we can see from Fig. D.3 the condition number of Galerkin matrix with an underlying uniform mesh behaves like $O(\sqrt{N})$.The condition numbers for the different mesh strategies are growing much faster than for the uniform mesh. For the graded meshes there holds the greater β is the faster the condition number grows.

The solution is obviously singular at the boundary of the boundary-domain Γ with strong singularities in the edges. The uniform mesh does not take the singular behavior into account which yields a lower convergence rate than for the different strategies. If we apply a graded mesh as described earlier we can improve the convergence rate in the energy norm of 0.25 for the uniform mesh to 0.73 for the graded mesh with $\beta = 4$ (see Fig. D.2). The more we take the singularity into account the better is the convergence rate. The local error is a product of the local

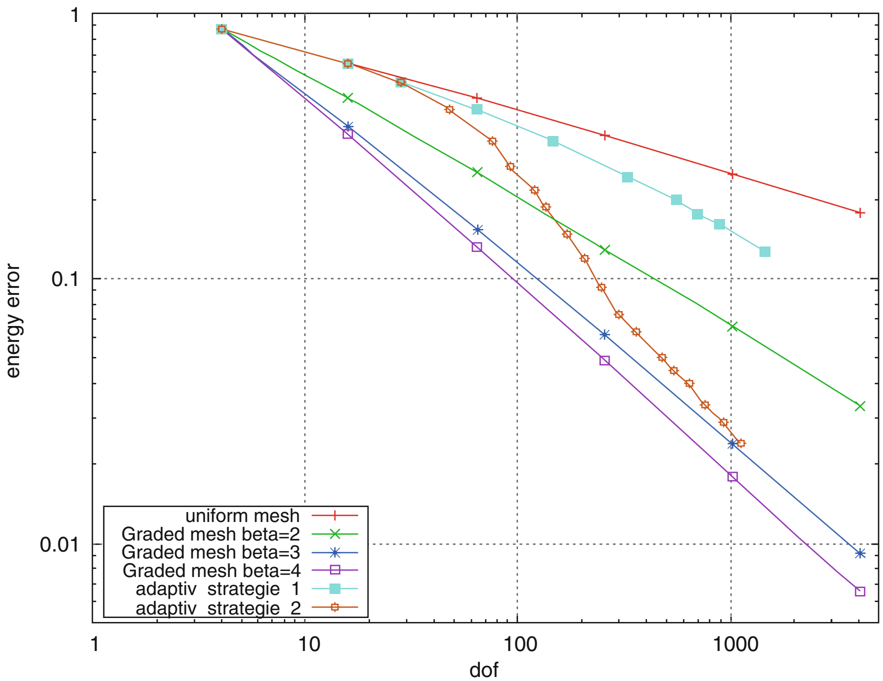


Fig. D.2 Energy error for different meshes

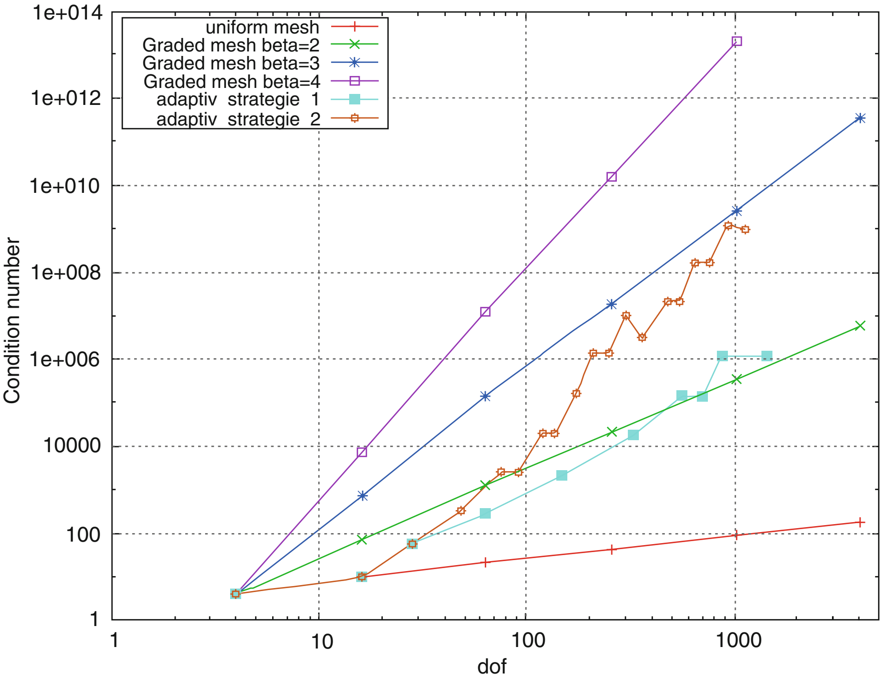


Fig. D.3 Condition number for different meshes

element size with the local error-behavior of the solution ψ. Therefore a reduction of the local element size will reduce the local error and thus the global error. As the reduction strategy of the first adaptive strategy is limited by no reduction or splitting into four equally sized elements we expect a convergence rate which is greater than of the uniform mesh and less than of a graded mesh. The second adaptive strategy has a broader reduction strategy and can therefore take the singular behavior better into account as the first strategy. However it is still worse than the graded strategy as it has no continuously, systematic, slow reduction of the elements close to the center of the boundary-domain.

D.3.1 Implementation of hp-BEM on Surfaces

In the following we report from [310]. The combination of geometric mesh refinement and h-p approximation with boundary element techniques gives a powerful tool for the approximate solution of boundary integral equations. In [235] an h-p Galerkin scheme for weakly singular and hypersingular integral equations on plane screens in $\mathbb{R}^3$ was analyzed and in [251] exponential convergence could be proved.

Although the singular integrals for plane surfaces in [235] can be evaluated analytically the assembly of the Galerkin matrix is extremely expensive. This becomes even worse if curved surfaces are considered and the entries of the Galerkin matrix have to be computed by a numerical quadrature rule. Here, an application of the h-p boundary element method has the advantage that the Galerkin error decays exponentially fast with the size of the Galerkin matrix, i.e. the number of Galerkin entries is kept low.

In this subsection we focus on the weakly singular integral equation on an open surface Γ, which corresponds to the direct single layer potential formulation of the Dirichlet problem for the homogeneous Laplace equation in $\mathbb{R}^3 \setminus \Gamma$. Our aim is now to define a quadrature rule which approximates the Galerkin entries exponentially fast with the number n of kernel evaluations. By increasing n at each h-p refinement step we may, hence, expect to preserve the exponential convergence of the Galerkin scheme while keeping the computational costs low.

The quadrature rules which we use are basically applications of Schwab's [372] graded quadrature rules for singular integrals to the inner *and* outer integrals in our Galerkin matrix. Schwab's rule can be applied directly to assemble collocation matrices or the inner integrals of the Galerkin entries. Based on the h-p approximation results in [17] and the interpolation property of e.g. Gaussian quadrature formulae exponential convergence could be proved [372]. For the outer integrals we need a similar rule which is designed to approximate the singularities of the single layer potential.

Let $G \subset \mathbb{R}^3$ be an open curved surface with parameter region $\Gamma = [-1, 1]^2$ and parameter function $\gamma : \Gamma \to G$. We assume that G satisfies a Lipschitz condition and that $\gamma(\partial\Gamma) = \partial G$. Let $V : \widetilde{H}^{-1/2}(G) \to H^{1/2}(G)$ be the single layer potential operator defined as

$$V\psi(x) = \frac{1}{4\pi} \int_\Gamma \frac{\psi(y)}{\|x - y\|} \, d\sigma_y \,.$$

Let $\widetilde{X}$ be a finite dimensional space of piecewise polynomial functions over Γ and let $X = \{\phi \circ \gamma^{-1} ; \phi \in \widetilde{X}\} \subset \widetilde{H}^{-1/2}(G)$. Let ψ_i, ψ_j be two basis functions in $\widetilde{X}$ and let Γ_i, Γ_j be elements in Γ with $\mathrm{supp}(\psi_i) \subset \overline{\Gamma}_i$ and $\mathrm{supp}(\psi_j) \subset \overline{\Gamma}_j$. Now, the entries of the Galerkin matrix and the right hand side vector are

$$\langle V\psi_j \circ \gamma^{-1}, \psi_i \circ \gamma^{-1} \rangle = \frac{1}{4\pi} \int_{\Gamma_i} \psi_i(x) \left(\int_{\Gamma_j} \frac{\psi_j(x)}{\|\gamma(x) - \gamma(y)\|} J(y) dy \right) J(x) dx \quad \text{(D.7)}$$

$$\langle f, \psi_i \circ \gamma^{-1} \rangle = \int_{\Gamma_i} \psi_i(x) f(\gamma(x)) J(x) dx \quad \text{(D.8)}$$

where $J(x) = \| \frac{\partial \gamma(x)}{\partial x_1} \times \frac{\partial \gamma(x)}{\partial x_2} \|$ and $\|.\|$ denotes the Euklidean norm in $\mathbb{R}^3$.

It could be shown in [251] for plane surfaces Γ that the h-p version of the boundary element method converges exponentially fast, see also Sect. 8.2. The

h-p meshes are geometrically graded towards $\partial\Gamma$ and the polynomial degrees of the test and trial functions in $x \in \Gamma$ are small if x is close to $\partial\Gamma$ and are increased perpendicular to $\partial\Gamma$ (for details see [235]). For plane surfaces, however, the integrals (D.7) can be evaluated analytically and the computational cost for the assembly of the Galerkin matrix grows only algebraically with the number N of h-p refinement steps, i.e. like $O(N^\alpha)$. We show that there is a quadrature rule which approximates the singular integrals (D.7) exponentially fast (with N) and which needs $O(N^\alpha)$ kernel evaluations ($\alpha \in \mathbb{N}$ fixed). Furthermore, we give (numerical) evidence that the h-p Galerkin method applied to (D.5) in combination with this quadrature rule leads to exponential convergence of the approximate solutions. To approximate both integrals we have to deal with point and edge singularites. The kernel $|\gamma(x) - \gamma(y)|^{-1}$ of the inner integral has obviously a point singularity at $y = x$ whereas the single layer potential has singular behaviour at $\gamma(\partial\,\mathrm{supp}(\psi))$.

For point singularities Schwab suggested the following rule for the approximation of the integral $\int_{\Gamma_0} \psi_0(x)\,dx$ where $\Gamma_0 = (0,1)^2$ and ψ_0 is singular at the origin: Given a fixed parameter $\sigma_1 \in (0,1)$ and an integer n one considers geometric subdivisions of Γ_0 into smaller rectangles $R_{l,k}$. We define $z_0 = 0$, $z_k = \sigma_1^{n-k}$, $1 \le k \le n$, and

$$\begin{aligned} R_{1,k} &= (z_{k-1}, z_k) \times (0, z_{k-1}) \quad \text{for } 2 \le k \le n\,, \\ R_{2,k} &= (z_{k-1}, z_k) \times (z_{k-1}, z_k) \quad \text{for } 1 \le k \le n\,, \\ R_{3,k} &= (0, z_{k-1}) \times (z_{k-1}, z_k) \quad \text{for } 2 \le k \le n\,. \end{aligned}$$

For fixed $\varrho_1, \varrho_2 \in \mathbb{N}_0$ let $Q_{l,k}$ denote the tensor product of the $(k+\varrho_1)$-point Gaussian quadrature rule in x_1-direction and the $(k+\varrho_2)$-point Gaussian quadrature rule in x_2-direction, scaled to $R_{l,k}$. Hence

$$Q_{l,k}\psi_0 \;\approx\; \int_{R_{l,k}} \psi_0(x)\,dx\,.$$

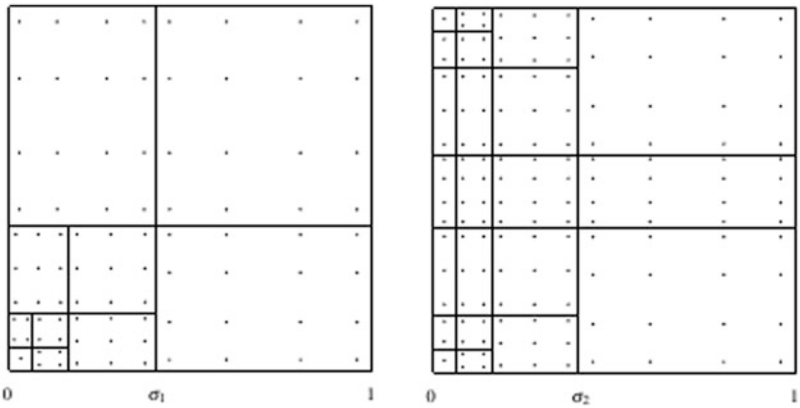
The composite quadrature rule $Q_n^{(1)}$ is now defined as

$$Q_n^{(1)}\psi_0 \;=\; Q_{2,1}\psi_0 + \sum_{k=2}^{n}\sum_{l=1}^{3} Q_{l,k}\psi_0 \;\approx\; \int_{\Gamma_0} \psi_0(x)\,dx\,.$$

The quadrature points and the subdivision of Γ_0 for $\sigma_1 = 0.4$, $n = 4$ and $\varrho_1 = \varrho_2 = 0$ are shown in Fig. D.4a.

Remark D.1 When approximating the Galerkin entries (D.7) we will choose $\varrho_k = p_k$ where p_k is the polynomial degree of ψ_j (or ψ_i) in x_k-direction ($k = 1, 2$).

For corner-edge singularities we consider again the reference element Γ_0. Let ϕ_0 have corner singularities at the origin $(0,0)$ and at the point $(x_1, x_2) = (0,1)$ and an edge singularity at $x_1 \equiv 0$. We use a geometric subdivision of Γ_0 towards the corners $(0,0)$ and $(0,1)$ and towards the corresponding edge with grading parameter



(a) $Q_n^{(1)}$ with $n = 4$ (b) $Q_n^{(2)}$ with $n = 4$

Fig. D.4 Subdivision of $[0, 1]^2$ and quadrature points for (**a**) and (**b**), where $\sigma_1 = \sigma_2 = 0.4$ and $\varrho_1 = \varrho_2 = 0$ [310]

$\sigma_2 \in (0, 1/2)$. This defines a quadrature rule $Q_n^{(2)}$. Hence, we have

$$Q_n^{(2)} \phi_0 \approx \int_{\Gamma_0} \phi_0(x)\, dx\,.$$

For an example see Fig. D.4b.

If the function ϕ_0 is singular at all four edges of Γ_0 we use a quadrature rule $Q_n^{(3)}$ with geometrical grading towards all the edges of Γ_0 and with grading parameter $\sigma_3 \in (0, 1/2)$.

To approximate the Galerkin entries (1.3) we have to deal with three critical cases where the kernel becomes singular:

i. Γ_i and Γ_j have a common node
ii. Γ_i and Γ_j have a common edge
iii. $\Gamma_i = \Gamma_j$

In the first case we use affine images of the quadrature rule $Q_n^{(1)}$ on Γ_i and Γ_j with grading towards the common node.

We define the integer $m = \lfloor n^{4/3} \rfloor$.

In the second case we use the affine image of $Q_m^{(2)}$ on Γ_i with grading towards the common edge and the common nodes. Let x_k denote the quadrature points of $Q_m^{(2)}$ on Γ_i. For any of these points x_k we consider the straight line L_k which contains x_k and which is orthogonal to the common edge E of Γ_i and Γ_j. The line L_k divides Γ_j into two rectangles $\Gamma_{j,k}^1$ and $\Gamma_{j,k}^2$. On each of these rectangles we apply the quadrature rule $Q_n^{(1)}$ with grading towards the point $L_k \cap E$ (see Fig. D.5).

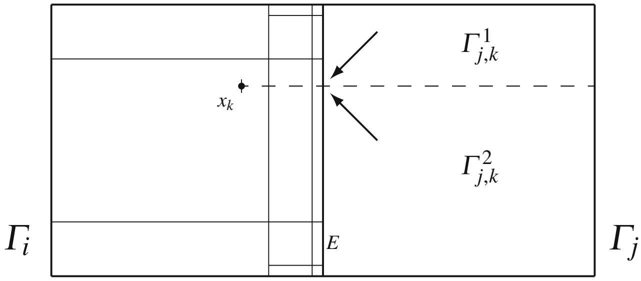


Fig. D.5 Composite quadrature for two elements Γ_i and Γ_j with common edge E. The arrows indicate direction of grading on Γ_j. The grading on Γ_j varies with the location of the quadrature points x_k in Γ_i [310]

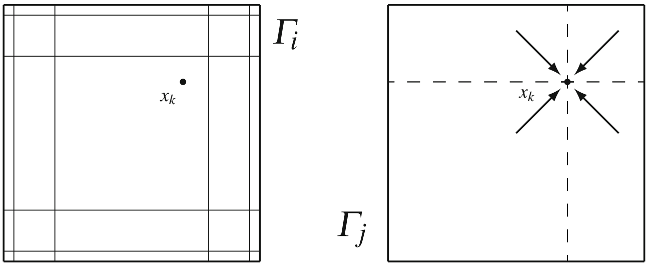


Fig. D.6 Composite quadrature for $\Gamma_i = \Gamma_j$. The grading for the inner quadrature (over Γ_j) varies with the location of the outer quadrature points $x_k \in \Gamma_i$ [310]

In the third case we use the affine image of $Q_m^{(3)}$ on Γ_i. For any quadrature point x_k which belongs to $Q_m^{(3)}$ we divide Γ_j into four rectangles with common node x_k and use the affine image of $Q_n^{(1)}$ on each of these rectangles (with grading towards x_k). See Fig. (D.6).

Next we prove exponential convergence of the quadrature rule introduced above for a simple example of two square elements in the (x_1, x_2)-plane with a common node. In the proof we restrict ourselves to piecewise constant test and trial functions, i.e. we have $\varrho_1 = \varrho_2 = 0$. Numerical results for higher polynomial degrees, for the case of two elements Γ_i and Γ_j with common edge and for the case $\Gamma_i = \Gamma_j$ are included in [310]: There the experimental results indicate exponential convergence in this case.

As a simple example we consider the elements $\Gamma_1 = (0,1)^2$ and $\Gamma_2 = (-1,0)^2$ and the parameter function $\gamma : [-1,1]^2 \to G$ defined as $\gamma(x_1, x_2) = (x_1, x_2, 0)$. Let $\psi_1, \psi_2 \in \widetilde{H}^{-1/2}(G)$ be defined as

$$\psi_j(x) = \begin{cases} 1 & \text{if } x \in \Gamma_j \times \{0\} \\ 0 & \text{if } x \in G \setminus (\Gamma_j \times \{0\}) \end{cases} \qquad (j = 1, 2)\,.$$

Hence,

$$V(\psi_1, \psi_2) := \frac{1}{4\pi} \int_{\Gamma_2} \int_{\Gamma_1} \frac{1}{|x-y|} \, dy \, dx \, . \tag{D.9}$$

For $0 < \sigma < 1$ and $n \in \mathbb{N}$ define $Q_1 = Q_n^{(1)}$ and let Q_2 be the affine image of $Q_n^{(1)}$ on Γ_2 with grading towards the origin. For $i \in \{1, 2\}$ let $x_k^{(j)}$ and $w_k^{(j)}$ be the knots and weights of the rule Q_j, i.e.

$$Q_j g \;=\; \sum_{k=1}^{M} w_k^{(j)} g(x_k^{(j)})$$

where $M = 1 + 3\sum_{i=2}^{n} i^2 = n^3 + \frac{3}{2}n^2 + \frac{1}{2}n - 2$.

The following result states exponential convergence of the composite quadrature rule $Q_2 Q_1$ applied to $|x-y|^{-1}$.

Theorem D.1 ([310]) *For any* $0 < \sigma < 1$ *there are constants* $c_1, c_2 > 0$ *such that*

$$\left| V(\psi_1, \psi_2) - \sum_{k=1}^{M} \sum_{j=1}^{M} w_k^{(1)} w_j^{(2)} |x_k^{(1)} - x_j^{(2)}|^{-1} \right| \;\le\; c_1 \, e^{-c_2 n} \tag{D.10}$$

for all $n \in \mathbb{N}$.

To prove the theorem we need the theory of countably normed spaces and the result from [372] on the exponential convergence of Q_j:

For a domain $A \subset \mathbb{R}^2 \setminus \{0\}$ and a parameter $0 < \beta < 1$ let $H_\beta^k(A)$ be the closure of $\mathscr{C}^\infty(\overline{A})$ with respect to the weighted Sobolev norm

$$\|g\|^2_{H_\beta^k(A)} \;=\; \sum_{j=0}^{k} \sum_{|\alpha|=j} \int_A |D^\alpha g(x)|^2 \phi^2_{\beta+j}(x) \, dx$$

where the weight function ϕ_s is defined as $\phi_s(x) = |x|^s$ $(s \in \mathbb{R})$.

The countably normed space $\mathscr{B}_\beta(A)$ is defined as the subspace of all functions g in $L^1(A) \cap \bigcap_{k=0}^{\infty} H_\beta^k(A)$ whose derivatives satisfy the growth condition

$$\left(\int_A |D^\alpha g(x)|^2 \phi^2_{\beta+j}(x) \, dx \right)^{1/2} \;\le\; D_g \, (d_g)^j \; j! \tag{D.11}$$

for all $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = j$. The constants $d_g \ge 1$ and $D_g > 0$ depend on A and g but not on j.

As an example consider the family of functions

$$k_x(y) = \frac{1}{|x-y|} \in \mathscr{B}_\beta(\Gamma_1) \text{ for } 0 < \beta < 1 \text{ uniformly for } x \in \Gamma_2 .$$

The following result was proved in [372]:

Lemma D.1 *Let $g \in \mathscr{B}_\beta(\Gamma_1)$ with $\beta > 0$ sufficiently small. Then, for any $0 < \sigma_1 < 1$ there exist constants $b_1, b_2 > 0$ independent of n such that*

$$\left| \int_{\Gamma_1} g(y)\, dy - Q_n^{(1)} g \right| \leq b_1 e^{-b_2 n} \tag{D.12}$$

where the constants b_1 and b_2 depend only on $\sigma_1, \beta, d_g, D_g$ and Γ_1.

We are now in the position to prove Theorem D.1.

Proof The triangle inequality yields

$$\begin{aligned} 4\pi \left| V(\psi_1, \psi_2) - \sum_{k=1}^{M} \sum_{j=1}^{M} w_k^{(1)} w_j^{(2)} |x_k^{(1)} - x_j^{(2)}|^{-1} \right| &\leq \\ \leq \underbrace{\left| \int_{\Gamma_1} \int_{\Gamma_2} \frac{1}{|x-y|}\, dy\, dx - Q_2^{(x)} \int_{\Gamma_1} \frac{1}{|x-y|}\, dy \right|}_{=\epsilon_1} & \\ + \underbrace{\left| Q_2^{(x)} \int_{\Gamma_1} \frac{1}{|x-y|}\, dy - Q_2^{(x)} Q_1^{(y)} \frac{1}{|x-y|} \right|}_{=\epsilon_2} . & \end{aligned}$$

We will estimate ϵ_2 and ϵ_1 seperately. By definition of Q_i and k_x we have

$$\epsilon_2 = \sum_{j=1}^{M} w_j^{(2)} \left| \int_{\Gamma_1} k_{x_j^{(2)}}(y)\, dy - Q_n^{(1)} k_{x_j^{(2)}} \right| .$$

From Lemma D.1 it follows that

$$\epsilon_2 \leq \sum_{j=1}^{M} w_j^{(2)} b_1 e^{-b_2 n}$$

where b_1 and b_2 are independent of j. Hence,

$$\epsilon_2 \leq |\Gamma_2|\, b_1 e^{-b_2 n} \leq b_1 e^{-b_2 n} . \tag{D.13}$$

To estimate ϵ_1 we have to show that $V\psi_1(x) \in \mathscr{B}_\beta(\Gamma_2)$ for all $0 < \beta < 1$. For plane rectangular elements the single layer potential can be calculated analytically [235]. For $x \in \Gamma_2$ we have:

$$\begin{aligned} V\psi_1(x) = & \ (y_1 - x_1)\,\text{arsinh}\frac{y_2 - x_2}{|y_1 - x_1|} + (y_2 - x_2)\,\text{arsinh}\frac{y_1 - x_1}{|y_2 - x_2|}\Bigg|_{y_1=0}^{1}\Bigg|_{y_2=0}^{1} \\ & = -x_1 \ln(-x_2 + |x|) - x_2 \ln(-x_1 + |x|) + g_1(x) \\ & = -(\cos\theta + \sin\theta)\, r \ln r + g_2(r,\theta) \end{aligned}$$

where (r,θ) are the usual polar co-ordinates and g_1, g_2 are analytic in Γ_2.

With an alternative formulation of the growth condition in polar co-ordinates [19] it can be shown easily that $V\psi_1 \in \mathscr{B}_\beta(\Gamma_2)$ for all $0 < \beta < 1$. Hence, from Lemma D.1 it follows that

$$\epsilon_1 \ \leq \ b_3\, e^{-b_4 n} \,. \tag{D.14}$$

From (D.13) and (D.14) we conclude (D.10) with $c_1 = (b_1 + b_3)/(4\pi)$ and $c_2 = \min\{b_2, b_4\}$. □

For further reading see [104].
A comprehensive list of numerical experiments can be found e.g. in the *Book of Numerical Experiments – BONE* which can be downloaded from the home page of M. Maischak, http://people.brunel.ac.uk/mastmmm/.

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Index

J. Gwinner, E. P. Stephan, *Advanced Boundary Element Methods*,
Springer Series in Computational Mathematics 52,
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