Appendix A
Maxwell's Equations, Equations of Motion, and Energy Balance in an Electromagnetic Field

A.1 Maxwell's Equations

Classical electrodynamics in vacuum is governed by the Maxwell equations. In the SI system of units, the Maxwell equations are

$$\nabla \cdot \boldsymbol{D} = \rho\,, \tag{A.1a}$$

$$\nabla \cdot \boldsymbol{B} = 0\,, \tag{A.1b}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\,, \tag{A.1c}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}\,, \tag{A.1d}$$

where ρ is the charge density, $\boldsymbol{j}$ is the current density, $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{B}/\mu_0$. Traditionally $\boldsymbol{B}$ is called the magnetic induction, and $\boldsymbol{H}$ is called the magnetic field, but in this book we refer to $\boldsymbol{B}$ as the magnetic field. The Maxwell equations are linear: a sum of two solutions, $\boldsymbol{E}_1$, $\boldsymbol{B}_1$ and $\boldsymbol{E}_2$, $\boldsymbol{B}_2$, is also a solution corresponding to the sum of densities $\rho_1 + \rho_2$, $\boldsymbol{j}_1 + \boldsymbol{j}_2$.

For a point charge q moving along a trajectory $\boldsymbol{r} = \boldsymbol{r}_0(t)$ the charge density and the current density are

$$\rho(\boldsymbol{r}, t) = q\delta(\boldsymbol{r} - \boldsymbol{r}_0(t))\,, \qquad \boldsymbol{j}(\boldsymbol{r}, t) = q\boldsymbol{v}(t)\delta(\boldsymbol{r} - \boldsymbol{r}_0(t))\,, \tag{A.2}$$

with $\boldsymbol{v}(t) = d\boldsymbol{r}_0(t)/dt$.

To find a particular solution of the Maxwell equations in a volume, proper boundary conditions should be specified at the volume boundary. On a surface of a good conducting metal the boundary condition requires the tangential component of the electric field to be equal to zero, $\boldsymbol{E}_t|_S = 0$.

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A.2 Wave Equations

In free space with no local charges and currents the electric field satisfies the wave equation,

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \frac{\partial^2 \boldsymbol{E}}{\partial x^2} - \frac{\partial^2 \boldsymbol{E}}{\partial y^2} - \frac{\partial^2 \boldsymbol{E}}{\partial z^2} = 0\,. \tag{A.3}$$

The same equation is valid for the magnetic field $\boldsymbol{B}$. A particular solution of Eq. (A.3) is a sinusoidal wave characterized by the frequency ω and the wave vector $\boldsymbol{k}$,

$$\boldsymbol{E} = \boldsymbol{E}_0 \sin(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})\,, \tag{A.4}$$

where $\boldsymbol{E}_0$ is a constant vector perpendicular to $\boldsymbol{k}$, $\boldsymbol{E}_0\cdot\boldsymbol{k} = 0$, and $\omega = ck$.

A.3 Vector and Scalar Potentials

It is often convenient to express the fields in terms of the *vector potential* $\boldsymbol{A}(\boldsymbol{r}, t)$ and the *scalar potential* $\phi(\boldsymbol{r}, t)$:

$$\begin{aligned} \boldsymbol{E} &= -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}\,, \\ \boldsymbol{B} &= \nabla\times\boldsymbol{A}\,. \end{aligned} \tag{A.5}$$

Substituting these equations into Maxwell's equations, we find that the second and third equations are satisfied identically. We only need to take care of the first and fourth equations.

A.4 Energy Balance and the Poynting Theorem

The electromagnetic field has an energy and momentum associated with it. The energy density of the field (energy per unit volume) is

$$u = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}) = \frac{\epsilon_0}{2}(E^2 + c^2 B^2)\,. \tag{A.6}$$

The Poynting vector,

$$\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H}\,, \tag{A.7}$$

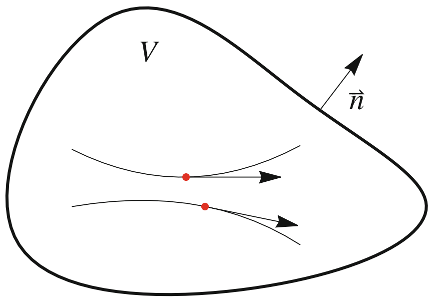

Fig. A.1 Point charges shown by red dots move inside volume V and interact through the electromagnetic field

gives the energy flow (energy per unit area per unit time) in the electromagnetic field. Consider charges that move inside a volume V enclosed by a surface A, see Fig. A.1. The Poynting theorem states

$$\frac{\partial}{\partial t}\int_V u\,dV = -\int_V \boldsymbol{j}\cdot\boldsymbol{E}\,dV - \int_A \boldsymbol{n}\cdot\boldsymbol{S}\,dA\,, \tag{A.8}$$

where $\boldsymbol{n}$ is the unit vector normal to the surface and directed outward. The left-hand side of this equation is the rate of change of the electromagnetic energy due to the interaction with moving charges. The first term on the right-hand side is the work done by the electric field on the moving charges. The second term describes the electromagnetic energy flow from the volume through the enclosing surface.

A.5 Photons

The quantum view on electromagnetic radiation is that the electromagnetic field is represented by photons. Each photon carries the energy $\hbar\omega$ and the momentum $\hbar\boldsymbol{k}$, where the vector $\boldsymbol{k}$ is the wavenumber which points to the direction of propagation of the radiation, $\hbar = 1.05\times10^{-34}$ J$\cdot$ sec is the Planck constant divided by 2π, and $k=\omega/c$.

Appendix B
Lorentz Transformations and the Relativistic Doppler Effect

B.1 Lorentz Transformation and Matrices

Consider two coordinate systems, K and K'. The system K' is moving with velocity v in the z direction relative to the system K (see Fig. B.1). The coordinates of an event in both systems are related by the Lorentz transformation

$$\begin{aligned} x &= x' \,, \\ y &= y' \,, \\ z &= \gamma(z' + \beta ct') \,, \\ t &= \gamma(t' + \beta z'/c) \,, \end{aligned} \tag{B.1}$$

where $\beta = v/c$, and $\gamma = 1/\sqrt{1-\beta^2}$.

The vector $(ct, \boldsymbol{r}) = (ct, x, y, z)$ is called a *4-vector*, and the above transformation is valid for any 4-vector quantity. The transformation from K to K' is also a Lorentz transformation, but the original frame K has a velocity $-v$ relative to the system K', so the inverse transformation is obtained from Eq. (B.1) by changing the sign of β:

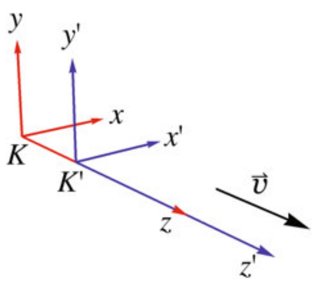

Fig. B.1 Laboratory frame K and a moving frame K'

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$$
\begin{aligned}
x' &= x\,, \\
y' &= y\,, \\
z' &= \gamma(z - \beta ct)\,, \\
t' &= \gamma(t - \beta z/c)\,.
\end{aligned}
\tag{B.2}
$$

The Lorentz transformation (B.1) can also be written in the matrix notation

$$
\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & c\beta\gamma \\ 0 & 0 & \frac{\beta\gamma}{c} & \gamma \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = L \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix}\,, \tag{B.3}
$$

where L denotes the 4×4 matrix in the middle of the equation. The advantage of using matrices is that consecutive transformations reduce to matrix multiplication.

B.2 Lorentz Contraction and Time Dilation

Two events occurring in the moving frame at the same point and separated by the time interval $\Delta t'$ will be measured by the lab observers as separated by Δt,

$$
\Delta t = \gamma \Delta t'\,. \tag{B.4}
$$

This is the effect of relativistic *time dilation*.

An object of length l' aligned in the moving frame with the z' axis will have the length l in the lab frame:

$$
l = \frac{l'}{\gamma}\,. \tag{B.5}
$$

This is the effect of relativistic *contraction*. The length in the direction transverse to the velocity is not changed.

B.3 Relativistic Doppler Effect

Consider a wave propagating in a moving frame K'. It has the time-space dependence:

$$
\propto \cos(\omega' t' - \boldsymbol{k}' \cdot \boldsymbol{r}')\,, \tag{B.6}
$$

where ω' is the frequency and $\boldsymbol{k}'$ is the wavenumber of the wave in K'. An observer that measures this wave in the reference frame K will see the time-space dependence that is obtained from the Lorentz transformation of coordinates and time in Eq. (B.6):

$$\begin{aligned}\cos(\omega' t' - \boldsymbol{k}' \cdot \boldsymbol{r}') &= \cos(\omega'\gamma(t - \beta z/c) - k_x{}'x - k_y{}'y - k_z{}'\gamma(z - \beta ct)) \\ &= \cos(\gamma(\omega' + k_z{}'\beta c)t - k_x{}'x - k_y{}'y - \gamma(k_z{}' + \omega'\beta/c)z)\,. \end{aligned} \tag{B.7}$$

We see that in the K frame this process is also a wave

$$\propto \cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{r})\,, \tag{B.8}$$

with the frequency and wavenumber

$$\begin{aligned} k_x &= k_x{}'\,, \\ k_y &= k_y{}'\,, \\ k_z &= \gamma(k_z{}' + \beta\omega'/c)\,, \\ \omega &= \gamma(\omega' + \beta c k_z{}')\,. \end{aligned} \tag{B.9}$$

Hence, the combination $(\omega, c\boldsymbol{k})$ is a 4-vector.

The above transformation is valid for any type of waves (electromagnetic, acoustic, plasma waves, etc.) Let us now apply it to electromagnetic waves in vacuum. For these waves we know that

$$\omega = ck\,. \tag{B.10}$$

Assume that an electromagnetic wave propagates at angle θ' in the frame K',

$$\cos\theta' = \frac{k_z'}{k'}\,, \tag{B.11}$$

and has a frequency ω' in that frame. What is the angle θ and the frequency ω of this wave in the lab frame? We can always choose the coordinate system such that $\boldsymbol{k} = (0, k_y, k_z)$, then

$$\tan\theta = \frac{k_y}{k_z} = \frac{k_y'}{\gamma(k_z{}' + \beta\omega'/c)} = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}\,. \tag{B.12}$$

In the limit $\gamma \gg 1$ almost all angles θ' (except for the angles very close to π) are transformed to angles $\theta \sim 1/\gamma$. This explains why radiation of an ultrarelativistic beam goes mostly in the forward direction, within an angle of the order of $1/\gamma$.

A general expression for how the Lorentz transformation changes the angle θ of an electromagnetic wave with respect to the direction of the velocity is:

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}\,, \qquad \sin\theta' = \frac{\sin\theta}{\gamma(1 - \beta\cos\theta)}\,. \tag{B.13}$$

For the frequency, a convenient formula relates ω with ω' and θ (not θ'). To derive it, we use the inverse Lorentz transformation

$$\omega' = \gamma(\omega - \beta c k_z) = \gamma(\omega - \beta c k \cos\theta)\,, \tag{B.14}$$

which gives

$$\omega = \frac{\omega'}{\gamma(1 - \beta\cos\theta)}\,. \tag{B.15}$$

Assuming a large γ and a small angle θ and using $\beta \approx 1 - 1/2\gamma^2$ and $\cos\theta = 1 - \theta^2/2$, we obtain

$$\omega = \frac{2\gamma\omega'}{1 + \gamma^2\theta^2}\,. \tag{B.16}$$

The radiation in the forward direction ($\theta = 0$) gets a large factor 2γ in the frequency transformation.

B.4 Lorentz Transformation of Fields

The electromagnetic field ($\boldsymbol{E}$, $\boldsymbol{B}$) is transformed from K' to K according to the following equations:

$$\begin{aligned} E_z &= E'_z\,, \qquad \boldsymbol{E}_\perp = \gamma\left(\boldsymbol{E}'_\perp - \boldsymbol{v}\times\boldsymbol{B}'\right)\,, \\ B_z &= B'_z\,, \qquad \boldsymbol{B}_\perp = \gamma\left(\boldsymbol{B}'_\perp + \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}'\right)\,, \end{aligned} \tag{B.17}$$

where $\boldsymbol{E}'_\perp$ and $\boldsymbol{B}'_\perp$ are the components of the electric and magnetic fields perpendicular to the velocity $\boldsymbol{v}$: $\boldsymbol{E}'_\perp = (E_x, E_y)$, $\boldsymbol{B}'_\perp = (B_x, B_y)$.

The electromagnetic potentials $(\phi/c, \boldsymbol{A})$ are transformed exactly as the 4-vector $(ct, \boldsymbol{r})$:

$$\begin{aligned} A_x &= A'_x\,, \\ A_y &= A'_y\,, \\ A_z &= \gamma\left(A'_z + \frac{v}{c^2}\phi'\right)\,, \\ \phi &= \gamma(\phi' + vA'_z)\,. \end{aligned} \tag{B.18}$$

B.5 Lorentz Transformation and Photons

It is often convenient, even in classical electrodynamics, to consider electromagnetic radiation as a collection of photons. How do we transform the parameters of a photon from K' to K? The answer is rather evident: the combination $(\boldsymbol{k}, \omega)$ constitutes a 4-vector and is transformed according to Eq. (B.9). This is of course in agreement with the fact that the pair $(\hbar\boldsymbol{k}, \hbar\omega)$ is the momentum-energy 4-vector for the photon. The number of photons in K' to K is the same — it is a relativistic invariant.

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