Introduction to Appendices A and B

These appendices reproduce the texts of two manuscripts in John von Neumann's handwriting preserved in the Herman H. Goldstine papers at the American Philosophical Society. In the first manuscript, transcribed in Appendix A, von Neumann defines a code for an EDVAC architecture that uses short delay lines. The second manuscript, transcribed in Appendix B, contains von Neumann's development of the code for a routine to mesh, or merge, two sorted sequences.

The manuscripts are undated. However, Von Neumann mentioned the issue of short delay lines in a letter to Haskell Curry on 20 August, 1945, and they are fully described in the EDVAC progress report subsequently written by Eckert and Mauchly (1945). This report also contains a summary of the code used to write the meshing routine, commenting that it 'is essentially that which von Neumann has proposed after trying out various coding methods on typical problems' (Eckert and Mauchly 1945, 75). The manuscripts can therefore be dated with some confidence to the late summer of 1945.

A small number of obvious slips in the manuscripts have been silently corrected in the transcriptions in Appendices A and B. The manuscripts are archived in boxes 20 and 21 of Goldstine's papers, respectively, and are reproduced here by kind permission of the American Philosophical Society.

M. Priestley, *Routines of Substitution*, SpringerBriefs in History of Computing,
https://doi.org/10.1007/978-3-319-91671-2

Appendix A
Von Neumann's Second EDVAC Code

1. For $x, \; y = 0, 1, 2, \cdots, \xi = 0, 1, \cdots, 2^5 - 1$, x determines y, ξ uniquely by virtue of the relation

$$x = 2^5 y + \xi.$$

Define accordingly

$$y = Q x, \; \xi = R x.$$

2. Time is measured by a variable $t = 0, 1, 2, \cdots$. Each t stands for the time interval $t, \; t + 1$. The length of this unit interval is in conventional units $2^5 \tau$, where τ is a pulse-time: $\tau \sim 10^{-6}$ sec.

3. *Words.*
A *word* is a sequence of 2^5 *pulses*. Each pulse occupies a pulse-time (τ), hence a word occupies a time unit of *2.* ($2^5 \tau$). The memory consists of

$$C = 2^5 A + B + 1$$

words, which are enumerated as follows:

(1) $2^5 A$ words $W x, \; x = 0, 1, \cdots, 2^5 A - 1$.
(x is to be taken mod $2^5 A$.)
(2) B words $W \bar{z}, \; z = 0, 1, \cdots, B - 1$.
(z is to be taken mod B.)
(3) 1 word σ.

Actually $A \sim 2^8$, $B \sim 2^5$ to 2^6. We assume that

$$A \leqq 2^8, \; B \leqq 2^6.$$

M. Priestley, *Routines of Substitution*, SpringerBriefs in History of Computing,
https://doi.org/10.1007/978-3-319-91671-2

4. *Gates.*
The device contains

$$D = A + B + 1$$

gates, which are enumerated as follows:

(1) A gates $G\,y,\ y = 0, 1, \cdots, A-1$.
(y is to be taken mod A.)
(2) B gates $G\,\overline{z},\ z = 0, 1, \cdots, B-1$.
(z is to be taken mod B.)
(3) 1 gate σ.

We must now discuss the following matters: *Access*, *Control*, *Operation*, *Substitution*.

5. *Access.*
At each time t each gate has *access* to exactly one word. Specifically:

(1) At each time t gate $G\,y$ has access to word $x = 2^5 y + R\,t$. (I.e. $Q\,x = y$, $R\,x = R\,t$.)
(2) Gate $G\,\overline{z}$ has always access to word $W\,\overline{z}$.
(3) Gate σ has always access to word σ.

6. This is a list of all possible words, or rather of the written symbols which denote them:

(0)	0	(3)	$\overline{z'} \to x' \mid r$	(7)	$\overline{z'}\ \omega\ \overline{z''}$
(1)	$x' \to \mathcal{C}$	(4)	$x' \to \overline{z'} \mid r$	(8)	$\to \overline{z'} \mid r$
(2)	$\overline{z'} \to \mathcal{C}$	(5)	$\overline{z'} \to \overline{z''}$	(9)	$\overline{z'} \to \quad \mid r$
		(6)	$\sigma \to \overline{z''}$	(10)	$\mathcal{N}\xi$

Here x' is a 14-binary-digit integer; $\overline{z'}$, $\overline{z''}$ are 6-binary-digit integers; r is a 5 digit binary integer, $r = 0$ being omissible; ω is one of the symbols enumerated in *8.*; ξ is a 30-binary-digit fraction with sign, $-1 \leqq \xi < 1$.

7. *Control.*
Assume that at time t the control organ $\mathcal{C}$ is connected to $\left\{\begin{array}{l} G\,y, \text{ i.e. to } W x \text{ with} \\ G\,\overline{z}, \text{ i.e. to } W\,\overline{z} \text{ - - - -} \end{array}\right\}$ $\left\{\begin{array}{l} x = 2^5 y + R\,t, \text{ whence } y = Q\,x,\quad R\,t = R\,x \\ \text{- -} \end{array}\right\}$. Then the following events will take place:

$\left\{\begin{matrix} Wx \\ W\overline{z} \end{matrix}\right\}$ is the word:	Events:
(0) 0	$\mathcal{C}$ disconnects, and at time $t+1$ connects to $\left\{\begin{matrix} G\,\underline{Q}(x+1), \text{ i.e. to } \overline{W(x+1)} \\ G\,\overline{z+1}, \text{ i.e. to } W\,\overline{z+1} \end{matrix}\right\}$.
(1) $x' \rightarrow \mathcal{C}$	$\mathcal{C}$ disconnects. Waiting until the first $t' > t$ with $R\,t' = R\,x'$. Then $\mathcal{C}$ connects to $G\,Q\,x'$, i.e. to Wx'.
(2) $\overline{z'} \rightarrow \mathcal{C}$	$\mathcal{C}$ disconnects, and at time $t+1$ connects to $G\,\overline{z'}$, i.e. to $W\,\overline{z'}$.
(3) $\overline{z'} \rightarrow x' \mid r$	$\mathcal{C}$ disconnects. Waiting until the first $t' > t$ with $R\,t' = R\,x'$. At each time $t'+s$, $s = 0, 1, \cdots, r$, for the duration of that time unit, $G\,\underline{Q}\,(x'+s)$ connects to $G\,\overline{z'+s}$, i.e. $\overline{W(x'+s)}$ to $W\,\overline{z'+s}$. At that time $W\,\overline{z'+s}$ is substituted into $W(x'+s)$. After the $s = 0, 1, \cdots, r$ have been exhausted, $\left\{\begin{matrix} \text{waiting until the first } t'' > t'+r \\ \text{time } t'' = t'+r+1 \text{ follows. - - -} \end{matrix}\right\}$ $\left\{\begin{matrix} \text{with } R\,t'' = R\,(t+1). \text{ This is } t'' = t'+2^5+1 \text{ or} \\ \text{- -} \end{matrix}\right\}$ $\left\{\begin{matrix} t'' = t'+2^6+1.) \\ \text{- - - - - - - - -} \end{matrix}\right\}$ Then $\mathcal{C}$ connects to $\left\{\begin{matrix} G\,\underline{Q}\,(x+1) \\ G\,\overline{z+1} \end{matrix}\right\}$, i.e. to $\left\{\begin{matrix} \overline{W(x+1)} \\ W\,\overline{z+1} \end{matrix}\right\}$.
(4) $x' \rightarrow \overline{z'} \mid r$	Same as (3), except that now $W(x'+s)$ is substituted for $W\,\overline{z'+s}$.
(5) $\overline{z'} \rightarrow \overline{z''}$	$\mathcal{C}$ disconnects. $G\,\overline{z'}$ connects with $G\,\overline{z''}$, i.e. $W\,\overline{z'}$ with $W\,\overline{z''}$. $W\,\overline{z'}$ is substituted into $W\,\overline{z''}$. Then, at time $t+1$, $\mathcal{C}$ is reconnected as in (0).
(6) $\sigma \rightarrow \overline{z''}$	Same as (5), except that $G\,\overline{z'}$ and $W\,\overline{z'}$ are replaced by σ and σ.
(7) $\overline{z'}\ \omega\ \overline{z''}$	$\mathcal{C}$ disconnects. $G\,\overline{z'}$, $G\,\overline{z''}$, i.e. $W\,\overline{z'}$, $W\,\overline{z''}$, connect with the inputs of the arithmetical organ $\mathcal{A}$, the output of $\mathcal{A}$ is σ. Between these three $\mathcal{A}$ performs the operation ω as described in *8*. Then $G\,\overline{z'}$, $G\,\overline{z''}$ disconnects. Waiting until the first $t' \geqq$ the completion of these operations $\left\{\begin{matrix} \text{for which } R\,t' = R\,(t+1) \\ \text{- - - - - - - - - - - - - - -} \end{matrix}\right\}$. Then $\mathcal{C}$ connects to $\left\{\begin{matrix} G\,\underline{Q}\,(x+1) \\ G\,\overline{z+1} \end{matrix}\right\}$, i.e. to $\left\{\begin{matrix} \overline{W(x+1)} \\ W\,\overline{x+1} \end{matrix}\right\}$.

(8)	$\rightarrow \overline{z'} \mid r$	$\mathcal{C}$ disconnects. At each time $t+1+s$, $s = 0, 1, \cdots, r$, for the duration of that time unit, $\left\{\begin{matrix} G\,Q\,(x+1+s) \\ G\,\overline{z+1+s} \end{matrix}\right\}$ connects to $G\,\overline{z'+s}$, i.e. $\left\{\begin{matrix} W(x+1+s) \\ W\,\overline{z+1+s} \end{matrix}\right\}$ to $W\,\overline{z'+s}$. At that time $\left\{\begin{matrix} W(x+1+s) \\ W\,\overline{z+1+s} \end{matrix}\right\}$ is substituted into $W\,\overline{z'+s}$. After the $s = 0, 1, \cdots, r$ have been exhausted, time $t+2+r$ follows. Then $\mathcal{C}$ connects to $\left\{\begin{matrix} G\,Q\,(x+2+r) \\ G\,\overline{z+2+r} \end{matrix}\right\}$, i.e. to $\left\{\begin{matrix} W(x+2+r) \\ W\,\overline{z+2+r} \end{matrix}\right\}$.
(9)	$\overline{z'} \rightarrow \quad \mid r$	Same as (8), except that now $W\overline{z'+s}$ is substituted into $\left\{\begin{matrix} W(x+1+s) \\ W\overline{z+1+s} \end{matrix}\right\}$.
(10)	$\mathcal{N}\xi$	Same as (0).

8. Operation.

When ω is called in by (7) in *7.*, $W\,\overline{z'}$, $W\,\overline{z''}$ must be of the form (10) in *7.*: $\mathcal{N}\xi'$, $\mathcal{N}\xi''$. σ is automatically of the form (10) in *7.*: $\mathcal{N}\xi^*$. Now ω operates as follows:

	Symbol	Replaces the ξ^* in the $\mathcal{N}\xi^*$ of σ by:
(C1)	$+$	$\xi' + \xi''$
(C2)	$-$	$\xi' - \xi''$
(C3)	$\times$	$\xi' + \xi''$
(C4)	$\div$	ξ'/ξ''
(C5)	$\sqrt{\ }$	$\sqrt{\xi'}$
(C6)	db	The decimal equivalent of ξ'.
(C7)	bd	The binary equivalent of the decimal interpretation of ξ'.
(H1)	$+_{\mathrm{h}}$	$\xi' + \xi'' + \xi^*$
(H2)	$-_{\mathrm{h}}$	$\xi' - \xi'' + \xi^*$
(H3)	$\times_{\mathrm{h}}$	$\xi'\xi'' + \xi^*$
(H4)	s	$\left\{\begin{matrix} \xi' \text{ for } \xi^* \geqq 0 \\ \xi'' \text{ for } \xi^* < 0 \end{matrix}\right.$

9. *Substitution.*
When a word w' is substituted into a word w'', the following will take place: (The categories (0)–(10) are those enumerated in *6.* and *7.*)

(A) w' is a word (0)–(9) (i.e. anything other than a word (10): $\mathcal{N}\xi$):
w'' is replaced by w' in its entirety.

(B) w' is a word (10): $\mathcal{N}\xi$:

(Ba) w'' is a word (0) or (10):
w'' is replaced by w' in its entirety.

(Bb) w'' is a word (1):
x' is replaced by digits 17–30 of ξ.

(Bc) w'' is a word (2):
z' is replaced by digits 11–16 of ξ.

(Bd) w'' is a word (3) or (4):
x' is replaced by digits 17–30 of ξ,
z' is replaced by digits 11–16 of ξ,
r is replaced by digits 6–10 of ξ.

(Be) w'' is a word (8) or (9):
z' is replaced by digits 11–16 of ξ.

(Bf) w'' is a word (5), (6) or (7):
No replacement is made.

Due to (Bb)–(Be) the 14-, 6-, 5- binary digit integers x', z', r appear in the replacing ξ as fractions $2^{-30}x'$, $2^{-16}z'$, $2^{-10}r$. In order to abbreviate, define

$$/x' = 2^{-30}x', \quad \overline{/}z' = 2^{-16}z', \quad \overline{\overline{/}}r = 2^{-10}r.$$

Appendix B
Von Neumann's Meshing Routine Manuscript

(1) *A p+1-complex*: $X^{(p)} = (x^0;\ x^1, \cdots, x^p)$ consists of the *main number*: x^0, and the *satellites*: $x^1, \cdots, x^p$. Throughout what follows $p = 1, 2, \cdots$ will be fixed. A complex $X^{(p)}$ *precedes* a complex $Y^{(p)}$: $X^{(p)} \leqslant Y^{(p)}$, if their main numbers are in this order: $x^0 \leqq y^0$.

An *n-sequence* of complexes: $\{X_0^{(p)}, \cdots, X_{n-1}^{(p)}\}$.

If $0', \cdots, (n-1)'$ is a permutation of $0, \cdots, (n-1)$, then the sequence $\{X_{0'}^{(p)}, \cdots, X_{(n-1)'}^{(p)}\}$ is a *permutation* of the sequence $\{X_0^{(p)}, \cdots, X_{n-1}^{(p)}\}$. A sequence $\{X_0^{(p)}, \cdots, X_{n-1}^{(p)}\}$ is *monotone* if its elements appear in their order of precedence: $X_0^{(p)} \leqslant X_1^{(p)} \leqslant \cdots \leqslant X_{n-1}^{(p)}$, i.e. $x_0^0 \leqq x_1^0 \leqq \cdots \leqq x_{n-1}^0$.

Every sequence $\{X_0^{(p)}, \cdots, X_{n-1}^{(p)}\}$ possesses a monotone permutation: $\{X_{0'}^{(p)}, \cdots, X_{(n-1)'}^{(p)}\}$ (at least one). Obtaining this monotone permutation is the operation of *sorting* the original sequence.

Given two (separately) monotone sequences $\{X_0^{(p)}, \cdots, X_{n-1}^{(p)}\}$ and $\{Y_0^{(p)}, \cdots, Y_{m-1}^{(p)}\}$, sorting the composite sequence $\{X_0^{(p)}, \cdots, X_{n-1}^{(p)}, Y_0^{(p)}, \cdots, Y_{m-1}^{(p)}\}$ is the operation of *meshing*.

(2) We wish to formulate code instructions for sorting and for meshing, and to see how much control-capacity they tie up and how much time they require. It is convenient to consider meshing first and sorting afterwards.

(3) Consider the operation of meshing the two (separately) monotone sequences $\{X_0^{(p)}, \cdots, X_{n-1}^{(p)}\}$ and $\{Y_0^{(p)}, \cdots, Y_{m-1}^{(p)}\}$.

M. Priestley, *Routines of Substitution*, SpringerBriefs in History of Computing,
https://doi.org/10.1007/978-3-319-91671-2

A natural procedure to achieve this is the following one:

Denote the meshed sequence by $\{Z_0^{(p)}, \cdots, Z_{n+m-1}^{(p)}\}$. Assume that the l first elements $Z_0^{(p)}, \cdots, Z_{l-1}^{(p)}$ have already been formed, $l = 0, 1, \cdots, n+m$. Assume that they consist of the n' (m') first elements of the X- (Y-) sequence: $X_0^{(p)}, \cdots, X_{n'-1}^{(p)}$ and $Y_0^{(p)}, \cdots, Y_{m'-1}^{(p)}$, with $n' = 0, 1, \cdots, n$ and $m' = 0, 1, \cdots, m$ and $n' + m' = l$.
Then the procedure is as follows:

(α) $n' < n, m' < m$:
Determine whether $x_{n'}^0 \leqq$ or $> y_{m'}^0$.
(α_1) $x_{n'}^0 \leqq y_{m'}^0$: $Z_l^{(p)} = X_{n'}^{(p)}$,
replace l, m', n' by $l+1, m', n'+1$.
(α_2) $x_{n'}^0 > y_{m'}^0$: $Z_l^{(p)} = Y_{m'}^{(p)}$,
replace l, m', n' by $l+1, m'+1, n'$.

(β) $n' < n, m' = m$:
Same as (α_1).

(γ) $n' = n, m' < m$:
Same as (α_2).

(δ) $n' = n, m' = m$:
The process is completed.

(4) In carrying out this process, the following observations apply:

(a) The process consists of steps which are enumerated by the index $l = 0, 1, \cdots, n+m$. It begins with $l = 0$, ends with $l = n+m$, and l increases by 1 at every step – hence there are $n+m+1$ steps.
(b) Each step is characterised not only by its l, but also by its n', m'. Since $l = n' + m'$, it is preferable to characterise it by n', m' alone, and to obtain l from the above formula. Thus the process begins with $(n', m') = (0, 0)$, ends with $(n', m') = (n, m)$, and at every step either n' or m' increases by 1 while the other remains constant.
(c) At the beginning of every step it is necessary to sense which of the 4 cases (α) – (δ) of (3) holds. (δ) terminates the procedure. (β), (γ) are related to (α): Indeed (β), (γ) correspond to (α_1), (α_2). Hence in the cases (β), (γ) one may replace $x_{n'}^0$, $y_{m'}^0$, (when they are being inspected) by 0, 0 or 0, -1 and then proceed as in (α).
(d) At the end of (α) (i.e. (α_1) or (α_2), by (c) equally for (β) or (γ)) the complex $X_{n'}^{(p)}$ ($Y_{m'}^{(p)}$) must be placed in the position of the complex $Z_l^{(p)}$. This amounts to transferring the elements of $X_{n'}^{(p)}$ ($Y_{m'}^{(p)}$), i.e. since x_n^0 (y_m^0) is already available, it amounts to transferring $x_{n'}^1, \cdots, x_{n'}^p$ ($y_{m'}^1, \cdots, y_{m'}^p$). This is an unbroken sequence of p elements, followed by $x_{n'+1}^0$ ($y_{m'+1}^0$). At the next step $x_{n'}^0$ ($y_{m'}^0$) will have to be replaced (for the next inspection) by $x_{n'+1}^0$ ($y_{m'+1}^0$), hence it is simplest to transfer at this point a sequence of $p+1$ elements, i.e. $x_{n'}^1, \cdots, x_{n'}^p, x_{n'+1}^0$ ($y_{m'}^1, \cdots, y_{m'}^p, y_{m'+1}^0$).

(e) The arrangement made at the end of (d) implies, that for $l \neq 0$ the quantities to be inspected for the step l, i.e. $x^0_{n'}$, $y^0_{m'}$, are already available at the beginning of the step. In the interest of homogeneity it is therefore desirable to have the same situation at the beginning of the step $l = 0$, i.e. $(n', m') = (0, 0)$. Hence the step $l = 0$ must be preceded by a preparatory step, say step —, which makes x^0_0, y^0_0 available.

(5) The remarks of (4) define the procedure more closely. Specifically:

(f) At the beginning of a step, say step (n', m'), the following quantities must be available, i.e. placed into short tanks: n', m', $x^0_{n'}$, $y^0_{m'}$. Denote the short tanks containing these quantities by $\overline{1_1}$, $\overline{2_1}$, $\overline{3_1}$, $\overline{4_1}$. Now the first operation must turn about determining which of the cases (α) – (δ) holds. This consists in determining which of $n' - n$, $m' - m$ are $\geqq 0$ or < 0. Hence n, m, too must be available, say in the short tanks $\overline{5_1}$, $\overline{6_1}$. According to which of the 4 cases holds, $\mathcal{C}$ must be sent to the place where its instructions begin, say the (long tank) words 1_α, 1_β, 1_γ, 1_δ. Their numbers must be available, i.e. in short tanks, say in the short tanks $\overline{7_1}$, $\overline{8_1}$, $\overline{9_1}$, $\overline{10_1}$. Finally, the order which will send $\mathcal{C}$ to 1_α – 1_δ must be in a short tank, say in the short tank $\overline{11_1}$.

(g) We now formulate a set of instructions to effect this 4-way decision between (α)–(δ). We state again the contents of the short tanks already assigned:

$\overline{1_1}$) $\mathcal{N}n'_{(-30)}$ $\overline{2_1}$) $\mathcal{N}m'_{(-30)}$ $\overline{3_1}$) $\mathcal{N}x^0_{n'}$ $\overline{4_1}$) $\mathcal{N}y^0_{m'}$
$\overline{5_1}$) $\mathcal{N}n_{(-30)}$ $\overline{6_1}$) $\mathcal{N}m_{(-30)}$ $\overline{7_1}$) $\mathcal{N}1_{\alpha(-30)}$ $\overline{8_1}$) $\mathcal{N}1_{\beta(-30)}$
$\overline{9_1}$) $\mathcal{N}1_{\gamma(-30)}$ $\overline{10_1}$) $\mathcal{N}1_{\delta(-30)}$ $\overline{11_1}$) $\ldots \to \mathcal{C}$

Now let the instructions occupy the (long tank) words $1_1, 2_1, \cdots$:

1_1)	$\overline{1_1} - \overline{5_1}$	σ) $\mathcal{N}n' - n_{(-30)}$	
2_1)	$\overline{9_1}$ s $\overline{7_1}$	σ) $\mathcal{N}\,{}^{1_\gamma}_{1_\alpha}{}_{(-30)}$	for $n' \substack{=\\<} n$
3_1)	$\sigma \to \overline{12_1}$	$\overline{12_1}$) $\mathcal{N}\,{}^{1_\gamma}_{1_\alpha}{}_{(-30)}$	for $n' \substack{=\\<} n$
4_1)	$\overline{1_1} - \overline{5_1}$	σ) $\mathcal{N}n' - n_{(-30)}$	
5_1)	$\overline{10_1}$ s $\overline{8_1}$	σ) $\mathcal{N}\,{}^{1_\delta}_{1_\beta}{}_{(-30)}$	for $n' \substack{=\\<} n$
6_1)	$\sigma \to \overline{13_1}$	$\overline{13_1}$) $\mathcal{N}\,{}^{1_\delta}_{1_\beta}{}_{(-30)}$	for $n' \substack{=\\<} n$
7_1)	$\overline{2_1} - \overline{6_1}$	σ) $\mathcal{N}m' - m_{(-30)}$	
8_1)	$\overline{13_1}$ s $\overline{12_1}$	σ) $\mathcal{N}\,{}^{\cdots\,\overline{13_1})\,\cdots}_{\cdots\,\overline{12_1})\,\cdots}$	for $m' \substack{=\\<} m$
		i.e.	
		σ) $\mathcal{N}\,{}^{1_\delta\;\;1_\beta}_{1_\gamma\;\;1_\alpha}{}_{(-30)}$	for $\substack{m'=m,n'=n \quad m'=m,n'<n \\ m'<m,n'=n \quad m'<m,n'<n}$ i.e. for $\substack{(\delta)\,(\beta)\\(\gamma)\,(\alpha)}$, respectively.
9_1)	$\sigma \to \overline{11_1}$	$\overline{11_1}$) $1_\alpha, 1_\beta, 1_\gamma, 1_\delta \to \mathcal{C}$ for (α), (β), (γ), (δ), respectively.	
10_1)	$\overline{11_1} \to \mathcal{C}$		

Now

$\overline{11_1})\ 1_\alpha, 1_\beta, 1_\gamma, 1_\delta \to \mathcal{C}$ for $(\alpha), (\beta), (\gamma), (\delta)$, respectively.

Thus at the end of this phase $\mathcal{C}$ is at $1_\alpha, 1_\beta, 1_\gamma, 1_\delta$, according to which case $(\alpha), (\beta), (\gamma), (\delta)$ holds.

(h) We now pass to the case (α). This has 2 subcases (α_1) and (α_2), according to whether $x^0_{n'} \geqq$ or $< y^0_{m'}$. According to which of the 2 subcases holds, $\mathcal{C}$ must be sent to the place where its instructions begin, say the (long tank) words $1_{\alpha_1}, 1_{\alpha_2}$. Their numbers must be available, say in the short tanks $\overline{1_2}$, $\overline{2_2}$.

(i) We now formulate a set of instructions to effect this 2-way decision between (α_1), (α_2). We state again the contents of the short tanks additionally assigned:

$\overline{1_2})\ \mathcal{N}1_{\alpha_1(-30)}$ $\overline{2_2})\ \mathcal{N}1_{\alpha_2(-30)}$

Now the instructions follow:

$1_\alpha)\ \overline{4_1} - \overline{3_1}$	$\sigma)\ \mathcal{N}y^0_{m'} - x^0_{n'}$
$2_\alpha)\ \overline{1_2}$ s $\overline{2_2}$	$\sigma)\ \mathcal{N}^{1_{\alpha_1}}_{1_{\alpha_2}}(-30)$ for $x^0_{n'} \underset{>}{\leqq} y^0_{m'}$ i.e. for $\substack{(\alpha_1)\\(\alpha_2)}$, respectively.
$3_\alpha)\ \sigma \to \overline{11_1}$	$\overline{11_1})\ 1_{\alpha_1}, 1_{\alpha_2} \to \mathcal{C}$ for $(\alpha_1), (\alpha_2)$, respectively.
$4_\alpha)\ \overline{11_1} \to \mathcal{C}$	

Now

$\overline{11_1})\ 1_{\alpha_1}, 1_{\alpha_2} \to \mathcal{C}$ for $(\alpha_1), (\alpha_2)$, respectively.

Thus at the end of this phase $\mathcal{C}$ is at $1_{\alpha_1}, 1_{\alpha_2}$, according to which case (α_1), (α_2) holds.

(j) Before turning to $(\alpha_1), (\alpha_2)$, let us dispose of the cases $(\beta), (\gamma)$ and (δ).

According to (c), the cases $(\beta), (\gamma)$ can be handled as follows:

Additional short tanks assigned:

$\overline{3_2})\ \mathcal{N}0$ $\overline{4_2})\ \mathcal{N}-1$

The instructions for (β):

$1_\beta)\ \overline{3_2} - \overline{3_2}$	$\sigma)\ \mathcal{N}0$
$2_\beta)\ 2_\alpha \to \mathcal{C}$	

and from here on like (α) with 0, 0 for $x^0_{n'}, y^0_{m'}$.

The instructions for (γ):

$1_\gamma)\ \overline{4_2} - \overline{3_2}$	$\sigma)\ \mathcal{N}-1$
$2_\gamma)\ 2_\alpha \to \mathcal{C}$	

and from here on like (α) with $0, -1$ for $x^0_{n'}, y^0_{m'}$.

(For both cases cf. (c).)

Assuming that after the conclusion of the procedure $\mathcal{C}$ is to be sent to the (long tank) word a, the instructions for (δ) are as follows:

1_δ) $a \rightarrow \mathcal{C}$

(k) We now pass to (α_1), (α_2). It is necessary to state at this point, where the complexes $X_0^{(p)}, \cdots, X_{n-1}^{(p)}$ and $Y_0^{(p)}, \cdots, Y_{m-1}^{(p)}$ are stored, and where the complexes $Z_0^{(p)}, \cdots, Z_{n+m-1}^{(p)}$ are to be placed. Let the X-complexes form a sequence which begins at the (long tank) word b, also the Y-complexes a sequence beginning at c, and the Z-complexes a sequence beginning at d. Since every complex consists of $p+1$ numbers, therefore $X_{n'}^{(p)}$ begins at $b+n'(p+1)$, $Y_{m'}^{(p)}$ begins at $c+m'(p+1)$, $Z_l^{(p)}$ begins at $d+l(p+1)$. Hence $x_{n'}^u$ is at $b+n'(p+1)+u$, $y_{m'}^u$ is at $c+m'(p+1)+u$, z_l^u is at $d+l(p+1)+u$. To conclude: The X complexes occupy the interval from b to $b+n(p+1)-1$, the Y complexes occupy the interval from c to $c+m(p+1)-1$, the Z complexes occupy the interval from d to $d+(n+m)(p+1)-1$. At the beginning of the (α_1) or (α_2) phase the following further quantities must be available, i.e. placed into short tanks: $b+n'(p+1)$, $c+m'(p+1)$, $d+l(p+1)$. It is also convenient to have $p+1$. Denote the short tanks containing these quantities by $\overline{1_3}$, $\overline{2_3}$, $\overline{3_3}$, $\overline{4_3}$. Hence these are the short tanks additionally assigned:

$\overline{1_3}$) $\mathcal{N} b+n'(p+1)_{(-30)}$
$\overline{2_3}$) $\mathcal{N} c+m'(p+1)_{(-30)}$
$\overline{3_3}$) $\mathcal{N} d+l(p+1)_{(-30)}$
$\overline{4_3}$) $\mathcal{N} p+1_{(-30)}$

Finally, the transfer of the complex $X_{n'}^{(p)}$ (in (α_1)) or $Y_{m'}^{(p)}$ (in (α_2)) to the place of the complex $Z_l^{(p)}$ must be channeled through the short tanks. According to (d), the numbers $x_{n'}^1, \cdots, x_{n'}^p, x_{n'+1}^0$ or the numbers $y_{m'}^1, \cdots, y_{m'}^p, y_{m'+1}^0$ must be brought in (from X or Y) and the numbers $x_{n'}^0, x_{n'}^1, \cdots, x_{n'}^p$ or $y_{m'}^0, y_{m'}^1, \cdots, y_{m'}^p$ must be taken out (to Z). Consequently the numbers $x_{n'}^0, x_{n'}^1, \cdots, x_{n'}^p, x_{n'+1}^0$ or the numbers $y_{m'}^0, y_{m'}^1, \cdots, y_{m'}^p, y_{m'+1}^0$ must be routed through the short tanks. It is clearly best to have them in the form of an unbroken sequence. The length of this sequence is $p+2$. Denote the short tanks which are designated to hold this sequence by $\overline{1_4}$, $\overline{2_4}$, $\cdots$, $\overline{(p+1)_4}$, $\overline{(p+2)_4}$.

We add: The primary function of (α_1) [(α_2)] is to move $x_{n'}^0, x_{n'}^1, \cdots, x_{n'}^p$ [$y_{m'}^0, y_{m'}^1, \cdots, y_{m'}^p$] into the (long tank) words $d+l(p+1), d+l(p+1)+1, \cdots, d+l(p+1)+p$. However, there is also a secondary function: It must prepare the conditions for step $l+1$. This means that it must replace the numbers n', $x_{n'}^0$, $b+n'(p+1)$, $d+l(p+1)$ [m', $y_{m'}^0$, $c+m'(p+1)$, $d+l(p+1)$] in the short tanks $\overline{1_1}$, $\overline{3_1}$, $\overline{1_3}$, $\overline{3_3}$ [$\overline{2_1}$, $\overline{4_1}$, $\overline{2_3}$, $\overline{3_3}$] by the numbers $n'+1$, $x_{n'+1}^0$, $b+(n'+1)(p+1)$, $d+(l+1)(p+1)$

$[m' + 1, y^0_{m'+1}, c + (m' + 1)(p + 1), d + (l + 1)(p + 1)]$.

To conclude: There are also two orders, affecting the transfers of X $[Y]$ into the short tanks, and of Z out of the short tanks, and these orders are best placed into short tanks, say $\overline{1_5}$, $\overline{2_5}$. They must be followed by an order returning $\mathcal{C}$ to the (α_1) or (α_2) sequence in long tanks. Hence this third order must be in $\overline{3_5}$, and it must depend on (α_1) or (α_2). I.e. it must be transfered into $\overline{3_5}$ from the (α_1) or (α_2) sequence.

(l) We now formulate 2 sets of instructions to carry out the tasks of (α_1) and (α_2), as formulated in (k).

Additional short tanks assigned:

$\overline{1_5}$) $\ldots \rightarrow \overline{1_4} \mid p+2$ $\quad$ $\overline{2_5}$) $\overline{1_4} \rightarrow \ldots \mid p+1$ $\quad$ $\overline{3_5}$) $\ldots$ $\quad$ $\overline{4_5}$) $\mathcal{N}1_{(-30)}$

The instructions for (α_1):

1_{α_1})	$\overline{1_3} \rightarrow \overline{1_5}$	$\overline{1_5}$) $b + n'(p+1) \rightarrow \overline{1_4} \mid p+2$
2_{α_1})	$\overline{3_3} \rightarrow \overline{2_5}$	$\overline{2_5}$) $\overline{1_4} \rightarrow d + l(p+1) \mid p+1$
3_{α_1})	$\vdash\!\!\to \overline{3_5}$	$\overline{3_5}$) $6_{\alpha_1} \rightarrow \mathcal{C}$
4_{α_1})	$6_{\alpha_1} \rightarrow \mathcal{C}$	
5_{α_1})	$\overline{1_5} \rightarrow \mathcal{C}$	

$\overline{1_5}$) $b + n'(p+1) \rightarrow \overline{1_4} \mid p+2$

$b + n'(p+1)$) $\mathcal{N}x^0_{n'}$	to	$\overline{1_4}$) $\mathcal{N}x^0_{n'}$
$b + n'(p+1) + 1$) $\mathcal{N}x^1_{n'}$	to	$\overline{2_4}$) $\mathcal{N}x^1_{n'}$
$\ldots$		$\ldots$
$b + n'(p+1) + p$) $\mathcal{N}x^p_{n'}$	to	$\overline{(p+1)_4}$) $\mathcal{N}x^p_{n'}$
$b + (n'+1)(p+1)$) $\mathcal{N}x^0_{n'+1}$	to	$\overline{(p+2)_4}$) $\mathcal{N}x^0_{n'+1}$

$\overline{2_5}$) $\overline{1_4} \rightarrow d + l(p+1) \mid p+1$

$\overline{1_4}$) $\mathcal{N}x^0_{n'}$	to	$d + l(p+1)$) $\mathcal{N}x^0_{n'}$
$\overline{2_4}$) $\mathcal{N}x^1_{n'}$	to	$d + l(p+1) + 1$) $\mathcal{N}x^1_{n'}$
$\ldots$		$\ldots$
$\overline{(p+1)_4}$) $\mathcal{N}x^p_{n'}$	to	$d + l(p+1) + p$) $\mathcal{N}x^p_{n'}$

$\overline{3_5}$) $6_{\alpha_1} \rightarrow \mathcal{C}$

6_{α_1})	$\overline{1_1} + \overline{4_5}$	σ) $\mathcal{N}n' + 1_{(-30)}$
7_{α_1})	$\sigma \rightarrow \overline{1_1}$	$\overline{1_1}$) $\mathcal{N}n' + 1_{(-30)}$
8_{α_1})	$\overline{(p+2)_4} \rightarrow \overline{3_1}$	$\overline{3_1}$) $\mathcal{N}x^0_{n'+1}$
9_{α_1})	$\overline{1_3} + \overline{4_3}$	σ) $\mathcal{N}b + (n'+1)(p+1)_{(-30)}$
10_{α_1})	$\sigma \rightarrow \overline{1_3}$	$\overline{1_3}$) $\mathcal{N}b + (n'+1)(p+1)_{(-30)}$
11_{α_1})	$\overline{3_3} + \overline{4_3}$	σ) $\mathcal{N}d + (l+1)(p+1)_{(-30)}$
12_{α_1})	$\sigma \rightarrow \overline{3_3}$	$\overline{3_3}$) $\mathcal{N}d + (l+1)(p+1)_{(-30)}$
13_{α_1})	$1_1 \rightarrow \mathcal{C}$	To begin step $l+1$ according to (g).

The instructions for (α_2):

1_{α_2})	$\overline{2_3} \to \overline{1_5}$	$\overline{1_5}$) $c + m'(p+1) \to \overline{1_4} \mid p+2$
2_{α_2})	$\overline{3_3} \to \overline{2_5}$	$\overline{2_5}$) $\overline{1_4} \to d + l(p+1) \mid p+1$
3_{α_2})	$\vdash\!\!\!\to \overline{3_5}$	$\overline{3_5}$) $6_{\alpha_2} \to \mathcal{C}$
4_{α_2})	$6_{\alpha_2} \to \mathcal{C}$	
5_{α_2})	$\overline{1_5} \to \mathcal{C}$	

$\overline{1_5}$) $c + m'(p+1) \to \overline{1_4} \mid p+2$

$c + m'(p+1)$) $\mathcal{N} y^0_{m'}$	to	$\overline{1_4}$) $\mathcal{N} y^0_{m'}$
$c + m'(p+1) + 1$) $\mathcal{N} y^1_{m'}$	to	$\overline{2_4}$) $\mathcal{N} y^1_{m'}$
...		...
$c + m'(p+1) + p$) $\mathcal{N} y^p_{m'}$	to	$\overline{(p+1)_4}$) $\mathcal{N} y^p_{m'}$
$c + (m'+1)(p+1)$) $\mathcal{N} y^0_{m'+1}$	to	$\overline{(p+2)_4}$) $\mathcal{N} y^0_{m'+1}$

$\overline{2_5}$) $\overline{1_4} \to d + l(p+1) \mid p+1$

$\overline{1_4}$) $\mathcal{N} y^0_{m'}$	to	$d + l(p+1)$) $\mathcal{N} y^0_{m'}$
$\overline{2_4}$) $\mathcal{N} y^1_{m'}$	to	$d + l(p+1) + 1$) $\mathcal{N} y^1_{m'}$
...		...
$\overline{(p+1)_4}$) $\mathcal{N} y^p_{m'}$	to	$d + l(p+1) + p$) $\mathcal{N} y^p_{m'}$

$\overline{3_5}$) $6_{\alpha_2} \to \mathcal{C}$

6_{α_2})	$\overline{2_1} + \overline{4_5}$	σ) $\mathcal{N} m' + 1_{(-30)}$
7_{α_2})	$\sigma \to \overline{2_1}$	$\overline{2_1}$) $\mathcal{N} m' + 1_{(-30)}$
8_{α_2})	$\overline{(p+2)_4} \to \overline{4_1}$	$\overline{4_1}$) $\mathcal{N} y^0_{m'+1}$
9_{α_2})	$\overline{2_3} + \overline{4_3}$	σ) $\mathcal{N} c + (m'+1)(p+1)_{(-30)}$
10_{α_2})	$\sigma \to \overline{2_3}$	$\overline{2_3}$) $\mathcal{N} c + (m'+1)(p+1)_{(-30)}$
11_{α_2})	$\overline{3_3} + \overline{4_3}$	σ) $\mathcal{N} d + (l+1)(p+1)_{(-30)}$
12_{α_2})	$\sigma \to \overline{3_3}$	$\overline{3_3}$) $\mathcal{N} d + (l+1)(p+1)_{(-30)}$
13_{α_2})	$1_1 \to \mathcal{C}$	To begin step $l+1$ according to (g).

(6) Let us restate, which short tanks are occupied at the beginning of step l, and how. This is the list:

$\overline{1_1}$) $\mathcal{N}n'_{(-30)}$	$\overline{2_1}$) $\mathcal{N}m'_{(-30)}$	$\overline{3_1}$) $\mathcal{N}x^0_{n'}$	$\overline{4_1}$) $\mathcal{N}y^0_{m'}$
$\overline{5_1}$) $\mathcal{N}n_{(-30)}$	$\overline{6_1}$) $\mathcal{N}m_{(-30)}$	$\overline{7_1}$) $\mathcal{N}1_{\alpha(-30)}$	$\overline{8_1}$) $\mathcal{N}1_{\beta(-30)}$
$\overline{9_1}$) $\mathcal{N}1_{\gamma(-30)}$	$\overline{10_1}$) $\mathcal{N}1_{\delta(-30)}$	$\overline{11_1}$) $\ldots \to \mathcal{C}$	
$\overline{1_2}$) $\mathcal{N}1_{\alpha_1(-30)}$	$\overline{2_2}$) $\mathcal{N}1_{\alpha_2(-30)}$		
$\overline{3_2}$) $\mathcal{N}0$	$\overline{4_2}$) $\mathcal{N}-1$		
$\overline{1_3}$) $\mathcal{N}b + n'(p+1)_{(-30)}$			
$\overline{2_3}$) $\mathcal{N}c + m'(p+1)_{(-30)}$			
$\overline{3_3}$) $\mathcal{N}d + l(p+1)_{(-30)}$			
$\overline{4_3}$) $\mathcal{N}p + 1_{(-30)}$			
$\overline{1_4}$) $\ldots$	$\overline{2_4}$) $\ldots$ $\cdots$	$\overline{(p+1)_4}$) $\ldots$	$\overline{(p+2)_4}$) $\ldots$
$\overline{1_5}$) $\ldots \to \overline{1_4} \mid p+2$	$\overline{2_5}$) $\overline{1_4} \to \ldots \mid p+1$	$\overline{3_5}$) $\ldots$	$\overline{4_5}$) $\mathcal{N}1_{(-30)}$

The first thing to note is, that this requires $11 + 4 + 4 + (p + 2) + 4 = p + 25$ short tanks. Hence, if the total number of short tanks is 32 [64], this gives the upper limit 7 [39] for p.

The second observation is, that these short tanks have the following contents when the sequence of steps $l = 0, 1, \cdots, n + m$ begins, i.e. at the beginning of the step $l = 0$. This is the list:

$\overline{1_1}$) $\mathcal{N}0$	$\overline{2_1}$) $\mathcal{N}0$	$\overline{3_1}$) $\mathcal{N}x^0_0$	$\overline{4_1}$) $\mathcal{N}y^0_0$
$\overline{5_1}$) $\mathcal{N}n_{(-30)}$	$\overline{6_1}$) $\mathcal{N}m_{(-30)}$	$\overline{7_1}$) $\mathcal{N}1_{\alpha(-30)}$	$\overline{8_1}$) $\mathcal{N}1_{\beta(-30)}$
$\overline{9_1}$) $\mathcal{N}1_{\gamma(-30)}$	$\overline{10_1}$) $\mathcal{N}1_{\delta(-30)}$	$\overline{11_1}$) $\ldots \to \mathcal{C}$	
$\overline{1_2}$) $\mathcal{N}1_{\alpha_1(-30)}$	$\overline{2_2}$) $\mathcal{N}1_{\alpha_2(-30)}$		
$\overline{3_2}$) $\mathcal{N}0$	$\overline{4_2}$) $\mathcal{N}-1$		
$\overline{1_3}$) $\mathcal{N}b_{(-30)}$	$\overline{2_3}$) $\mathcal{N}c_{(-30)}$	$\overline{3_3}$) $\mathcal{N}d_{(-30)}$	$\overline{4_3}$) $\mathcal{N}p + 1_{(-30)}$
$\overline{1_4}$) $\ldots$	$\overline{2_4}$) $\ldots$ $\cdots$	$\overline{(p+1)_4}$) $\ldots$	$\overline{(p+2)_4}$) $\ldots$
$\overline{1_5}$) $\ldots \to \overline{1_4} \mid p+2$	$\overline{2_5}$) $\overline{1_4} \to \ldots \mid p+1$	$\overline{3_5}$) $\ldots$	$\overline{4_5}$) $\mathcal{N}1_{(-30)}$

Of these $p + 25$ short tanks the following must form unbroken sequences: $\overline{1_5}, \overline{2_5}, \overline{3_5}$ because of their rôle in (l) (between 5_{α_1} and 6_{α_1}, and between 5_{α_2} and 6_{α_2}); $\overline{1_4}, \overline{2_4}, \cdots, \overline{(p+1)_4}, \overline{(p+2)_4}$ because of their rôle in (l) (at $\overline{1_5}$ and $\overline{2_5}$, in the two intervals mentioned above).

These are $3 + (p + 2) = p + 5$ short tanks. Of these $p + 3$, namely $\overline{3_5}$ and $\overline{1_4}, \overline{2_4}, \cdots, \overline{(p+1)_4}, \overline{(p+2)_4}$ require no preliminary substitution; 2, namely $\overline{1_5}, \overline{2_5}$ have a fixed content.

The remaining $(p + 25) - (p + 5) = 20$ short tanks can be classified as follows: 12, namely $\overline{1_1}$, $\overline{2_1}$, $\overline{7_1}$, $\overline{8_1}$, $\overline{9_1}$, $\overline{10_1}$, $\overline{11_1}$, $\overline{1_2}$, $\overline{2_2}$, $\overline{3_2}$, $\overline{4_2}$, $\overline{4_5}$ have a fixed content; 6, namely $\overline{5_1}$, $\overline{6_1}$, $\overline{1_3}$, $\overline{2_3}$, $\overline{3_3}$, $\overline{4_3}$ have to be substituted from the general data of the problem (they contain n, m, b, c, d, $p + 1$); 2, namely $\overline{3_1}$, $\overline{4_1}$ have to be substituted from the sequences X, Y (they contain x_0^0, y_0^0). It is desirable that all short tanks with a fixed content form an unbroken sequence, so that they can be substituted by one order. I.e. the 14 given here and the 2 given above must form an unbroken sequence. These 2 last are $\overline{1_5}$, $\overline{2_5}$, as noted still earlier, they must be followed by $\overline{3_5}$. This gives an unbroken sequence of $12 + 2 + 1 = 15$ short tanks.

Finally, it is desirable to have the 6 short tanks with n, m, b, c, d, $p + 1$ at the beginning, and the 2 with x_0^0, y_0^0 immediately afterwards. Also, to have the sequence of indefinite length $(p + 2)$ at the end.

This gives the following final assignment of the $p + 25$ short tanks used:

$\overline{1})$	$\cdots$	$\overline{5_1})$	$\mathcal{N} n_{(-30)}$
$\overline{2})$	$\cdots$	$\overline{6_1})$	$\mathcal{N} m_{(-30)}$
$\overline{3})$	$\cdots$	$\overline{1_3})$	$\mathcal{N} b_{(-30)}$
$\overline{4})$	$\cdots$	$\overline{2_3})$	$\mathcal{N} c_{(-30)}$
$\overline{5})$	$\cdots$	$\overline{3_3})$	$\mathcal{N} d_{(-30)}$
$\overline{6})$	$\cdots$	$\overline{4_3})$	$\mathcal{N} p + 1_{(-30)}$
$\overline{7})$	$\cdots$	$\overline{3_1})$	$\mathcal{N} x_0^0$
$\overline{8})$	$\cdots$	$\overline{4_1})$	$\mathcal{N} y_0^0$
$\overline{9})$	$\cdots$	$\overline{1_1})$	$\mathcal{N} 0$
$\overline{10})$	$\cdots$	$\overline{2_1})$	$\mathcal{N} 0$
$\overline{11})$	$\cdots$	$\overline{7_1})$	$\mathcal{N} 1_{\alpha(-30)}$
$\overline{12})$	$\cdots$	$\overline{8_1})$	$\mathcal{N} 1_{\beta(-30)}$
$\overline{13})$	$\cdots$	$\overline{9_1})$	$\mathcal{N} 1_{\gamma(-30)}$
$\overline{14})$	$\cdots$	$\overline{10_1})$	$\mathcal{N} 1_{\delta(-30)}$
$\overline{15})$	$\cdots$	$\overline{11_1})$	$\ldots \to \mathcal{C}$
$\overline{16})$	$\cdots$	$\overline{1_2})$	$\mathcal{N} 1_{\alpha_1(-30)}$
$\overline{17})$	$\cdots$	$\overline{2_2})$	$\mathcal{N} 1_{\alpha_2(-30)}$
$\overline{18})$	$\cdots$	$\overline{3_2})$	$\mathcal{N} 0$
$\overline{19})$	$\cdots$	$\overline{4_2})$	$\mathcal{N} -1$
$\overline{20})$	$\cdots$	$\overline{4_5})$	$\mathcal{N} 1_{(-30)}$
$\overline{21})$	$\cdots$	$\overline{1_5})$	$\ldots \to \overline{24} \mid p + 2$
$\overline{22})$	$\cdots$	$\overline{2_5})$	$\overline{24} \to \ldots \mid p + 1$
$\overline{23})$	$\cdots$	$\overline{3_5})$	$\ldots$
$\overline{24})$	$\cdots$	$\overline{1_4})$	$\ldots$
$\overline{25})$	$\cdots$	$\overline{2_4})$	$\ldots$
		$\cdots$	
$\overline{p + 24})$	$\cdots$	$\overline{p + 1_4})$	$\ldots$
$\overline{p + 25})$	$\cdots$	$\overline{p + 2_4})$	$\ldots$

(7) We now come to the step — mentioned in (e). We foresaw there that — would have to substitute x_0^0, y_0^0 into the proper short tanks (as we saw in (6), into $\overline{7}$,

$\overline{8}$). We see now, however, that — has to take care of the substitution into all short tanks. More precisely: No substitutions into $\overline{23}$ and $\overline{24}$, $\overline{25}$, $\cdots$, $\overline{p+24}$, $\overline{p+25}$ are needed. And $\overline{1}$, $\cdots$, $\overline{6}$ should be substituted when the problem is set up as such. Hence the short tanks left for the step — are $\overline{7}$, $\overline{8}$ and $\overline{10}$, $\cdots$, $\overline{22}$.

We will substitute $\overline{7}$, $\overline{8}$ first and $\overline{10}$, $\cdots$, $\overline{22}$ afterwards. During the first operation the short tanks $\overline{1}$, $\cdots$, $\overline{6}$ are already occupied as indicated above (by n, m, b, c, d, $p+1$), while $\overline{9}$, $\cdots$ are still available. Hence we can use $\overline{9}$, $\cdots$ while carrying out the first step, the substitution of $\overline{7}$, $\overline{8}$, and substitute $\overline{9}$, $\cdots$, or more precisely $\overline{9}$, $\cdots$, $\overline{22}$, in the final form only subsequently, as a second step.

Actually it is desirable to place during the first step, the substitution of $\overline{7}$, $\overline{8}$, some orders into short tanks. In accordance with what was said above, we use for this $\overline{9}$, $\cdots$.

We now formulate the instructions which carry out all these substitutions. Let these instructions occupy the (long tank) words 1_0, 2_0, $\cdots$:

1_0) $+\!\!\triangleright\ \overline{9} \mid 2$	
2_0) $\ldots\ +\!\!\triangleright\ \overline{7}$	$\overline{9}$) $\ldots\ +\!\!\triangleright\ \overline{7}$
3_0) $\ldots\ +\!\!\triangleright\ \overline{8}$	$\overline{10}$) $\ldots\ +\!\!\triangleright\ \overline{8}$
4_0) $8_0 \rightarrow \mathcal{C}$	$\overline{11}$) $8_0 \rightarrow \mathcal{C}$
5_0) $\overline{3} \rightarrow \overline{9}$	$\overline{9}$) $b\ +\!\!\triangleright\ \overline{7}$
6_0) $\overline{4} \rightarrow \overline{10}$	$\overline{10}$) $c\ +\!\!\triangleright\ \overline{8}$
7_0) $\overline{9} \rightarrow \mathcal{C}$	
$\overline{9}$) $b\ +\!\!\triangleright\ \overline{7}$	$\overline{7}$) $\mathcal{N}x_0^0$
$\overline{10}$) $c\ +\!\!\triangleright\ \overline{8}$	$\overline{8}$) $\mathcal{N}y_0^0$
$\overline{11}$) $8_0 \rightarrow \mathcal{C}$	
8_0) $+\!\!\triangleright\ \overline{9} \mid 13$	
9_0) $\mathcal{N}0$	$\overline{9}$) $\mathcal{N}0$
10_0) $\mathcal{N}0$	$\overline{10}$) $\mathcal{N}0$
11_0) $\mathcal{N}1_{\alpha(-30)}$	$\overline{11}$) $\mathcal{N}1_{\alpha(-30)}$
12_0) $\mathcal{N}1_{\beta(-30)}$	$\overline{12}$) $\mathcal{N}1_{\beta(-30)}$
13_0) $\mathcal{N}1_{\gamma(-30)}$	$\overline{13}$) $\mathcal{N}1_{\gamma(-30)}$
14_0) $\mathcal{N}1_{\delta(-30)}$	$\overline{14}$) $\mathcal{N}1_{\delta(-30)}$
15_0) $\ldots \rightarrow \mathcal{C}$	$\overline{15}$) $\ldots \rightarrow \mathcal{C}$
16_0) $\mathcal{N}1_{\alpha_1(-30)}$	$\overline{16}$) $\mathcal{N}1_{\alpha_1(-30)}$
17_0) $\mathcal{N}1_{\alpha_2(-30)}$	$\overline{17}$) $\mathcal{N}1_{\alpha_2(-30)}$
18_0) $\mathcal{N}0$	$\overline{18}$) $\mathcal{N}0$

19_0) $\mathcal{N}-1$	$\overline{19}$) $\mathcal{N}-1$
20_0) $\ldots \to \overline{24} \mid p+2$	$\overline{20}$) $\ldots \to \overline{24} \mid p+2$
21_0) $\overline{24} \to \ldots \mid p+1$	$\overline{21}$) $\overline{24} \to \ldots \mid p+1$
22_0) $\mathcal{N}1_{(-30)}$	$\overline{22}$) $\mathcal{N}1_{(-30)}$
23_0) $1_1 \to \mathcal{C}$	To begin step 0 according to (g).

(8) We now have a complete list of instructions:

First, in short tanks $\overline{1}$, ..., $\overline{6}$, as described at the end of (6).

Second, in long tanks the words $1_0, \cdots, 23_0$ (cf. (7)); $1_1, \cdots, 10_1$ (cf. (g)); $1_\alpha, \cdots, 4_\alpha$ (cf. (i)); $1_\beta, 2_\beta$ (cf. (j)); $1_\gamma, 2_\gamma$ (cf. (j)); 1_δ (cf. (j)); $1_{\alpha_1}, \cdots, 13_{\alpha_1}$ (cf. (l)); $1_{\alpha_2}, \cdots, 13_{\alpha_2}$ (cf. (l)).

Let us consider the second category of instructions, i.e. the words in long tanks, more closely. The first thing to note is, that this requires $23 + 10 + 4 + 2 + 2 + 1 + 13 + 13 = 68$ (long tank) words. The second observation is, that according to (j) $1_\beta, 2_\beta$ as well as $1_\gamma, 2_\gamma$ are always followed by $1_\alpha, \ldots, 4_\alpha$, and according to (i) $1_\alpha, \ldots, 4_\alpha$ are always followed by $1_{\alpha_1}, \ldots, 13_{\alpha_1}$ or $1_{\alpha_2}, \ldots, 13_{\alpha_2}$. Hence it is reasonable to make the final assignment of numbers to these (long tank) words in such a way, that these precedences are maintained.

Actually it is best to delay the final assignment of numbers, for reasons which will appear in (9). We make, however, a secondary assignment of numbers as follows:

$1', \cdots, 23'$	to	$1_0, \cdots, 23_0$
$24', \cdots, 33'$	to	$1_1, \cdots, 10_1$
$34', 35'$	to	$1_\beta, 2_\beta$
$36', 37'$	to	$1_\gamma, 2_\gamma$
$38', \cdots, 41'$	to	$1_\alpha, \cdots, 4_\alpha$
$42', \cdots, 54'$	to	$1_{\alpha_1}, \cdots, 13_{\alpha_1}$
$55', \cdots, 67'$	to	$1_{\alpha_2}, \cdots, 13_{\alpha_2}$
$68'$	to	1_δ

(9) The assignment of numbers at the end of (8), makes 1_α, 1_β, 1_γ, 1_δ, 1_{α_1}, 1_{α_2} equal to $38'$, $34'$, $36'$, $68'$, $42'$, $55'$. These numbers occur in 11_0, 12_0, 13_0, 14_0, 16_0, 17_0, i.e. in $11'$, $12'$, $13'$, $14'$, $16'$, $17'$. Hence the content of these 6 (long tank) words depends explicitely on the final assignment of (long tank word) numbers to $1', \ldots, 68'$.

If this final assignment were made now, in a rigid form, then the (long tank) words $11', \ldots, 14'$, $16'$, $17'$ could be formulated accordingly, and the instructions would be completed. It is, however, preferable to have these instructions in such a form that they can begin anywhere, i.e. that their first (long tank) word can be chosen freely.

Let this first (long tank) word be e, i.e. e is $1'$. Hence $e, \ldots, e+67$ should correspond to $1', \ldots, 68'$. However, it is worth while to deviate from this simple sequential correspondence for the following reasons:

(A) In $1_0, \cdots, 23_0$ (i.e. $1', \cdots, 23'$) the passage of $\mathcal{C}$ from $\overline{11}$ to 8_0 involves a delay of about one long tank, if 8_0 follows immediately upon 7_0: Indeed 7_0 is followed by $\overline{9}$, $\overline{10}$, $\overline{11}$. Hence it is better to intercalate 3 words between 7_0 and 8_0 to time correctly for $\overline{9}$, $\overline{10}$, $\overline{11}$, plus, say, 1 word for the long tank switching in $\overline{11}$. I.e., there should be 4 (empty) words between 7_0 and 8_0, i.e. $7'$ and $8'$.

(B) In $1_{\alpha_1}, \cdots, 13_{\alpha_1}$ (i.e. $42', \cdots, 54'$) there exists the same situation as in (A) between 5_{α_1} and 6_{α_1}, where $\overline{1_5}$, $\overline{2_5}$, $\overline{3_5}$ (i.e. $\overline{21}$, $\overline{22}$, $\overline{23}$) are intercalated. In order to avoid a delay of about one long tank, it is again necessary to intercalate $3+1=4$ (empty) words between 5_{α_1} and 6_{α_1}, i.e. $46'$ and $47'$.

(C) In $1_{\alpha_2}, \cdots, 13_{\alpha_2}$ (i.e. $55', \cdots, 67'$) between 5_{α_2} and 6_{α_2} the situation is exactly the same as in (B). Hence it is again advisable to intercalate 4 (empty) words between 5_{α_2} and 6_{α_2}, i.e. $59'$ and $60'$.

(D) 10_1 (i.e. $33'$) sends $\mathcal{C}$ to $\overline{11_1}$ (i.e. $\overline{15}$), and this in turn sends $\mathcal{C}$ to 1_α or 1_β or 1_γ or 1_δ (i.e. $38'$ or $34'$ or $36'$ or $68'$). In order to avoid a delay of about one long tank, it is necessary to intercalate 1 word after 10_1 to time correctly for $\overline{11_1}$, plus, say, 1 word for the long tank switching in $\overline{11_1}$. I.e. there should be 2 (empty) words after 10_1, i.e. $33'$.

Taking these matters into account, the following final assignment of numbers obtains:

$e, \cdots, e+6$	to	$1', \cdots, 7'$	to	$1_0, \cdots, 7_0$
$e+7, \cdots, e+10$	to	empty (cf. (A))		
$e+11, \cdots, e+26$	to	$8', \cdots, 23'$	to	$8_0, \cdots, 23_0$
$e+27, \cdots, e+36$	to	$24', \cdots, 33'$	to	$1_1, \cdots, 10_1$
$e+37, e+38$	to	empty (cf. (D))		
$e+39, e+40$	to	$34', 35'$	to	$1_\beta, 2_\beta$
$e+41, e+42$	to	$36', 37'$	to	$1_\gamma, 2_\gamma$
$e+43, \cdots, e+46$	to	$38', \cdots, 41'$	to	$1_\alpha, \cdots, 4_\alpha$
$e+47, \cdots, e+51$	to	$42', \cdots, 46'$	to	$1_{\alpha_1}, \cdots, 5_{\alpha_1}$
$e+52, \cdots, e+55$	to	empty (cf. (B))		
$e+56, \cdots, e+63$	to	$47', \cdots, 54'$	to	$6_{\alpha_1}, \cdots, 13_{\alpha_1}$
$e+64, \cdots, e+68$	to	$55', \cdots, 59'$	to	$1_{\alpha_2}, \cdots, 5_{\alpha_2}$
$e+69, \cdots, e+72$	to	empty (cf. (C))		
$e+73, \cdots, e+80$	to	$60', \cdots, 67'$	to	$6_{\alpha_2}, \cdots, 13_{\alpha_2}$
$e+81$	to	$68'$	to	1_δ

Hence the (long tank) words 11′, 12′, 13′, 14′, 16′, 17′ become

$e + 14, e + 15, e + 16, e + 17, e + 19, e + 20,$

and they contain the numbers 38′, 34′, 36′, 68′, 42′, 55′, and these become

$e + 43, e + 39, e + 41, e + 81, e + 47, e + 64.$

We rewrite these (long tank) words:

$e + 14)\ \mathcal{N}e + 43_{(-30)}$
$e + 15)\ \mathcal{N}e + 39_{(-30)}$
$e + 16)\ \mathcal{N}e + 41_{(-30)}$
$e + 17)\ \mathcal{N}e + 81_{(-30)}$
$e + 19)\ \mathcal{N}e + 47_{(-30)}$
$e + 20)\ \mathcal{N}e + 64_{(-30)}$

(10) Disregarding for the time being the 6 substitutions required to produce the 6 (long tank) words enumerated at the end of (9), the total system of instructions, at the present stage, is this:

(I) The 82 (long tank) words $e, \cdots, e + 81$ of (9).
(II) The 6 short tanks $\overline{1}, \cdots, \overline{6}$ of (6).

The quantities which actually determine the problem, as a function of the X, Y, are these:

(*) n, m, b, c, d, p, a, e. (For a cf. (1_δ) in (j), for e cf. (9).)

Of these the 6 first, n, m, b, c, d, p, are given in (II), but p occurs again in (I). The others a, e, occur in (I) only. So we must discuss how the occurrences of

(**) p, a, e

in (I) are to be taken care of.

p occurs in 20_0, 21_0 (cf. (7)), i.e. $e + 23$, $e + 24$ (cf. (9)). a occurs in 1_δ (cf. (j)), i.e. $e + 81$ (cf. (9)). The occurrences of e have been summarized at the end of (9).

We rewrite the (long tank) words which contain these additional substitutions:

$e + 23)\ \ldots \to \overline{24} \mid p + 2$
$e + 24)\ \overline{24} \to \ldots \mid p + 1$

and

$e + 81)\ a \to \mathcal{C}$

(11) The complete system of instructions, as derived in what preceded, can also be formulated as follows:

The 82 (long tank) words of (I) in (10), namely $e, \cdots, e+81$, contain only fixed symbols, except for certain occurrences of the 3 variables of (**) in (10), namely p, a, e, in the 9 words enumerated at the end of (9) and the end of (10). Assume, that $e, \cdots, e+81$ are stated, with blanks ... in place of these 3 variables in the 9 words in question. Call this group of 82 words G_{82}.
Then, after G_{82} has been placed in the long tanks, in an unbroken sequence beginning at e, the following further steps are necessary:

First, 6 substitutions into short tanks, according to (II) in (10), and 9 substitutions into long tanks, according to the end of (9) and the end of (10). We restate these $6+9=15$ substitutions:

$\overline{1}$) $\mathcal{N}n_{(-30)}$	$e+14$) $\mathcal{N}e+43_{(-30)}$	$e+23$) $\ldots \to \overline{24} \mid p+2$
$\overline{2}$) $\mathcal{N}m_{(-30)}$	$e+15$) $\mathcal{N}e+39_{(-30)}$	$e+24$) $\overline{24} \to \ldots \mid p+1$
$\overline{3}$) $\mathcal{N}b_{(-30)}$	$e+16$) $\mathcal{N}e+41_{(-30)}$	$e+81$) $a \to \mathcal{C}$
$\overline{4}$) $\mathcal{N}c_{(-30)}$	$e+17$) $\mathcal{N}e+81_{(-30)}$	
$\overline{5}$) $\mathcal{N}d_{(-30)}$	$e+19$) $\mathcal{N}e+47_{(-30)}$	
$\overline{6}$) $\mathcal{N}p+1_{(-30)}$	$e+20$) $\mathcal{N}e+64_{(-30)}$	

Denote this group by S_{15}.

After the substitutions S_{15} have been carried out, $\mathcal{C}$ can be sent at any time to e. This will cause the meshing to take place as desired, and after its completion send $\mathcal{C}$ to a.

The following final remark should be added: G_{82}, as defined above, contains only fixed symbols, i.e. it is a fixed routine. With a suitable choice of S_{15} it will, therefore, cause any desired meshing process to take place. Thus G_{82} can be stored permanently outside the machine, and it may be fed into the machine as a 'sub routine', as a part of the instructions of any more extensive problem, which contains one or more meshing operations. Then S_{15} must be part of the 'main routine' of that problem, it may be effected there in several parts if desired. If, in particular, the problem contains several meshing operations, only those parts of S_{15} need be repeated, in which those operations differ. And since G_{82} contains no explicit reference to its own position, i.e. to e, therefore G_{82} can be placed anywhere in the long tanks, it is only necessary that the 'main routine' take care of the proper e (by means of its S_{15}). This 'mobility' within the long tanks is, of course, an absolute necessity for 'sub routines' which are suited for use in a flexible general logical scheme of 'main routines' and (possibly multiple and interchangeable) 'sub routines'.

(12) To conclude, we must estimate the duration of a meshing process according to the instructions which we derived.

We will not count the time in effecting S_{15}, hence we begin when $\mathcal{C}$ reaches e. We follow the list of words $e, \cdots, e+81$ given in (9):

The process begins with the step — of (7), i.e. $1_0, \cdots, 23_0$, i.e. $e, \cdots, e+26$. Apart from 26 words $= \frac{26}{32}$ ms $= .81$ ms, there are the following delays: $\overline{9}$, $\overline{10}$ each averages $\frac{1}{2}$ tank $= .5$ ms; $\overline{11}$ is 1 word $= \frac{1}{32}$ ms $= .03$ ms. The total is $.81 + .5 + .5 + .03 = 1.84$ ms.

Now consider a step $l = 0, 1, \ldots, n+m$. Its make up is as follows: It begins with $1_1, \cdots, 10_1$ of (g), i.e. $e+27, \cdots, e+38$. These are 12 words $= \frac{12}{32}$ ms $= .38$ ms, and no other delays. From here on the process splits, according to which of the 4 cases (α), (β), (γ), (δ) obtains.

Consider (α) first. It begins with $1_\alpha, \cdots, 4_\alpha$ of (i), i.e. $e+43, \cdots, e+46$. Apart from 4 words $= \frac{4}{32}$ ms $= .13$ ms, there are the following delays: At the beginning of this sequence 7 words (from $e+36$ to $e+43$) $= \frac{7}{32}$ ms $= .22$ ms; from the time of $\overline{11}$ (which follows upon $e+46$, and hence is $e+47$) until the beginning of (α_1) or of (α_2) ($e+47$ or $e+64$), i.e. nothing or 17 words, averaging $\frac{1}{2}$ 17 words $= \frac{1}{2}\,\frac{17}{32}$ ms $= .27$ ms. This totals $.13 + .22 + .27 = .62$ ms. Next there is (α_1) $[(\alpha_2)]$, consisting of $1_{\alpha_1}, \cdots, 13_{\alpha_1}$ $[1_{\alpha_2}, \cdots, 13_{\alpha_2}]$ of (l), i.e. $e+47, \cdots, e+63$ $[e+64, \cdots, e+80]$. Apart from 17 words $= \frac{17}{32}$ ms $=$ .53 ms, there are the following delays: $\overline{1_5}$ averages $p+1$ words and $\frac{1}{2}$ tank; $\overline{2_5}$ averages $p+2$ words and $\frac{1}{2}$ tank; after 13_{α_1} $[13_{\alpha_2}]$ (i.e. $e+63$ $[e+80]$) there is a delay until 1_1 ($e+27$), since this delay is to be taken modulo entire tank, i.e. modulo 32 words, it amounts to 28 [11] words, i.e. an average of $\frac{1}{2}(28+11) = 18\frac{1}{2}$ words. This totals $(p+2)+(p+1)+18\frac{1}{2} = 2p + 21\frac{1}{2}$ words and $\frac{1}{2} + \frac{1}{2} = 1$ tank, i.e. $\frac{2p+21\frac{1}{2}}{32} + 1$ ms $= \frac{p}{16} + .67$ ms. The grand total for (α) is therefore $.62 + (\frac{p}{16} + .67)$ ms $= \frac{p}{16} + 1.29$ ms.

Consider next (β), (γ). These differ from (α) only inasmuch as they replace 1_α by 1_β, 2_β (2_γ, 3_γ) of (j). In either case, there is, with actual operation and delays, a direct sequence from 10_1 to 2_α, i.e. from $e+36$ to $e+44$. Hence their duration is the same as α.

Consider finally (δ). This involves the delay from 1_{10} to 1_δ of (j), i.e. from $e+36$ to $e+47$, and the word 1_δ, i.e. $e+47$, itself. This amounts to 12 words $= \frac{12}{32}$ ms $= .38$ ms.

Now of the $n + m + 1$ steps $l = 0, 1, \ldots, n + m$ all but the last one, $n + m$, are (α) or (β) or (γ); $n + m$ is (δ). Hence there are $n + m$ lasting $.38 + (\frac{p}{16} + 1.29)$ ms $= \frac{p}{16} + 1.67$ ms and 1 lasting $.38 + .38$ ms $= .76$ ms. The total duration of the entire meshing process is therefore this: $1.84 + (n + m)(\frac{p}{16} + 1.67) + .76$ ms $= 2.60 + (n + m)(\frac{p}{16} + 1.67)$ ms. For $p = 1$ this is $2.60 + (n + m)1.78$ ms, for $p = 7$ it is $2.60 + (n + m)2.11$ ms, for $p = 39$ it is $2.60 + (n + m)4.11$ ms. (Concerning these p values consider the first part of (6).)

References

Agar, J. (1998). Digital patina: Texts, spirit and the first computer. *History and Technology*, *15*(1–2), 121–135.

Ahmed, H. (2013). *Cambridge computing: The first 75 years*. London: Third Millenium.

Aspray, W. (1990). *John von Neumann and the origins of modern computing*. The MIT Press.

Barany, M. J. (2018). The officer's three names: The formal, familiar, and bureaucratic in the transnational history of scientific fellowships. http://mbarany.com/ThreeNames.pdf (downloaded 7 June 2018).

Bullynck, M., & de Mol, L. (2010). Setting-up early computer programs: D. H. Lehmer's ENIAC computation. *Archive for Mathematical Logic*, *49*(2), 123–146.

Burks, A. W. (1945). Notes on meetings with Dr. von Neumann, March–April 1945. AWB-IUPUI.

Burks, A. W. (1989). Manuscript page headed 'From H. H. Goldstine Sept '68'. Comments dated 10 February 1989. AWB-IUPUI.

Burks, A. W. (1998). Draft material for Chap. 6 of an unpublished book. AWB-IUPUI.

Burks, A. W., Goldstine, H. H., & von Neumann, J. (1946). Preliminary discussion of the logical design of an electronic computing instrument. The Institute for Advanced Study, 28 June 1946

Campbell-Kelly, M. (2011). From theory to practice: The invention of programming, 1947–1951. In C. B. Jones & J. L. Lloyd (Eds.), *Dependable and historic computing: Essays dedicated to Brian Randell on the occasion of his 75th birthday*. Berlin: Springer.

Campbell-Kelly, M., & Williams, M . R. (Eds.). (1985). *The Moore School lectures: Theory and techniques for design of electronic digital computers*. Charles Babbage Institute reprint series for the history of computing (Vol. 9). The MIT Press.

Cunningham, L. E. (1945). Proposed set-up for the punch-card equipment of the ENIAC. 4 April 1945. HHG-APS, box 23.

Curry, H. B. (1929). An analysis of logical substitution. *American Journal of Mathematics*, *51*(3), 363–384.

Curry, H. B. (1945). Letter to Herman H. Goldstine, 26 September 1945. HHG-APS, box 21.

Curry, H. B. (1950). A program composition technique as applied to inverse interpolation. U. S. Naval Ordnance Laboratory, memorandum 10337. 29 March 1950.

Curry, H. B. & Wyatt, W. A. (1946). A study of inverse interpolation of the Eniac. Ballistic Research Laboratories Report No. 615. Aberdeen Proving Ground, Maryland, 19 August 1946.

de Mol, L., Carlé, M., & Bullynck, M. (2015). Haskell before Haskell: An alternative lesson in practical logics of the ENIAC. *Journal of Logic and Computation*, *25*(4), 1011–1046.

Dyson, G. (2012). *Turing's cathedral: The origins of the digital universe*. Allen Lane.

Eames, C., & Eames, R. (1973). *A computer perspective*. Harvard University Press.

M. Priestley, *Routines of Substitution*, SpringerBriefs in History of Computing,
https://doi.org/10.1007/978-3-319-91671-2

Eckert, J. P. & Mauchly, J. W. (1945). Automatic high-speed computing: A progress report on the EDVAC. Moore School of Electrical Engineering, University of Pennsylvania, 30 September 1945.

Eckert, J. P., Mauchly, J. W., & Warren, S. R. (1945). PY Summary Report No. 1. Moore School of Electrical Engineering, University of Pennsylvania, 31 March 1945.

Ensmenger, N. (2016). The multiple meanings of a flowchart. *Information & Culture*, *51*(3), 321–351.

Gamma, E., Helm, R., Johnson, R., & Vlissides, J. (1995). *Design patterns: Elements of reusable object-oriented software*. Addison-Wesley.

Goldstine, H. H. (1944). Report of a conference on computing devices at the Ballistic Research Laboratory on 26 January 1944. ETE-UP, Plaintiff's Trial Exhibit No. 1934.

Goldstine, H. H. (1945a). Letter to Calvin Mooers, 15 November 1945. HHG-APS, box 21.

Goldstine, H. H. (1945b). Letter to Haskell B. Curry, 3 October 1945. HHG-APS, box 21.

Goldstine, H. H. (1945c). Letter to John von Neumann, 13 April 1945. HHG-APS, box 9.

Goldstine, H. H. & von Neumann, J. (1946). Draft version of Goldstine and von Neumann (1947-8). JVN-LOC, box 33, folder 7.

Goldstine, H. H. & von Neumann, J. (1947–8). Planning and coding problems for an Electronic Computing Instrument. Vol. 1, 1 April 1947; vol. 2, 15 April 1948; vol. 3, 16 August 1948. The Institute for Advanced Study.

Grier, D. A. (1996). The ENIAC, the verb 'to program' and the emergence of digital computers. *IEEE Annals of the History of Computing*, *18*(1), 51–55.

Grier, D. A. (2005). *When computers were human*. Princeton University Press.

Haigh, T., Priestley, M., & Rope, C. (2014). Reconsidering the stored-program concept. *IEEE Annals of the History of Computing*, *36*(1), 2–30.

Haigh, T., Priestley, M., & Rope, C. (2016). *ENIAC in action: Making and remaking the modern computer*. The MIT Press.

Hartree, D. R. (1949). *Calculating instruments and machines*. The University of Illinois Press.

Harvard (1946). *A manual of operation for the Automatic Sequence Controlled Calculator*. The annals of the Computation Laboratory of Harvard University (Vol. 1). Harvard University Press.

IAS (1945). Minutes of E. C. meetings #1 – #4. 12, 19, 21 and 26 November 1945. Institute for Advanced Study. HHG-APS, box 27.

IBM (1936). *Machine methods of accounting*. New York: International Business Machines Corporation.

IBM (1945). *IBM Electric punched card accounting machines principles of operation: Collator type 077*. New York: International Business Machines Corporation.

Jackson, J. B. & Metropolis, N. (1954). The MANIAC. Report LA-1725, Los Alamos, New Mexico, 15 December 1951. Revised 16 July 1954.

Jimmy (n.d.). Letter to Herman Goldstine, undated. HHG-APS, box 51.

Knuth, D. E. (1970). Von Neumann's first computer program. *Computing Surveys*, *2*(4), 247–260.

Knuth, D . E. (1973). *The art of computer programming, volume 3: Sorting and searching*. Addison-Wesley.

Kullback, S. (1942). The British GC & CS. 1 August 1942. Box 1134, entry 9032 (Historic Cryptographic Collection), RG 457, US National Archives and Records Administration, College Park, MD.

Mahoney, M. S. (2005). The histories of computing(s). *Interdisciplinary Science Reviews*, *30*(2), 119–135.

Mauchly, J. W. (1944a). Notes from a conversation with Mr. Madow of the Census Bureau. 9 October 1944. JWM-UP box 10, folder 12.

Mauchly, J. W. (1944b). Notes on a conversation with Lieutenant-Colonel S. Kullback, lecturer in statistics, George Washington University, Wednesday, October 11, 1944. JWM-UP box 10, folder 12.

Mauchly, J. W. (1945a). Notes on possible meteorological use of high speed sorting and computing devices. 14 April 1945. JWM-UP box 10, folder 12.

Mauchly, J. W. (1945b). Notes on some problems requiring high speed sorting and/or computing. 14 April 1945. JWM-UP box 10, folder 12.

Mauchly, J. W. (1945c). Notes on sorting problems occurring in cryptanalysis. 14 April 1945. JWM-UP box 10, folder 12.

Mauchly, J. W. (1946a). Automatic sorting. Unpublished typescript, April 1946. HML-UV, box 261.

Mauchly, J. W. (1946b). Sorting and collating. In (Campbell-Kelly and Williams 1985), 271–297.

McClung Fleming, E. (1974). Artifact Study. *Winterthur Portfolio*, *9*, 153–173.

Montfort, N., Baudoin, P., Bell, J., Bogost, I., Douglass, J., Marino, M. C., Mateas, M., Reas, C., Sample, M., & Vawter, N. (2013). `10 PRINT CHR(205.5+RND(1)); : GOTO 10`. The MIT Press.

Moore School (1943). The ENIAC: A report covering work until 31 December 1943. Moore School of Electrical Engineering, University of Pennsylvania.

Moore School (1944). The ENIAC: Progress report covering work from 1 January to 30 June 1944. Moore School of Electrical Engineering, University of Pennsylvania.

Nofre, D., Priestley, M., & Alberts, G. (2014). When technology became language: The origins of the linguistic conception of computer programming, 1950–1960. *Technology and Culture*, *55*(1), 40–75.

Norberg, A. L. (2005). *Computers and commerce: A study of technology and management at Eckert-Mauchly Computer Company, Engineering Research Assiciates, and Remington Rand, 1946-1957*. The MIT Press.

Priestley, M. (2017). AI and the origins of the functional programming language style. *Minds and Machines*, *27*(3), 449–472.

Priestley, M. (2018). The mathematical origins of modern computing. In S. O. Hansson (Ed.), in *Technology and mathematics: Philosophical and historical investigations*. Berlin: Springer. (forthcoming).

Rohrhuber, J. (2018). Sans-papiers as first-class citizens. Unpublished chapter.

Simondon, G. (2012). *Du mode d'existence des objets techniques*. Aubier.

Stibitz, G. R. (1944). Report to Applied Mathematics Panel on conference at Aberdeen on 26 January. HHG-APS, box 20.

Strachey, C. (2000). Fundamental concepts in programming languages. *Higher-Order and Symbolic Computation*, *13*, 11–49.

Truesdell, L. E. (1965). The development of punch card tabulation in the Bureau of the Census 1890–1940. U.S. Department of Commerce, Bureau of the Census.

Turing, A. M. (1936). On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society, Second Series*, *42*, 230–265.

Turing, A. M. (1950). Computing machinery and intelligence. *Mind*, *59*, 433–60.

von Neumann, J. (1944). Letter to J. R. Oppenheimer, 1 August 1944. Los Alamos National Laboratory Archives, LA-UR-12-24686.

von Neumann, J. (1945a). First draft of a report on the EDVAC, 30 June 1945. Moore School of Electrical Engineering, University of Pennsylvania.

von Neumann, J. (1945b). Letter to Frank Aydelotte. 1 November 1945. JVN-LOC box 12, folder 1.

von Neumann, J. (1945c). Letter to Haskell B. Curry, 20 August 1945. ETE-UP, Plaintiff's Trial Exhibit No. 3450.

von Neumann, J. (1945d). Letter to Herman Goldstine, 12 February 1945. HHG-APS, box 9.

von Neumann, J. (1945e). Letter to Herman Goldstine, 8 May 1945. HHG-APS, box 21.

von Neumann, J. (1945f). Letter to Howard Aiken *et al.*, 12 January 1945. JVN-LOC, box 7, folder 14.

von Neumann, J. (1945g). Memorandum on mechanical computing devices. 30 January 1945. HHG-APS box 21.

von Neumann, J. (1945h). Memorandum on the program of the high-speed computer project. 8 November 1945. HHG-APS box 21.

von Neumann, J. (1945i). Unpublished manuscript describing the development of a meshing routine. HHG-APS box 21. Reproduced in Appendix B of this book.

von Neumann, J. (1946). Letter to Herman Goldstine, 16 September 1946. HHG-APS, box 21.

von Neumann, J. (1947). Letter to Herman Goldstine, 2 March 1947. HHG-APS, box 20.

Wilkes, M. V., Wheeler, D. J., & Gill, S. (1951). *The preparation of programs for an electronic digital computer*. Addison-Wesley Press, Inc.

Williams, S. B. (1944). Calculating system. Bell Telephone Laboratories, 29 March 1944. HHG-APS, box 20.

Winder, R. (2005). *Bloody foreigners: The story of immigration to Britain*. Abacus.

AWB-IUPUI Arthur W. Burks papers, Institute for American Thought, Indiana University–Purdue University Indianapolis.

ETE-UP ENIAC Trial Exhibits, Master Collection, 1964–1973, University of Pennsylvania Archives and Records Center, Philadelphia.

HHG-APS Herman Heine Goldstine Papers, Mss. Ms. Coll. 19, American Philosophical Society, Philadelphia.

HML-UV Sperry-Rand Corporation, UNIVAC Division Records, Accession 182.5.I, Hagley Museum and Library, Wilmington, Delaware.

JVN-LOC John von Neumann and Klara Dan Von Neumann Papers, Library of Congress, Washington DC.

JWM-UP John W. Mauchly papers, Ms. Coll. 925, University of Pennsylvania, Philadelphia.

Index

M. Priestley, *Routines of Substitution*, SpringerBriefs in History of Computing,
https://doi.org/10.1007/978-3-319-91671-2