# Appendix A
# Prerequisites

This appendix recalls some notation and results that are needed throughout the book.

## A.1 Linear Algebra

We first review some facts from linear algebra and matrix analysis, see [151] for an advanced course.

For a $(m \times d)$-matrix $A \in \mathbb{R}^{m\times d}$ we denote by $A^j_k$ the element in the $j$'th row and $k$'th column ($j = 1, \dots, m$; $k = 1, \dots, d$). In this book the matrix space $\mathbb{R}^{m\times d}$ always comes equipped with the **Frobenius matrix (inner) product**

$$A : B := \operatorname{tr}(A^T B) = \operatorname{tr}(AB^T) = \sum_{j,k} A^j_k B^j_k, \qquad A, B \in \mathbb{R}^{m\times d},$$

which is just the Euclidean product if we identify such matrices with vectors in $\mathbb{R}^{md}$. This inner product induces the **Frobenius matrix norm**

$$|A| := \sqrt{\sum_{j,k} (A^j_k)^2}, \qquad A \in \mathbb{R}^{m\times d}.$$

While of course all norms on the finite-dimensional space $\mathbb{R}^{m\times d}$ are equivalent, some finer arguments require us to specify a matrix norm; if nothing else is stated, we always use the Frobenius norm.

The Frobenius norm can also be expressed as

$$|A| = \sqrt{\sum_i \sigma_i(A)^2}, \qquad A \in \mathbb{R}^{m\times d},$$



F. Rindler, *Calculus of Variations*, Universitext,
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where $\sigma_i(A) \geq 0$ is the $i$'th singular value of $A$, $i = 1, \dots, \min\{d, m\}$. For this, recall that every matrix $A \in \mathbb{R}^{m\times d}$ has a **(real) singular value decomposition**

$$A = P\Sigma Q^T$$

for orthogonal matrices $P \in \mathbb{R}^{m\times m}$, $Q \in \mathbb{R}^{d\times d}$ ($P^{-1} = P^T$, $Q^{-1} = Q^T$), and a diagonal matrix

$$\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_{\min\{d,m\}}) = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_{\min\{d,m\}} \\ \hline 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{m\times d}$$

with only positive diagonal entries

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0 = \sigma_{r+1} = \sigma_{r+2} = \dots = \sigma_{\min\{d,m\}},$$

called the **singular values** of $A$, where $r$ is the rank of $A$.

From the above expression using the singular values, it follows immediately that the Frobenius norm is orthogonally invariant, that is, for all $A \in \mathbb{R}^{m\times d}$ and all orthogonal $P \in \mathbb{R}^{m\times m}$, $Q \in \mathbb{R}^{d\times d}$ it holds that

$$|PA| = |A| = |AQ|.$$

A special matrix in $\mathbb{R}^{m\times d}$ is the **tensor product** of the vectors $a \in \mathbb{R}^m$, $b \in \mathbb{R}^d$, which is defined as

$$a \otimes b := ab^T \in \mathbb{R}^{m\times d}.$$

Occasionally, we will also use $a \otimes b$ for $a$ a column vector and $b$ a row-vector to denote the matrix product $ab$. While technically incorrect, this notation emphasizes that the result is a *matrix*. We recall the following elementary fact: Let $A \in \mathbb{R}^{m\times d}$. Then, rank $A \leq 1$ if and only if there exist vectors $a \in \mathbb{R}^m$, $b \in \mathbb{R}^d$ such that $A = a \otimes b$. The tensor product also interacts well with the Frobenius norm:

$$|a \otimes b| = |a| \cdot |b|, \qquad a \in \mathbb{R}^m,\ b \in \mathbb{R}^d,$$

where in $\mathbb{R}^m$ and $\mathbb{R}^d$ we use the usual Euclidean norm.

A fundamental inequality involving the determinant is the **Hadamard inequality**: Let $A \in \mathbb{R}^{d\times d}$ with columns $A_j \in \mathbb{R}^d$ ($j = 1, \dots, d$). Then,

$$|\det A| \le \prod_{j=1}^{d} |A_j| \le |A|^d.$$

An analogous formula holds with the rows of $A$.

For $A \in \mathbb{R}^{d\times d}$ the **cofactor matrix** $\operatorname{cof} A \in \mathbb{R}^{d\times d}$ of $A$ is the matrix whose $(j,k)$'th entry is $(-1)^{j+k} M_{\neg k}^{\neg j}(A)$ with $M_{\neg k}^{\neg j}(A)$ being the $(j,k)$**-minor** of $A$, i.e., the determinant of the matrix that originates from $A$ by deleting the $j$'th row and the $k$'th column. For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$\operatorname{cof} A = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

One important formula (and another way to define the cofactor matrix) is

$$\frac{\mathrm{d}}{\mathrm{d}A} \det A = \operatorname{cof} A.$$

Furthermore, **Cramer's rule** entails that

$$A(\operatorname{cof} A)^T = (\operatorname{cof} A)^T A = (\det A)\mathrm{Id}, \qquad A \in \mathbb{R}^{d\times d},$$

where Id denotes the identity matrix. In particular, if $A$ is invertible,

$$A^{-1} = \frac{(\operatorname{cof} A)^T}{\det A}. \tag{A.1}$$

From this we deduce that $\operatorname{cof} A$ is invertible if $A$ is. Sometimes, the matrix $(\operatorname{cof} A)^T$ is called the **adjugate matrix** to $A$ in the literature (we will not use this terminology here, however).

One particular consequence of Cramer's rule is **Jacobi's formula**, which says that for any continuously differentiable function $A(t)\colon \mathbb{R} \to \mathbb{R}^{d\times d}$ it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} \det A(t) = \operatorname{cof} A(t) : \frac{\mathrm{d}A(t)}{\mathrm{d}t} = \operatorname{tr}\Big[(\operatorname{cof} A(t))^T \frac{\mathrm{d}A(t)}{\mathrm{d}t}\Big].$$

In particular, if $A(t) = A_0 + tB$ ($A_0, B \in \mathbb{R}^{d\times d}$), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \det[A_0 + tB] = \operatorname{tr}\big[(\operatorname{cof} A_0)^T B + t(\operatorname{cof} B)^T B\big] = (\operatorname{cof} A_0) : B + td \det B.$$

As a consequence, we derive that if $\frac{\mathrm{d}}{\mathrm{d}t}\det[A_0 + tB]$ is constant, then necessarily $\det B = 0$.

The **special orthogonal group** $\mathrm{SO}(d)$ is defined as

$$\mathrm{SO}(d) := \left\{ Q \in \mathbb{R}^{d\times d} \;:\; Q \text{ invertible}, \, Q^{-1} = Q^T, \det Q = 1 \right\}.$$

It has the following useful property, which can be verified via (A.1):

$$\operatorname{cof} Q = Q \qquad \text{for all } Q \in \mathrm{SO}(d).$$

Any $Q \in \mathrm{SO}(2) \subset \mathbb{R}^{2\times 2}$ (a rotation) has the form

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \in [0, 2\pi)$. For the convex hull $\mathrm{SO}(2)^{**}$ of $\mathrm{SO}(2)$ one may compute

$$\mathrm{SO}(2)^{**} = \left\{ A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \;:\; a^2 + b^2 \leq 1 \right\}.$$

Moreover, a Taylor expansion and the fact that the Lie algebra of the Lie group SO(2) is the vector space of all skew-symmetric matrices yields

$$\operatorname{dist}(\mathrm{Id} + A, \mathrm{SO}(2)) \leq \frac{1}{2}|A + A^T| + C|A|^2 \tag{A.2}$$

for some $C > 0$.

We also recall a special case of the theorem on the **Jordan normal form** for real $(2 \times 2)$-matrices: Let $A \in \mathbb{R}^{2\times 2}$. Then, there exists an invertible matrix $S \in \mathbb{R}^{2\times 2}$ such that

$$S^{-1}AS = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \qquad \text{or} \qquad S^{-1}AS = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

with $a, c \in \mathbb{R}$ and $b \in \{0, 1\}$

Finally, any square matrix $A \in \mathbb{R}^{d\times d}$ has a real **polar decomposition**

$$A = QS,$$

where $Q$ is orthogonal ($Q^{-1} = Q^T$) and $S$ is symmetric and positive definite.

In a few instances we also deal with fourth-order **tensors** $\mathbf{T} = \mathbf{T}^{ik}_{jl}$ ($i, k = 1, \ldots, m$; $j, l = 1, \ldots, d$). They define bilinear forms on $\mathbb{R}^{m\times d}$ via

$$A : \mathbf{T}B := \sum_{i,k} \sum_{j,l} \mathbf{T}^{ik}_{jl} A^i_j B^k_l.$$

We call $\mathbf{T}$ symmetric and positive definite if the corresponding bilinear form has these properties.

## A.2 Functional Analysis

We assume that the reader has a solid foundation in the basic notions of functional analysis such as Banach spaces and their duals, weak/weak* convergence (and topology), reflexivity, and weak/weak* compactness. Note that in this book we mostly only need convergence of sequences and rarely more advanced topological concepts. One very thorough reference for most of this material is [74].

Let $X$ be a Banach space. The application of $x^* \in X^*$ to $x \in X$ is often expressed via the **duality pairing** $\langle x, x^*\rangle := x^*(x)$. We write $x_j \rightharpoonup x$ in $X$ for **weak convergence**, that is, $\langle x_j, x^*\rangle \to \langle x, x^*\rangle$ for all $x^* \in X^*$, and $x_j^* \overset{*}{\rightharpoonup} x^*$ in $X^*$ for **weak* convergence**, that is, $\langle x, x_j^*\rangle \to \langle x, x^*\rangle$ for all $x \in X$. The weak* topology is metrizable on norm-bounded sets in the dual to a separable Banach space. Likewise, in reflexive and separable Banach spaces the weak topology is metrizable on norm-bounded sets. In this context we note that in Banach spaces topological weak compactness is equivalent to sequential weak compactness by the Eberlein–Šmulian theorem. Also recall that the norm in a Banach space is lower semicontinuous with respect to weak convergence: If $x_j \rightharpoonup x$ in $X$, then $\|x\| \le \liminf_{j\to\infty} \|x_j\|$.

**Theorem A.1** (**Hahn–Banach separation theorem**). *Let $X$ be a Banach space and let $K, F \subset X$ be disjoint, non-empty, and convex subsets of $X$ such that $K$ is compact and $F$ is closed. Then, $K$ and $F$ can be separated by a hyperplane, that is, there exists an $x^* \in X^*$ such that*

$$\sup_{x\in K} \langle x, x^*\rangle < \inf_{x\in F} \langle x, x^*\rangle.$$

**Theorem A.2** (**Weak compactness**). *Let $X$ be a separable, reflexive Banach space. Then, norm-bounded sets in $X$ are sequentially weakly precompact.*

**Theorem A.3** (**Banach–Alaoglu**). *Let $X$ be a separable Banach space. Then, norm-bounded sets in the dual space $X^*$ are weakly* sequentially precompact.*

Weak convergence can be "improved" to strong convergence in the following way (see Section I.1.2 in [106] for a proof):

**Lemma A.4** (**Mazur**). *Let $x_j \rightharpoonup x$ in a Banach space $X$. Then, there exists a sequence $(y_j) \subset X$ of convex combinations,*

$$y_j = \sum_{n=j}^{N(j)} \theta_n^{(j)} x_n, \qquad \theta_n^{(j)} \in [0, 1], \qquad \sum_{n=j}^{N(j)} \theta_n^{(j)} = 1$$

*such that $y_j \to x$ in $X$.*

## A.3 Measure Theory

We assume that the reader is familiar with the notion of Lebesgue- and Borel-measurability, negligible sets, and $L^p$-spaces; a good introduction is [236]. For the $d$-dimensional Lebesgue measure we write $\mathscr{L}^d$ or $\mathscr{L}^d_x$ if we want to stress the integration variable. Often, however, the Lebesgue measure of a Borel- or Lebesgue-measurable set $A \subset \mathbb{R}^d$ is simply denoted by $|A|$. We also write $\omega_d$ for the volume of the $d$-dimensional unit ball.

The **indicator function** of a subset $A \in \mathbb{R}^d$ is

$$\mathbb{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise}, \end{cases} \qquad x \in \mathbb{R}^d.$$

In the following we recall some basic results that are needed throughout the book.

**Lemma A.5** *Let $B \subset \mathbb{R}^d$ be a Borel set. A function $f : B \to \mathbb{R}^N$ is Lebesgue-measurable if and only if there exists a sequence of simple functions*

$$f_j := \sum_{k=1}^{K(j)} v_k^{(j)} \mathbb{1}_{E_k^{(j)}}$$

*such that*

$$f_j \to f \quad \text{pointwise} \qquad \text{as } j \to \infty,$$

*where $K(j) \in \mathbb{N}$, the $E_k^{(j)} \subset B$ are Lebesgue-measurable sets with $\bigcup_{k=1}^{K(j)} E_k^{(j)} = B$, and $v_k^{(j)} \in \mathbb{R}^N$ for all $j, k$.*

**Lemma A.6** **(Fatou).** *Let $f_j : \mathbb{R}^d \to [0, +\infty]$, $j \in \mathbb{N}$, be Lebesgue-measurable functions. Then,*

$$\int \liminf_{j\to\infty} f_j(x)\, dx \le \liminf_{j\to\infty} \int f_j(x)\, dx.$$

**Lemma A.7** **(Monotone convergence).** *Let $f_j : \mathbb{R}^d \to [0, +\infty]$, $j \in \mathbb{N}$, be Lebesgue-measurable functions with $f_j(x) \uparrow f(x)$ for almost every $x \in \Omega$. Then, $f : \mathbb{R}^d \to [0, +\infty]$ is measurable and*

$$\int f(x)\, dx = \lim_{j\to\infty} \int f_j(x)\, dx.$$

**Lemma A.8** *If $f_j \to f$ (strongly) in $L^p(\mathbb{R}^d)$, $p \in [1, \infty]$, that is,*

$$\|f_j - f\|_{L^p} \to 0 \qquad \text{as } j \to \infty,$$

*then there exists a subsequence (not explicitly labeled; we do not choose different indices for subsequences if the original sequence is discarded at the same time) such that $f_j \to f$ pointwise almost everywhere.*

**Theorem A.9 (Lebesgue dominated convergence theorem).** *Let $f_j : \mathbb{R}^d \to \mathbb{R}^N$, $j \in \mathbb{N}$, be Lebesgue-measurable functions such that there exists an* $\mathrm{L}^p$*-integrable majorant* $g \in \mathrm{L}^p(\mathbb{R}^d)$ *for some* $p \in [1, \infty)$, *that is,*

$$|f_j| \leq g \qquad \text{for all } j \in \mathbb{N}.$$

*If $f_j \to f$ pointwise almost everywhere for some $f : \mathbb{R}^d \to \mathbb{R}$, then also $f_j \to f$ in* $\mathrm{L}^p$*; in particular,* $f \in \mathrm{L}^p(\mathbb{R}^d)$.

The following strengthening of Lebesgue's theorem is often useful in the calculus of variations:

**Theorem A.10 (Pratt).** *Let $f_j : \mathbb{R}^d \to \mathbb{R}^N$, $j \in \mathbb{N}$, be Lebesgue-measurable functions. If $f_j \to f$ pointwise almost everywhere (or in measure) and there exists a sequence $(g_j) \subset \mathrm{L}^1(\mathbb{R}^d)$ with $g_j \to g$ in $\mathrm{L}^1$ such that $|f_j| \leq g_j$, then $f_j \to f$ in* $\mathrm{L}^1$.

The following convergence theorem is of fundamental significance:

**Theorem A.11 (Vitali).** *Let $\Omega \subset \mathbb{R}^d$ be bounded and let $(f_j) \subset \mathrm{L}^p(\Omega; \mathbb{R}^m)$,* $p \in [1, \infty)$. *Assume furthermore that the following two conditions hold:*

*(i)* No oscillations: *$f_j \to f$ in measure, that is, for all $\delta > 0$,*

$$\Big|\Big\{ x \in \Omega \,:\, |f_j(x) - f(x)| > \delta \Big\}\Big| \to 0 \qquad \text{as } j \to \infty.$$

*(ii)* No concentrations: *the family $\{f_j\}_j$ is* $\mathrm{L}^p$*-equiintegrable.*

*Then, $f_j \to f$ in* $\mathrm{L}^p$.

Here, $\{f_j\}_j \subset \mathrm{L}^p(\Omega; \mathbb{R}^m)$ is called $\mathrm{L}^p$**-equiintegrable** if one of the following equivalent conditions is satisfied:

(i) $\displaystyle\lim_{R\uparrow\infty} \sup_{j\in\mathbb{N}} \int_{\{|f_j|>R\}} |f_j|^p \,\mathrm{d}x = 0;$

(ii) $\displaystyle\lim_{R\uparrow\infty} \limsup_{j\to\infty} \int_{\{|f_j|>R\}} |f_j|^p \,\mathrm{d}x = 0;$

(iii) for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all Borel sets $B \subset \Omega$ with $|B| < \delta$ we have

$$\sup_{j\in\mathbb{N}} \int_B |f_j|^p \,\mathrm{d}x < \varepsilon.$$

**Theorem A.12 (Dunford–Pettis).** *Let $\Omega \subset \mathbb{R}^d$ be bounded and open. A norm-bounded family $\{f_j\}_{j\in\mathbb{N}} \subset \mathrm{L}^1(\Omega)$ is equiintegrable if and only if it is weakly sequentially precompact in* $\mathrm{L}^1(\Omega)$.

We remark that the usual formulation of the Dunford–Pettis theorem only mentions *topological* precompactness. The statement above follows by also utilizing the Eberlein–Šmulian theorem (see Chapter V in [74]).

**Theorem A.13** **(Egorov).** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Borel set and let $f_j \colon \Omega \to \mathbb{R}^N$, $j \in \mathbb{N}$, be Lebesgue-measurable functions. If $f_j \to f$ pointwise almost everywhere, then for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ with $|\Omega \setminus K_\varepsilon| \leq \varepsilon$ and such that $f_j \to f$ uniformly in $K_\varepsilon$.*

**Theorem A.14** **(Radon–Riesz).** *Let $p \in (1, \infty)$ and let $(f_j) \subset \mathrm{L}^p(\Omega; \mathbb{R}^m)$ with $f_j \rightharpoonup f$ (weak convergence) as well as $\|f_j\|_{\mathrm{L}^p} \to \|f\|_{\mathrm{L}^p}$. Then, $f_j \to f$ in $\mathrm{L}^p$.*

The following *covering theorem* is a handy tool for several constructions, see Theorem 2.19 in [15] for a proof:

**Theorem A.15** **(Vitali covering theorem).** *Let $\Omega, D \subset \mathbb{R}^d$ be open and bounded. Then, there exist $a_k \in \Omega$, $r_k > 0$, where $k \in \mathbb{N}$, such that we may write $\Omega$ as the disjoint union*

$$\Omega = Z \cup \bigcup_{k=1}^{\infty} \overline{D(a_k, r_k)}, \qquad D(a_k, r_k) := a_k + r_k D,$$

*with $Z \subset \Omega$ a Lebesgue-negligible set ($|Z| = 0$). Moreover, if for almost every $x \in \Omega$ we are given a real number $r(x) > 0$, then we may additionally require of the cover that $r_k < r(a_k)$ for all $k \in \mathbb{N}$.*

**Theorem A.16** **(Lusin).** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Borel set and let $f \colon \Omega \to \mathbb{R}^N$ be Lebesgue-measurable. Then, for every $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $|\Omega \setminus K| \leq \varepsilon$ and $f|_K$ is continuous.*

**Theorem A.17** **(Sard).** *Let $f \colon \mathbb{R}^d \to \mathbb{R}$ be $d$ times continuously differentiable. Then,*

$$\mathscr{L}^1(f(S)) = 0, \qquad \text{where} \qquad S := \bigl\{ x \in \mathbb{R}^d \;:\; \nabla f(x) = 0 \bigr\}.$$

We also need other measures than Lebesgue measure on subsets of $\mathbb{R}^N$ (this will usually be either $\mathbb{R}^d$ or a matrix space $\mathbb{R}^{m \times d}$, which is identified with $\mathbb{R}^{md}$). All of these abstract measures will be **Borel measures**, that is, they are defined on the **Borel $\sigma$-algebra** $\mathfrak{B}(\mathbb{R}^N)$ of $\mathbb{R}^N$, which is the smallest $\sigma$-algebra that contains all the open sets. All positive Borel measures defined on $\mathbb{R}^N$ that do not take the value $+\infty$ are collected in the set $\mathscr{M}^+(\mathbb{R}^N)$ of **(finite) positive Radon measures**; its subclass of **probability measures** is $\mathscr{M}^1(\mathbb{R}^N)$. We remark that all $\sigma$-finite measures on $\mathbb{R}^N$ are in fact **inner regular**, meaning that for every Borel set $B \subset \mathbb{R}^N$ it holds that

$$\mu(B) = \sup\bigl\{ \mu(K) \;:\; K \subset B \text{ compact} \bigr\}.$$

A **local Radon measure** $\mu$ is a set function $\mu \colon \mathfrak{B}(\mathbb{R}^N) \to [0, +\infty]$ such that $\mu$ restricted to (subsets of) any compact set is a finite Radon measure. In this case we write $\mu \in \mathscr{M}^+_{\mathrm{loc}}(\mathbb{R}^N)$. We also use $\mathscr{M}^+(U)$, $\mathscr{M}^+_{\mathrm{loc}}(U)$, $\mathscr{M}^1(U)$ for the subset of measures that only charge $U \subset \mathbb{R}^N$, that is, the measure of the complement of $U$ is zero. A good reference for (advanced) measure theory is [15].

For $h \colon \mathbb{R}^N \to \mathbb{R}$ and $\mu \in \mathscr{M}^+(\mathbb{R}^N)$ we define the **duality pairing**

$$\langle h, \mu \rangle := \int h(A) \,\mathrm{d}\mu(A)$$

whenever this integral makes sense. The following notation is convenient for the **barycenter** of a (finite, positive) Borel measure $\mu \in \mathscr{M}^+(\mathbb{R}^N)$:

$$[\mu] := \langle \mathrm{id}, \mu \rangle = \int A \,\mathrm{d}\mu(A).$$

We also define the **support** of $\mu \in \mathscr{M}^+(\mathbb{R}^N)$ by

$$\operatorname{supp} \mu := \left\{ x \in \mathbb{R}^N \;\colon\; \mu(B(x,r)) > 0 \text{ for all } r > 0 \right\},$$

where $B(x_0, r) \subset \mathbb{R}^N$ is the ball with center $x_0$ and radius $r > 0$. The **restriction** of a Borel measure $\mu \in \mathscr{M}^+(\mathbb{R}^N)$ to a Borel set $A \subset \mathbb{R}^N$ is

$$(\mu \llcorner A)(B) := \mu(A \cap B) \qquad \text{for any Borel set } B \subset \mathbb{R}^N.$$

Probability measures and convex functions interact well:

**Lemma A.18** (**Jensen inequality**). *For all probability measures $\mu \in \mathscr{M}^1(\mathbb{R}^N)$ and all convex $h \colon \mathbb{R}^N \to \mathbb{R}$ it holds that*

$$h([\mu]) \le \int h(A) \, d\mu(A).$$

We say that a sequence $(\mu_j) \subset \mathscr{M}^+(\mathbb{R}^N)$ **converges weakly*** in $\mathscr{M}^+(\mathbb{R}^N)$ to $\mu \in \mathscr{M}^+(\mathbb{R}^N)$, in symbols "$\mu_j \overset{*}{\rightharpoonup} \mu$", if $\langle \psi, \mu_j \rangle \to \langle \psi, \mu \rangle$ for all $\psi \in \mathrm{C}_0(\mathbb{R}^N)$. We speak of **local weak* convergence** if $\langle \psi, \mu_j \rangle \to \langle \psi, \mu \rangle$ for all $\psi \in \mathrm{C}_c(\mathbb{R}^N)$. A sequence $(\mu_j) \subset \mathscr{M}^+(\mathbb{R}^N)$ with $\sup_j \mu_j(\mathbb{R}^N) < \infty$ has a weakly* converging subsequence by Theorem A.2.

We also recall a useful convergence lemma:

**Lemma A.19** *Let $\mu_j \overset{*}{\rightharpoonup} \mu$ in $\mathscr{M}^+_{\mathrm{loc}}(\mathbb{R}^N)$. Then for every lower semicontinuous function $g \colon \mathbb{R}^N \to [0, \infty]$ it holds that*

$$\int g \, d\mu \le \liminf_{j \to \infty} \int g \, d\mu_j,$$

*and for every upper semicontinuous function $h \colon \mathbb{R}^N \to [0, \infty)$ with compact support it holds that*

$$\int h \, d\mu \ge \limsup_{j \to \infty} \int h \, d\mu_j.$$

*In particular, for $U \subset \mathbb{R}^N$ open and $K \subset \mathbb{R}^N$ compact,*

$$\mu(U) \leq \liminf_{j\to\infty} \mu_j(U) \qquad \textit{and} \qquad \mu(K) \geq \limsup_{j\to\infty} \mu_j(K).$$

Similar definitions and statements apply to $\mathscr{M}^+(U)$, $\mathscr{M}^+_{\mathrm{loc}}(U)$, $\mathscr{M}^1(U)$.

Finally, we recall a very useful "continuity" property of measurable functions:

**Theorem A.20** *Let $f \in \mathrm{L}^1(\mathbb{R}^N, \mu)$, that is, $f$ is $\mu$-integrable, where $\mu \in \mathscr{M}^+(\mathbb{R}^N)$. Then, $\mu$-almost every $x_0 \in \mathbb{R}^N$ is a* ***Lebesgue point*** *of $f$ with respect to $\mu$, that is,*

$$\lim_{r\downarrow 0} \fint_{B(x_0,r)} |f(x) - f(x_0)| \, d\mu(x) = 0,$$

*where* $\fint_{B(x_0,r)} := |B(x_0,r)|^{-1} \int_{B(x_0,r)}$.

We denote by $\mathscr{H}^s$ the $s$-dimensional Hausdorff measure, $0 \leq s < \infty$. For the definition of this measure and the associated notion of **$\mathscr{H}^s$-rectifiable sets** (which is not important for most of this book), we refer to [15].

## A.4 Vector Measures

In this section we exhibit a few aspects of the theory of **vector (Radon) measures** (often just called "measures" in this book), which are $\sigma$-additive set functions $\mu \colon \mathfrak{B}(\mathbb{R}^d) \to \mathbb{R}^N$ (in particular, $\mu(\emptyset) = 0$). All such $\mu$ are collected in the space $\mathscr{M}(\mathbb{R}^d; \mathbb{R}^N)$; likewise define $\mathscr{M}(\Omega; \mathbb{R}^N)$ and $\mathscr{M}(\overline{\Omega}; \mathbb{R}^N)$ for an open set $\Omega \subset \mathbb{R}^d$. We will also use **local vector measures**, defined analogously to the above, which are collected in the set $\mathscr{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$.

If for the target dimension we have $N = 1$, then we simply write $\mathscr{M}(\Omega)$ instead of $\mathscr{M}(\Omega; \mathbb{R})$; the elements of this space are called **signed (Radon) measures**, but in this case we also usually just speak of "measures".

The **total variation measure** of $\mu \in \mathscr{M}(\mathbb{R}^d; \mathbb{R}^N)$ is the positive measure $|\mu| \in \mathscr{M}^+(\mathbb{R}^d)$ defined as

$$|\mu|(B) := \sup \left\{ \sum_{k=1}^{\infty} |\mu(B_k)| \;:\; B = \bigcup_{k=1}^{\infty} B_k \text{ as a disjoint union of Borel sets} \right\}.$$

It can be shown that for all open sets $U \subset \mathbb{R}^d$ it holds that

$$|\mu|(U) = \sup \left\{ \int \psi \cdot \mathrm{d}\mu \;:\; \psi \in \mathrm{C}_c(U; \mathbb{R}^N), \;\; \|\psi\|_\infty \leq 1 \right\}, \qquad \text{(A.3)}$$

where the dot "·" indicates that $\mu$'s values are to be scalar-multiplied with the values of $\psi$ (e.g. $\int (v_0 \mathbb{1}_B)\cdot d\mu = v_0 \cdot \mu(B)$ for $v_0 \in \mathbb{R}^N$ and $B$ a Borel set). See Proposition 1.47 in [15] for a proof. We also set $\operatorname{supp} \mu := \operatorname{supp} |\mu|$.

There is an alternative, dual, view on vector measures, expressed in the following important theorem:

**Theorem A.21** **(Riesz representation theorem).** *The space of vector Radon measures $\mathscr{M}(\mathbb{R}^d;\mathbb{R}^N)$ is isometrically isomorphic to the dual space $\mathrm{C}_0(\mathbb{R}^d;\mathbb{R}^N)^*$ via the duality pairing*

$$\langle \varphi, \mu \rangle = \int \varphi \cdot d\mu, \qquad \varphi \in \mathrm{C}_0(\mathbb{R}^d;\mathbb{R}^N),\ \ \mu \in \mathscr{M}(\mathbb{R}^d;\mathbb{R}^N).$$

As an easy, often convenient, consequence, we can *define* measures through their *action* on $\mathrm{C}_0(\mathbb{R}^d;\mathbb{R}^N)$. Note, however, that we then need to check the boundedness $|\langle \varphi, \mu \rangle| \le \|\varphi\|_\infty$ for all $\varphi \in \mathrm{C}_0(\mathbb{R}^d;\mathbb{R}^N)$.

If an element $\mu \in \mathrm{C}_0(\mathbb{R}^d)^*$ is additionally positive, that is, $\langle \varphi, \mu \rangle \ge 0$ for $\varphi \ge 0$, and normalized, that is, $\langle \mathbb{1}, \mu \rangle = 1$ (here, $\mathbb{1} = 1$ on the whole space), then the $\mu$ from the Riesz representation theorem is a *probability measure*, $\mu \in \mathscr{M}^1(\mathbb{R}^N)$.

The weak* convergence of vector measures is defined exactly as for positive measures, namely by considering vector measures as elements of $\mathrm{C}_0(\mathbb{R}^d;\mathbb{R}^N)^*$. Sometimes, for a norm-bounded sequence $(v_j) \subset \mathrm{L}^1(\Omega;\mathbb{R}^N)$, we will say that "$v_j \overset{*}{\rightharpoonup} \mu$ in $\mathscr{M}(\Omega;\mathbb{R}^N)$" when really we mean $v_j \mathscr{L}^d \llcorner \Omega \overset{*}{\rightharpoonup} \mu$ in $\mathscr{M}(\Omega;\mathbb{R}^N)$. A sequence $(\mu_j) \subset \mathscr{M}(\Omega;\mathbb{R}^N)$ with $\sup_j |\mu_j|(\Omega) < \infty$ has a weakly* converging subsequence by Theorem A.2.

The following lemma is proved in Proposition 1.62 (b) of [15].

**Lemma A.22** *Let $\mu_j \overset{*}{\rightharpoonup} \mu$ in $\mathscr{M}(\mathbb{R}^d;\mathbb{R}^N)$ and assume that $|\mu_j| \overset{*}{\rightharpoonup} \Lambda \in \mathscr{M}^+(\mathbb{R}^d)$. If $K \subset \mathbb{R}^d$ is compact and $\Lambda(\partial K) = 0$, then $\mu_j(K) \to \mu(K)$. Moreover, if $h\colon \mathbb{R}^d \to \mathbb{R}$ is a bounded Borel function with compact support and a $\Lambda$-negligible set of discontinuity points, then*

$$\int h \, d\mu_j \to \int h \, d\mu.$$

Of fundamental importance is the following theorem:

**Theorem A.23** **(Besicovitch differentiation theorem).** *Given $\mu \in \mathscr{M}(\mathbb{R}^d;\mathbb{R}^N)$ and $\nu \in \mathscr{M}^+(\mathbb{R}^d)$, for $\nu$-almost every $x_0 \in \mathbb{R}^d$ in the support of $\nu$, the limit*

$$\frac{d\mu}{d\nu}(x_0) := \lim_{r \downarrow 0} \frac{\mu(B(x_0,r))}{\nu(B(x_0,r))}$$

*exists in $\mathbb{R}^N$ and is called the* ***Radon–Nikodým derivative*** *of $\mu$ with respect to $\nu$, Moreover, the* ***Lebesgue–Radon–Nikodým decomposition*** *of $\mu$ is given as*

$$\mu = \frac{d\mu}{d\nu}\nu + \mu^s.$$

*Here, $\mu^s = \mu \llcorner E$ is* ***singular*** *with respect to $\nu$ (that is, $\mu^s$ is concentrated on a $\nu$-negligible set), where*

$$E := (\mathbb{R}^d \setminus \operatorname{supp} \nu) \cup \left\{ x \in \operatorname{supp} \nu \ : \ \lim_{r \downarrow 0} \frac{|\mu|(B(x,r))}{\nu(B(x,r))} = \infty \right\}.$$

Finally, for a Borel measure $\mu \in \mathscr{M}(\Omega; \mathbb{R}^N)$ and a surjective Borel map $\varphi \colon \Omega \to \Omega' \subset \mathbb{R}^n$, we define the **push-forward measure** of $\mu$ under $\varphi$ via

$$\varphi_{\#}\mu := \mu \circ \varphi^{-1} \in \mathscr{M}(\Omega'; \mathbb{R}^N).$$

We have the following transformation formula for any $g \colon \Omega' \to \mathbb{R}$:

$$\int_{\Omega'} g \, \mathrm{d}(\varphi_{\#}\mu) = \int_{\Omega} g \circ \varphi \, \mathrm{d}\mu,$$

provided these integrals are defined.

## A.5 Sobolev and Other Function Spaces

We give a brief overview of Sobolev spaces, see [176] or [111] for more detailed accounts and proofs.

In all of the following we assume that $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain, that is, $\Omega$ is open, bounded, connected, and has a boundary that is the union of finitely many Lipschitz manifolds. We also let $p \in [1, \infty]$, unless otherwise indicated. As usual, we denote by $\mathrm{C}(\Omega) = \mathrm{C}^0(\Omega)$, $\mathrm{C}^k(\Omega)$, $k = 1, 2, \ldots$, the spaces of continuous and $k$ times continuously differentiable functions. The spaces $\mathrm{C}^k(\overline{\Omega})$ contain the $\mathrm{C}^k(\Omega)$-functions such that all $l$'th-order derivatives for $l \leq k$ can be continuously extended to $\overline{\Omega}$. As norms in these spaces we have

$$\|u\|_{\mathrm{C}^k} := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_\infty, \qquad u \in \mathrm{C}^k(\Omega), \qquad k = 0, 1, 2, \ldots,$$

where $\|\bullet\|_\infty$ is the supremum norm. Here, the sum is over all **multi-indices** $\alpha \in (\mathbb{N} \cup \{0\})^d$ with $|\alpha| := \alpha_1 + \cdots + \alpha_d \leq k$, and

$$\partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}$$

is the $\alpha$-derivative operator.

Similarly, we define the linear space $\mathrm{C}^\infty(\Omega)$ of infinitely-often differentiable functions, but this cannot be equipped with a complete norm. A subscript "$_c$" indicates that all functions $u$ in the respective function space (e.g. $\mathrm{C}_c^\infty(\Omega)$) must have their **support**

$$\operatorname{supp} u := \overline{\{ x \in \Omega \,:\, u(x) \neq 0 \}}$$

compactly contained in $\Omega$ (so, $\operatorname{supp} u \subset \Omega$ and $\operatorname{supp} u$ compact). For the compact containment of a bounded set $A$ in an open set $B$ we write $A \Subset B$, which means that $\overline{A} \subset B$. In this way, the previous condition could be written as $\operatorname{supp} u \Subset \Omega$. We denote by $\mathrm{C}_0^k(\Omega)$ the closure of $\mathrm{C}_c^\infty(\Omega)$ in $\mathrm{C}^k(\Omega)$. All the spaces $\mathrm{C}_0^k(\Omega)$ are separable.

For $k \in \mathbb{N}$ a positive integer and $p \in [1, \infty]$, the **Sobolev space** $\mathrm{W}^{k,p}(\Omega)$ is defined to contain all functions $u \in \mathrm{L}^p(\Omega)$ such that the **weak derivative** $\partial^\alpha u$ exists and lies in $\mathrm{L}^p(\Omega)$ for all multi-indices $\alpha \in (\mathbb{N} \cup \{0\})^d$ with $|\alpha| \leq k$. This means that for every such $\alpha$, there is a (unique) function $v_\alpha \in \mathrm{L}^p(\Omega)$ satisfying

$$\int v_\alpha \cdot \psi \, \mathrm{d}x = (-1)^{|\alpha|} \int u \cdot \partial^\alpha \psi \, \mathrm{d}x \qquad \text{for all } \psi \in \mathrm{C}_c^\infty(\Omega),$$

and we write $\partial^\alpha u$ for this $v_\alpha$. The uniqueness follows from the Fundamental Lemma 3.10 of the calculus of variations. Clearly, if $u \in \mathrm{C}^k(\Omega)$, then all $k$'th-order weak derivatives coincide with their classical counterparts. As norm in $\mathrm{W}^{k,p}(\Omega)$, $p \in [1, \infty)$, we use

$$\|u\|_{\mathrm{W}^{k,p}} := \left( \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{\mathrm{L}^p}^p \right)^{1/p}, \qquad u \in \mathrm{W}^{k,p}(\Omega).$$

For $p = \infty$, we set

$$\|u\|_{\mathrm{W}^{k,\infty}} := \max_{|\alpha| \leq k} \|\partial^\alpha u\|_{\mathrm{L}^\infty}, \qquad u \in \mathrm{W}^{k,\infty}(\Omega).$$

Under these norms, the sets $\mathrm{W}^{k,p}(\Omega)$ become Banach spaces.

For $u \in \mathrm{W}^{1,p}(\Omega)$ we further define the **weak gradient** and **weak divergence**,

$$\nabla u := (\partial_1 u, \partial_2 u, \dots, \partial_d u), \qquad \operatorname{div} u := \partial_1 u + \partial_2 u + \cdots + \partial_d u.$$

Concerning the boundary values of Sobolev functions we have:

**Theorem A.24** **(Trace).** *For $p \in [1, \infty]$ there exists a linear* ***trace operator***

$$\operatorname{tr}_\Omega \colon \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}^p(\partial\Omega)$$

*such that*

$$\operatorname{tr}_\Omega(\varphi) = \varphi|_{\partial\Omega} \qquad \textit{if } \varphi \in \mathrm{C}(\overline{\Omega}).$$

*We write $\operatorname{tr}_\Omega(u)$ simply as $u|_{\partial\Omega}$. For $p \in (1, \infty)$ the operator $\operatorname{tr}_\Omega$ is bounded and weakly continuous between $\mathrm{W}^{1,p}(\Omega)$ and $\mathrm{L}^p(\partial\Omega)$.*

For $p \in (1, \infty)$ denote the image of $\mathrm{W}^{1,p}(\Omega)$ under $\mathrm{tr}_\Omega$ by $\mathrm{W}^{1-1/p,p}(\partial\Omega)$, which is called the **trace space** of $\mathrm{W}^{1,p}(\Omega)$. The norms on $\mathrm{W}^{1-1/p,p}(\partial\Omega)$ involve fractional derivatives, see [176] for details. For $p = 1$, the trace space is $\mathrm{L}^1(\partial\Omega, \mathscr{H}^{d-1} \llcorner \partial\Omega)$, which we will denote by just $\mathrm{L}^1(\partial\Omega)$.

We write $\mathrm{W}^{1,p}_0(\Omega)$ for the linear subspace of $\mathrm{W}^{1,p}(\Omega)$ consisting of all $\mathrm{W}^{1,p}$-functions with zero boundary values (in the sense of trace). More generally, we use $\mathrm{W}^{1,p}_g(\Omega)$ with $g \in \mathrm{W}^{1-1/p,p}(\partial\Omega)$ (with the convention $\mathrm{W}^{0,1}(\partial\Omega) = \mathrm{L}^1(\partial\Omega)$ in the case $p = 1$), for the affine subspace of all $\mathrm{W}^{1,p}$-functions with boundary trace $g$.

The following are some properties of Sobolev spaces, stated for simplicity only for the first-order space $\mathrm{W}^{1,p}(\Omega)$.

**Theorem A.25** **(Extension).** *Every $u \in \mathrm{W}^{1,p}(\Omega)$ can be extended to $\bar{u} \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ with $\|\bar{u}\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \leq C\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}$, where $C = C(\Omega, p) > 0$ is a constant.*

**Theorem A.26** **(Poincaré inequalities).** *Let $u \in \mathrm{W}^{1,p}(\Omega)$.*

*(i) If $u|_{\partial\Omega} = 0$, then*

$$\|u\|_{\mathrm{L}^p} \leq C\|\nabla u\|_{\mathrm{L}^p},$$

*where $C = C(\Omega, p) > 0$ is a constant.*

*(ii) Setting $[u]_\Omega := \fint_\Omega u\,dx$, it furthermore holds that*

$$\|u - [u]_\Omega\|_{\mathrm{L}^p} \leq C\|\nabla u\|_{\mathrm{L}^p},$$

*where $C = C(\Omega, p) > 0$ is a constant.*

**Theorem A.27** **(Sobolev embedding).** *Let $u \in \mathrm{W}^{1,p}(\Omega)$.*

*(i) If $p < d$, then $u \in \mathrm{L}^{p^*}(\Omega)$, where*

$$p^* := \frac{dp}{d-p},$$

*and there is a constant $C = C(\Omega, p) > 0$ such that*

$$\|u\|_{\mathrm{L}^{p^*}} \leq C\|u\|_{\mathrm{W}^{1,p}}.$$

*(ii) If $p = d$, then $u \in \mathrm{L}^q(\Omega)$ for all $1 \leq q < \infty$ and*

$$\|u\|_{\mathrm{L}^q} \leq C\|u\|_{\mathrm{W}^{1,p}},$$

*where $C = C(\Omega, p, q) > 0$ is a constant.*

*(iii) If $p > d$, then $u \in \mathrm{C}(\Omega)$ and*

$$\|u\|_\infty \leq C\|u\|_{\mathrm{W}^{1,p}},$$

*where $C = C(\Omega, p) > 0$ is a constant.*

The second part can in fact be made more precise by considering embeddings into Hölder spaces, see Section 5.6.3 in [111] for details.

**Theorem A.28** **(Rellich–Kondrachov).** *Let* $(u_j) \subset \mathrm{W}^{1,p}(\Omega)$ *with* $u_j \rightharpoonup u$ *in* $\mathrm{W}^{1,p}$.

*(i) If* $p < d$, *then* $u_j \to u$ *in* $\mathrm{L}^q(\Omega)$ *for any* $q < p^* = dp/(d-p)$.
*(ii) If* $p = d$, *then* $u_j \to u$ *in* $\mathrm{L}^q(\Omega)$ *for any* $q < \infty$.
*(iii) If* $p > d$, *then* $u_j \to u$ *uniformly (i.e., in the supremum norm).*

**Theorem A.29** **(Density).** *For every* $p \in [1, \infty)$, $u \in \mathrm{W}^{1,p}(\Omega)$, *and all* $\varepsilon > 0$ *there exists a map* $v \in (\mathrm{W}^{1,p} \cap \mathrm{C}^\infty)(\Omega)$ *with* $v|_{\partial\Omega} = u|_{\partial\Omega}$ *and* $\|u - v\|_{\mathrm{W}^{1,p}} < \varepsilon$. *Moreover, there also exists a countably piecewise affine* $w \in (\mathrm{W}^{1,p} \cap \mathrm{C})(\Omega)$ *with* $w|_{\partial\Omega} = u|_{\partial\Omega}$ *and* $\|u - w\|_{\mathrm{W}^{1,p}} < \varepsilon$.

Here, a map $w\colon D \to \mathbb{R}$ is called **countably piecewise affine** if there exists a disjoint partition of $\Omega$ into countably many open sets $D_k$ ($k \in \mathbb{N}$), up to a negligible set, i.e., $\Omega = Z \cup \bigcup_k D_k$, where $|Z| = 0$, such that $w|_{D_k}$ is affine.

For $0 < \gamma \leq 1$ a function $u\colon \Omega \to \mathbb{R}$ is $\gamma$**-Hölder-continuous function**, in symbols $u \in \mathrm{C}^{0,\gamma}(\Omega)$, if

$$\|u\|_{\mathrm{C}^{0,\gamma}} := \|u\|_\infty + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\gamma} < \infty.$$

Functions in $\mathrm{C}^{0,1}(\Omega)$ are called **Lipschitz continuous**. We construct the higher-order spaces $\mathrm{C}^{k,\gamma}(\Omega)$ analogously.

**Theorem A.30** **(Rademacher).** *Let* $\Omega \subset \mathbb{R}^d$ *be an open, bounded, and convex set. Then, the space* $\mathrm{W}^{1,\infty}(\Omega)$ *consists precisely of all Lipschitz maps on* $\Omega$, *the Lipschitz constant is equal to the* $\mathrm{W}^{1,\infty}$*-norm, and for* $u \in \mathrm{W}^{1,\infty}(\Omega)$ *the classical gradient* $\nabla u$ *exists almost everywhere in* $\Omega$ *and agrees with the weak gradient.*

Next, we define a **family of mollifiers** as follows: Let $\eta \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ be radially symmetric and positive. Then, the family $(\eta_\delta)_{\delta>0}$ is defined as follows:

$$\eta_\delta(x) := \frac{1}{\delta^d}\eta\left(\frac{x}{\delta^d}\right), \qquad x \in \mathbb{R}^d.$$

For $u \in \mathrm{W}^{k,p}(\mathbb{R}^d)$, where $k \in \mathbb{N} \cup \{0\}$, and $p \in [1, \infty]$, we define the **mollification** $u_\delta \in \mathrm{W}^{k,p}(\mathbb{R}^d)$ of $u$ as the **convolution** between $\eta_\delta$ and $u$, i.e.,

$$u_\delta(x) := (\eta_\delta \star u)(x) := \int \eta_\delta(x-y)u(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^d.$$

**Lemma A.31** *For every* $p \in [1, \infty)$, *if* $u \in \mathrm{W}^{1,p}(\mathbb{R}^d)$, *then* $u_\delta \to u$ *in* $\mathrm{W}^{1,p}$ *as* $\delta \downarrow 0$.

Analogous results also hold for continuously differentiable functions.

**Lemma A.32** (**Young's inequality for convolutions**). *Let $u \in \mathrm{L}^p(\mathbb{R}^d)$, $v \in \mathrm{L}^q(\mathbb{R}^d)$ and let $p, q, r \in [1, \infty]$ be such that*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

*Then,*

$$\|u \star v\|_{\mathrm{L}^r} \le \|u\|_{\mathrm{L}^p} \cdot \|v\|_{\mathrm{L}^q}.$$

Finally, all the above notions and theorems continue to hold for vector-valued functions $u = (u^1, \dots, u^m)^T : \Omega \to \mathbb{R}^m$ and in this case we set

$$\nabla u := \begin{pmatrix} \partial_1 u^1 & \partial_2 u^1 & \cdots & \partial_d u^1 \\ \partial_1 u^2 & \partial_2 u^2 & \cdots & \partial_d u^2 \\ \vdots & \vdots & & \vdots \\ \partial_1 u^m & \partial_2 u^m & \cdots & \partial_d u^m \end{pmatrix}.$$

We use the spaces $\mathrm{C}(\Omega; \mathbb{R}^m)$, $\mathrm{C}^k(\Omega; \mathbb{R}^m)$, $\mathrm{W}^{k,p}(\Omega; \mathbb{R}^m)$, $\mathrm{C}^{k,\gamma}(\Omega; \mathbb{R}^m)$ with analogous definitions as in the scalar-valued case; for matrices like $\nabla u$ we use the Frobenius matrix norm and similarly for higher-order tensors.

Occasionally, we employ *local* versions of the spaces defined above, namely $\mathrm{C}_{\mathrm{loc}}(\Omega)$, $\mathrm{C}^k_{\mathrm{loc}}(\Omega)$, $\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega)$, $\mathrm{C}^{k,\gamma}_{\mathrm{loc}}(\Omega)$, where the defining norm is only finite on every compact subset of $\Omega$.

Finally, we quote the following two classical results about extensions of functions:

**Theorem A.33** (**Tietze**). *Let $X$ be a metric space, let $F \subset X$ be closed, and assume that $f : F \to \mathbb{R}^m$ is continuous. Then, $f$ can be extended to a continuous $\bar{f} : X \to \mathbb{R}^m$. If $f$ is bounded, then $\bar{f}$ can also be chosen as bounded.*

**Theorem A.34** (**Kirszbraun**). *Let $\Omega \subset \mathbb{R}^d$ and let $f : \Omega \to \mathbb{R}^m$ be a Lipschitz continuous map. Then, $f$ can be extended to $\bar{f} : \mathbb{R}^d \to \mathbb{R}^m$ with the same Lipschitz constant as $f$.*

## A.6 Harmonic Analysis

In this book we only need a few basics of Fourier analysis and the Mihlin multiplier theorem. A thorough introduction can be found in [138, 139].

Define for $u \in \mathrm{L}^1(\mathbb{R}^d)$ (or vector-valued $u$) the **Fourier transform** $\hat{u} = \mathscr{F} u \in \mathrm{L}^\infty(\mathbb{R}^d)$ as follows:

$$\hat{u}(\xi) := \mathscr{F} u(\xi) := \int_{\mathbb{R}^d} u(x) \mathrm{e}^{-2\pi \mathrm{i} x \cdot \xi} \,\mathrm{d}x, \qquad \xi \in \mathbb{R}^d.$$

We also define the **inverse Fourier transform** $\check{v} = \mathscr{F}^{-1} v$ for $v \in \mathrm{L}^1(\mathbb{R}^d)$ to be

$$\check{u}(x) := \mathscr{F}^{-1}v(x) := \int_{\mathbb{R}^d} v(\xi)\mathrm{e}^{2\pi \mathrm{i} x\cdot\xi}\,\mathrm{d}x, \qquad x \in \mathbb{R}^d.$$

One can extend $\mathscr{F}$, $\mathscr{F}^{-1}$ to the space $\mathrm{L}^2(\mathbb{R}^d)$ via the **Plancherel identity**,

$$\|\hat{u}\|_{\mathrm{L}^2} = \|u\|_{\mathrm{L}^2}. \tag{A.4}$$

Moreover, we have the **Parseval relation**

$$\int u \cdot \overline{v}\,\mathrm{d}x = \int \hat{u} \cdot \overline{\hat{v}}\,\mathrm{d}\xi \tag{A.5}$$

for all $u, v \in \mathrm{L}^2(\mathbb{R}^d)$; the same relations hold for $\mathbb{C}^N$-valued functions.

The following is a classical result concerning the $(\mathrm{L}^p \to \mathrm{L}^p)$-boundedness of Fourier multiplier operators, see, for instance, [38, 138] (Theorem 6.1.6) for a proof.

**Theorem A.35** (**Mihlin multiplier theorem**). *Let $m \in \mathrm{C}^{\lfloor d/2\rfloor+1}(\mathbb{R}^d \setminus \{0\}; \mathbb{C})$ satisfy*

$$|\partial^\alpha m(\xi)| \le K|\xi|^{-|\alpha|}, \qquad \xi \in \mathbb{R}^d \setminus \{0\},$$

*for all multi-indices $\alpha \in \mathbb{N}_0^d$ with $|\alpha| := |\alpha_1| + \cdots + |\alpha_d| \le \lfloor d/2\rfloor + 1$ ($\lfloor t\rfloor$ denotes the largest integer less than or equal to $t \in \mathbb{R}$) and some $K > 0$. Then,*

$$Tu := \mathscr{F}^{-1}[m(\xi)\hat{u}(\xi)],$$

*which for $u \in \mathrm{L}^2(\mathbb{R}^d)$ is well-defined via the Plancherel identity (A.4), extends to a bounded operator $T\colon \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$, which satisfies the estimate*

$$\|T\|_{\mathrm{L}^p\to\mathrm{L}^p} \le C\max\{p, (p-1)^{-1}\}K,$$

*where $C = C(d) > 0$ is a constant. Furthermore, for $p = 1$ the weak-type estimate*

$$\left|\left\{\, x \in \mathbb{R}^d \;:\; |(Tu)(x)| \ge t \,\right\}\right| \le \frac{CK}{t}\|u\|_{\mathrm{L}^1}$$

*holds for all $t > 0$ and a constant $C = C(d) > 0$.*

As a special case, the conclusions of the preceding theorem hold for any positively 0-homogeneous smooth multiplier $m\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$.

We will also use the (centered) **maximal function** $Mf\colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ of $f\colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, which is defined as

$$(Mf)(x_0) := \sup_{r>0} \fint_{B(x_0,r)} |f(x)|\,\mathrm{d}x, \qquad x_0 \in \mathbb{R}^d.$$

We quote the following results about the maximal function, whose proofs can be found in [138, 177, 247] (in particular, (iii) is essentially contained in Lemma 1.68 of [177]):

**Theorem A.36** *The following statements are true:*

*(i) If* $p \in (1, \infty]$*, then*

$$\|Mf\|_{\mathrm{L}^p} \le C\|f\|_{\mathrm{L}^p},$$

*where* $C = C(d, p) > 0$ *is a constant.*

*(ii) If* $p \in [1, \infty)$*, then the weak-type estimate*

$$\big|\big\{\, x \in \mathbb{R}^d \;:\; |Mf| \ge t \,\big\}\big| \le \frac{C}{t^p} \int_{\{|f| \ge t/2\}} |f|^p \, dx \le \frac{C}{t^p} \|u\|^p_{\mathrm{L}^p}$$

*holds for all* $t > 0$ *and a constant* $C = C(d) > 0$*.*

*(iii) For every* $K > 0$ *and* $f \in \mathrm{W}^{1,p}(\mathbb{R}^d; \mathbb{R}^m)$*,* $p \in (1, \infty]$*, the maximal function* $Mf$ *is Lipschitz continuous on the set* $\{M(|f| + |\nabla f|) < K\}$ *and its Lipschitz constant is bounded by* $CK$*, where* $C = C(d, m, p) > 0$ *is a constant.*

# References

1. Acerbi, E., Fusco, N.: Semicontinuity problems in the calculus of variations. Arch. Ration. Mech. Anal. **86**, 125–145 (1984)
2. Acerbi, E., Fusco, N.: A regularity theorem for minimizers of quasiconvex integrals. Arch. Ration. Mech. Anal. **99**, 261–281 (1987)
3. Acerbi, E., Fusco, N.: An approximation lemma for $W^{1,p}$ functions. Material Instabilities in Continuum Mechanics (Edinburgh. 1985–1986), Oxford Science Publications, pp. 1–5. Oxford University Press, Oxford (1988)
4. Alberti, G.: Rank one property for derivatives of functions with bounded variation. Proc. R. Soc. Edinb. Sect. A **123**, 239–274 (1993)
5. Alberti, G., Csörnyei, M., Preiss, D.: Structure of null sets in the plane and applications. In: Proceedings of the Fourth European Congress of Mathematics (Stockholm, 2004), pp. 3–22. European Mathematical Society (2005)
6. Alibert, J.J., Bouchitté, G.: Non-uniform integrability and generalized Young measures. J. Convex Anal. **4**, 129–147 (1997)
7. Alibert, J.J., Dacorogna, B.: An example of a quasiconvex function that is not polyconvex in two dimensions. Arch. Ration. Mech. Anal. **117**, 155–166 (1992)
8. Allaire, G.: Shape Optimization by the Homogenization Method. Applied Mathematical Sciences, vol. 146. Springer, Berlin (2002)
9. Allard, W.K.: On the first variation of a varifold. Ann. Math. **95**, 417–491 (1972)
10. Almgren Jr., F.J.: Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. Ann. Math. **87**, 321–391 (1968)
11. Almgren, Jr., F.J.: Plateau's Problem: An Invitation to Varifold Geometry, Revised Edition. Student Mathematical Library, vol. 13. American Mathematical Society, New York (2001)
12. Ambrosio, L., Coscia, A., Dal Maso, G.: Fine properties of functions with bounded deformation. Arch. Ration. Mech. Anal. **139**, 201–238 (1997)
13. Ambrosio, L., Dal Maso, G.: On the relaxation in $\mathrm{BV}(\Omega; R^m)$ of quasi-convex integrals. J. Funct. Anal. **109**, 76–97 (1992)
14. Ambrosio, L., De Giorgi, E.: Un nuovo tipo di funzionale del calcolo delle variazioni. Atti Acc. Naz. dei Lincei, Rend. Cl. Sc. Fis. Mat. Natur. **82**, 199–210 (1988)
15. Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free-Discontinuity Problems. Oxford Mathematical Monographs. Oxford University Press, Oxford (2000)
16. Arroyo-Rabasa, A., De Philippis, G., Rindler, F.: Lower semicontinuity and relaxation of linear-growth integral functionals under PDE constraints. Adv. Calc. Var. (2017). To appear, arXiv:1701.02230

 
F. Rindler, *Calculus of Variations*, Universitext,
https://doi.org/10.1007/978-3-319-77637-8

17. Astala, K., Faraco, D.: Quasiregular mappings and Young measures. Proc. R. Soc. Edinb. Sect. A **132**, 1045–1056 (2002)
18. Astala, K., Iwaniec, T., Prause, I., Saksman, E.: Burkholder integrals, Morrey's problem and quasiconformal mappings. J. Am. Math. Soc. **25**, 507–531 (2012)
19. Attouch, H., Buttazzo, G., Michaille, G.: Variational Analysis in Sobolev and BV Spaces - Applications to PDEs and Optimization. Society for Industrial and Applied Mathematics, MPS-SIAM Series on Optimization (2006)
20. Aubin, J.P., Cellina, A.: Differential Inclusions. Set-Valued Maps and Viability Theory. Grundlehren der mathematischen Wissenschaften, vol. 264. Springer, Berlin (1984)
21. Aumann, R.J., Hart, S.: Bi-convexity and bi-martingales. Israel J. Math. **54**, 159–180 (1986)
22. Auslender, A., Teboulle, M.: Asymptotic Cones and Functions in Optimization and Variational Inequalities. Springer Monographs in Mathematics. Springer, Berlin (2003)
23. Baía, M., Krömer, S., Kružík, M.: Generalized $W^{1,1}$-Young measures and relaxation of problems with linear growth. SIAM J. Math. Anal. **50**, 1076–1119 (2018)
24. Balder, E.J.: A general approach to lower semicontinuity and lower closure in optimal control theory. SIAM J. Control Optim. **22**, 570–598 (1984)
25. Ball, J.M.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Ration. Mech. Anal. **63**, 337–403 (1976/77)
26. Ball, J.M.: Global invertibility of Sobolev functions and the interpenetration of matter. Proc. R. Soc. Edinb. Sect. A **88**, 315–328 (1981)
27. Ball, J.M.: A version of the fundamental theorem for Young measures. PDEs and Continuum Models of Phase Transitions (Nice, 1988). Lecture Notes in Physics, vol. 344, pp. 207–215. Springer, Berlin (1989)
28. Ball, J.M.: Some open problems in elasticity. Geometry. Mechanics, and Dynamics, pp. 3–59. Springer, Berlin (2002)
29. Ball, J.M., Currie, J.C., Olver, P.J.: Null Lagrangians, weak continuity, and variational problems of arbitrary order. J. Funct. Anal. **41**, 135–174 (1981)
30. Ball, J.M., James, R.D.: Fine phase mixtures as minimizers of energy. Arch. Ration. Mech. Anal. **100**, 13–52 (1987)
31. Ball, J.M., James, R.D.: Proposed experimental tests of a theory of fine microstructure and the two-well problem. Phil. Trans. R. Soc. Lond. A **338**, 389–450 (1992)
32. Ball, J.M., James, R.D.: Incompatible sets of gradients and metastability. Arch. Ration. Mech. Anal. **218**, 1363–1416 (2015)
33. Ball, J.M., Kirchheim, B., Kristensen, J.: Regularity of quasiconvex envelopes. Calc. Var. Partial Differ. Equ. **11**, 333–359 (2000)
34. Ball, J.M., Mizel, V.J.: One-dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation. Arch. Ration. Mech. Anal. **90**, 325–388 (1985)
35. Ball, J.M., Murat, F.: Remarks on Chacon's biting lemma. Proc. Am. Math. Soc. **107**, 655–663 (1989)
36. Beck, L.: Elliptic Regularity Theory. A First Course. Lecture Notes of the Unione Matematica Italiana, vol. 19. Springer, Berlin (2016)
37. Weak lower semicontinuity of integral functionals and applications: Benešová, B., Kruží k, M. SIAM Rev. **59**, 703–766 (2017)
38. Bergh, J., Löfström, J.: Interpolation Spaces. Grundlehren der mathematischen Wissenschaften, vol. 223. Springer, Berlin (1976)
39. Berliocchi, H., Lasry, J.M.: Intégrandes normales et mesures paramétrées en calcul des variations. Bull. Soc. Math. France **101**, 129–184 (1973)
40. Bernoulli, J.: Problema novum ad cujus solutionem Mathematici invitantur. In: Acta Eruditorum, p. 269. Lipsiæ, apud J. Grossium & J.F. Gletitschium (1696). https://hdl.handle.net/2027/mdp.39015067096282?urlappend=%3Bseq=285. Hathi Trust Digital Library. Accessed 15 May 2017
41. Bhattacharya, K.: Microstructure of Martensite. Oxford Series on Materials Modelling. Oxford University Press, Oxford (2003)

42. Bhattacharya, K., Dolzmann, G.: Relaxation of some multi-well problems. Proc. R. Soc. Edinb. Sect. A **131**, 279–320 (2001)
43. Bhattacharya, K., Firoozye, N.B., James, R.D., Kohn, R.V.: Restrictions on microstructure. Proc. R. Soc. Edinb. Sect. A **124**, 843–878 (1994)
44. Bildhauer, M.: Convex Variational Problems. Lecture Notes in Mathematics, vol. 1818. Springer, Berlin (2003)
45. Blåsjö, V.: The isoperimetric problem. Am. Math. Monthly **112**, 526–566 (2005)
46. Bojarski, B., Iwaniec, T.: Analytical foundations of the theory of quasiconformal mappings in $\mathbf{R}^n$. Ann. Acad. Sci. Fenn. Ser. A I Math. **8**, 257–324 (1983)
47. Bouchitté, G., Fonseca, I., Malý, J.: The effective bulk energy of the relaxed energy of multiple integrals below the growth exponent. Proc. R. Soc. Edinb. Sect. A **128**(3), 463–479 (1998)
48. Bouchitté, G., Fonseca, I., Mascarenhas, L.: A global method for relaxation. Arch. Ration. Mech. Anal. **145**, 51–98 (1998)
49. Bousquet, P., Mariconda, C., Treu, G.: On the Lavrentiev phenomenon for multiple integral scalar variational problems. J. Funct. Anal. **266**, 5921–5954 (2014)
50. Braides, A.: Homogenization of some almost periodic coercive functional. Rend. Accad. Naz. Sci. XL Mem. Mat. **9**, 313–321 (1985)
51. Braides, A.: Γ-Convergence for Beginners. Oxford Lecture Series in Mathematics and its Applications, vol. 22. Oxford University Press, Oxford (2002)
52. Braides, A., Defranceschi, A.: Homogenization of Multiple Integrals. Oxford Lecture Series in Mathematics and its Applications, vol. 12. Oxford University Press, Oxford (1998)
53. Brooks, J.K., Chacon, R.V.: Continuity and compactness of measures. Adv. Math. **37**, 16–26 (1980)
54. Cahn, J.W., Hilliard, J.E.: Free energy of a nonuniform system. I. Interfacial free energy. J. Chem. Phys. **28**, 258–267 (1958)
55. Campanato, S.: Hölder continuity of the solutions of some nonlinear elliptic systems. Adv. Math. **48**, 16–43 (1983)
56. Carstensen, C., Roubíček, T.: Numerical approximation of Young measures in non-convex variational problems. Numer. Math. **84**, 395–415 (2000)
57. Castaing, C., de Fitte, P.R., Valadier, M.: Young Measures on Topological Spaces: With Applications in Control Theory and Probability Theory. Mathematics and its Applications. Kluwer, Dordrecht (2004)
58. Cellina, A.: On the differential inclusion $x' \in [-1, +1]$. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **69**, 1–6 (1980)
59. Cellina, A., Perrotta, S.: On a problem of potential wells. J. Convex Anal. **2**, 103–115 (1995)
60. Chandler, R.E.: Hausdorff Compactifications. Lecture Notes in Pure and Applied Mathematics, vol. 23. Marcel Dekker (1976)
61. Chaudhuri, N., Müller, S.: Rigidity estimate for two incompatible wells. Calc. Var. Partial Differ. Equ. **19**, 379–390 (2004)
62. Chermisi, M., Conti, S.: Multiwell rigidity in nonlinear elasticity. SIAM J. Math. Anal. **42**, 1986–2012 (2010)
63. Chlebík, M., Kirchheim, B.: Rigidity for the four gradient problem. J. Reine Angew. Math. **551**, 1–9 (2002)
64. Ciarlet, P.G.: Mathematical Elasticity, vol. 1: Three Dimensional Elasticity. North-Holland (1988)
65. Ciarlet, P.G., Geymonat, G.: Sur les lois de comportement en élasticité non linéaire compressible. C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers. Sci. Terre **295**, 423–426 (1982)
66. Ciarlet, P.G., Gratie, L., Mardare, C.: Intrinsic methods in elasticity: a mathematical survey. Discret. Contin. Dyn. Syst. **23**, 133–164 (2009)
67. Ciarlet, P.G., Nečas, J.: Injectivity and self-contact in nonlinear elasticity. Arch. Ration. Mech. Anal. **97**, 171–188 (1987)
68. Clarke, F.: Functional Analysis, Calculus of Variations and Optimal Control. Graduate Texts in Mathematics, vol. 264. Springer, Berlin (2013)

69. Conti, S., Dolzmann, G., Kirchheim, B.: Existence of Lipschitz minimizers for the three-well problem in solid-solid phase transitions. Ann. Inst. H. Poincaré Anal. Non Linéaire **24**, 953–962 (2007)
70. Conti, S., Dolzmann, G., Kirchheim, B., Müller, S.: Sufficient conditions for the validity of the Cauchy-Born rule close to $\mathrm{SO}(n)$. J. Eur. Math. Soc. **8**, 515–530 (2006)
71. Conti, S., Faraco, D., Maggi, F.: A new approach to counterexamples to $L^1$ estimates: Korn's inequality, geometric rigidity, and regularity for gradients of separately convex functions. Arch. Ration. Mech. Anal. **175**(2), 287–300 (2005)
72. Conti, S., Fonseca, I., Leoni, G.: A $\Gamma$-convergence result for the two-gradient theory of phase transitions. Commun. Pure Appl. Math. **55**, 857–936 (2002)
73. Conti, S., Schweizer, B.: Rigidity and gamma convergence for solid-solid phase transitions with SO(2) invariance. Commun. Pure Appl. Math. **59**, 830–868 (2006)
74. Conway, J.B.: A Course in Functional Analysis. Graduate Texts in Mathematics, vol. 96, 2nd edn. Springer, Berlin (1990)
75. Dacorogna, B.: Weak Continuity and Weak Lower Semicontinuity for Nonlinear Functionals. Lecture Notes in Mathematics, vol. 922. Springer, Berlin (1982)
76. Dacorogna, B.: Direct Methods in the Calculus of Variations. Applied Mathematical Sciences, vol. 78, 2nd edn. Springer, Berlin (2008)
77. Dacorogna, B.: Introduction to the Calculus of Variations, 2nd edn. Imperial College Press, London (2009)
78. Dacorogna, B., Marcellini, P.: A counterexample in the vectorial calculus of variations. Material Instabilities in Continuum Mechanics (Edinburgh. 1985–1986), Oxford Science Publications, pp. 77–83. Oxford University Press, Oxford (1988)
79. Dacorogna, B., Marcellini, P.: General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases. Acta Math. **178**, 1–37 (1997)
80. Dacorogna, B., Marcellini, P.: Implicit Partial Differential Equations. Progress in Nonlinear Differential Equations and their Applications, vol. 37. Birkhäuser (1999)
81. Dafermos, C.M.: Hyperbolic Conservation Laws in Continuum Physics. Grundlehren der mathematischen Wissenschaften, vol. 325, 3rd edn. Springer, Berlin (2010)
82. Dal Maso, G.: An Introduction to $\Gamma$-Convergence. Progress in Nonlinear Differential Equations and their Applications, vol. 8. Birkhäuser (1993)
83. Dal Maso, G., DeSimone, A., Mora, M.G., Morini, M.: Time-dependent systems of generalized Young measures. Netw. Heterog. Media **2**, 1–36 (2007)
84. Dal Maso, G., DeSimone, A., Mora, M.G., Morini, M.: Globally stable quasistatic evolution in plasticity with softening. Netw. Heterog. Media **3**, 567–614 (2008)
85. Dal Maso, G., DeSimone, A., Mora, M.G., Morini, M.: A vanishing viscosity approach to quasistatic evolution in plasticity with softening. Arch. Ration. Mech. Anal. **189**, 469–544 (2008)
86. Dal Maso, G., Modica, L.: Integral functionals determined by their minima. Rend. Sem. Mat. Univ. Padova **76**, 255–267 (1986)
87. De Giorgi, E.: Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **3**, 25–43 (1957)
88. De Giorgi, E.: Un esempio di estremali discontinue per un problema variazionale di tipo ellittico. Boll. Un. Mat. Ital. **1**, 135–137 (1968)
89. De Giorgi, E., Franzoni, T.: Su un tipo di convergenza variazionale. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **58**, 842–850 (1975)
90. De Lellis, C.: A note on Alberti's rank-one theorem. Transport Tations and Multi-D Hyperbolic Conservation Laws (Bologna, 2005). Lecture Notes of the Unione Matematica Italiana, vol. 5, pp. 61–74. Springer, Berlin (2008)
91. De Lellis, C., Székelyhidi Jr., L.: The Euler equations as a differential inclusion. Ann. Math. **170**, 1417–1436 (2009)
92. De Philippis, G., Rindler, F.: On the structure of $\mathscr{A}$-free measures and applications. Ann. Math. **184**, 1017–1039 (2016)

93. De Philippis, G., Rindler, F.: Characterization of generalized Young measures generated by symmetric gradients. Arch. Ration. Mech. Anal. **224**, 1087–1125 (2017)
94. Delladio, S.: Lower semicontinuity and continuity of functions of measures with respect to the strict convergence. Proc. R. Soc. Edinb. Sect. A **119**, 265–278 (1991)
95. Demoulini, S.: Young measure solutions for nonlinear evolutionary systems of mixed type. Ann. Inst. H. Poincaré Anal. Non Linéaire **14**, 143–162 (1997)
96. Diestel, J., Uhl, Jr., J.J.: Vector Measures. Mathematical Surveys, vol. 15. American Mathematical Society (1977)
97. Dinculeanu, N.: Vector Measures. International Series of Monographs in Pure and Applied Mathematics, vol. 95. Pergamon Press, Oxford (1967)
98. DiPerna, R.J.: Compensated compactness and general systems of conservation laws. Trans. Am. Math. Soc. **292**, 383–420 (1985)
99. DiPerna, R.J., Majda, A.J.: Oscillations and concentrations in weak solutions of the incompressible fluid equations. Commun. Math. Phys. **108**, 667–689 (1987)
100. Dolzmann, G.: Variational Methods for Crystalline Microstructure - Analysis and Computation. Lecture Notes in Mathematics, vol. 1803. Springer, Berlin (2003)
101. Dolzmann, G., Kirchheim, B., Müller, S., Šverák, V.: The two-well problem in three dimensions. Calc. Var. Partial Differ. Equ. **10**, 21–40 (2000)
102. Dolzmann, G., Kristensen, J.: Higher integrability of minimizing Young measures. Calc. Var. Partial Differ. Equ. **22**, 283–301 (2005)
103. Dolzmann, G., Müller, S.: Microstructures with finite surface energy: the two-well problem. Arch. Ration. Mech. Anal. **132**, 101–141 (1995)
104. Duggin, M.J., Rachinger, W.A.: The nature of the martensite transformation in a coppernickel-aluminium alloy. Acta Metallurgica **12**, 529–535 (1964)
105. E, W., Ming, P.: Cauchy–Born rule and the stability of crystalline solids: static problems. Arch. Ration. Mech. Anal. **183**, 241–297 (2007)
106. Ekeland, I., Temam, R.: Convex Analysis and Variational Problems. North-Holland (1976)
107. Enami, K., Nenno, S.: Memory effect in Ni-36.8 at. pct. Al martensite. Metall. Trans. **2**, 1487 (1971)
108. Ericksen, J.L.: On the Cauchy-Born rule. Math. Mech. Solids **13**, 199–220 (2008)
109. Evans, L.C.: Quasiconvexity and partial regularity in the calculus of variations. Arch. Ration. Mech. Anal. **95**, 227–252 (1986)
110. Evans, L.C.: Weak Convergence Methods for Nonlinear Partial Differential Equations. CBMS Regional Conference Series in Mathematics, vol. 74. American Mathematical Society (1990)
111. Evans, L.C.: Partial Differential Equations. Graduate Studies in Mathematics, vol. 19, 2nd edn. American Mathematical Society (2010)
112. Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics, 2nd edn. CRC Press, Boca Raton (2015)
113. Falconer, K.: Fractal Geometry, 3rd edn. Wiley, New Jersey (2014)
114. Faraco, D., Székelyhidi Jr., L.: Tartar's conjecture and localization of the quasiconvex hull in $\mathbb{R}^{2\times 2}$. Acta Math. **200**, 279–305 (2008)
115. Federer, H.: Geometric Measure Theory. Grundlehren der mathematischen Wissenschaften, vol. 153. Springer, Berlin (1969)
116. Federer, H., Fleming, W.H.: Normal and integral currents. Ann. Math. **72**, 458–520 (1960)
117. Ferriero, A.: The Lavrentiev phenomenon in the Calculus of Variations. Ph.D. thesis, Universitá degli Studi di Milano-Bicocca (2004)
118. Ferriero, A.: A direct proof of the Tonelli's partial regularity result. Discret. Contin. Dyn. Syst. **32**, 2089–2099 (2012)
119. Folland, G.B.: Real Analysis. Pure and Applied Mathematics, 2nd edn. Wiley, New Jersey (1999)
120. Fonseca, I.: The lower quasiconvex envelope of the stored energy function for an elastic crystal. J. Math. Pures Appl. **9**(67), 175–195 (1988)
121. Fonseca, I., Kružík, M.: Oscillations and concentrations generated by $\mathscr{A}$-free mappings and weak lower semicontinuity of integral functionals. ESAIM Control Optim. Calc. Var. **16**, 472–502 (2010)

122. Fonseca, I., Leoni, G.: Modern Methods in the Calculus of Variations: $L^p$ Spaces. Springer, Berlin (2007)
123. Fonseca, I., Müller, S.: Quasi-convex integrands and lower semicontinuity in $L^1$. SIAM J. Math. Anal. **23**, 1081–1098 (1992)
124. Fonseca, I., Müller, S.: Relaxation of quasiconvex functionals in BV$(\Omega, R^p)$ for integrands $f(x, u, \nabla u)$. Arch. Ration. Mech. Anal. **123**, 1–49 (1993)
125. Fonseca, I., Müller, S., Pedregal, P.: Analysis of concentration and oscillation effects generated by gradients. SIAM J. Math. Anal. **29**, 736–756 (1998)
126. Francfort, G.A.: An introduction to $H$-measures and their applications. In: Variational Problems in Materials Science (Trieste, 2004). Progress in Nonlinear Differential Equations and their Applications, vol. 68, pp. 85–110. Birkhäuser (2006)
127. Friesecke, G., James, R.D., Müller, S.: A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. Commun. Pure Appl. Math. **55**, 1461–1506 (2002)
128. Friesecke, G., Theil, F.: Validity and failure of the Cauchy-Born hypothesis in a two-dimensional mass-spring lattice. J. Nonlinear Sci. **12**, 445–478 (2002)
129. Gelfand, I.M., Fomin, S.V.: Calculus of Variations. Prentice-Hall, Revised English edition translated and edited by Richard A. Silverman (1963)
130. Gérard, P.: Microlocal defect measures. Commun. Partial Differ. Equ. **16**, 1761–1794 (1991)
131. Giaquinta, M., Hildebrandt, S.: Calculus of Variations. I – The Lagrangian Formalism. Grundlehren der mathematischen Wissenschaften, vol. 310. Springer, Berlin (1996)
132. Giaquinta, M., Hildebrandt, S.: Calculus of Variations. II – The Hamiltonian Formalism. Grundlehren der mathematischen Wissenschaften, vol. 311. Springer, Berlin (1996)
133. Giaquinta, M., Martinazzi, L.: An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs. Lecture Notes. Scuola Normale Superiore di Pisa (New Series), vol. 11, 2nd edn. Edizioni della Normale, Pisa (2012)
134. Giaquinta, M., Modica, G., Souček, J.: Cartesian Currents in the Calculus of Variations. I. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 37. Springer, Berlin (1998)
135. Giaquinta, M., Modica, G., Souček, J.: Cartesian Currents in the Calculus of Variations. II. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 38. Springer, Berlin (1998)
136. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Grundlehren der mathematischen Wissenschaften, vol. 224. Springer, Berlin (1998)
137. Giusti, E.: Direct Methods in the Calculus of Variations. World Scientific, Singapore (2003)
138. Grafakos, L.: Classical Fourier Analysis. Graduate Texts in Mathematics, vol. 249, 3rd edn. Springer, Berlin (2014)
139. Grafakos, L.: Modern Fourier Analysis. Graduate Texts in Mathematics, vol. 250, 3rd edn. Springer, Berlin (2014)
140. Gratwick, R.: Singular sets and the Lavrentiev phenomenon. Proc. R. Soc. Edinb. Sect. A **145**, 513–533 (2015)
141. Gratwick, R.: Variations, approximation, and low regularity in one dimension. Calc. Var. Partial Differ. Equ. (2017). To appear
142. Gratwick, R., Preiss, D.: A one-dimensional variational problem with continuous Lagrangian and singular minimizer. Arch. Ration. Mech. Anal. **202**, 177–211 (2011)
143. Gratwick, R., Sychev, M.A., Tersenov, A.S.: Regularity and singularity phenomena for one-dimensional variational problems with singular ellipticity. Pure Appl. Funct. Anal. **1**, 397–416 (2016)
144. Gromov, M.: Convex integration of differential relations. I. Izv. Akad. Nauk SSSR Ser. Mat. **37**, 329–343 (1973)
145. Gromov, M.: Partial Differential Relations. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 9. Springer, Berlin (1986)
146. Gurtin, M.E.: On phase transitions with bulk, interfacial, and boundary energy. Arch. Ration. Mech. Anal. **96**, 243–264 (1986)
147. Gurtin, M.E.: Some results and conjectures in the gradient theory of phase transitions. In: Metastability and Incompletely Posed Problems (Minneapolis, Minn., 1985). The IMA Volumes in Mathematics and its Applications, vol. 3, pp. 135–146. Springer, Berlin (1987)

148. Henao, D., Mora-Corral, C.: Invertibility and weak continuity of the determinant for the modelling of cavitation and fracture in nonlinear elasticity. Arch. Ration. Mech. Anal. **197**, 619–655 (2010)
149. Hilbert, D.: Mathematische Probleme – Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900. Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse pp. 253–297 (1900)
150. Hildebrandt, S., Tromba, A.: Mathematics and Optimal Form. Scientific American Library (1985)
151. Horn, R.A., Johnson, C.R.: Matrix Analysis, 2nd edn. Cambridge University Press, Cambridge (2013)
152. Isett, P.: A Proof of Onsager's Conjecture. arXiv:1608.08301
153. James, R.D.: Displacive phase transformations in solids. J. Mech. Phys. Solids **34**, 359–394 (1986)
154. Jodeit Jr., M., Olver, P.J.: On the equation grad $f = M \operatorname{grad} g$. Proc. R. Soc. Edinb. Sect. A **116**, 341–358 (1990)
155. Kałamajska, A., Kružík, M.: Oscillations and concentrations in sequences of gradients. E-SAIM Control Optim. Calc. Var. **14**, 71–104 (2008)
156. Kinderlehrer, D.: Remarks about equilibrium configurations of crystals. Material Instabilities in Continuum Mechanics (Edinburgh. 1985–1986), Oxford Science Publications, pp. 217–241. Oxford University Press, Oxford (1988)
157. Kinderlehrer, D., Pedregal, P.: Characterizations of Young measures generated by gradients. Arch. Ration. Mech. Anal. **115**, 329–365 (1991)
158. Kinderlehrer, D., Pedregal, P.: Gradient Young measures generated by sequences in Sobolev spaces. J. Geom. Anal. **4**, 59–90 (1994)
159. Kirchheim, B.: Lipschitz minimizers of the 3-well problem having gradients of bounded variation. Preprint 12, Max-Planck-Institut für Mathematik in den Naturwissenschaften (1998)
160. Kirchheim, B.: Rigidity and Geometry of Microstructures. Lecture notes 16, Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig (2003)
161. Kirchheim, B., Kristensen, J.: Automatic convexity of rank-1 convex functions. C. R. Math. Acad. Sci. Paris **349**, 407–409 (2011)
162. Kirchheim, B., Kristensen, J.: On rank-one convex functions that are homogeneous of degree one. Arch. Ration. Mech. Anal. **221**, 527–558 (2016)
163. Kristensen, J.: Lower semicontinuity of quasi-convex integrals in BV. Calc. Var. Partial Differ. Equ. **7**(3), 249–261 (1998)
164. Kristensen, J.: Lower semicontinuity in spaces of weakly differentiable functions. Math. Ann. **313**, 653–710 (1999)
165. Kristensen, J.: On the non-locality of quasiconvexity. Ann. Inst. H. Poincaré Anal. Non Linéaire **16**, 1–13 (1999)
166. Kristensen, J., Melcher, C.: Regularity in oscillatory nonlinear elliptic systems. Math. Z. **260**, 813–847 (2008)
167. Kristensen, J., Mingione, G.: The singular set of Lipschitzian minima of multiple integrals. Arch. Ration. Mech. Anal. **184**, 341–369 (2007)
168. Kristensen, J., Rindler, F.: Characterization of generalized gradient Young measures generated by sequences in $W^{1,1}$ and BV. Arch. Ration. Mech. Anal. **197**, 539–598 (2010); Erratum: Vol. 203, pp. 693–700 (2012)
169. Kristensen, J., Rindler, F.: Relaxation of signed integral functionals in BV. Calc. Var. Partial Differ. Equ. **37**, 29–62 (2010)
170. Kružík, M.: On the composition of quasiconvex functions and the transposition. J. Convex Anal. **6**, 207–213 (1999)
171. Kružík, M., Roubíček, T.: Explicit characterization of $L^p$-Young measures. J. Math. Anal. Appl. **198**, 830–843 (1996)
172. Kružík, M., Roubíček, T.: On the measures of DiPerna and Majda. Math. Bohem. **122**, 383–399 (1997)

173. Kuiper, N.H.: On $C^1$-isometric imbeddings. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 58 =. Indag. Math. **17**(545–556), 683–689 (1955)
174. Larsen, C.J.: Quasiconvexification in $W^{1,1}$ and optimal jump microstructure in BV relaxation. SIAM J. Math. Anal. **29**, 823–848 (1998)
175. Lavrentiev, M.: Sur quelques problèmes du calcul des variations. Ann. Mat. Pura. Appl. **4**, 7–28 (1926)
176. Leoni, G.: A First Course in Sobolev Spaces. Graduate Studies in Mathematics, vol. 105. American Mathematical Society (2009)
177. Malý, J., Ziemer, W.P.: Fine Regularity of Solutions of Elliptic Partial Differential Equations. Mathematical Surveys and Monographs, vol. 51. Applied Mathematical Sciences (1997)
178. Manià, B.: Sopra un essempio di Lavrentieff. Boll. Un. Mat. Ital. **13**, 147–153 (1934)
179. Marcellini, P.: Periodic solution and homogenization of nonlinear variational problems. Ann. Mat. Pura. Appl. **117**, 481–498 (1978)
180. Marcus, M., Mizel, V.J.: Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems. Bull. Am. Math. Soc. **79**, 790–795 (1973)
181. Massaccesi, A., Vittone, D.: An elementary proof of the rank-one theorem for BV functions. J. Eur. Math. Soc. (2016). To appear, arXiv:1601.02903
182. Matos, J.P.: Young measures and the absence of fine microstructures in a class of phase transitions. Eur. J. Appl. Math. **3**, 31–54 (1992)
183. Mattila, P.: Geometry of Sets and Measures in Euclidean Spaces. Cambridge Studies in Advanced Mathematics, vol. 44. Cambridge University Press, Cambridge (1995)
184. McShane, E.J.: Generalized curves. Duke Math. J. **6**, 513–536 (1940)
185. McShane, E.J.: Necessary conditions in generalized-curve problems of the calculus of variations. Duke Math. J. **7**, 1–27 (1940)
186. McShane, E.J.: Sufficient conditions for a weak relative minimum in the problem of Bolza. Trans. Am. Math. Soc. **52**, 344–379 (1942)
187. Milton, G.W.: The Theory of Composites. Cambridge Monographs on Applied and Computational Mathematics, vol. 6. Cambridge University Press, Cambridge (2002)
188. Mingione, G.: Regularity of minima: an invitation to the dark side of the calculus of variations. Appl. Math. **51**, 355–426 (2006)
189. Modica, L.: The gradient theory of phase transitions and the minimal interface criterion. Arch. Ration. Mech. Anal. **98**, 123–142 (1987)
190. Modica, L., Mortola, S.: The Γ-convergence of some functionals. Preprint 77-7, Instituto Matematico "Leonida Tonelli", University of Pisa (1977)
191. Mooney, C., Savin, O.: Some singular minimizers in low dimensions in the calculus of variations. Arch. Ration. Mech. Anal. **221**, 1–22 (2016)
192. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation. I – Basic Theory. Grundlehren der mathematischen Wissenschaften, vol. 330. Springer, Berlin (2006)
193. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation. II – Applications. Grundlehren der mathematischen Wissenschaften, vol. 331. Springer, Berlin (2006)
194. Morrey Jr., C.B.: On the solutions of quasi-linear elliptic partial differential equations. Trans. Am. Math. Soc. **43**, 126–166 (1938)
195. Morrey Jr., C.B.: Quasi-convexity and the lower semicontinuity of multiple integrals. Pac. J. Math. **2**, 25–53 (1952)
196. Morrey, Jr., C.B.: Multiple Integrals in the Calculus of Variations. Grundlehren der mathematischen Wissenschaften, vol. 130. Springer, Berlin (1966)
197. Moser, J.: A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. Commun. Pure Appl. Math. **13**, 457–468 (1960)
198. Moser, J.: On Harnack's theorem for elliptic differential equations. Commun. Pure Appl. Math. **14**, 577–591 (1961)
199. Müller, S.: Homogenization of nonconvex integral functionals and cellular elastic materials. Arch. Ration. Mech. Anal. **99**, 189–212 (1987)
200. Müller, S.: On quasiconvex functions which are homogeneous of degree 1. Indiana Univ. Math. J. **41**, 295–301 (1992)

201. Müller, S.: Rank-one convexity implies quasiconvexity on diagonal matrices. Internat. Math. Res. Notices **20**, 1087–1095 (1999)
202. Müller, S.: A sharp version of Zhang's theorem on truncating sequences of gradients. Trans. Am. Math. Soc. **351**, 4585–4597 (1999)
203. Müller, S.: Variational models for microstructure and phase transitions. Calculus of Variations and Geometric Evolution Problems (Cetraro, 1996). Lecture Notes in Mathematics, vol. 1713, pp. 85–210. Springer, Berlin (1999)
204. Müller, S., Spector, S.J.: An existence theory for nonlinear elasticity that allows for cavitation. Arch. Ration. Mech. Anal. **131**, 1–66 (1995)
205. Müller, S., Šverák, V.: Attainment results for the two-well problem by convex integration. In: Geometric Analysis and the Calculus of Variations, pp. 239–251. International Press, Somerville (1996)
206. Müller, S., Šverák, V.: Convex integration with constraints and applications to phase transitions and partial differential equations. J. Eur. Math. Soc. **1**, 393–422 (1999)
207. Müller, S., Šverák, V.: Convex integration for Lipschitz mappings and counterexamples to regularity. Ann. Math. **157**, 715–742 (2003)
208. Müller, S., Sychev, M.A.: Optimal existence theorems for nonhomogeneous differential inclusions. J. Funct. Anal. **181**, 447–475 (2001)
209. Murat, F.: Compacité par compensation. Ann. Sc. Norm. Super. Pisa Cl. Sci. **5**, 489–507 (1978)
210. Murat, F.: Compacité par compensation. II. In: Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), pp. 245–256. Pitagora Editrice Bologna (1979)
211. Murat, F.: Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant. Ann. Sc. Norm. Super. Pisa Cl. Sci. **8**, 69–102 (1981)
212. Nash, J.: $C^1$ isometric imbeddings. Ann. Math. **60**, 383–396 (1954)
213. Nash, J.: Continuity of solutions of parabolic and elliptic equations. Am. J. Math. **80**, 931–954 (1958)
214. Nečas, J.: On regularity of solutions to nonlinear variational inequalities for second-order elliptic systems. Rend. Mat. **8**, 481–498 (1975)
215. Nesi, V., Milton, G.W.: Polycrystalline configurations that maximize electrical resistivity. J. Mech. Phys. Solids **39**, 525–542 (1991)
216. Nguetseng, G.: A general convergence result for a functional related to the theory of homogenization. SIAM J. Math. Anal. **20**, 608–623 (1989)
217. Noether, E.: Invariante Variationsprobleme, pp. 235–257. Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (1918)
218. Olver, P.J.: Applications of Lie Groups to Differential Equations. Graduate Texts in Mathematics, vol. 107, 2nd edn. Springer, Berlin (1993)
219. O'Neil, T.: A measure with a large set of tangent measures. Proc. Am. Math. Soc. **123**, 2217–2220 (1995)
220. Ornstein, D.: A non-inequality for differential operators in the $L_1$ norm. Arch. Ration. Mech. Anal. **11**, 40–49 (1962)
221. Pedregal, P.: Laminates and microstructure. Eur. J. Appl. Math. **4**, 121–149 (1993)
222. Pedregal, P.: Parametrized Measures and Variational Principles. Progress in Nonlinear Differential Equations and their Applications, vol. 30. Birkhäuser (1997)
223. Pompe, W.: The quasiconvex hull for the five-gradient problem. Calc. Var. Partial Differ. Equ. **37**, 461–473 (2010)
224. Preiss, D.: Geometry of measures in $R^n$: distribution, rectifiability, and densities. Ann. Math. **125**, 537–643 (1987)
225. Reshetnyak, Y.G.: Liouville's conformal mapping theorem under minimal regularity hypotheses. Sibirsk. Mat. Ž. **8**, 835–840 (1967)
226. Reshetnyak, Y.G.: The weak convergence of completely additive vector-valued set functions. Sibirsk. Mat. Ž. **9**, 1386–1394 (1968)

227. Rindler, F.: Lower semicontinuity for integral functionals in the space of functions of bounded deformation via rigidity and Young measures. Arch. Ration. Mech. Anal. **202**, 63–113 (2011)
228. Rindler, F.: Lower semicontinuity and Young measures in BV without Alberti's Rank-One theorem. Adv. Calc. Var. **5**, 127–159 (2012)
229. Rindler, F.: A local proof for the characterization of Young measures generated by sequences in BV. J. Funct. Anal. **266**, 6335–6371 (2014)
230. Rindler, F., Shaw, G.: Liftings, Young Measures, and Lower Semicontinuity. arXiv:1708.04165
231. Rindler, F., Shaw, G.: Strictly continuous extensions of functionals with linear growth to the space BV. Q. J. Math. **66**, 953–978 (2015)
232. Rockafellar, R.T.: Convex Analysis. Princeton Mathematical Series, vol. 28. Princeton University Press, Princeton (1970)
233. Rockafellar, R.T., Wets, R.J.B.: Variational Analysis. Grundlehren der mathematischen Wissenschaften, vol. 317. Springer, Berlin (1998)
234. Rogers, L.C.G., Williams, D.: Diffusions, Markov Processes, and Martingales. vol. 1: Foundations. Cambridge Mathematical Library. Cambridge University Press, Cambridge (2000). Reprint of the second edition (1994)
235. Roubíček, T.: Relaxation in Optimization Theory and Variational Calculus. De Gruyter Series in Nonlinear Analysis and Applications, vol. 4. De Gruyter (1997)
236. Rudin, W.: Real and Complex Analysis, 3rd edn. McGraw-Hill, New York (1986)
237. Schauder, J.: über lineare elliptische Differentialgleichungen zweiter Ordnung. Math. Z. **38**, 257–282 (1934)
238. Schauder, J.: Numerische Abschätzungen in elliptischen linearen Differentialgleichungen. Studia Math. **5**, 34–42 (1937)
239. Scheffer, V.: Regularity and Irregularity of Solutions to Nonlinear Second Order Elliptic Regularity and Irregularity of Solutions to Nonlinear Second Order Elliptic Systems and Inequalities. Ph.D. thesis, Princeton University (1974)
240. Scorza Dragoni, G.: Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile. Rend. Sem. Mat. Univ. Padova **17**, 102–106 (1948)
241. Seiner, H., Glatz, O., Landa, M.: A finite element analysis of the morphology of the twinned-to-detwinned interface observed in microstructure of the Cu-Al-Ni shape memory alloy. Int. J. Solids Struct. **48**, 2005–2014 (2011)
242. Serrin, J.: On the definition and properties of certain variational integrals. Trans. Am. Math. Soc. **101**, 139–167 (1961)
243. Shnirelman, A.: On the nonuniqueness of weak solution of the Euler equation. Commun. Pure Appl. Math. **50**, 1261–1286 (1997)
244. Simon, L.: Schauder estimates by scaling. Calc. Var. Partial Differ. Equ. **5**, 391–407 (1997)
245. Spadaro, E.N.: Non-uniqueness of minimizers for strictly polyconvex functionals. Arch. Ration. Mech. Anal. **193**, 659–678 (2009)
246. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
247. Stein, E.M.: Harmonic Analysis. Princeton University Press, Princeton (1993)
248. Šverák, V.: Regularity properties of deformations with finite energy. Arch. Ration. Mech. Anal. **100**, 105–127 (1988)
249. Šverák, V.: On regularity for the Monge-Ampère equation without convexity assumptions, Technical report. Heriot-Watt University (1991)
250. Šverák, V.: Quasiconvex functions with subquadratic growth. Proc. R. Soc. Lond. Ser. A **433**, 723–725 (1991)
251. Šverák, V.: New examples of quasiconvex functions. Arch. Ration. Mech. Anal. **119**, 293–300 (1992)
252. Šverák, V.: Rank-one convexity does not imply quasiconvexity. Proc. R. Soc. Edinb. Sect. A **120**, 185–189 (1992)
253. Šverák, V.: On Tartar's conjecture. Ann. Inst. H. Poincaré Anal. Non Linéaire **10**, 405–412 (1993)

254. Šverák, V.: On the problem of two wells. Microstructure and Phase Transition. The IMA Volumes in Mathematics and its Applications, vol. 54, pp. 183–189. Springer, Berlin (1993)
255. Šverák, V., Yan, X.: A singular minimizer of a smooth strongly convex functional in three dimensions. Calc. Var. Partial Differ. Equ. **10**, 213–221 (2000)
256. Šverák, V., Yan, X.: Non-Lipschitz minimizers of smooth uniformly convex functionals. Proc. Natl. Acad. Sci. USA **99**, 15269–15276 (2002)
257. Sychev, M.A.: Comparing Various Methods of Resolving Nonconvex Variational Problems. Preprint 66, SISSA (1998)
258. Sychev, M.A.: A new approach to Young measure theory, relaxation and convergence in energy. Ann. Inst. H. Poincaré Anal. Non Linéaire **16**, 773–812 (1999)
259. Sychev, M.A.: Characterization of homogeneous gradient Young measures in case of arbitrary integrands. Ann. Sc. Norm. Super. Pisa Cl. Sci. **29**, 531–548 (2000)
260. Sychev, M.A.: Comparing two methods of resolving homogeneous differential inclusions. Calc. Var. Partial Differ. Equ. **13**, 213–229 (2001)
261. Sychev, M.A.: A few remarks on differential inclusions. Proc. R. Soc. Edinb. Sect. A **136**, 649–668 (2006)
262. Sychev, M.A.: Comparing various methods of resolving differential inclusions. J. Convex Anal. **18**, 1025–1045 (2011)
263. Székelyhidi Jr., L.: From isometric embeddings to turbulence. In: HCDTE Lecture Notes. Part II. Nonlinear Hyperbolic PDEs, Dispersive and Transport Equations. AIMS Series on Applied Mathematics, vol. 7, pp. 195–255. American Institute of Mathematical Sciences (AIMS) (2013)
264. Székelyhidi Jr., L.: The regularity of critical points of polyconvex functionals. Arch. Ration. Mech. Anal. **172**, 133–152 (2004)
265. Székelyhidi Jr., L., Wiedemann, E.: Young measures generated by ideal incompressible fluid flows. Arch. Ration. Mech. Anal. **206**, 333–366 (2012)
266. Tang, Q.: Almost-everywhere injectivity in nonlinear elasticity. Proc. R. Soc. Edinb. Sect. A **109**, 79–95 (1988)
267. Tartar, L.: Compensated compactness and applications to partial differential equations. In: Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. IV. Research Notes in Mathematics, vol. 39, pp. 136–212. Pitman (1979)
268. Tartar, L.: The compensated compactness method applied to systems of conservation laws. In: Systems of Nonlinear Partial Differential Equations (Oxford, 1982). NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 111, pp. 263–285. Reidel (1983)
269. Tartar, L.: $H$-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations. Proc. R. Soc. Edinb. Sect. A **115**, 193–230 (1990)
270. Tartar, L.: On mathematical tools for studying partial differential equations of continuum physics: $H$-measures and Young measures. In: Developments in Partial Differential Equations and Applications to Mathematical Physics (Ferrara, 1991), pp. 201–217. Plenum (1992)
271. Tartar, L.: Some remarks on separately convex functions. Microstructure and Phase Transition. The IMA Volumes in Mathematics and its Applications, vol. 54, pp. 191–204. Springer, Berlin (1993)
272. Tartar, L.: Beyond Young measures. Meccanica **30**, 505–526 (1995)
273. Temam, R.: Mathematical Problems in Plasticity. Gauthier-Villars (1985)
274. Temam, R., Strang, G.: Functions of bounded deformation. Arch. Ration. Mech. Anal. **75**, 7–21 (1980)
275. Tonelli, L.: Sur un méthode directe du calcul des variations. Rend. Circ. Mat. Palermo **39**, 233–264 (1915)
276. Tonelli, L.: La semicontinuità nel calcolo delle variazioni. Rend. Circ. Mat. Palermo **44**, 167–249 (1920)
277. Tonelli, L.: Opere scelte. Vol II: Calcolo delle variazioni. Edizioni Cremonese, Rome (1961)
278. Van der Vorst, R.C.A.M.: Variational identities and applications to differential systems. Arch. Ration. Mech. Anal. **116**, 375–398 (1992)

279. Wigner, E.P.: The unreasonable effectiveness of mathematics in the natural sciences. Commun. Pure Appl. Math. **13**, 1–14 (1960). Richard Courant Lecture in Mathematical Sciences Delivered at New York University, May 11, 1959
280. Young, L.C.: Generalized curves and the existence of an attained absolute minimum in the calculus of variations. C. R. Soc. Sci. Lett. Varsovie, Cl. III **30**, 212–234 (1937)
281. Young, L.C.: Generalized surfaces in the calculus of variations. Ann. Math. **43**, 84–103 (1942)
282. Young, L.C.: Generalized surfaces in the calculus of variations. II. Ann. Math. **43**, 530–544 (1942)
283. Young, L.C.: Lectures on the Calculus of Variations and Optimal Control Theory, 2nd edn. Chelsea (1980)
284. Zhang, K.: A construction of quasiconvex functions with linear growth at infinity. Ann. Sc. Norm. Super. Pisa Cl. Sci. **19**, 313–326 (1992)
285. Ziemer, W.P.: Weakly Differentiable Functions. Graduate Texts in Mathematics, vol. 120. Springer, Berlin (1989)

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