Appendix A
Gaussian Random Variables and Processes

A.1 General Characteristics

Let X denote a random variable. Then, X is a *standard Gaussian or normal random variable* if it has a mean $E[X] = 0$, variance $\sigma^2 = E\left[X^2\right] > 0$, and density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right). \tag{A.1}$$

To verify that $f_X(x)$ is a legitimate density function, we prove that

$$\int_{-\infty}^{\infty} f(x)\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-x^2)\, dx = 1. \tag{A.2}$$

The first equality follows from the substitution of (A.1) and a change of variable. Squaring the second integral, applying Fubini's theorem (Section C.1) to equate the result to a double integral, changing variables, and applying Fubini's theorem to perform successive integrations, we obtain

$$\begin{aligned}
\left[\int_{-\infty}^{\infty} \exp(-x^2)\, dx\right]^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-(x^2+y^2)\right] dxdy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} \exp(-\rho^2)\rho d\rho d\theta \\
&= \pi
\end{aligned} \tag{A.3}$$

which proves the second equality of (A.2). The standard Gaussian distribution function is

D. Torrieri, *Principles of Spread-Spectrum Communication Systems*,
https://doi.org/10.1007/978-3-319-70569-9

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} \exp\left(-\frac{y^2}{2}\right) dy. \tag{A.4}$$

Finite moments of the standard Gaussian density function exist for any non-negative integer k. An integration by parts indicates that

$$\begin{aligned} E\left[X^k\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k \exp\left(-\frac{x^2}{2}\right) dx \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{k-1} \frac{d}{dx} \exp\left(-\frac{x^2}{2}\right) dx \\ &= \frac{k-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{k-2} \exp\left(-\frac{x^2}{2}\right) dx \\ &= (k-1)\, E\left[X^{k-2}\right], \quad k \geq 2. \end{aligned} \tag{A.5}$$

Since $E[X] = 0$ and $E\left[X^2\right] = 1$, it follows by induction that

$$E\left[X^{2k+1}\right] = 0, \quad k \geq 0. \tag{A.6}$$

$$E\left[X^{2k}\right] = (2k-1)(2k-3)\ldots 1, \quad k \geq 1. \tag{A.7}$$

The *characteristic function* (Appendix C.2) of a random variable X is defined as $E\left[e^{juX}\right]$, where $j = \sqrt{-1}$ and $-\infty < u < \infty$. Therefore, the *characteristic function of the standard Gaussian random variable* is

$$h_s(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \exp(jux)\, dx. \tag{A.8}$$

To evaluate this integral, we apply Cauchy's integral theorem to a contour integral in the complex plane over the rectangle with vertices $(-x_0, 0)$, $(x_0, 0)$, $(x_0, x_0 + ju)$, and $(-x_0, -x_0 + ju)$. The complex integration variable is z, and the integrals over the vertical sides of the rectangle become negligible as $x_0 \to \infty$. The integral from $(-x_0, 0)$ to $(x_0, 0)$ approaches $h_s(u)$ as $x_0 \to \infty$. Since there are no singularities within the transformed rectangle,

$$\begin{aligned} 0 &= \lim_{x_0 \to \infty} \oint \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \exp(juz)\, dz \\ &= h_s(u) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \\ &= h_s(u) - \exp\left(-\frac{u^2}{2}\right) \end{aligned} \tag{A.9}$$

where the final equality is obtained by applying (A.2). Thus, the characteristic function of the standard Gaussian random variable is $\exp\left(-u^2/2\right)$, and

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \exp(jux)\, dx = \sqrt{2\pi} \exp\left(-\frac{u^2}{2}\right). \tag{A.10}$$

If X is a standard Gaussian random variable, then $Y = \mu + \sigma X$ is a Gaussian or normal random variable with mean μ and variance σ^2. By including the constant random variable as a standard Gaussian random variable with $\sigma = 0$, we have $\sigma^2 \geq 0$. The characteristic function of Y is $h(u) = E\left[e^{juY}\right] = E\left[e^{ju(\mu+\sigma X)}\right]$. Using (A.10), we obtain the *characteristic function of a Gaussian random variable*:

$$h(u) = \exp\left(ju\mu - \frac{\sigma^2 u^2}{2}\right). \tag{A.11}$$

Since the distribution function of a random variable is uniquely determined by the characteristic function (Appendix C.2), *a necessary and sufficient condition that a random variable is Gaussian is that its characteristic function takes the form of* (A.11).

The *joint characteristic function* of the random variables $X_1, \ldots, X_n$ is defined as

$$h(\mathbf{u}) = E\left[\exp\left(j\mathbf{u}^T\mathbf{X}\right)\right] = E\left[\exp\left(j\sum_{k=1}^{n} u_k X_k\right)\right] \tag{A.12}$$

where $\mathbf{u} = [u_1\, u_2\, \ldots\, u_n]^T$. If the random variables are independent, zero-mean Gaussian random variables with $var(X_i) = \sigma_i^2$, then

$$\begin{aligned} h(\mathbf{u}) &= \prod_{i=1}^{n} E\left[\exp(ju_i X_i)\right] \\ &= \prod_{k=1}^{n} \exp\left(-\frac{\sigma_k^2 u^2}{2}\right). \end{aligned} \tag{A.13}$$

An $n \times 1$ random column vector $\mathbf{X} = [X_1\, \ldots\, X_n]^T$ has components that are random variables. Let $\boldsymbol{\mu} = E[\mathbf{X}]$ denote the mean vector, and let $\mathbf{K}$ denote the $n \times n$ *covariance matrix*

$$\mathbf{K} = E\left[(\mathbf{X}-\boldsymbol{\mu})(\mathbf{X}-\boldsymbol{\mu})^T\right] \tag{A.14}$$

which is symmetric. Since $\mathbf{x}^T\mathbf{K}\mathbf{x} = E\left[\left[(\mathbf{X}-\boldsymbol{\mu})^T\mathbf{x}\right]^2\right]$, $\mathbf{K}$ is nonnegative definite. *A Gaussian random vector X* is defined as one with a characteristic function of the form

$$h(\mathbf{u}) = E\left[\exp\left(j\mathbf{u}^T\mathbf{X}\right)\right] = \exp\left(j\mathbf{u}^T\boldsymbol{\mu} - \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u}\right) \tag{A.15}$$

where $\boldsymbol{\mu}$ is the mean vector, and $\mathbf{K}$ is the symmetric, nonnegative-definite covariance matrix. The components of a Gaussian random vector $X_1, \ldots, X_n$ are called *jointly Gaussian random variables.*

Theorem A1 *The random vector* $\mathbf{X} = \mathbf{AY} + \boldsymbol{\mu}$ *is a Gaussian random vector if* $\mathbf{Y}$ *has components that are independent zero-mean Gaussian random variables.*

Proof If $\mathbf{Y}$ is an $n \times 1$ Gaussian random vector with components that are independent zero-mean Gaussian random variables, then it has a mean $\mathbf{0}$ with an $n \times n$ diagonal covariance matrix $\mathbf{D}$. If $\mathbf{X} = \mathbf{AY} + \boldsymbol{\mu}$, then the characteristic function of $\mathbf{X}$ is

$$\begin{aligned} E\left[\exp\left(j\mathbf{u}^T\mathbf{X}\right)\right] &= \exp\left(j\mathbf{u}^T\boldsymbol{\mu}\right) E\left[\exp\left(j\mathbf{u}^T\mathbf{AY}\right)\right] \\ &= \exp\left(j\mathbf{u}^T\boldsymbol{\mu} - \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u}\right), \quad \mathbf{K} = \mathbf{ADA}^T. \end{aligned} \tag{A.16}$$

Since the characteristic function of $\mathbf{X}$ has the form of (A.15), $\mathbf{X}$ is a Gaussian random vector. □

Theorem A2 *An* $n \times 1$ *Gaussian random vector* $\mathbf{X}$ *with an* $n \times 1$ *mean vector* μ *and an* $n \times n$ *covariance matrix* $\mathbf{K}$ *can be expressed as* $\mathbf{X} = \mathbf{AY} + \mu$*, where the components of* $\mathbf{Y}$ *are independent Gaussian random variables with zero mean, and* $\mathbf{A}$ *is an* $n \times n$ *orthogonal matrix.*

Proof Since the $n \times n$ matrix $\mathbf{K}$ is a symmetric nonnegative definite, it can be diagonalized by an orthogonal matrix (Appendix G). Let $\mathbf{A}$ denote an $n \times n$ orthogonal matrix, such that $\mathbf{A}^T\mathbf{KA} = \mathbf{D}$, where $\mathbf{D}$ is an $n \times n$ diagonal matrix with diagonal elements equal to the $\{\lambda_k\}$, which are the eigenvalues of $\mathbf{K}$. Define $\mathbf{Y} = \mathbf{A}^T(\mathbf{X} - \boldsymbol{\mu})$. Then, $\mathbf{Y}$ has a mean $\mathbf{0}$, covariance $\mathbf{A}^T\mathbf{KA} = \mathbf{D}$, and characteristic function

$$\begin{aligned} E\left[\exp\left(j\mathbf{u}^T\mathbf{Y}\right)\right] &= \exp\left(-\frac{1}{2}\mathbf{u}^T\mathbf{D}\mathbf{u}\right) = \exp\left(-\frac{1}{2}\sum_{k=1}^{n}\lambda_k u_k^2\right) \\ &= \prod_{k=1}^{n}\exp\left(-\frac{1}{2}\lambda_k u_k^2\right). \end{aligned} \tag{A.17}$$

By the uniqueness of the characteristic function, (A.17), and (A.13), the $\{Y_k\}$ are independent, and Y_k is a zero-mean Gaussian random variable with variance λ_k. The orthogonality of $\mathbf{A}$ implies that $\mathbf{A}^T = \mathbf{A}^{-1}$, and hence $\mathbf{X} = \mathbf{AY} + \boldsymbol{\mu}$. □

Theorem A3 *If* $\mathbf{K}$ *is invertible, an* $n \times 1$ *Gaussian random vector* $\mathbf{X}$ *with an* $n \times 1$ *mean vector* $\boldsymbol{\mu}$ *and an* $n \times n$ *covariance matrix* $\mathbf{K}$ *has a density function*

$$f_{\mathbf{X}}(\mathbf{x}) = (2\pi)^{-n/2}(\det \mathbf{K})^{-1/2} \exp\left[-\frac{1}{2}(\mathbf{X}-\boldsymbol{\mu})^T\mathbf{K}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right] \tag{A.18}$$

which indicates that a Gaussian density function is completely determined by its mean vector and its covariance matrix.

Proof If the symmetric nonnegative-definite matrix $\mathbf{K}$ is invertible, it is positive definite and every eigenvalue λ_k of $\mathbf{K}$ is positive (Appendix G). Let $\mathbf{A}$ denote an $n \times n$ orthogonal matrix, such that $\mathbf{A}^T\mathbf{K}\mathbf{A} = \mathbf{D}$, where $\mathbf{D}$ is an $n \times n$ diagonal matrix with positive diagonal elements equal to the $\{\lambda_k\}$. Then, Theorem A2, (A.17), and (A.13) imply that $\mathbf{Y} = \mathbf{A}^T(\mathbf{X}-\boldsymbol{\mu})$ has a density function

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= \prod_{k=1}^{n}(2\pi\lambda_k)^{-1/2}\exp\left(-\frac{y_k^2}{2\lambda_k}\right) \\ &= (2\pi)^{-n/2}(\det \mathbf{D})^{-1/2}\exp\left(-\frac{\mathbf{y}^T\mathbf{D}^{-1}\mathbf{y}}{2}\right). \end{aligned} \tag{A.19}$$

Since $\mathbf{A}$ is orthogonal, the Jacobian of the transformation $\mathbf{X} = \mathbf{A}\mathbf{Y}+\boldsymbol{\mu}$ is $\left|\det \mathbf{A}^T\right| = \left|\det \mathbf{A}^{-1}\right| = 1$. Since $\mathbf{A}^T\mathbf{K}\mathbf{A} = \mathbf{D}$ implies that $\mathbf{A}\mathbf{D}^{-1}\mathbf{A}^T = \mathbf{K}^{-1}$ and $\det \mathbf{D} = \det \mathbf{K}$, the density function of $\mathbf{X}$ is given by (A.18). □

If $\mathbf{K}$ is singular, then $\mathbf{X}$ does not have a density function.

Theorem A4 *If* $\mathbf{X}$ *is a Gaussian random vector,* $\mathbf{B}$ *is an arbitrary* $n \times n$ *matrix, and* $\mathbf{Z} = \mathbf{B}\mathbf{X}$*, then* $\mathbf{Z}$ *is a Gaussian random vector. Thus, the linear transformation of a Gaussian random vector is itself a Gaussian random vector.*

Proof According to Theorem A1, $\mathbf{X}$ can be expressed as $\mathbf{X} = \mathbf{A}\mathbf{Y}+\boldsymbol{\mu}$, where $\mathbf{Y}$ is a vector with components that are independent zero-mean Gaussian random variables. Then $\mathbf{Z} = \mathbf{B}\mathbf{A}\mathbf{Y} + \mathbf{B}\boldsymbol{\mu}$, which is a Gaussian random vector by Theorem A1. □

It is important to note that *even if* $\mathbf{X}$ *has Gaussian components,* $\mathbf{X}$ *may not be a Gaussian random vector*. Thus, $\mathbf{Z} = \mathbf{B}\mathbf{X}$ may not be a Gaussian random vector if $\mathbf{X}$ has Gaussian components but is not a Gaussian random vector.

Theorem A5 *The components of a Gaussian random vector* $\mathbf{X}$ *are independent random variables if and only if they are uncorrelated and have positive variances.*

Proof If the component random variables of a Gaussian random vector $\mathbf{X}$ are uncorrelated, then $\mathbf{K}$ is diagonal. If all variances are positive, then the diagonal elements of $\mathbf{K}$ are positive. Since $\mathbf{K}$ is invertible, Theorem A3 indicates that the density function of $\mathbf{X}$ is the product of the density functions of its components, and hence the components are independent. Conversely, if the components of a Gaussian random vector $\mathbf{X}$ are independent random variables, then (A.14) implies that $\mathbf{K}$ is diagonal, and hence the components are uncorrelated. □

A *complex* $n \times 1$ *random vector* has the form $\mathbf{X} = \mathbf{X}_1 + j\mathbf{X}_2$, where $\mathbf{X}_1$ and $\mathbf{X}_2$ are real-valued random vectors with means $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ respectively. Thus, $E[\mathbf{X}] = \boldsymbol{\mu} = \boldsymbol{\mu}_1 + j\boldsymbol{\mu}_2$. Let $\mathbf{K}_i$ denote the $n \times n$ covariance matrix of $\mathbf{X}_i$, $i = 1, 2$. We define the $n \times n$ cross-covariance matrices of $\mathbf{X}_1$ and $\mathbf{X}_2$ as

$$\mathbf{K}_{12} = E\left[(\mathbf{X}_1 - \boldsymbol{\mu}_1)(\mathbf{X}_2 - \boldsymbol{\mu}_2)^T\right], \ \mathbf{K}_{21} = E\left[(\mathbf{X}_2 - \boldsymbol{\mu}_2)(\mathbf{X}_1 - \boldsymbol{\mu}_1)^T\right]. \tag{A.20}$$

The $n \times n$ covariance matrix of $\mathbf{X}$ is

$$\begin{aligned}\mathbf{K} &= E\left[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^H\right] \\ &= \mathbf{K}_1 + \mathbf{K}_2 + j(\mathbf{K}_{21} - \mathbf{K}_{12}).\end{aligned} \tag{A.21}$$

The density function of $\mathbf{X}$ is defined as the density function of the $2n \times 1$ real-valued vector $\mathbf{X}_c = [\mathbf{X}_1 \ \mathbf{X}_2]^T$. The $2n \times 2n$ covariance matrix of $\mathbf{X}_c$ is

$$\mathbf{K}_c = \begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_2 \end{bmatrix}. \tag{A.22}$$

A complex $n \times 1$ random vector $\mathbf{X}$ is *circularly symmetric* if

$$E\left[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T\right] = \mathbf{0}. \tag{A.23}$$

Expanding this equation in terms of its real and imaginary parts, we find that circular symmetry implies that

$$\mathbf{K}_1 = \mathbf{K}_2, \ \mathbf{K}_{21} = -\mathbf{K}_{12}. \tag{A.24}$$

A complex $n \times 1$ random vector $\mathbf{X}$ is a *complex Gaussian random vector* if the $2n \times 1$ real-valued vector $\mathbf{X}_c = [\mathbf{X}_1 \mathbf{X}_2]^T$ is a Gaussian random vector. Let $\mathbf{D}$ denote a real-valued diagonal matrix with positive elements. If a complex Gaussian random vector $\mathbf{X}$ has covariance matrix $\mathbf{K} = \mathbf{D}$, then

$$\mathbf{D} = \mathbf{K}_1 + \mathbf{K}_2, \ \mathbf{K}_{21} = \mathbf{K}_{12}. \tag{A.25}$$

If $\mathbf{X}$ is also circularly symmetric, then (A.22), (A.24), and (A.25) imply that

$$\mathbf{K}_c = \begin{bmatrix} \frac{1}{2}\mathbf{D} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathbf{D} \end{bmatrix} \tag{A.26}$$

which indicates that the components of $\mathbf{X}_c$ are uncorrelated and hence are independent Gaussian random variables.

A stochastic process $X(t) = \{X_t, \ t \in T\}$ is called a *Gaussian process* if every finite linear combination of the form

$$Y = \sum_{i=1}^{N} a_i X_{t_i} \tag{A.27}$$

is a Gaussian random variable.

A zero-mean stochastic process $n(t)$ is called a *white process* if its autocorrelation is

$$E[n(t)n(t+\tau)] = \frac{N_0}{2}\delta(\tau) \tag{A.28}$$

where $\delta(\tau)$ is the Dirac delta function, and $N_0/2$ is the two-sided power spectral density of this process. A white process is an idealization of a physical process because it requires an infinite bandwidth and power and zero correlation time. However, consider noise that has a flat spectrum across the passband of a band-limiting filter in a receiver. Since the filter blocks the noise spectrum beyond the filter passband, the hypothetical existence of the blocked spectrum does not affect the noise in the filter output. Consequently, a white process provides the standard mathematical model for thermal, shot, and environmental noise.

A.2 Central Limit Theorem

The *central limit theorem* establishes conditions under which the sum of many random variables has an approximately normal or Gaussian distribution. The proof exploits the following fundamental theorem. *Let* $F_n(x)$ *and* $F(x)$ *denote distribution functions with characteristic functions* $h_n(u)$ *and* $h(u)$ respectively. *A necessary and sufficient condition for* $F_n(x) \to F(x)$ *is that* $h_n(u) \to h(u)$ *for each* u [6, 9].

In the proof of the central limit theorem, Taylor series are needed. Let $f^{(n)}(x)$ denote the nth derivative of $f(x)$.

Taylor's Theorem for Complex-Valued Functions Let $f(x)$ denote a complex-valued function of a real variable x with $n+1$ continuous derivatives on an open interval, including the origin. Then, for all x in the interval,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)\, x^k}{k!} + R_n(x) \tag{A.29}$$

where the remainder is

$$R_n(x) = \frac{x^{n+1}}{n!} \int_0^1 f^{(n+1)}(xy)\,(1-y)^n\, dy. \tag{A.30}$$

If $\left|f^{(n+1)}(u)\right| \le c$ for all u on the open interval, then

$$|R_n(x)| \le \frac{c\,|x|^{n+1}}{(n+1)!}\,. \tag{A.31}$$

Proof Integrating by parts the integral in (A.30), we obtain

$$R_n(x) = -\frac{f^{(n)}(0)\,x^n}{n!} + R_{n-1}(x)\,.$$

Repeated substitutions into the right-hand side of this equation and the final substitution of $R_0(x) = f(x) - f(0)$ proves (A.29). Substituting $\left|f^{(n+1)}(u)\right| \leq c$ and

$$\int_0^1 (1-y)^n\,dy = \frac{1}{n+1}$$

into (A.30) proves (A.31). □

Applying Taylor's theorem to a series expansion of $\exp(jx)$ about the origin $x = 0$, we obtain

$$\exp(jx) = \sum_{k=0}^{n} \frac{(jx)^k}{k!} + \frac{\theta\,|x|^{n+1}}{(n+1)!}\,, \quad |\theta| \leq 1 \tag{A.32}$$

where $j = \sqrt{-1}$ and y is real-valued.

Taylor's Theorem for Analytic Functions Let $f(z)$ denote an analytic function over an open disk $\mathcal{D}$ including the origin in the complex plane. Then, for every $z \in \mathcal{D}$, we have the Taylor series

$$f(z) = \sum_{k=1}^{\infty} \frac{f^{(k)}(0)\,z^k}{k!}\,. \tag{A.33}$$

Proof Let C denote a circle centered at the origin and within $\mathcal{D}$. Using Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi j} \oint_C \frac{f(z_1)}{z_1 - z} dz_1$$

where z lies within C, z_1 is the integration variable, and the integration path is counterclockwise around C. From the definition of a derivative in the complex plane and Cauchy's integral formula, we find that

$$f^{(k)}(z) = \frac{k!}{2\pi j} \oint_C \frac{f(z_1)}{(z_1 - z)^{k+1}} dz_1. \tag{A.34}$$

From the formula for a finite geometric sum, we obtain

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k, \quad |t| < 1$$

where the convergence is uniform. Since $|z/z_1| < 1$, (A.34) and the geometric sum imply that

$$f(z) = \frac{1}{2\pi j}\oint_C \frac{f(z_1)}{z_1}\sum_{n=0}^{\infty}\left(\frac{z}{z_1}\right)^k dz_1$$
$$= \frac{1}{2\pi j}\sum_{n=0}^{\infty} z^k \oint_C \frac{f(z_1)}{z_1^{k+1}} dz_1$$

where the interchange of the order of integration and summation is valid because of the uniform convergence. Substitution of (A.34) with $z = 0$ into this equation yields (A.33). □

A Taylor series for the principal branch of $\ln(1+z)$ is

$$\ln(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^k}{k}$$
$$= z + z^2 \sum_{k=2}^{\infty} \frac{(-1)^{k+1} z^{k-2}}{k}, \quad |z| < 1. \tag{A.35}$$

We define

$$\zeta = \frac{z^2}{|z|^2}\sum_{k=2}^{\infty} \frac{(-1)^{k+1} z^{k-2}}{k}. \tag{A.36}$$

If $|z| \le 1/2$, then

$$|\zeta| \le \sum_{k=2}^{\infty} \frac{|z|^{k-2}}{k} \le \frac{1}{2}\sum_{k=2}^{\infty} |z|^{k-2} \le 1, \quad |z| \le 1/2. \tag{A.37}$$

Therefore,

$$\ln(1+z) = z + \zeta |z|^2, \quad |z| \le 1/2, \ |\zeta| \le 1. \tag{A.38}$$

Central Limit Theorem *Suppose that for each n, the sequence* $X_1, X_2, \ldots, X_n$ *is independent, and that each* X_k *has finite mean* m_k*, finite variance* σ_k^2*, and distribution function* $F_k(x)$*. Let* $S_n = X_1 + X_2 + \ldots + X_n$ *and* $T_n = (S_n - E[S_n])/s_n$*, where* $s_n^2 = var(S_n) = \sum_{k=1}^{n} \sigma_k^2$*. If for every positive* ϵ*,*

$$\sum_{k=1}^{n} \frac{1}{s_n^2} \int_{|x-m_k| \ge \epsilon s_n} (x - m_k)^2 \, dF_k(x) \to 0 \ \ as \ \ n \to \infty$$

then T_n converges in distribution to a standard Gaussian random variable with distribution given by (A.4).

Proof We assume in the proof that $m_k = 0$ with no loss of generality because $S_n - E[S_n] = \sum_{k=1}^{\infty} (X_k - m_k)$, and $E[X_k - m_k] = 0$.

Let h_k and ϕ_n denote the characteristic functions of X_k and T_n, respectively. The independence of each X_k implies that

$$\phi_n(u) = E\left[e^{juT_n}\right] = \prod_{k=1}^{n} E\left[e^{juX_k/s_n}\right] = \prod_{k=1}^{n} h_k\left(\frac{u}{s_n}\right).$$

Since the convergence of the characteristic functions determines the convergence of the distribution functions, the theorem is proved if it is shown that $\phi_n(u) \to \exp\left(-u^2/2\right)$, which is equivalent to showing that

$$\ln(\phi_n(u)) = \sum_{k=1}^{n} \ln\left(h_k\left(\frac{u}{s_n}\right)\right) \to -u^2/2 \text{ as } n \to \infty. \tag{A.39}$$

The partitioning of the defining integral gives

$$h_k\left(\frac{u}{s_n}\right) = \int_{|x|<\epsilon s_n} e^{jux/s_n} dF_k(x) + \int_{|x|\geq\epsilon s_n} e^{jux/s_n} dF_k(x)$$

for each positive ϵ. Substituting (A.32) with $n = 2$ and $n = 1$ into the first and second integrals, respectively, and using $m_k = E[X_k] = 0$, we obtain

$$h_k\left(\frac{u}{s_n}\right) = 1 + \frac{u^2}{2}\theta_1\alpha_{nk} - \frac{u^2}{2}\beta_{nk} + \frac{|u|^3}{6s_n^3}\theta_2 \int_{|x|<\epsilon s_n} |x|^3 \, dF_k(x) \tag{A.40}$$

where $|\theta_1|, |\theta_2| \leq 1$ and

$$\alpha_{nk} = \frac{1}{s_n^2}\int_{|x|\geq\epsilon s_n} x^2 dF_k(x), \quad \beta_{nk} = \frac{1}{s_n^2}\int_{|x|<\epsilon s_n} x^2 dF_k(x). \tag{A.41}$$

Since $|x|^3 < \epsilon s_n x^2$ when $|x| < \epsilon s_n$, (A.40) may be expressed as

$$h_k\left(\frac{u}{s_n}\right) = 1 + \gamma_{nk} \tag{A.42}$$

where

$$\gamma_{nk} = \frac{u^2}{2}\theta_1\alpha_{nk} - \frac{u^2}{2}\beta_{nk} + \frac{|u|^3}{6}\epsilon\theta_3\beta_{nk} \tag{A.43}$$

and $|\theta_3| < 1$. If $|\gamma_{n,k}| \leq 1/2$, the application of (A.38) yields

$$\sum_{k=1}^{n} \ln\left(h_k\left(\frac{u}{s_n}\right)\right) = \sum_{k=1}^{n} \gamma_{nk} + \zeta \sum_{k=1}^{n} |\gamma_{nk}|^2 \tag{A.44}$$

where $|\zeta| \leq 1$.

From the hypothesis of the theorem, as $n \to \infty$,

$$\sum_{k=1}^{n} \alpha_{nk} \to 0, \quad \alpha_{nk} \to 0. \tag{A.45}$$

It follows from (A.41) that

$$\sum_{k=1}^{\infty} (\alpha_{nk} + \beta_{nk}) = \frac{1}{s_n^2} \sum_{k=1}^{\infty} \sigma_k^2 = 1.$$

This equation and (A.45) imply that, as $n \to \infty$,

$$\sum_{k=1}^{n} \beta_{nk} \to 1 \tag{A.46}$$

and hence

$$\sum_{k=1}^{n} \gamma_{nk} \to -u^2/2 + \frac{|u|^3}{6}\theta_3\epsilon \text{ as } n \to \infty. \tag{A.47}$$

Since (A.41) indicates that $0 \leq \beta_{nk} < \epsilon^2$, we obtain

$$\max_k |\gamma_{nk}| < \frac{u^2}{2}\epsilon^2 + \frac{|u|^3}{6}\epsilon^3 \tag{A.48}$$

for sufficiently large n. Thus, $|\gamma_{nk}| \leq 1/2$ for all k and u if n is sufficiently large and ϵ is sufficiently small. Using (A.48), (A.43), (A.45), and (A.46), we obtain

$$\sum_{k=1}^{n} |\gamma_{nk}|^2 \leq \max_k |\gamma_{nk}| \sum_{k=1}^{n} |\gamma_{nk}| < \left(\frac{u^2}{2}\epsilon^2 + \frac{|u|^3}{6}\epsilon^3\right)\left(\frac{u^2}{2} + \frac{|u|^3}{6}\epsilon\right) \tag{A.49}$$

for sufficiently large n. Thus, for any positive δ, (A.44), (A.47), and (A.49) imply that

$$\left|\sum_{k=1}^{n} \ln\left(h_k\left(\frac{u}{s_n}\right)\right) + u^2/2\right| < \delta$$

if ϵ is chosen sufficiently small and n is sufficiently large, which proves (A.39). □

Corollary A1 *Suppose that the sequence $X_1, X_2, \ldots, X_n$ is independent for each n and identically distributed so that each X_k has a finite mean m, finite variance $\sigma^2 > 0$, and a distribution function $F(x)$. Let $S_n = X_1 + X_2 + \ldots + X_n$ and $T_n = (S_n - nm)/\sigma\sqrt{n}$. Then, T_n converges in distribution to a standard Gaussian random variable.*

Proof Since $s_n^2 = n\sigma^2$, $m_k = m$, and $F_k(x) = F(x)$,

$$\sum_{k=1}^{n} \frac{1}{s_n^2} \int_{|x-m_k|\geq\epsilon s_n} (x-m_k)^2\, dF_k(x) = \frac{1}{\sigma^2} \int_{|x-m|\geq\epsilon\sigma\sqrt{n}} (x-m)^2\, dF(x)$$

which converges to zero by Lebesgue's dominated convergence theorem, as σ^2 is finite and positive, and $\left\{|x-m| \geq \epsilon\sigma\sqrt{n}\right\}$ converges to the empty set as $n \to \infty$. □

To prove the next corollary, we use Chebyshev's inequality (Section 4.3).

Corollary A2 *Suppose that the sequence $X_1, X_2, \ldots, X_n$ is independent for each n, and that each X_k has a finite mean m_k, distribution function $F_k(x)$, and is uniformly bounded with $|X_k - m_k| < M$ for all k. Let $S_n = X_1 + X_2 + \ldots + X_n$ and $T_n = (S_n - E[S_n])/s_n$, where $s_n^2 = var(S_n) = \sum_{k=1}^{n} \sigma_k^2$. If $s_n \to \infty$, then T_n converges in distribution to a standard Gaussian random variable.*

Proof Using $|X_k - m_k| < M$ for all k and Chebyshev's inequality, we obtain

$$\sum_{k=1}^{n} \frac{1}{s_n^2} \int_{|x-m_k|\geq\epsilon s_n} (x-m_k)^2\, dF_k(x) \leq \sum_{k=1}^{n} \frac{4M^2}{s_n^2} P\{|x-m_k| \geq \epsilon s_n\}$$

$$\leq \sum_{k=1}^{n} \frac{4M^2\sigma_k^2}{s_n^4\epsilon^2} = \frac{4M^2}{s_n^2\epsilon^2} \to 0. \ \square$$

To prove the next corollary, we use the indicator function and the Cauchy-Schwarz inequality for random variables. The *indicator function* I_A of a set A is the function on the sample space Ω that assumes the value 1 on A and 0 on the complement of A. Substituting $x = X/\sqrt{E[X^2]}$ and $y = Y/\sqrt{E[Y^2]}$ into the inequality $2xy \leq x^2 + y^2$, and then taking the expected value of both sides of the resulting equation, we obtain the *Cauchy-Schwarz inequality*:

$$E[XY] \leq \sqrt{E[X^2]\, E[Y^2]}. \tag{A.50}$$

Corollary A3 *Suppose that the sequence* $X_1, X_2, \ldots, X_n$ *is independent for each n, and that each* X_k *has finite mean* m_k*, finite variance* σ_k^2*, and distribution function* $F_k(x)$*. Let* $S_n = X_1 + X_2 + \ldots + X_n$ *and* $T_n = (S_n - E[S_n])/s_n$*, where* $s_n^2 = var(S_n) = \sum_{k=1}^{n} \sigma_k^2$*. If* $s_n^3/n \to \infty$*, and the fourth central moment are uniformly bounded so that* $E\left[(X_k - m_k)^4\right] < M^2$ *for all k, then* T_n *converges in distribution to a standard Gaussian random variable.*

Proof Applying the Cauchy-Schwarz inequality, we obtain

$$E\left[(X_k - m_k)^2\right] = \sigma_k^2 \leq M.$$

Applying the Cauchy-Schwarz inequality, Chebyshev's inequality, $\sigma_k^2 \leq M$, and $s_n^3/n \to \infty$, we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{s_n^2} \int_{|x-m_k| \geq \epsilon s_n} (x - m_k)^2 \, dF_k(x) &= \sum_{k=1}^{n} \frac{1}{s_n^2} E\left[(X_k - m_k)^2 I_{\{|x-m_k| \geq \epsilon s_n\}}\right] \\
&\leq \sum_{k=1}^{n} \frac{M}{s_n^2} \sqrt{P\left[|x - m_k| \geq \epsilon s_n\right]} \\
&\leq \sum_{k=1}^{n} \frac{M^{3/2}}{s_n^3 \epsilon} = \frac{M^{3/2}/\epsilon}{s_n^3/n} \to 0. \quad \square
\end{aligned}$$

Appendix B
Moment-Generating Function and Laplace Transform

B.1 Moment-Generating Function

The *moment-generating function* of the random variable X with distribution function $F(x)$ is defined as

$$M(s) = E\left[e^{sX}\right] = \int_{-\infty}^{\infty} e^{sx} dF(x) \tag{B.1}$$

for all real-valued s for which the integral is finite. Thus, the moment-generating function is the two-sided Laplace transform restricted to real values of s. If $s_0 > 0$ and $M(s)$ is defined throughout $(-s_0, s_0)$, then $\exp(|sx|)$ is integrable for $|s| < s_0$. Since its series expansion indicates that $\exp(|sx|) \geq |sx|^k/k!, k \geq 0$, X has finite moments of all orders.

If $s_0 > 0$ and $M(s)$ are defined throughout $(-s_0, s_0)$, the difference quotient

$$\frac{M(s+\Delta s) - M(s)}{\Delta s} = \int_{-\infty}^{\infty} \frac{e^{(s+\Delta s)x} - e^{sx}}{\Delta s} dF(x), \ s \in (-s_0, s_0) \tag{B.2}$$

is finite when $s + \Delta s \in (-s_0, s_0)$ and $s_0 > 0$. Taking the limit of both sides of this equation as $\Delta s \to 0$ and applying the Lebesgue-dominated convergence theorem, we find that the derivative of $M(s)$ is

$$M'(s) = \int_{-\infty}^{\infty} x e^{sx} dF(x), \ s \in (-s_0, s_0) \,. \tag{B.3}$$

Since finite moments of all orders exist, the preceding derivation can be extended successively to higher-order derivatives of $M(s)$. The kth derivative is

$$M^{(k)}(s) = \int_{-\infty}^{\infty} x^k e^{sx} dF(x), \ s \in (-s_0, s_0) \,. \tag{B.4}$$

D. Torrieri, *Principles of Spread-Spectrum Communication Systems*,
https://doi.org/10.1007/978-3-319-70569-9

Thus, if $M(s)$ exists in some neighborhood of 0, then the kth moment of X is

$$E\left[X^k\right] = M^{(k)}(0) \tag{B.5}$$

which indicates that the moment-generating function is aptly named.

B.2 Laplace Transform

The *Laplace transform* of a nonnegative random variable X with a distribution function $F(x)$ concentrated on $[0, \infty)$ is defined as

$$\mathcal{L}(s) = E\left[e^{-sX}\right] = \int_0^\infty e^{-sx} dF(x),\ Re(s) \geq 0 \tag{B.6}$$

where s is a complex variable. The moment-generating function of a nonnegative random variable is obtained from its Laplace transform by replacing the complex variable s with the real variable $-s$. In the following exposition, we restrict attention to $\mathcal{L}(s)$ for the nonnegative real variable $s \geq 0$ because it simplifies the analysis.

Theorem B1 *If $F(x)$ is the distribution function of a nonnegative random variable X, then the kth derivative of the Laplace transform $\mathcal{L}(s)$ is*

$$\mathcal{L}^{(k)}(s) = (-1)^k \int_0^\infty x^k e^{-sx} dF(x),\ k \geq 0,\ s > 0. \tag{B.7}$$

If X has a kth moment, then

$$E\left(X^k\right) = (-1)^k \mathcal{L}^{(k)}\left(0^+\right). \tag{B.8}$$

Proof The proof is by mathematical induction. Equation (B.7) is true for $k = 0$ by definition (B.6). Assuming that it is true for $k = n$ and applying Taylor's theorem (Appendix A.2) to e^{-hx}, we find that for $s + h > 0$,

$$\begin{aligned}\frac{\mathcal{L}^{(n)}(s+h) - \mathcal{L}^{(n)}(s)}{h} &= (-1)^n \int_0^\infty x^n e^{-sx} \frac{e^{-hx} - 1}{h} dF(x) \\ &= (-1)^{n+1} \int_0^\infty x^{n+1} e^{-sx} dF(x) + R\end{aligned} \tag{B.9}$$

where

$$|R| \leq \frac{h}{2} \int_0^\infty x^{n+2} e^{-sx} dF(x). \tag{B.10}$$

The right-hand side of (B.10) is finite if $s > 0$. Therefore, taking the limit as $h \to 0$ in (B.9), we verify (B.7) for $k = n + 1$. If X has a kth moment, then taking the limit of (B.7) as $s \to 0$ from the right and applying the dominated convergence theorem, we obtain (B.8). □

Theorem B2 *The Laplace transform $\mathcal{L}(s)$ of a random variable X uniquely determines its distribution function $F(x)$.*

Proof For positive y, (B.7) implies that

$$\sum_{k=0}^{\lfloor sy \rfloor} \frac{(-1)^k}{k!} s^k \mathcal{L}^{(k)}(s) = \int_0^\infty H(s, y, x)\, dF(x),\ s > 0$$

where

$$H(s, y, x) = \sum_{k=0}^{\lfloor sy \rfloor} e^{-sx} \frac{(sx)^k}{k!},\ s > 0.$$

Let Z denote a discrete random variable mean $\lambda > 0$, variance λ, and Poisson distribution

$$P[Z = k] = e^{-\lambda} \frac{\lambda^k}{k!},\ k = 0, 1, \ldots$$

If $t > 0$,

$$P[Z \le \lambda t] = \sum_{k=0}^{\lfloor \lambda t \rfloor} e^{-\lambda} \frac{\lambda^k}{k!} \tag{B.11}$$

which implies that

$$\lim_{\lambda \to \infty} \sum_{k=0}^{\lfloor \lambda t \rfloor} e^{-\lambda} \frac{\lambda^k}{k!} = 1,\ t > 1. \tag{B.12}$$

Chebyshev's inequality (Section 4.3) yields

$$P[|Z - \lambda| \ge \lambda \epsilon] \le \frac{1}{\lambda \epsilon^2}$$

and hence

$$\lim_{\lambda \to \infty} P[|Z - \lambda| \ge \lambda \epsilon] = 0. \tag{B.13}$$

Therefore, Z has a value concentrated in $[\lambda(1-\epsilon), \lambda(1+\epsilon)]$ for $\epsilon > 0$ as $\lambda \to \infty$, and hence (B.11) implies that

$$\lim_{\lambda\to\infty} \sum_{k=0}^{\lfloor \lambda t \rfloor} e^{-\lambda} \frac{\lambda^k}{k!} = 0, \; t < 1. \tag{B.14}$$

Application of (B.12) and (B.14) indicates that if $y > x \geq 0$, then $H(s, y, x) \to 1$ as $s \to \infty$; if $0 \leq y < x$, then $H(s, y, x) \to 0$ as $s \to \infty$. Therefore, the dominated convergence theorem indicates that at all continuity points of $F(y)$, we have

$$\lim_{s\to\infty} \sum_{k=0}^{\lfloor sy \rfloor} \frac{(-1)^k}{k!} s^k \mathcal{L}^{(k)}(s) = \int_0^y dF(x) = F(y). \tag{B.15}$$

Since $F(y)$ is right continuous, (B.15) determines $F(y)$ as a function of $\mathcal{L}(s)$. If the Laplace transform is the same for distributions $F_1(y)$ and $F_2(y)$, then (B.15) indicates that $F_1(y) = F_2(y)$.□

Theorem B3 *The Laplace transform $\mathcal{L}_t(s)$ of a sum of independent nonnegative random variables X_1 and X_2 with Laplace transforms $\mathcal{L}_1(s)$ and $\mathcal{L}_2(s)$, respectively, is*

$$\mathcal{L}_t(s) = \mathcal{L}_1(s)\,\mathcal{L}_2(s). \tag{B.16}$$

Proof Independence of the random variables implies that

$$\mathcal{L}_t(s) = E\left[e^{-s(X_1+X_2)}\right] = E\left[e^{-sX_1}\right] E\left[e^{-sX_2}\right] = \mathcal{L}_1(s)\,\mathcal{L}_2(s). \; \square$$

Appendix C
Fourier Transform and Characteristic Function

C.1 Fourier Transform

The Fourier transform of a complex-valued, integrable function $g(x)$ is defined as

$$\mathcal{F}(g) = \int_{-\infty}^{\infty} e^{-j2\pi fx} g(x)\, dx \tag{C.1}$$

where $j = \sqrt{-1}$ and $-\infty < f < \infty$. Since integration is a linear operation,

$$\mathcal{F}(ag + bh) = a\mathcal{F}(g) + b\mathcal{F}(h) \tag{C.2}$$

for integrable functions $g(x)$ and $h(x)$ and constants a and b. The following is the inversion theorem most commonly used.

Theorem C1 *If $g(x)$ is a bounded, continuous, and integrable function, and its Fourier transform $\mathcal{F}(g) = \widehat{g}(f)$ is an integrable function, then*

$$g(x) = \int_{-\infty}^{\infty} e^{j2\pi fx} \widehat{g}(f)\, df. \tag{C.3}$$

Proof The substitution of the identity $\lim_{\sigma\to\infty}(e^{-2\pi^2 f^2/\sigma^2}) = 1$ and (C.1) into (C.3). Further evaluation yields

$$\begin{aligned}\int_{-\infty}^{\infty} e^{j2\pi fx}\widehat{g}(f)\, df &= \lim_{\sigma\to\infty}\int_{-\infty}^{\infty} e^{j2\pi fx} e^{-2\pi^2 f^2/\sigma^2}\int_{-\infty}^{\infty} e^{-j2\pi fz} g(z)\, dz df \\ &= \lim_{\sigma\to\infty}\int_{-\infty}^{\infty} g(z)\left[\int_{-\infty}^{\infty} e^{j2\pi f(x-z)} e^{-2\pi^2 f^2/\sigma^2} df\right] dz\end{aligned}$$

D. Torrieri, *Principles of Spread-Spectrum Communication Systems*,
https://doi.org/10.1007/978-3-319-70569-9

$$= \frac{1}{\sqrt{2\pi}} \lim_{\sigma\to\infty} \int_{-\infty}^{\infty} g(z)\,\sigma \exp\left[-\frac{\sigma^2 (z-x)^2}{2}\right] dz$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{\sigma\to\infty} [g\left(x+\frac{u}{\sigma}\right)] \exp\left(-\frac{u^2}{2}\right) du$$
$$= g(x).$$

In the first equality, the integrability of $\widehat{g}(f)$ and the dominated convergence theorem justify taking the limit outside the outer integral. In the second equality, the integrability of $g(x)$ and Fubini's theorem (see below) justify the interchange of the order of integration. The third equality follows from a change of integration variable and (A.10) of Appendix A.1. The fourth equality results from a change in the integration variable followed by application of the dominated convergence theorem to justify taking the limit inside the integral. The final equality is obtained by taking the limit and then applying (A.2). □

The convolution of functions g and h is the function $g \star h$ defined by

$$(g \star h)(x) = \int_{-\infty}^{\infty} g(x-y)\, h(y)\, dy. \tag{C.4}$$

Convolution Theorem

(a) If g and h are bounded and integrable with Fourier transforms $\mathcal{F}(g)$ and $\mathcal{F}(h)$, respectively, then

$$\mathcal{F}(g * h) = \mathcal{F}(g)\,\mathcal{F}(h).$$

(b) If $\mathcal{F}(g)$ and $\mathcal{F}(h)$ are bounded and integrable, then

$$\mathcal{F}(gh) = \mathcal{F}(g) * \mathcal{F}(h).$$

Proof

(a) Since g and h are bounded and integrable, $g*h$ is integrable. Therefore, Fubini's theorem justifies the following interchange of the order of integration, and we obtain

$$\mathcal{F}(g * h) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(x-y)\, h(y)\, dy\right] e^{-j2\pi fx} dx$$
$$= \int_{-\infty}^{\infty} e^{-j2\pi fy} h(y) \left[\int_{-\infty}^{\infty} e^{-j2\pi f(x-y)} g(x-y)\, dx\right] dy$$
$$= \mathcal{F}(g)\,\mathcal{F}(h)$$

where the final equality results from a change in the integration variable.

(b) If $\mathcal{F}(g)$ and $\mathcal{F}(h)$ are bounded and integrable, then $\mathcal{F}(g) * \mathcal{F}(h)$ is integrable. Let $\mathcal{F}^{-1}(\cdot)$ denote the inverse Fourier transform. A derivation almost identical to the preceding one yields

$$\begin{aligned}\mathcal{F}^{-1}[\mathcal{F}(g) * \mathcal{F}(h)] &= \mathcal{F}^{-1}[\mathcal{F}(g)]\,\mathcal{F}^{-1}[\mathcal{F}(h)] \\ &= gh\,.\end{aligned}$$

Taking the Fourier transform of both sides of this equation completes the proof. □

Parseval's Identities If g, $\widehat{g}(f)$, h, and $\widehat{h}(f)$ are bounded and integrable, then

$$\int_{-\infty}^{\infty} g(x)\,h^*(x)\,dx = \int_{-\infty}^{\infty} \widehat{g}(f)\,\widehat{h}^*(f)\,df$$

and

$$\int_{-\infty}^{\infty} |g(x)|^2\,dx = \int_{-\infty}^{\infty} |\widehat{g}(f)|^2\,df.$$

Proof Since $g(x)$ and $\widehat{h}(f)$ are bounded and integrable,

$$\int_{-\infty}^{\infty} \left| e^{-j2\pi fx}\widehat{h}^*(f)\,g(x)\right| df = |g(x)| \int_{-\infty}^{\infty} \left|\widehat{h}(f)\right| df < \infty.$$

Therefore, Fubini's theorem justifies the following interchange of the order of integration, and we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} g(x)\,h^*(x)\,dx &= \int_{-\infty}^{\infty} g(x)\left[\int_{-\infty}^{\infty} e^{-j2\pi fx}\widehat{h}^*(f)\,df\right]dx \\ &= \int_{-\infty}^{\infty} \widehat{h}^*(f)\left[\int_{-\infty}^{\infty} e^{-j2\pi fx} g(x)\,dx\right]df \\ &= \int_{-\infty}^{\infty} \widehat{g}(f)\,\widehat{h}^*(f)\,df.\end{aligned}$$

The second identity of the theorem follows by setting $g = h$. □

Application of Fubini's Theorem

Fubini's theorem is applied several times in this section and elsewhere in this book. This theorem, which is proved using measure theory [6, 9, 114], states that under certain conditions, a double integral may be evaluated as either of two iterated integrals:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\,dx\right]dy$$
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right]dx. \tag{C.5}$$

To apply the theorem, we compute or bound one of the iterated integrals with $|f(x,y)|$ in place of $f(x,y)$. If the result is finite, then the double integral of $|f(x,y)|$ is finite, which implies that the double integral of $f(x,y)$ may be computed as either of the two iterated integrals.

C.2 Characteristic Function

The *characteristic function* of the random variable X with distribution function $F(x)$ is

$$h(u) = E\left[e^{juX}\right] = \int_{-\infty}^{\infty} e^{jux}dF(x) \tag{C.6}$$

where $j = \sqrt{-1}$ and $-\infty < u < \infty$. Since $|\exp(jux)| \leq 1$, the characteristic function always exists, and $|h(u)| \leq h(0) = 1$. Since

$$|h(u+h) - h(u)| \leq \int_{-\infty}^{\infty} \left|e^{juh} - 1\right| dF(x) \tag{C.7}$$

application of the bounded convergence theorem indicates that $h(u)$ is continuous. If a continuous density function $f(x)$ exists, then $h(u)$ is the conjugate *Fourier transform* of the density function:

$$h(u) = \int_{-\infty}^{\infty} e^{jux} f(x)\,dx. \tag{C.8}$$

The principal advantage of the characteristic function relative to the Laplace transform is that the characteristic function is applicable to a random variable that may take negative values.

The usefulness of the characteristic function depends on the fact that it uniquely determines the distribution function from which it is derived. To prove this fact, we need to evaluate the integral

$$I_1 = P\int_{-\infty}^{\infty} \frac{e^{jx}}{x}dy = \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} \frac{e^{jx}}{x}dy + \int_{+\epsilon}^{\infty} \frac{e^{jx}}{x}dy\right) \tag{C.9}$$

where P denotes the Cauchy principal value, which is defined to avoid the singularity at the origin when the integral is a Riemann integral. However, if the integral is considered a Lebesgue integral, then the singularity has measure 0, and does not have to be avoided. Since $\cos(x)/x$ is an odd function and $\sin(x)/x$ is an even function, I_1 reduces to

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} \frac{e^{jx}}{x} dx \\ &= 2j \int_0^{\infty} \frac{\sin(x)}{x} dx. \end{aligned} \tag{C.10}$$

The integrand is uniformly bounded, and the Lebesgue integral exists because

$$\int_{(n-1)\pi}^{n\pi} \frac{\sin(x)}{x} dx \tag{C.11}$$

alternates in sign for each positive integer n, and its absolute value decreases to zero.

Theorem C2

$$P \int_{-\infty}^{\infty} \frac{e^{jx}}{x} dx = 2j \int_0^{\infty} \frac{\sin(x)}{x} dx = j\pi. \tag{C.12}$$

Proof We apply Cauchy's integral theorem to a contour integral of $\exp(-jz)/z$, where z is a complex variable. The contour includes a large semicircle C_1 with a radius c in the upper complex plane and a small semicircle C_2 with a radius ϵ traversed clockwise around the pole at the origin. Since there are no singularities within the contour,

$$\int_{-c}^{-\epsilon} \frac{e^{jx}}{x} dx + \int_{\epsilon}^{c} \frac{e^{jx}}{x} dx + \int_{C_1} \frac{e^{jz}}{z} dz + \int_{C_2} \frac{e^{jz}}{z} dz = 0. \tag{C.13}$$

After changing the integration variable in the third integral by substituting $z = ce^{j\theta}$ and then applying the dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{c\to\infty} \int_{C_1} \frac{e^{jz}}{z} dz &= \lim_{c\to\infty} \int_0^{-\pi} je^{jc\cos\theta - c\sin\theta} d\theta \\ &= 0. \end{aligned}$$

After changing the integration variable in the fourth integral by substituting $z = \epsilon e^{j\theta}$ and then applying the dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{\epsilon\to 0} \int_{C_2} \frac{e^{jz}}{z} dz &= \lim_{\epsilon\to 0} \int_{-\pi}^{0} je^{j\epsilon\cos\theta - \epsilon\sin\theta} d\theta \\ &= -j\pi. \end{aligned}$$

Using these results and taking the limit of (C.13) as $c \to \infty$ and $\epsilon \to 0$, we obtain (C.12). □

Theorem C3 *If the distribution function $F(x)$ has the characteristic function $h(u)$, then*

$$F(b) - F(a) = \lim_{L \to \infty} \int_{-L}^{L} \frac{e^{-jua} - e^{-jub}}{j2\pi u} h(u)\, du \tag{C.14}$$

for all points a and $b > a$ at which $F(x)$ is continuous, and the distribution function is uniquely determined. If the characteristic function is integrable, then the distribution function has a continuous density function given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-jux} h(u)\, du. \tag{C.15}$$

Proof Let

$$I_L = \int_{-L}^{L} \frac{e^{-jua} - e^{-jub}}{j2\pi u} h(u)\, du. \tag{C.16}$$

Substituting (C.6) into this integral gives

$$I_L = \int_{-L}^{L} \frac{e^{-jua} - e^{-jub}}{j2\pi u} \left[\int_{-\infty}^{\infty} e^{-jux} dF(x) \right] du. \tag{C.17}$$

To interchange the order of integration on the right-hand side of (C.17), we first observe that

$$\begin{aligned} \left| \frac{e^{-jua} - e^{-jub}}{j2\pi u} e^{jux} \right| = \left| \frac{e^{-jua} - e^{-jub}}{j2\pi u} \right| &= \left| \int_a^b \frac{e^{-ju\mathrm{y}}}{2\pi} dy \right| \\ &\le \frac{b-a}{2\pi} \end{aligned} \tag{C.18}$$

and

$$\int_{-L}^{L} \left[\int_{-\infty}^{\infty} \frac{b-a}{2\pi} dF(x) \right] du = \frac{L(b-a)}{\pi} < \infty$$

which indicates that Fubini's theorem is applicable to (C.17). Interchanging the order of integration and then changing integration variables, we obtain

$$I_L = \int_{-\infty}^{\infty} G_L(x)\, dF(x)$$

where

$$G_L(x) = \int_{-L(x-a)}^{L(x-a)} \frac{e^{jy}}{j2\pi y}dy - \int_{-L(x-b)}^{L(x-b)} \frac{e^{jy}}{j2\pi y}dy.$$

By Theorem C2, each of these integrals is bounded by $1/2$, and hence $|G_L(x)| \leq 1$. Therefore, the bounded convergence theorem implies that

$$\lim_{L\to\infty} I_L = \int_{-\infty}^{\infty} \lim_{L\to\infty} G_L(x)\, dF(x). \tag{C.19}$$

Applying Theorem C2, we find that for $a < b$,

$$\lim_{L\to\infty} G_L(x) = \begin{cases} 0 & x < a \text{ or } x > b \\ 1/2 & x = a \text{ or } x = b \\ 1 & a < x < b. \end{cases}$$

Substituting this equation into (C.19), evaluating the integral, and then equating the result with the limit of (C.16), we obtain (C.14) for all points a and $b > a$ at which $F(x)$ is continuous. Since $F(x)$ is right continuous, $h(u)$ determines $F(x)$ everywhere. Thus, the characteristic function uniquely determines the distribution function.

If $h(u)$ is integrable, then (C.14) and (C.18) indicate that

$$F(b) - F(a) \leq \frac{(b-a)}{2\pi} \int_{-\infty}^{\infty} |h(u)|\, du$$

and hence $F(x)$ is continuous. If $f(x)$ is defined by (C.15), application of the dominated convergence theorem proves that $f(x)$ is continuous.

Applying Fubini's theorem to interchange the order of integration, we find that

$$\begin{aligned}\int_a^x f(y)\, dy &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_a^x e^{-juy} dy\right] h(u)\, du \\ &= \lim_{L\to\infty} \int_{-L}^{L} \frac{e^{-jua} - e^{-jux}}{j2\pi u} h(u)\, du \\ &= F(x) - F(a).\end{aligned}$$

By the continuity of $f(x)$, this equation implies that the derivative of $F(x)$ is $f(x)$. Since $F(x)$ is monotonically increasing, $f(x)$ is nonnegative everywhere; hence, $f(x)$ is the density function for $F(x)$. □

Let $h^{(k)}(u)$ denote the kth derivative of $h(u)$ with respect to u. The following theorem enables the calculation of the kth moment of a random variable without the restrictive condition in Theorem B2 of Appendix B.2.

Theorem C4 *If the random variable X has a distribution function $F(x)$ and a characteristic function $h(u)$, and if $E\left[|X|^k\right] < \infty$ for a positive integer k, then*

$$h^{(k)}(u) = \int_{-\infty}^{\infty} (jx)^k e^{jux} dF(x) \tag{C.20}$$

and

$$E\left[X^k\right] = j^{-k} h^{(k)}(0) . \tag{C.21}$$

Proof Since $\left|(jx)^k e^{jux}\right| = |x|^k$ and $E\left[|X|^k\right] < \infty$, we can differentiate the right-hand side of (C.20) k times under the integral sign, which proves (C.20). Using the dominated convergence theorem and taking the limit in (C.6) proves (C.21). □

From definition (C.6) and an evaluation similar to that in Theorem B3 of Appendix B.2, it follows that *the characteristic function $h_t(u)$ of the sum of independent random variables X_1 and X_2 with characteristic functions $h_1(u)$ and $h_2(u)$*, respectively, *is*

$$h_t(u) = h_1(u) h_2(u) . \tag{C.22}$$

Appendix D
Signal Characteristics

D.1 Bandpass Signals

A *bandpass signal* has its power spectrum in a spectral band surrounding a carrier frequency, which is usually at the center of the band. The *Hilbert transform* provides the basis for signal representations that facilitate the analysis of bandpass signals and systems. Let P denote the Cauchy principal value of an integral. The Hilbert transform of a function $g(t)$ is defined as

$$H[g(t)] = \hat{g}(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{g(u)}{t-u}\,du$$
$$= \frac{1}{\pi} \lim_{\epsilon \to 0} \left[\int_{-\infty}^{t-\epsilon} \frac{g(u)}{t-u}\,du + \int_{t+\epsilon}^{\infty} \frac{g(u)}{t-u}\,du \right] \tag{D.1}$$

provided that the integral exists as a principal value.

Since (D.1) has the form of the convolution of $g(t)$ with $1/\pi t$, $\hat{g}(t)$ results from passing $g(t)$ through a linear filter with an impulse response equal to $1/\pi t$. The transfer function of the filter is given by the Fourier transform of $1/\pi t$. For this function, the Fourier transform is

$$\mathcal{F}\left[\frac{1}{\pi t}\right] = P \int_{-\infty}^{\infty} \frac{e^{-j2\pi f t}}{\pi t}\,dt \tag{D.2}$$

where $j = \sqrt{-1}$. Changing variables and applying Theorem C2 of Appendix C.2, we obtain

$$\mathcal{F}\left[\frac{1}{\pi t}\right] = \frac{-\text{sgn}(f)}{\pi} P \int_{-\infty}^{\infty} \frac{e^{jx}}{x}\,dx$$
$$= -j\,\text{sgn}(f) \tag{D.3}$$

D. Torrieri, *Principles of Spread-Spectrum Communication Systems*,
https://doi.org/10.1007/978-3-319-70569-9

where *sgn*(f) is the *signum function* defined by

$$\operatorname{sgn}(f) = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0. \end{cases} \tag{D.4}$$

Let $G(f) = \mathcal{F}[g(t)]$ and $\hat{G}(f) = \mathcal{F}[\hat{g}(t)]$. Applying the convolution theorem of Fourier analysis (Appendix C.1) to (D.1) and then substituting (D.3), we obtain

$$\hat{G}(f) = -j \operatorname{sgn}(f) G(f). \tag{D.5}$$

Because $H[\hat{g}(t)]$ results from passing $g(t)$ through two successive filters, each with a transfer function $-j \operatorname{sgn}(f)$,

$$H[\hat{g}(t)] = -g(t) \tag{D.6}$$

provided that $G(0) = 0$.

Equation (D.5) indicates that taking the Hilbert transform corresponds to introducing a phase sift of $-\pi/2$ radians for all positive frequencies and $+\pi/2$ radians for all negative frequencies. Consequently,

$$H[\cos 2\pi f_c t] = \sin 2\pi f_c t \tag{D.7}$$

$$H[\sin 2\pi f_c t] = -\cos 2\pi f_c t. \tag{D.8}$$

These relations can be formally verified by taking the Fourier transform of the left-hand side of (D.7) or (D.8), applying (D.5), and then taking the inverse Fourier transform of the result. If $G(f) = 0$ for $|f| > W$ and $f_c > W$, the same method yields

$$H[g(t) \cos 2\pi f_c t] = g(t) \sin 2\pi f_c t \tag{D.9}$$

$$H[g(t) \sin 2\pi f_c t] = -g(t) \cos 2\pi f_c t. \tag{D.10}$$

A *bandpass signal* is a signal with a Fourier transform that is negligible except for $f_c - W/2 \leq |f| \leq f_c + W/2$, where $0 \leq W < 2f_c$ and f_c is the center frequency. If $W << f_c$, the bandpass signal is often called a *narrowband signal*. A complex-valued signal with a Fourier transform that is nonzero only for $f > 0$ is called an *analytic signal*.

Consider a bandpass signal $g(t)$ with Fourier transform $G(f)$. The analytic signal $g_a(t)$ associated with $g(t)$ is defined to be the signal with Fourier transform

$$G_a(f) = [1 + \operatorname{sgn}(f)]G(f) \tag{D.11}$$

which is zero for $f \leq 0$ and is confined to the band $|f - f_c| \leq W/2$ when $f > 0$. The inverse Fourier transform of $G_a(f)$ and (D.5) imply that

$$g_a(t) = g(t) + j\hat{g}(t). \tag{D.12}$$

The *complex envelope* of $g(t)$ is defined by

$$g_l(t) = g_a(t)e^{-j2\pi f_c t} \tag{D.13}$$

where f_c is the center frequency if $g(t)$ is a bandpass signal. Since the Fourier transform of $g_l(t)$ is $G_a(f + f_c)$, which occupies the band $|f| \leq W/2$, the complex envelope is a baseband signal that may be regarded as an *equivalent lowpass representation* of $g(t)$. Equations (D.12) and (D.13) imply that $g(t)$ can be expressed in terms of its complex envelope as

$$g(t) = \text{Re}[g_l(t)e^{j2\pi f_c t}]. \tag{D.14}$$

The complex envelope can be decomposed as

$$g_l(t) = g_c(t) + jg_s(t) \tag{D.15}$$

where $g_c(t)$ and $g_s(t)$ are real-valued functions. Therefore, (D.14) yields

$$g(t) = g_c(t)\cos(2\pi f_c t) - g_s(t)\sin(2\pi f_c t). \tag{D.16}$$

Since the two sinusoidal carriers are in phase quadrature, $g_c(t)$ and $g_s(t)$ are called the *in-phase* and *quadrature* components of $g(t)$, respectively. These components are lowpass signals confined to $|f| \leq W/2$.

Applying Parseval's identity from Fourier analysis (Appendix C.1) and then (D.5), we obtain

$$\int_{-\infty}^{\infty} \hat{g}^2(t)\, dt = \int_{-\infty}^{\infty} |\hat{G}(f)|^2\, df = \int_{-\infty}^{\infty} |G(f)|^2\, df = \int_{-\infty}^{\infty} g^2(t)\, dt. \tag{D.17}$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} |g_l(t)|^2\, dt &= \int_{-\infty}^{\infty} |g_a(t)|^2\, dt = \int_{-\infty}^{\infty} g^2(t)\, dt + \int_{-\infty}^{\infty} \hat{g}^2(t)\, dt \\ &= 2\int_{-\infty}^{\infty} g^2(t)\, dt = 2\mathcal{E}\end{aligned} \tag{D.18}$$

where $\mathcal{E}$ denotes the energy of the bandpass signal $g(t)$.

D.2 Stationary Stochastic Processes

A stochastic process is called *wide-sense stationary* if its mean is independent of the sampling time, and its autocorrelation depends only on the time difference between samples. Consider a real-valued, wide-sense-stationary stochastic process $n(t)$ that is a zero-mean with autocorrelation

$$R_n(\tau) = E[n(t)n(t+\tau)] \tag{D.19}$$

where $E[x]$ denotes the expected value of x. The Hilbert transform of this process is the real-valued stochastic process defined by

$$\hat{n}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{n(u)}{t-u} du \tag{D.20}$$

where we assume that the Cauchy principal value of the integral exists for almost every sample function of $n(t)$. This equation indicates that $\hat{n}(t)$ is a zero-mean stochastic process. The zero-mean processes $n(t)$ and $\hat{n}(t)$ are *jointly wide-sense stationary* if their correlation and cross-correlation functions are not functions of t.

An application of (D.20) and (D.19) gives the cross-correlation of $n(t)$ and $\hat{n}(t)$:

$$\begin{aligned} R_{n\hat{n}}(\tau) &= E[n(t)\hat{n}(t+\tau)] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R_n(u)}{\tau-u} du \\ &= \hat{R}_n(\tau). \end{aligned} \tag{D.21}$$

An application of this result and (D.6) yields the autocorrelation

$$\begin{aligned} R_{\hat{n}}(\tau) &= E[\hat{n}(t)\hat{n}(t+\tau)] \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{R}_n(t+\tau-u)}{t-u} du = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{R}_n(x)}{\tau-x} dx \\ &= R_n(\tau). \end{aligned} \tag{D.22}$$

Equations (D.19), (D.21), and (D.22) indicate that $n(t)$ and $\hat{n}(t)$ are jointly wide-sense stationary. Since $n(t)$ is wide-sense stationary, $R_n(\tau)$ is an even function. It then follows from (D.22) that $R_{\hat{n}}(\tau)$ is an even function. Equation (D.21) and a change in the integration variable indicate that $\widehat{R}_n(\tau)$ and $R_{n\hat{n}}(\tau)$ are odd functions. Thus,

$$\begin{aligned} &R_n(-\tau) = R_n(\tau),\ R_{\hat{n}}(-\tau) = R_{\hat{n}}(\tau) \\ &\hat{R}_n(-\tau) = -\hat{R}_n(\tau),\ R_{n\hat{n}}(-\tau) = -R_{n\hat{n}}(\tau). \end{aligned} \tag{D.23}$$

The *analytic signal* associated with $n(t)$ is the complex-valued zero-mean process defined by

$$n_a(t) = n(t) + j\hat{n}(t). \tag{D.24}$$

The autocorrelation of the analytic signal is defined as

$$R_a(\tau) = E[n_a^*(t)n_a(t+\tau)] \tag{D.25}$$

where the asterisk denotes the complex conjugate. Using (D.19) and (D.21) to (D.25), we obtain

$$R_a(\tau) = 2R_n(\tau) + 2j\hat{R}_n(\tau) \tag{D.26}$$

which establishes the wide-sense stationarity of the analytic signal.

Since (D.19) indicates that $R_n(\tau)$ is an even function, (D.21) yields

$$R_{n\hat{n}}(0) = \hat{R}_n(0) = 0 \tag{D.27}$$

which indicates that $n(t)$ and $\hat{n}(t)$ are uncorrelated. Equations (D.22), (D.26), and (D.27) yield

$$R_{\hat{n}}(0) = R_n(0) = 1/2R_a(0). \tag{D.28}$$

The *complex envelope* of $n(t)$ or the *equivalent lowpass representation* of $n(t)$ is the zero-mean stochastic process defined by

$$n_l(t) = n_a(t)e^{-j2\pi f_c t} \tag{D.29}$$

where f_c is an arbitrary frequency usually chosen as the center or carrier frequency of $n(t)$. The complex envelope can be decomposed as

$$n_l(t) = n_c(t) + jn_s(t) \tag{D.30}$$

where $n_c(t)$ and $n_s(t)$ are real-valued, zero-mean stochastic processes.

Equations (D.24) and (D.29) imply that

$$n(t) = Re[n_l(t)e^{j2\pi f_c t)}]. \tag{D.31}$$

The substitution of (D.30) into (D.31) gives an in-phase and quadrature representation:

$$n(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t). \tag{D.32}$$

Substituting (D.24) and (D.30) into (D.29) we find that

$$n_c(t) = n(t)\cos(2\pi f_c t) + \hat{n}(t)\sin(2\pi f_c t) \tag{D.33}$$

$$n_s(t) = \hat{n}(t)\cos(2\pi f_c t) - n(t)\sin(2\pi f_c t). \tag{D.34}$$

Using (D.33), (D.34), (D.19), (D.21), (D.22), (D.23), and trigonometric identities, we obtain the autocorrelations of $n_c(t)$ and $n_s(t)$, which are

$$R_c(\tau) = E[n_c(t)n_c(t+\tau)] = R_n(\tau)\cos(2\pi f_c \tau) + \hat{R}_n(\tau)\sin(2\pi f_c \tau) \tag{D.35}$$

$$R_s(\tau) = E[n_s(t)n_s(t+\tau)] = R_c(\tau) \tag{D.36}$$

and the cross-correlations

$$R_{cs}(\tau) = E[n_c(t)n_s(t+\tau)] = \hat{R}_n(\tau)\cos(2\pi f_c \tau) - R_n(\tau)\sin(2\pi f_c \tau) \tag{D.37}$$

$$R_{sc}(\tau) = E[n_s(t)n_c(t+\tau)] = -R_{cs}(\tau). \tag{D.38}$$

These equations show explicitly that if $n(t)$ is wide-sense stationary, then $n_c(t)$ and $n_s(t)$ are jointly wide-sense stationary with identical autocorrelation functions. From (D.23), we obtain

$$\begin{aligned} R_c(-\tau) &= R_c(\tau),\; R_s(-\tau) = R_s(\tau) \\ R_{cs}(-\tau) &= -R_{cs}(\tau),\; R_{sc}(-\tau) = -R_{sc}(\tau). \end{aligned} \tag{D.39}$$

Since

$$R_c(0) = R_s(0) = R_n(0),\; R_{cs}(0) = 0 \tag{D.40}$$

the variances of $n(t)$, $n_c(t)$, and $n_s(t)$ are all equal, and $n_c(t)$ and $n_s(t)$ are uncorrelated.

Equations (D.30) and (D.39) imply that

$$E[n_l(t)n_l(t+\tau)] = 0. \tag{D.41}$$

A complex-valued, zero-mean stochastic process that satisfies this equation is called a *circularly symmetric* process. Thus, *the complex envelope of a zero-mean, wide-sense stationary process is a circularly symmetric process.* The autocorrelation of a complex envelope is defined as

$$R_l(\tau) = E[n_l^*(t)n_l(t+\tau)]. \tag{D.42}$$

Substituting (D.29) and (D.26) into (D.42), we obtain

$$R_l(\tau) = 2e^{-j2\pi f_c \tau}\left[R_n(\tau) + j\hat{R}_n(\tau)\right] \tag{D.43}$$

which shows that $n_l(t)$ is a zero-mean, wide-sense-stationary process. Since $R_n(\tau)$ and $\hat{R}_n(\tau)$ are real-valued,

$$R_n(\tau) = \frac{1}{2}Re\left[R_l(\tau)e^{j2\pi f_c \tau}\right]. \tag{D.44}$$

Power Spectral Density

The *power spectral density* (PSD) of a signal is the Fourier transform of its autocorrelation. Let $S_n(f)$, $S_c(f)$, and $S_s(f)$ denote the PSDs of $n(t)$, $n_c(t)$, and $n_s(t)$, respectively. We assume that $S_n(f)$ occupies the band $f_c - W/2 \leq |f| \leq f_c + W/2$ and that $f_c > W/2 \geq 0$. Taking the Fourier transform of (D.35), using (D.5), and simplifying, we obtain

$$S_c(f) = S_s(f) = \begin{cases} S_n(f - f_c) + S_n(f + f_c), & |f| \leq W/2 \\ 0, & |f| > W/2. \end{cases} \tag{D.45}$$

Similarly, the cross-spectral density of $n_c(t)$ and $n_s(t)$ can be derived by taking the Fourier transform of (D.37) and using (D.5). After simplification, the result is

$$S_{cs}(f) = \begin{cases} j[S_n(f - f_c) - S_n(f + f_c)], & |f| \leq W/2 \\ 0, & |f| > W/2. \end{cases} \tag{D.46}$$

Since $R_n(\tau)$ is an even function, $S_n(f)$ is a real-valued, even function. If $S_n(f)$ is *locally symmetric* about f_c so that

$$\begin{aligned} S_n(f_c + f) &= S_n(f_c - f), \\ &= S_n(f - f_c), \quad |f| \leq W/2 \end{aligned} \tag{D.47}$$

then (D.46) indicates that $S_{cs}(f) = 0$, which implies that

$$R_{cs}(\tau) = 0 \tag{D.48}$$

for all τ. Thus, $n_c(t)$ and $n_s(t + \tau)$ are uncorrelated for all τ when $S_n(f)$ is locally symmetric.

The PSD of $n_l(t)$, which we denote as $S_l(f)$, can be derived by calculating the Fourier transform of (D.43), and using (D.5). If $S_n(f)$ occupies the band $f_c - W/2 \leq |f| \leq f_c + W/2$ and $f_c > W/2 \geq 0$, then

$$S_l(f) = \begin{cases} 4S_n(f + f_c), & |f| \leq W/2 \\ 0, & |f| > W/2 \end{cases} \tag{D.49}$$

which indicates that $S_l(f)$ is a real-valued function. Therefore, expanding the right-hand side of (D.44) by using $Re[z] = (z + z^*)/2$ and then taking the Fourier transform yields

$$S_n(f) = \frac{1}{4}S_l(f - f_c) + \frac{1}{4}S_l(-f - f_c). \tag{D.50}$$

White Gaussian Noise

The communication channel is often modeled as an *additive white Gaussian noise* (AWGN) *channel* for which the noise in the receiver is a zero-mean, white Gaussian process with autocorrelation

$$R_n(\tau) = E[n(t)n(t+\tau)] = \frac{N_0}{2}\delta(\tau) \tag{D.51}$$

and the *two-sided noise PSD*

$$S_n(f) = \frac{N_0}{2}. \tag{D.52}$$

The Hilbert transform $\hat{n}(t)$ is the limit of Riemann sums that are linear combinations of $n(t)$. Therefore, Theorem A1 of Appendix A.1 implies that if $n(t)$ is a zero-mean, white Gaussian process, then $\hat{n}(t)$ and $n(t)$ are zero-mean jointly Gaussian processes. Equations (D.33) and (D.34) and Theorem A4 of Appendix A.1 then imply that $n_c(t)$ and $n_s(t)$ are zero-mean jointly Gaussian processes. Since (D.40) shows that they are uncorrelated for a specific value of t, $n_c(t)$ and $n_s(t)$ are statistically independent, zero-mean Gaussian random variables with equal variances.

D.3 Downconverter

Most modern digital receivers use a *downconverter* to convert the received signal into a filtered baseband signal. The main components of a downconverter for the desired signal $s(t)$ with carrier or center frequency f_c are shown in Figure D.1 (a).

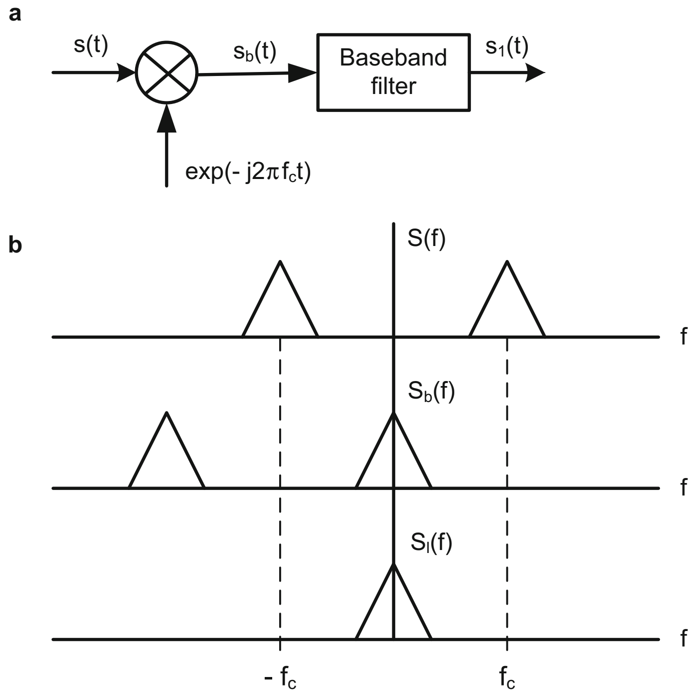

Fig. D.1 Envelope extraction: (**a**) downconverter and (**b**) associated spectra

Consider the received signal

$$r(t) = s(t) + n(t) \tag{D.53}$$

where $n(t)$ is the noise. Let $h(t)$ denote the impulse response of the baseband filter. The output of the baseband filter due to $s(t)$ is

$$s_1(t) = \int_{-\infty}^{\infty} s(\tau)e^{-j2\pi f_c \tau} h(t-\tau)\, d\tau. \tag{D.54}$$

Using (D.14) to express $s(t)$ in terms of its complex envelope $s_l(\tau)$ and then substituting $Re(x) = (x + x^*)/2$, where x^* denotes the complex conjugate of x, we obtain

$$s_1(t) = \frac{1}{2}\int_{-\infty}^{\infty} s_l(\tau)h(t-\tau)\, d\tau + \frac{1}{2}\int_{-\infty}^{\infty} s_l(\tau)h(t-\tau)e^{-j4\pi f_c \tau} d\tau. \tag{D.55}$$

The second term is the Fourier transform of $s_l(\tau)h(t-\tau)$ evaluated at frequency $-2f_c$. Assuming that $s_l(\tau)$ and $h(t-\tau)$ have transforms confined to $|f| < f_c$, their product has a transform confined to $|f| < 2f_c$, and the second term in (D.55) vanishes, leaving

$$s_1(t) = \frac{1}{2}\int_{-\infty}^{\infty} s_l(\tau)h(t-\tau)\,d\tau. \tag{D.56}$$

This convolution is equivalent to the multiplication of the corresponding Fourier transforms. Therefore, if the Fourier transform of $h(t)$ is a constant over the passband of $s_l(t)$, then $s_1(t)$ is proportional to $s_l(t)$, which indicates that the complex envelope of the received signal can be extracted by the downconverter. The spectra of the received signal $s(t)$, the input to the baseband filter $s_b(t) = s(t)\exp(-j2\pi f_c t)$, and the complex envelope $s_l(t)$ are depicted in Figure D.1 (b).

The downconverter alters the character of the noise $n(t)$ entering it. The complex-valued noise at the output of the downconverter is

$$z(t) = \int_{-\infty}^{\infty} n(u)e^{-j2\pi f_c u}h(t-u)du. \tag{D.57}$$

We assume that the noise is a zero-mean, white Gaussian process. The approximating Riemann sums of the real and imaginary parts of the integral in (D.57) are sums of independent zero-mean Gaussian random variables. Therefore, the real and imaginary components of $z(t)$ are *jointly zero-mean Gaussian random variables* (Appendix A.1), and $z(t)$ is zero-mean. The autocorrelation of a wide-sense-stationary, complex-valued process $z(t)$ is defined as

$$R_z(\tau) = E[z^*(t)z(t+\tau)]. \tag{D.58}$$

Substituting (D.57), interchanging the expectation and integration operations, using (D.51) to evaluate one of the integrals, and then changing variables, we obtain

$$R_z(\tau) = \frac{N_0}{2}\int_{-\infty}^{\infty} h^*(u)h(u+\tau)du. \tag{D.59}$$

Equations (D.57) and (D.51) imply that

$$E[z(t)z(t+\tau)] = \frac{N_0}{2}e^{-j4\pi f_c t}\int_{-\infty}^{\infty} e^{j4\pi f_c u}h(u+\tau)h(u)\,du. \tag{D.60}$$

The integrand is the Fourier transform of $h(u+\tau)h(u)$ evaluated at frequency $-2f_c$. Assuming that $h(u)$ has a Fourier transform confined to $|f| < f_c$, $h(u+\tau)h(u)$ has a Fourier transform confined to $|f| < 2f_c$, and

$$E[z(t)z(t+\tau)] = 0. \tag{D.61}$$

A complex-valued stochastic process $z(t)$ that satisfies (D.61) is called a *circularly symmetric* process.

Let $z_R(t)$ and $z_I(t)$ denote the real and imaginary parts of $z(t)$, respectively. Since $z(t)$ is zero-mean, $z_R(t)$ and $z_I(t)$ are zero-mean. Setting $\tau = 0$ in (D.61), equating (D.58) and (D.59) for $\tau = 0$, and then solving the preceding equations, we obtain

$$E[(z_R(t))^2] = E[(z_I(t))^2] = \frac{1}{2}E[|z(t)|^2] = \frac{N_0}{4}\int_{-\infty}^{\infty} |h(u)|^2 \, du \tag{D.62}$$

$$E[(z_R(t)z_I(t))] = 0 \tag{D.63}$$

which proves that $z_R(t)$ and $z_I(t)$ are zero-mean, independent Gaussian processes with the same variance.

D.4 Sampling Theorem

The Fourier transform and the inverse Fourier transform of a bounded, complex-valued, continuous-time signal $x(t)$ are

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt \tag{D.64}$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} df. \tag{D.65}$$

A sample of this signal is $x_n = x(nT)$, where $1/T$ is the sampling rate and n is an integer. The *discrete-time Fourier transform (DTFT) is defined as*

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x_n e^{-jn\theta} \tag{D.66}$$

if the infinite sum converges to a finite value for all values of θ. A sufficient condition for convergence is

$$\sum_{n=-\infty}^{\infty} |x_n| < \infty \tag{D.67}$$

which implies that $X(e^{j\theta})$ is uniformly convergent over $[-\pi, \pi]$. Equation (D.66) indicates that $X(e^{j\theta})$ is a periodic function of θ with period 2π.

Let $\mathcal{Z}$ denote the set of all integers. The *inverse DTFT* of $X(e^{j\theta})$ is

$$x_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})e^{jn\theta} d\theta, \; n \in \mathcal{Z} \tag{D.68}$$

which is verified by the direct substitution of (D.66) into the integrand and an evaluation. Since the complex exponentials constitute a complete set of orthonormal basis functions over $[-\pi, \pi]$, a uniqueness property can be shown [6]: if $X_1(\theta)$ and $X_2(\theta)$ have the same inverse DTFT, then $X_1(\theta) = X_2(\theta)$ for almost all θ.

The *sampling theorem* relates $x(t)$ and x_n in the frequency domain, indicating that $X(e^{j2\pi fT})$ is the sum of shifted versions of $X(f)$.

Sampling Theorem *The discrete-time and continuous-time Fourier transforms of a bounded continuous-time signal* $x(t)$ *are related by*

$$X(e^{j2\pi fT}) = \frac{1}{T}\sum_{i=-\infty}^{\infty} X\left(f - \frac{i}{T}\right) \tag{D.69}$$

if the series converges uniformly over the interval $f \in [-1/2T, +1/2T]$.

Proof Consider a bounded continuous-time signal $x(t)$ with Fourier transform $X(f)$ and $x_n = x(nT)$. According to the uniqueness property, the theorem can be proved by showing that the inverse DTFT of the right side of (D.69) for all $n \in \mathcal{Z}$ is x_n. Let I_n denote this inverse DTFT for $n \in \mathcal{Z}$. Using the uniform convergence to interchange the summation and integration, we obtain

$$\begin{aligned} I_n &= \int_{-1/2T}^{1/2T} \left[\sum_{i=-\infty}^{\infty} X\left(f - \frac{i}{T}\right)\right] e^{j2\pi nfT} df \\ &= \sum_{i=-\infty}^{\infty} \int_{-1/2T}^{1/2T} X\left(f - \frac{i}{T}\right) e^{j2\pi nfT} df, \ n \in \mathcal{Z}. \end{aligned} \tag{D.70}$$

Changing the integration variable in this equation and using $e^{j2\pi ni} = 1$ when n and i are integers, we obtain

$$I_n = \sum_{i=-\infty}^{\infty} \int_{(2i-1)/2T}^{(2i+1)/2T} X(f) e^{j2\pi nfT} df \tag{D.71}$$

$$= \int_{-\infty}^{\infty} X(f) e^{j2\pi nfT} df \tag{D.72}$$

$$= x(nT) = x_n, \ n \in \mathcal{Z} \tag{D.73}$$

which completes the proof. □

If a continuous-time signal $x(t)$ with Fourier transform $X(f)$ is *bandlimited* so that $X(f) = 0$ for $|f| > 1/2T$, then (D.69) indicates that

$$X(e^{j2\pi fT}) = \frac{1}{T} X(f), \quad -\frac{1}{2T} \le f \le \frac{1}{2T}. \tag{D.74}$$

Substitution of this equation into (D.65) and the use of (D.66) yield

$$\begin{aligned} x(t) &= \int_{-1/2T}^{1/2T} X(f)e^{j2\pi ft}df = T\int_{-1/2T}^{1/2T} X(e^{j2\pi fT})e^{j2\pi ft}df \\ &= T\int_{-1/2T}^{1/2T} \sum_{n=-\infty}^{\infty} x_n e^{-j2\pi fnT} e^{j2\pi ft}df \\ &= T\sum_{n=-\infty}^{\infty} x_n \int_{-1/2T}^{1/2T} e^{j2\pi f(t-nT)}df. \end{aligned} \tag{D.75}$$

Evaluating the final integral, we obtain the formula for the exact reconstruction of a bandlimited $x(t)$ from its samples:

$$x(t) = \sum_{n=-\infty}^{\infty} x_n\, sinc\left(\frac{t-nT}{T}\right), \quad X(f) = 0 \text{ for } |f| > 1/2T \tag{D.76}$$

where $sinc\,(x) = \sin\,(\pi x)\,/\pi x$. This equation implies that a signal with a one-sided bandwidth W that is sampled at the rate $1/T > 2W$ is uniquely defined by its samples. If the sampling rate is not high enough that $X(f) = 0$ for $|f| > 1/2T$, then the terms in the sum in (D.69) overlap, which is called *aliasing,* and the samples may not specify a unique continuous-time signal.

Appendix E
Probability Distribution Functions

E.1 Chi-Squared Distribution

Consider the random variable

$$Z = \sum_{i=1}^{N} X_i^2 \tag{E.1}$$

where the $\{X_i\}$ are independent Gaussian random variables with means $\{m_i\}$ and common variance σ^2. The random variable Z is said to have a *noncentral chi-squared (also chi-square) distribution* with N degrees of freedom and a *noncentral parameter*

$$\lambda = \sum_{i=1}^{N} m_i^2. \tag{E.2}$$

To derive the density function of Z, we first note that each X_i has the density function

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-m_i)^2}{2\sigma^2}\right]. \tag{E.3}$$

From elementary probability, the density function of $Y_i = X_i^2$ is

$$f_{Y_i}(x) = \frac{1}{2\sqrt{x}}[f_{X_i}(\sqrt{x}) + f_{X_i}(-\sqrt{x})]\, u(x) \tag{E.4}$$

where $u(x)$ is the unit step function: $u(x) = 1$, $x \geq 0$, and $u(x) = 0, x < 0$. Substituting (E.3) into (E.4), expanding the exponentials, and simplifying, we obtain

D. Torrieri, *Principles of Spread-Spectrum Communication Systems*,
https://doi.org/10.1007/978-3-319-70569-9

the density function

$$f_{Y_i}(x) = \frac{1}{\sqrt{2\pi x}\sigma} \exp\left(-\frac{x+m_i^2}{2\sigma^2}\right) \cosh\left(\frac{m_i\sqrt{x}}{\sigma^2}\right) u(x). \tag{E.5}$$

Characteristic functions, moment-generating functions, and Laplace transforms all uniquely determine distribution functions. Since Y_i assumes only nonnegative values, it is convenient to use the Laplace transform. The *Laplace transform* of a continuous nonnegative random variable Y is defined as (Appendix B.2)

$$\mathcal{L}(s) = E[e^{-sY}] = \int_0^\infty f_Y(x)e^{-sx}dx \tag{E.6}$$

where $f_Y(x)$ is the density function of Y. To evaluate the Laplace transform of $Y_i = X_i^2$, we substitute (E.5) into (E.6), change the integration variable by setting $y = \sqrt{(1+2\sigma^2 s)x/2\sigma^2}$, expand the range of integration because $\cosh(\cdot)$ is an even function, and obtain

$$\mathcal{L}_i(s) = \frac{1}{\sqrt{(1+2\sigma^2 s)\pi}} \int_{-\infty}^\infty \exp\left(-\frac{m_i^2}{2\sigma^2} - y^2\right) \cosh\left(\frac{ym_i\sqrt{2}}{\sigma\sqrt{1+2\sigma^2 s}}\right) dy. \tag{E.7}$$

Using $\cosh(z) = (e^z + e^{-z})/2$, separating the integral into two integrals, completing the squares in the arguments of the exponentials, and then observing that a Gaussian density function must integrate to unity, we obtain the Laplace transform of Y_i:

$$\mathcal{L}_i(s) = \frac{\exp\left(\frac{-sm_i^2}{1+2\sigma^2 s}\right)}{(1+2\sigma^2 s)^{1/2}}, \quad Re(s) > -\frac{1}{2\sigma^2}\ . \tag{E.8}$$

The Laplace transform of a sum of independent random variables is equal to the product of the individual Laplace transforms. Because Z is the sum of the $\{Y_i\}$, the Laplace transform of Z is

$$\mathcal{L}_Z(s) = \frac{\exp\left(\frac{-s\lambda}{1+2\sigma^2 s}\right)}{(1+2\sigma^2 s)^{N/2}}, \quad Re(s) > -\frac{1}{2\sigma^2} \tag{E.9}$$

where we have used (E.2).

The noncentral chi-squared density with N degrees of freedom and noncentral parameter λ is the unique density (see Section B.2) with the Laplace transform given by (E.9). As shown subsequently, this density is

$$f_Z(x) = \frac{1}{2\sigma^2}\left(\frac{x}{\lambda}\right)^{(N-2)/4} \exp\left[-\left(\frac{x+\lambda}{2\sigma^2}\right)\right] I_{N/2-1}\left(\frac{\sqrt{x\lambda}}{\sigma^2}\right) u(x) \tag{E.10}$$

where $I_n(\cdot)$ is the modified Bessel function of the first kind and order n (Appendix H.3). Thus, the noncentral chi-squared distribution is

$$F_Z(x) = \int_0^x \frac{1}{2\sigma^2}\left(\frac{y}{\lambda}\right)^{(N-2)/4} \exp\left(-\frac{y+\lambda}{2\sigma^2}\right) I_{N/2-1}\left(\frac{\sqrt{y\lambda}}{\sigma^2}\right) dy\ , \quad x \geq 0. \tag{E.11}$$

To prove that $F_Z(\infty) = 1$, and hence that $F_Z(x)$ is a legitimate distribution function, we first need to establish an integral identity:

$$\int_0^\infty x^{(N-2)/4} \exp(-x)\, I_{N/2-1}\left(\sqrt{4xa}\right) dx = a^{(N-2)/4} \exp(a)\ , \quad a \geq 0. \tag{E.12}$$

To prove (E.12), we substitute (H.13) of Appendix H.3 into the integral, apply the monotone convergence theorem to interchange the integral and infinite summation, apply (H.1), simplify, and then recognize the power series expansion of $\exp(a)$. A change of variables in (E.12) then proves that $F_Z(\infty) = 1$.

To prove that $f_Z(x)$ is given by (E.10), we substitute (E.10) into (E.6), change variables, and use (E.12) to obtain (E.9).

If N is even so that $N/2$ is an integer, then a change of variables in (E.11) yields

$$F_Z(x) = 1 - Q_{N/2}\left(\frac{\sqrt{\lambda}}{\sigma}\ ,\ \frac{\sqrt{x}}{\sigma}\right), \quad x \geq 0 \tag{E.13}$$

where $Q_m(\alpha, \beta)$ is the *generalized Marcum Q-function* (Appendix H.4). The moments of Z can be obtained by using (E.1) and the properties of independent Gaussian random variables. The mean and variance of Z are

$$E[Z] = N\sigma^2 + \lambda,\ \ \sigma_z^2 = 2N\sigma^4 + 4\lambda\sigma^2 \tag{E.14}$$

where σ^2 is the common variance of the $\{X_i\}$. Alternatively, the moments of Z can be obtained by applying Theorem B1 of Appendix B.2 and (E.9).

From (E.9), it follows that the sum of two independent noncentral chi-squared random variables with N_1 and N_2 degrees of freedom, noncentral parameters λ_1 and λ_2, respectively, and the same parameter σ^2 is a noncentral chi-squared random variable with $N_1 + N_2$ degrees of freedom and noncentral parameter $\lambda_1 + \lambda_2$.

E.2 Central Chi-Squared Distribution

To determine the density function of Z when the $\{X_i\}$ have zero-means, we substitute (H.13) of Appendix H.3 into (E.10) and then take the limit as $\lambda \to 0$.

We obtain the *central chi-squared density* with N degrees of freedom:

$$f_Z(x) = \frac{1}{(2\sigma^2)^{N/2}\Gamma(N/2)} x^{N/2-1} \exp\left(-\frac{x}{2\sigma^2}\right) u(x). \tag{E.15}$$

The central chi-squared distribution with N degrees of freedom, which is concentrated on the positive x-axis, is

$$F_Z(x) = \frac{1}{(2\sigma^2)^{N/2}\Gamma(N/2)} \int_0^x y^{N/2-1} \exp\left(-\frac{y}{2\sigma^2}\right) dy, \ x \geq 0. \tag{E.16}$$

In terms of the incomplete gamma function $\gamma(a, b)$ defined by (H.6), we have

$$F_Z(x) = \frac{\gamma\left(\frac{N}{2}, \frac{x}{2\sigma^2}\right)}{\Gamma(N/2)} u(x). \tag{E.17}$$

The moments of Z may be obtained by direct integration using (E.15) and (H.1) of Appendix H.1. The mean and variance of Z are

$$E[Z] = N\sigma^2, \ \ \sigma_z^2 = 2N\sigma^4. \tag{E.18}$$

If N is even, so that $N/2$ is an integer, then integrating (E.16) by parts $N/2 - 1$ times yields

$$F_Z(x) = 1 - \exp\left(-\frac{x}{2\sigma^2}\right) \sum_{i=0}^{N/2-1} \frac{1}{i!} \left(\frac{x}{2\sigma^2}\right)^i, \ \ x \geq 0. \tag{E.19}$$

If $N = 1$, then changing the integration variable to $z = \sqrt{y}$ in (E.16) and using (H.20), we obtain

$$F_Z(x) = \left[1 - 2Q\left(\frac{\sqrt{x}}{\sigma}\right)\right] u(x), \ N = 1. \tag{E.20}$$

E.3 Rice Distribution

Consider the random variable

$$R = \sqrt{X_1^2 + X_2^2} \tag{E.21}$$

where X_1 and X_2 are independent Gaussian random variables with means m_1 and m_2, respectively, and a common variance σ^2. The distribution function of R must satisfy $F_R(r) = F_Z(r^2)$, where $Z = X_1^2 + X_2^2$ has a chi-squared distribution with two degrees of freedom. Therefore, (E.13) with $N = 2$ implies that R has a Rice distribution:

$$F_R(r) = 1 - Q_1\left(\frac{\sqrt{\lambda}}{\sigma}, \frac{r}{\sigma}\right), \quad r \geq 0 \tag{E.22}$$

where $\lambda = m_1^2 + m_2^2$. The *Rice density*, which may be obtained by differentiation of (E.22), is

$$f_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2 + \lambda}{2\sigma^2}\right) I_0\left(\frac{r\sqrt{\lambda}}{\sigma^2}\right) u(r). \tag{E.23}$$

The moments of even order can be derived from (E.21) and the moments of the independent Gaussian random variables. The second moment is

$$E[R^2] = 2\sigma^2 + \lambda. \tag{E.24}$$

In general, moments of the Rice distribution are given by an integration over the density function in (E.23). Substituting (H.13) of Appendix H.3 into the integrand, interchanging the summation and integration, changing the integration variable, and using (H.1) of Appendix H.1, we obtain a series that is recognized as a special case of the confluent hypergeometric function. Thus,

$$E[R^n] = (2\sigma^2)^{n/2} \exp\left(-\frac{\lambda}{2\sigma^2}\right) \Gamma\left(1 + \frac{n}{2}\right) {}_1F_1\left(1 + \frac{n}{2}, 1; \frac{\lambda}{2\sigma^2}\right), \quad n \geq 0 \tag{E.25}$$

where ${}_1F_1(\alpha, \beta; x)$ is the *confluent hypergeometric function defined by* (H.27) of Appendix H.5.

The Rice density often arises in the context of a transformation of variables. Let X_1 and X_2 represent independent Gaussian random variables with common variance σ^2 and means λ and zero respectively. Let R and Θ be implicitly defined by $X_1 = R\cos\Theta$ and $X_2 = R\sin\Theta$. Then (E.21) and $\Theta = \tan^{-1}(X_2/X_1)$ describe a transformation of variables. Therefore, the joint density function of R and Θ is

$$f_{R,\Theta}(r, \theta) = \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2 - 2r\lambda\cos\theta + \lambda^2}{2\sigma^2}\right), \quad r \geq 0, \quad |\theta| \leq \pi. \tag{E.26}$$

The density function of R is obtained by integration over θ. Using (H.15) of Appendix H.3, this density function reduces to the Rice density (E.23). The density function of the angle Θ is obtained by integrating (E.26) over r. Completing the square of the argument in (E.26), changing variables, and using the definition of the Gaussian Q-function or complementary error function (Appendix H.4), we obtain

$$f_\Theta(\theta) = \frac{1}{2\pi}\exp\left(-\frac{\lambda^2}{2\sigma^2}\right) + \frac{\lambda\cos\theta}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\lambda^2\sin^2\theta}{2\sigma^2}\right)\left[1 - Q\left(\frac{\lambda\cos\theta}{\sigma}\right)\right],$$
$$|\theta| \le \pi. \tag{E.27}$$

Since (E.26) cannot be written as the product of (E.23) and (E.27), the random variables R and Θ are not independent.

E.4 Rayleigh Distribution

A Rayleigh-distributed random variable is defined by (E.21) when X_1 and X_2 are independent Gaussian random variables with zero-means and a common variance σ^2. Since $F_R(r) = F_Z(r^2)$, where Z has a central chi-squared distribution with two degrees of freedom, (E.19) with $N = 2$ implies that the *Rayleigh distribution* is

$$F_R(r) = 1 - \exp\left(-\frac{r^2}{2\sigma^2}\right), \quad r \ge 0. \tag{E.28}$$

The *Rayleigh density*, which may be obtained by differentiation of (E.28), is

$$f_R(r) = \frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)u(r). \tag{E.29}$$

By a change of the variable in the defining integral, any moment of R can be expressed in terms of the gamma function:

$$E[R^n] = (2\sigma^2)^{n/2}\Gamma\left(1 + \frac{n}{2}\right). \tag{E.30}$$

Using the properties of the gamma function (Appendix H.1), we obtain the mean and the variance of a Rayleigh-distributed random variable:

$$E[R] = \sqrt{\frac{\pi}{2}}\sigma, \;\; \sigma_R^2 = \left(2 - \frac{\pi}{2}\right)\sigma^2. \tag{E.31}$$

Since X_1 and X_2 have zero means, the joint density function of the random variables $R = \sqrt{X_1^2 + X_2^2}$ and $\Theta = \tan^{-1}(X_2/X_1)$ is given by (E.26) with $\lambda = 0$. Therefore,

$$f_{R,\Theta}(r,\theta) = \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right), \quad r \ge 0, \quad |\theta| \le \pi. \tag{E.32}$$

Integration over θ yields (E.29), and integration over r yields the uniform density function:

$$f_\Theta(\theta) = \frac{1}{2\pi}\,, \quad |\theta| \leq \pi. \tag{E.33}$$

Since (E.32) equals the product of (E.29) and (E.33), the random variables R and Θ are independent. In terms of these random variables, $X_1 = R\cos\Theta$ and $X_2 = R\sin\Theta$. A straightforward calculation using the independence and density functions of R and Θ verifies that X_1 and X_2 are zero-mean, independent, Gaussian random variables with common variance σ^2.

E.5 Exponential Distribution

A random variable X has the *exponential distribution* with parameter $\alpha > 0$ if it has the density function

$$f_X(x) = \alpha e^{-\alpha x} u(x). \tag{E.34}$$

The corresponding distribution function is

$$F_X(x) = 1 - e^{-\alpha x},\ x \geq 0. \tag{E.35}$$

Integrations using (H.3) and (H.2) yield

$$E[X] = \frac{1}{\alpha},\ \ var(X) = \frac{1}{\alpha^2}\,. \tag{E.36}$$

Since the square of a Rayleigh-distributed random variable may be expressed as $R^2 = X_1^2 + X_2^2$, where X_1 and X_2 are zero-mean, independent, Gaussian random variables with common variance σ^2, R^2 has a central chi-squared distribution with two degrees of freedom. Therefore, (E.15) with $N = 2$ indicates that the square of a Rayleigh-distributed random variable has an exponential density with mean $2\sigma^2$.

E.6 Gamma Distribution

A random variable X has the *gamma distribution* with parameters $\alpha, \beta > 0$ if it has the density

$$f(x; \alpha, \beta) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x} u(x). \tag{E.37}$$

Successive integrations by parts yield the corresponding distribution function:

$$F(x;\alpha,\beta) = [1 - \Gamma(\beta,\alpha x)]\, u(x) \tag{E.38}$$

where the incomplete gamma function is defined by (H.5). Integrations of (E.37) using (H.3) and (H.2) determine the moments of X. The mean and variance are

$$E[X] = \frac{\beta}{\alpha}, \quad var(X) = \frac{\beta}{\alpha^2}\,. \tag{E.39}$$

Let $\{f(x;\alpha,\beta), \beta > 0\}$ denote the family of gamma densities with the same parameter $\alpha > 0$ but different values of $\beta > 0$. Let $Z = X_1 + X_2$ denote the sum of two independent random variables with density functions $f(x;\alpha,\beta_1)$ and $f(x;\alpha,\beta_2)$ respectively. From elementary probability theory, it follows that the density function of Z is determined by the convolution of these two density functions:

$$\begin{aligned}
[f(\cdot;\alpha,\beta_1) * f(\cdot;\alpha,\beta_2)](x) &= \int_0^x f(y;\alpha,\beta_1) f(x-y;\alpha,\beta_2)\, dy \\
&= \frac{\alpha^{\beta_1+\beta_2} e^{-\alpha x}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^x y^{\beta_1-1}(x-y)^{\beta_2-1}\, dy \\
&= \frac{\alpha^{\beta_1+\beta_2} x^{\beta_1+\beta_2-1} e^{-\alpha x}}{\Gamma(\beta_1)\Gamma(\beta_2)} B(\beta_1,\beta_2)
\end{aligned} \tag{E.40}$$

where the star $\star$ denotes the convolution operation, and the beta function $B(\cdot,\cdot)$ is defined by (H.10) of Appendix H.2. Substituting (H.11) into (E.40) and using (E.37), we obtain

$$[f(\cdot;\alpha,\beta_1) * f(\cdot;\alpha,\beta_2)](x) = f(x;\alpha,\beta_1+\beta_2) \tag{E.41}$$

which indicates that the family of gamma densities $\{f(x;\alpha,\beta), \beta > 0\}$ is closed under convolution. By mathematical induction, if

$$Z = \sum_{i=1}^{N} X_i \tag{E.42}$$

is the sum of N independent random variables, and X_i has a gamma density with parameters α and β_i, then Z has a gamma density with parameters α and $\sum_{i=1}^{N} \beta_i$.

Let Z in (E.42) denote the sum of N independent, exponentially distributed random variables with the same parameter $\alpha > 0$. Since the exponential density is equal to $f(x;\alpha,1)$, (E.41) implies that the density function of Z is the gamma density $f(x;\alpha,N)$.

Equations (E.5) and (H.4) indicate that a central chi-squared density with *one* degree of freedom is equal to $f\left(x; 1/2\sigma^2, 1/2\right)$. Therefore, if Z in (E.42) denotes the sum of N independent central chi-squared random variables with the same parameter σ^2, then the application of (E.41) provides another proof that the density function of Z is the gamma density $f\left(x; 1/2\sigma^2, N/2\right)$, as indicated by (E.15).

Appendix F
Orthonormal Functions and Parameter Estimation

F.1 Deterministic Functions

A complete normed vector space is one in which every Cauchy sequence converges at a member of the vector space. A *Hilbert space* is a complete vector space with an inner product and a norm defined by the inner product. The *signal space* $L^2[0, T]$ is the Hilbert space of complex-valued functions $f(t)$ such that $|f(t)|^2$ is integrable over $[0, T]$. The inner product of functions $f(t)$ and $g(t)$ in $L^2[0, T]$ is

$$\langle f(t), g(t)\rangle = \int_0^T f(t)g^*(t)dt \tag{F.1}$$

where the asterisk denotes the complex conjugate. The norm of $f(t) \in L^2[0, T]$ is denoted by $\|f(t)\|$, and its square is

$$\|f(t)\|^2 = \langle f(t), f(t)\rangle = \int_0^T |f(t)|^2 dt < \infty. \tag{F.2}$$

These definitions indicate that the norm is nonnegative,

$$\langle f(t), g(t)\rangle = \langle g(t), f(t)\rangle^* \tag{F.3}$$

and

$$\langle f(t) + h(t), g(t)\rangle = \langle f(t), g(t)\rangle + \langle h(t), g(t)\rangle. \tag{F.4}$$

Functions that are equal almost everywhere are considered equivalent, and $\|f(t)\| = 0$ implies that $f(t) = 0$.

D. Torrieri, *Principles of Spread-Spectrum Communication Systems*,
https://doi.org/10.1007/978-3-319-70569-9

Let $\lambda = -\langle f(t), g(t)\rangle / \|f(t)\|$, where $\|f(t)\| \neq 0$. Then

$$
\begin{aligned}
0 &\leq \|\lambda f(t) + g(t)\|^2 \\
&= \left| \lambda \|f(t)\| + \frac{\langle f(t), g(t)\rangle}{\|f(t)\|} \right|^2 - \frac{|\langle f(t), g(t)\rangle|^2}{\|f(t)\|^2} + \|g(t)\|^2 \\
&= -\frac{|\langle f(t), g(t)\rangle|^2}{\|f(t)\|^2} + \|g(t)\|^2
\end{aligned}
\tag{F.5}
$$

which implies the *Cauchy-Schwarz inequality for an inner product*:

$$|\langle f(t), g(t)\rangle| \leq \|f(t)\| \, \|g(t)\| \, . \tag{F.6}$$

This inequality remains valid when $\|f(t)\| = 0$.

Let $\{\phi_i(t)\}$ denote a countable set of functions $\phi_i(t), \phi_i(t), \ldots$ in $L^2[0, T]$. The set $\{\phi_i(t)\}$ is orthonormal if

$$\langle \phi_i(t), \phi_k(t)\rangle = \delta_{ik}, \quad i \neq k \tag{F.7}$$

where $\delta_{ii} = 1$, and $\delta_{ik} = 0, k \neq i$. An orthonormal subset B of $L^2[0, T]$ is a *basis* for $L^2[0, T]$ if B is not a proper subset of any other orthonormal subset of $L^2[0, T]$. The subspace $S(B)$ *spanned* by B is the smallest closed subspace of $L^2[0, T]$ containing all the orthonormal basis functions of B. An orthonormal basis B is *complete* if $S(B) = L^2[0, T]$.

Using functional analysis [60], many specific complete orthonormal bases can be constructed. An example of an orthonormal basis in $L^2[0, T]$ comprises the complex exponential functions in a Fourier series representation of a function; that is, the basis is

$$\left\{ \sqrt{1/T} \exp(j2\pi kt/T), \; k \geq 1 \right\}.$$

An example of a real-valued orthonormal basis in $L^2[0, T]$ comprises the sine and cosine functions in a Fourier series representation of a function; that is, the basis is

$$\left\{ \sqrt{2/T} \sin(2\pi kt/T), \; \sqrt{2/T} \cos(2\pi kt/T), \; k \geq 1 \right\}.$$

A countable sequence of functions $\{f_n(t), n \geq 1\}$ in $L^2[0, T]$ converges to a function $f(t)$ in $L^2[0, T]$ if $\|f_n(t) - f(t)\| \to 0$ as $n \to \infty$. Consider a complete set of *orthonormal basis functions* $\{\phi_i(t)\}$ for $L^2[0, T]$. We define the finite expansion of $f(t) \in L^2[0, T]$ in terms of the first N basis functions as

$$S(f,N) = \sum_{i=1}^{N} f_i \phi_i(t) \tag{F.8}$$

where the expansion coefficients are defined as

$$f_i = \langle f(t), \phi_i(t) \rangle, \ i \geq 1. \tag{F.9}$$

It can be shown [6, 60] that $\|f(t) - S(f,N)\| \to 0$ as $N \to \infty$, and hence

$$f(t) = \lim_{N\to\infty} \sum_{i=1}^{N} f_i \phi_i(t) = \sum_{i=1}^{\infty} f_i \phi_i(t). \tag{F.10}$$

Suppose that $f(t)$, $g(t) \in L^2[0,T]$, $\|f(t) - S(f,N)\| \to 0$ as $N \to \infty$, and $\|g(t) - S(g,N)\| \to 0$ as $N \to \infty$. The linearity of the inner product gives

$$\begin{aligned}\langle f(t), g(t) \rangle &= \langle f(t) - S(f,N), g(t) \rangle - \langle f(t) - S(f,N), g(t) - S(g,N) \rangle \\ &\quad + \langle f(t), g(t) - S(g,N) \rangle + \langle S(f,N), S(g,N) \rangle. \end{aligned} \tag{F.11}$$

Applying Cauchy-Schwarz inequality successively to the first three terms on the right and using $\|f(t) - S(f,N)\| \to 0$ and $\|g(t) - S(g,N)\| \to 0$ as $N \to \infty$, it follows that each of these terms approaches zero as $N \to \infty$. Therefore, taking $N \to \infty$ and using (F.8), we obtain

$$\begin{aligned}\langle f(t), g(t) \rangle &= \lim_{N\to\infty} \left\langle \sum_{i=1}^{N} f_i \phi_i(t), \sum_{i=1}^{N} g_i \phi_i(t) \right\rangle \\ &= \lim_{N\to\infty} \sum_{i=1}^{N} f_i g_i^* \\ &= \sum_{i=1}^{\infty} f_i g_i^* \end{aligned} \tag{F.12}$$

and hence

$$\|f(t)\|^2 = \sum_{i=1}^{\infty} |f_i|^2. \tag{F.13}$$

The *Gram-Schmidt orthonormalization* procedure starts with a countable set of linearly independent functions $\{g_i(t)\}$ and then uses them in the construction of a complete set of orthonormal basis functions $\{\phi_i(t)\}$, $i = 1, 2, \ldots$, in $L^2[0,T]$. In the procedure, each new function $\phi_i(t)$, $i \leq N$, is a linear combination of $g_i(t)$ and the previously constructed $\{\phi_k(t)\}$, $k < i \leq N$. The linear combinations are selected to ensure orthonormality. The defining equations of the procedure are

$$\phi_i(t) = \alpha_i \left[g_i(t) - \sum_{k=1}^{N} \langle g_i\,(t)\,, \phi_k\,(t)\rangle\, \phi_k(t) \right] \tag{F.14}$$

where

$$\alpha_i = \left[\|g_i\,(t)\|^2 - \sum_{k=1}^{N} [\langle g_i\,(t)\,, \phi_k\,(t)\rangle]^2 \right]^{-1/2}. \tag{F.15}$$

F.2 White Gaussian Noise

The *Dirac delta function* $\delta\,(t, s)$ is defined as the nonstandard function such that for any continuous function $h\,(s)$, we have

$$\int_{-\infty}^{\infty} h\,(s)\,\delta\,(t-s)\,ds = h\,(t)\,. \tag{F.16}$$

For each $t, s \in [0, T]$, we expand $\delta\,(t-s)$ in terms of a complete set of orthonormal basis functions as

$$\delta\,(t-s) = \sum_{i=1}^{\infty} \delta_i\,(t)\,\phi_i(s) \tag{F.17}$$

where

$$\delta_i\,(t) = \langle \delta\,(t-s)\,, \phi_i(s)\rangle = \int_0^T \delta\,(t-s)\,\phi_i^*(s)ds. \tag{F.18}$$

Applying (F.16) to evaluate the integral and substituting the result into (F.17), we obtain the identity

$$\delta\,(t-s) = \sum_{i=1}^{\infty} \phi_i^*(t)\phi_i(s). \tag{F.19}$$

Consider zero-mean, white Gaussian noise $n\,(t)$ over $[0, T]$. Its autocorrelation function is

$$E[n(t)n(s)] = \frac{N_0}{2}\delta\,(t-s)\,. \tag{F.20}$$

Given a complete set of orthonormal basis functions $\{\phi_i(t)\}$ in $L^2[0,T]$, we define the projection of $n(t)$ onto the subspace spanned by the first N basis functions as

$$S(n,N,t)=\sum_{i=1}^{N}n_i\phi_i(t) \tag{F.21}$$

where

$$n_i=\langle n(t),\phi_i(t)\rangle=\int_0^T n(t)\phi_i^*(t)dt,\ i\geq 1. \tag{F.22}$$

Using $E[n(t)]=0$, (F.16), (F.20), and (F.7) in (F.22), we obtain

$$E[n_i]=0,\ E\left[|n_i|^2\right]=\frac{N_0}{2},\ i\geq 1 \tag{F.23}$$

$$E\left[n_i n_k^*\right]=0,\ i\neq k. \tag{F.24}$$

The white Gaussian noise may be represented by the expansion

$$n(t)=\sum_{i=1}^{\infty}n_i\phi_i(t) \tag{F.25}$$

in the sense that $E[S^*(n,N,t)\,S(n,N,s)]\to\delta(t-s)$ as $N\to\infty$. To prove this result, we calculate

$$\begin{aligned}\lim_{N\to\infty}E[S^*(n,N,t)\,S(n,N,s)]&=\lim_{N\to\infty}E[\sum_{i=1}^{N}n_i^*\phi_i^*(t)\sum_{i=1}^{N}n_k\phi_k(s)]\\&=\frac{N_0}{2}\sum_{i=1}^{\infty}\phi_i^*(t)\phi_i(s)\\&=\delta(t-s)\end{aligned} \tag{F.26}$$

where (F.24) and (F.23) are used in the second equality, and (F.20) is used in the third equality.

The approximating Lebesgue or Riemann sums of the real and imaginary parts of the integral in (F.22) are sums of independent, zero-mean Gaussian random variables because the noise is white. Therefore, the real and imaginary components of the $\{n_i\}$ are *jointly zero-mean Gaussian random variables* (Theorem A4, Appendix A.1). If the basis functions are real-valued, then the $\{n_i\}$ are real-valued, (F.24) implies that they are uncorrelated, and hence they are statistically independent of each other.

F.3 Estimation of Waveform Parameters

Consider the estimation of waveform parameters that are components of the vector $\boldsymbol{\theta}$. The observed signal is

$$r(t) = s(t, \boldsymbol{\theta}) + n(t), \quad 0 \leq t \leq T \tag{F.27}$$

where the real-valued signal $s(t, \boldsymbol{\theta})$ has a known waveform except for $\boldsymbol{\theta}$, and $n(t)$ is zero-mean, white Gaussian noise with autocorrelation function given by (F.20). For all values of the waveform parameters, $s(t, \boldsymbol{\theta})$ belongs to the signal space $L^2[0, T]$ of complex-valued functions f such that $|f|^2$is integrable over $[0, T]$.

Let $\{\phi_i(t)\}$ denote a complete set of real-valued orthonormal basis functions in $L^2[0, T]$. The orthonormal expansions of the functions in (F.27) are

$$r(t) = \sum_{i=1}^{\infty} r_i\phi_i(t), \;\; s(t, \boldsymbol{\theta}) = \sum_{i=1}^{\infty} s_i(\boldsymbol{\theta})\,\phi_i(t), \;\; n(t) = \sum_{i=1}^{\infty} n_i\phi_i(t) \tag{F.28}$$

with coefficients

$$r_i = \langle r(t), \phi_i(t)\rangle = \int_0^T r(t)\phi_i(t)dt = s_i(\boldsymbol{\theta}) + n_i \tag{F.29}$$

$$s_i(\boldsymbol{\theta}) = \langle s(t, \boldsymbol{\theta}), \phi_i(t)\rangle = \int_0^T s(t, \boldsymbol{\theta})\,\phi_i(t)dt \tag{F.30}$$

$$n_i = \langle n(t), \phi_i(t)\rangle = \int_0^T n(t)\phi_i(t)dt. \tag{F.31}$$

Each n_i is a statistically independent, real-valued, Gaussian random variable, and (F.23) indicates that the conditional density function of r_i is

$$f(r_i|\boldsymbol{\theta}) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(r_i - s_i(\boldsymbol{\theta}))^2}{N_0}\right]. \tag{F.32}$$

To avoid convergence problems arising from the assumption of white noise, we initially consider the $N \times 1$ vector $\mathbf{r} = [r_1\, r_2\, \ldots\, r_N]^T$. Applying (F.32) and the statistical independence of the r_i, we obtain the log-likelihood function

$$\ln f(\mathbf{r}|\boldsymbol{\theta}, N) = -\ln\sqrt{\pi N_0} - \frac{1}{N_0}\sum_{i=1}^{N} r_i^2 + \frac{2}{N_0}\sum_{i=1}^{N} r_i s_i(\boldsymbol{\theta}) - \frac{1}{N_0}\sum_{i=1}^{N} s_i^2(\boldsymbol{\theta}). \tag{F.33}$$

The first sum may not converge, but it is irrelevant to the maximum-likelihood estimation and may be dropped. The factor N_0 then also becomes irrelevant and may be similarly dropped. Taking $N \to \infty$, and applying (F.12), we obtain the *sufficient statistic for maximum-likelihood estimation of* $\boldsymbol{\theta}$:

$$\Lambda_s\left[r(t)\right] = 2\int_0^T r(t)s(t,\boldsymbol{\theta})dt - \int_0^T s^2(t,\boldsymbol{\theta})dt. \tag{F.34}$$

If $s\left(t,\boldsymbol{\theta}\right)$ depends on a random vector ϕ, we base the maximum-likelihood estimation on the *average log-likelihood ratio* defined as $E_\phi\left[\ln f\left(\mathbf{r}|\boldsymbol{\theta},N\right)\right]$, where $E_\phi\left[\cdot\right]$ denotes the expected value with respect to ϕ. Therefore, the sufficient statistic becomes

$$\Lambda_a\left[r(t)\right] = E_\phi\left[2\int_0^T r(t)s(t,\boldsymbol{\theta})dt - \int_0^T s^2(t,\boldsymbol{\theta})dt\right] \tag{F.35}$$

and the maximum-likelihood estimator is

$$\widehat{\theta} = \arg\max_{\boldsymbol{\theta}} \Lambda_a\left[r(t)\right]. \tag{F.36}$$

F.4 Cramer-Rao Inequality

The Cramer-Rao inequality provides a lower bound on the variance of an unbiased estimator. Consider a random vector $\mathbf{X}$ and a single unknown parameter θ. The conditional density function of $\mathbf{X}$ is $f\left(\mathbf{x}|\theta\right)$, and the partial derivative $f^{(1)}\left(\mathbf{x}|\theta\right)$ with respect to θ exists for all $\left(\mathbf{x},\theta\right)$. Assume that there is an integrable function $g\left(\mathbf{x}\right)$ such that $\left|\widehat{\theta}\left(\mathbf{x}\right)f^{(1)}\left(\mathbf{x}|\theta\right)\right| \leq g\left(\mathbf{x}\right)$ for all $\left(\mathbf{x},\theta\right)$ and that $\int_{-\infty}^{\infty} g\left(\mathbf{x}\right)d\mathbf{x} < \infty$. Let $\widehat{\theta}\left(\mathbf{X}\right)$ denote an estimator of θ. This estimator is *unbiased* if

$$E\left[\widehat{\theta}\left(\mathbf{X}\right)\right] = \theta = \int_{-\infty}^{\infty} \widehat{\theta}\left(\mathbf{x}\right)f\left(\mathbf{x}|\theta\right)d\mathbf{x}. \tag{F.37}$$

Differentiating both sides of the second equality with respect to θ, we obtain

$$1 = \int_{-\infty}^{\infty} \widehat{\theta}\left(\mathbf{x}\right)\frac{\partial}{\partial\theta}f\left(\mathbf{x}|\theta\right)d\mathbf{x} = \int_{-\infty}^{\infty} \widehat{\theta}\left(\mathbf{x}\right)\frac{\partial \ln f\left(\mathbf{x}|\theta\right)}{\partial\theta}f\left(\mathbf{x}|\theta\right)d\mathbf{x} \tag{F.38}$$

where the interchange of the derivative and integration is valid because of the assumption about $g\left(\mathbf{x}\right)$. Since $f\left(\mathbf{x}|\theta\right)$ integrates to unity,

$$\begin{aligned} 0 &= \frac{\partial}{\partial\theta}\int_{-\infty}^{\infty} f\left(\mathbf{x}|\theta\right)d\mathbf{x} \\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial\theta}f\left(\mathbf{x}|\theta\right)d\mathbf{x} \\ &= \int_{-\infty}^{\infty} \frac{\partial \ln f\left(\mathbf{x}|\theta\right)}{\partial\theta}f\left(\mathbf{x}|\theta\right)d\mathbf{x}. \end{aligned} \tag{F.39}$$

Combining (F.38) and (F.39) and using the Cauchy-Schwarz inequality for random variables (Appendix A.2) yields

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} \left[\widehat{\theta}(\mathbf{x}) - \theta\right] \frac{\partial \ln f(\mathbf{x}|\theta)}{\partial \theta} f(\mathbf{x}|\theta)\, d\mathbf{x} \\ &= E\left\{\left[\widehat{\theta}(\mathbf{X}) - \theta\right] \frac{\partial \ln f(\mathbf{X}|\theta)}{\partial \theta}\right\} \\ &= \left[var\left(\widehat{\theta}\right)\right]^2 E\left\{\left[\frac{\partial \ln f(\mathbf{X}|\theta)}{\partial \theta}\right]^2\right\}^{1/2} \end{aligned} \tag{F.40}$$

which implies *Cramer-Rao inequality for the variance of an unbiased estimator*:

$$var\left(\widehat{\theta}\right) \geq \left(E\left\{\left[\frac{\partial \ln f(\mathbf{X}|\theta)}{\partial \theta}\right]^2\right\}\right)^{-1}. \tag{F.41}$$

If the second partial derivative $f^{(2)}(\mathbf{x}|\theta)$ with respect to θ exists for all $(\mathbf{x}, \theta)$, there is an integrable function $h(\mathbf{x})$ such that $\left|f^{(2)}(\mathbf{x}|\theta)\right| \leq h(\mathbf{x})$ for all $(\mathbf{x}, \theta)$, and $\int_{-\infty}^{\infty} h(\mathbf{x})\, d\mathbf{x} < \infty$, then differentiating the final integral in (F.39) gives

$$0 = \int_{-\infty}^{\infty} \left\{\frac{\partial^2 \ln f(\mathbf{x}|\theta)}{\partial \theta^2} + \left[\frac{\partial \ln f(\mathbf{x}|\theta)}{\partial \theta}\right]^2\right\} f(\mathbf{x}|\theta)\, d\mathbf{x}. \tag{F.42}$$

Combining this equation with the lower bound in (F.41), we obtain an alternative version of Cramer-Rao inequality:

$$var\left(\widehat{\theta}\right) \geq -\left(E\left[\frac{\partial^2 \ln f(\mathbf{X}|\theta)}{\partial \theta^2}\right]\right)^{-1}. \tag{F.43}$$

We calculate the Cramer-Rao inequality for the observed signal of (F.27), assuming that $\partial s(t, \theta)/\partial \theta$ is in $L^2[0, T]$ and that each coefficient $s_i(\theta)$ of $s(t, \theta)$ is differentiable for all θ. Differentiating (F.33), we obtain

$$\frac{\partial \ln f(\mathbf{r}|\theta, N)}{\partial \theta} = \frac{2}{N_0}\left[\sum_{i=1}^{N} r_i \frac{\partial s_i(\theta)}{\partial \theta} - \sum_{i=1}^{N} s_i(\theta) \frac{\partial s_i(\theta)}{\partial \theta}\right]. \tag{F.44}$$

Let $S(s, N, \theta)$ and $S\left(s^{(1)}, N, \theta\right)$ denote the finite expansions of $s(t, \theta)$ and $\partial s(t, \theta)/\partial \theta$, respectively, in terms of the first N basis functions. Then

$$\frac{\partial S(s, N, \theta)}{\partial \theta} = \sum_{i=1}^{N} \frac{\partial s_i(\theta)}{\partial \theta} \phi_i(t) = S\left(s^{(1)}, N, \theta\right) \tag{F.45}$$

where the orthonormality of the basis functions is used to prove the second equality. Therefore,

$$\frac{\partial s(t,\theta)}{\partial \theta} = \lim_{N\to\infty} S\left(s^{(1)}, N, \theta\right) = \sum_{i=1}^{\infty} \frac{\partial s_i(\theta)}{\partial \theta}\phi_i(t). \tag{F.46}$$

Taking the limit of (F.44) as $N \to \infty$, applying (F.46) and (F.12), and then substituting (F.27) gives

$$\begin{aligned}\frac{\partial \ln f(\mathbf{r}|\theta)}{\partial \theta} &= \lim_{N\to\infty} \frac{\partial \ln f(\mathbf{r}|\theta,N)}{\partial \theta} \\ &= \frac{2}{N_0}\left[\int_0^T r(t)\frac{\partial s(t,\theta)}{\partial \theta}dt - \int_0^T s(t,\theta)\frac{\partial s(t,\theta)}{\partial \theta}dt\right] \\ &= \frac{2}{N_0}\int_0^T n(t)\frac{\partial s(t,\theta)}{\partial \theta}dt. \end{aligned} \tag{F.47}$$

Applying (F.41) and (F.20), we find that the Cramer-Rao inequality for a waveform parameter θ is

$$var\left(\widehat{\theta}\right) \geq \frac{N_0}{2}\left\{\int_0^T \left[\frac{\partial s(t,\theta)}{\partial \theta}\right]^2 dt\right\}^{-1}. \tag{F.48}$$

Appendix G
Hermitian Positive-Definite Matrices

An $n \times n$ matrix $\mathbf{A}$ has an eigenvector $\mathbf{u}$ and an eigenvalue λ if $\mathbf{Au} = \lambda\mathbf{u}$. A set of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are *orthonormal* if

$$\mathbf{u}_i^H \mathbf{u}_k = 0, \; i \neq k, \; \text{and} \; \|\mathbf{u}_i\|^2 = \mathbf{u}_i^H \mathbf{u}_i = 1 \tag{G.1}$$

where $\| \cdot \|$ denotes the *Euclidean norm* of a vector, and the superscript H denotes the conjugate transpose. A *unitary matrix* $\mathbf{U}$ is an $n \times n$ matrix with orthonormal column vectors. Therefore, $\mathbf{U}$ has rank n, $\mathbf{U}^H\mathbf{U} = \mathbf{I}$, and

$$\mathbf{U}^{-1} = \mathbf{I}\mathbf{U}^{-1} = \mathbf{U}^H\mathbf{U}\mathbf{U}^{-1} = \mathbf{U}^H \tag{G.2}$$

where $\mathbf{I}$ denotes the identity matrix. A unitary matrix with real-valued elements is called an *orthogonal matrix*.

The $n \times n$ matrix $\mathbf{A}$ has of a *complete set of n orthonormal eigenvectors* $\mathbf{u}_1, \ldots, \mathbf{u}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ if there are n orthonormal eigenvectors satisfying

$$\mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i, \; 1 \leq i \leq n. \tag{G.3}$$

If an $n \times n$ matrix $\mathbf{A}$ has of a complete set of n orthonormal eigenvectors, and $\mathbf{U}$ is a unitary matrix with column vectors equal to $\mathbf{u}_1, \ldots, \mathbf{u}_n$, then $\mathbf{U}^H\mathbf{A}\mathbf{u}_i = \lambda_i\mathbf{U}^H\mathbf{u}_i$, $1 \leq i \leq n$, and hence the diagonal matrix of eigenvalues is

$$\Lambda = \mathbf{U}^H\mathbf{A}\mathbf{U}. \tag{G.4}$$

Conversely, if an $n \times n$ matrix $\mathbf{A}$ and an $n \times n$ unitary matrix $\mathbf{U}$ satisfy (G.4) for some diagonal matrix $\boldsymbol{\Lambda}$ with diagonal elements $\lambda_1, \ldots, \lambda_n$, then an application of (G.2) proves that $\mathbf{A}$ has of a *complete set of n orthonormal eigenvectors* $\mathbf{u}_1, \ldots, \mathbf{u}_n$ with

D. Torrieri, *Principles of Spread-Spectrum Communication Systems*,
https://doi.org/10.1007/978-3-319-70569-9

corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Applying (G.2) to (G.4) twice, we obtain the *spectral decomposition*

$$\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^H = \mathbf{U}\Lambda\mathbf{U}^{-1}. \tag{G.5}$$

An $n \times n$ matrix $\mathbf{A}$ is *diagonalizable* if $\mathbf{B}^{-1}\mathbf{A}\mathbf{B} = \mathbf{D}$ for some nonsingular matrix $\mathbf{B}$ and diagonal matrix $\mathbf{D}$. Thus, if an $n \times n$ matrix $\mathbf{A}$ and an $n \times n$ unitary matrix $\mathbf{U}$ satisfy (G.5) for some diagonal matrix $\boldsymbol{\Lambda}$ with diagonal elements $\lambda_1, \ldots, \lambda_n$, then $\mathbf{A}$ is diagonalizable.

An $n \times n$ *Hermitian matrix* $\mathbf{A}$ is a matrix satisfying $\mathbf{A}^H = \mathbf{A}$. A Hermitian matrix with real-valued elements is called a *symmetric matrix.*

Theorem *An* $n \times n$ *Hermitian matrix* $\mathbf{A}$ *has a complete set of orthonormal eigenvectors and can be diagonalized.*

Proof The proof is by mathematical induction. The result is true if $n = 1$. Assume that the hypothesis is true for $k \times k$ Hermitian matrices, and let $\mathbf{A}$ denote a $(k + 1) \times (k + 1)$ Hermitian matrix. This matrix has at least one eigenvector $\mathbf{u}_1$ with a unit norm. Let λ_1 denote the corresponding eigenvalue. Using the Gram-Schmidt process, we determine orthonormal vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{k+1}$ that constitute the basis for a complex $(k + 1)$-dimensional vector space. Let $\mathbf{Z}$ denote the $(k + 1) \times (k + 1)$ unitary matrix with these orthonormal vectors as its columns. The $(k + 1) \times (k + 1)$ matrix $\mathbf{Z}^H\mathbf{A}\mathbf{Z}$ is Hermitian, and its first column is

$$\mathbf{Z}^H\mathbf{A}\mathbf{u}_1 = \lambda_1\mathbf{Z}^H\mathbf{u}_1 = \lambda_1\mathbf{e}_1 \tag{G.6}$$

where $\mathbf{e}_1 = [1\ 0\ \ldots\ 0]^H$. Therefore, as $\mathbf{Z}^H\mathbf{A}\mathbf{Z}$ is a Hermitian matrix, it must have the form

$$\mathbf{Z}^H\mathbf{A}\mathbf{Z} = \begin{bmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} \end{bmatrix} \tag{G.7}$$

where $\mathbf{Y}$ is a $k \times k$ Hermitian matrix. According to the induction hypothesis, a $k \times k$ unitary matrix $\mathbf{V}_1$ exists such that $\mathbf{V}_1^H\mathbf{Y}\mathbf{V}_1 = \mathbf{D}$, where $\mathbf{D}$ is a $k \times k$ diagonal matrix with its ith diagonal component equal to λ_{i+1}. Let $\mathbf{V}$ denote the $(k + 1) \times (k + 1)$ unitary matrix

$$\mathbf{V} = \begin{bmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_1 \end{bmatrix}. \tag{G.8}$$

The $(k + 1) \times (k + 1)$ matrix $\mathbf{U} = \mathbf{Z}\mathbf{V}$ is unitary because $\mathbf{U}^H\mathbf{U} = \mathbf{V}^H\mathbf{Z}^H\mathbf{Z}\mathbf{V} = \mathbf{I}$, and

$$\begin{aligned} \mathbf{U}^H\mathbf{A}\mathbf{U} = \mathbf{V}^H\mathbf{Z}^H\mathbf{A}\mathbf{Z}\mathbf{V} &= \begin{bmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_1^H\mathbf{Y}\mathbf{V}_1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} = \boldsymbol{\Lambda} \end{aligned} \tag{G.9}$$

where $\mathbf{\Lambda}$ is a $(k+1) \times (k+1)$ diagonal matrix with its ith diagonal component equal to λ_i. Therefore, the columns of $\mathbf{U}$ comprise a complete set of orthonormal eigenvectors, and $\mathbf{A}$ is diagonalizable. □

For an $n \times n$ Hermitian matrix $\mathbf{A}$, implies that

$$\mathbf{u}_i^H \mathbf{A} \mathbf{u}_i = \lambda_i \|\mathbf{u}_i\|^2, \ 1 \leq i \leq n. \tag{G.10}$$

The left-hand side of this equation is real as it is identical to its conjugate transpose, and hence each λ_i is real. Therefore, *the eigenvalues of a Hermitian matrix are real.*

An $n \times n$ Hermitian matrix is *positive semidefinite* if for all $n \times 1$ column vectors $\mathbf{x}$,

$$\mathbf{x}^H \mathbf{A} \mathbf{x} \geq \mathbf{0} \tag{G.11}$$

and it is *positive definite* if

$$\mathbf{x}^H \mathbf{A} \mathbf{x} > \mathbf{0}, \mathbf{x} \neq \mathbf{0}. \tag{G.12}$$

If $\mathbf{Y}$ is an $n \times 1$ random vector, then the $n \times n$ matrix $E\left[\mathbf{Y}\mathbf{Y}^H\right]$ is positive semidefinite because $\mathbf{x}^H E\left[\mathbf{Y}\mathbf{Y}^H\right] \mathbf{x} = E\left[\left|\mathbf{Y}^H \mathbf{x}\right|^2\right] \geq 0$ for all $n \times 1$ column vectors $\mathbf{x}$.

Equation (G.10) indicates that a *Hermitian positive-semidefinite matrix has non-negative eigenvalues,* and that a *Hermitian positive-definite matrix has positive eigenvalues.* If a Hermitian positive-definite matrix were singular, then $\lambda = 0$ would be one or more of its eigenvalues. Therefore, since all its eigenvalues are positive, *a Hermitian positive-definite matrix is invertible*.

If $\mathbf{B}$ is an $n \times n$ Hermitian positive-definite matrix and $\mathbf{A}$ is an $n \times n$ invertible matrix, then $\mathbf{x}^H \mathbf{A}^H \mathbf{B} \mathbf{A} \mathbf{x} = (\mathbf{A}\mathbf{x})^H \mathbf{B} \mathbf{A} \mathbf{x} \neq \mathbf{0}$ because $\mathbf{A}\mathbf{x} \neq \mathbf{0}$. Therefore, $\mathbf{A}^H \mathbf{B} \mathbf{A}$ *is an* $n \times n$ *Hermitian positive-definite matrix.*

The *trace of a matrix* is equal to the sum of its diagonal elements. From the definitions of matrix multiplication and the trace, it follows that $tr\,(\mathbf{AB}) = tr\,(\mathbf{BA})$ for compatible matrices $\mathbf{A}$ and $\mathbf{B}$. If $\mathbf{A}$ is a Hermitian positive-definite matrix, then its spectral decomposition and positive eigenvalues imply that

$$\begin{aligned} tr\,(\mathbf{A}) &= tr\left(\mathbf{U}\Lambda\mathbf{U}^H\right) = tr\left(\Lambda\mathbf{U}^H\mathbf{U}\right) \\ &= tr\,(\mathbf{\Lambda}) > 0. \end{aligned} \tag{G.13}$$

Therefore, if $\mathbf{A}$ is a Hermitian positive-definite matrix, then $tr\,(\mathbf{A}) > 0$. Similarly, if $\mathbf{A}$ is a Hermitian positive-semidefinite matrix, then $tr\,(\mathbf{A}) \geq 0$.

Appendix H
Special Functions

H.1 Gamma Functions

The *gamma function* is defined as

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy, \quad \text{Re}(x) > 0. \tag{H.1}$$

An integration by parts indicates that

$$\Gamma(1 + x) = x\Gamma(x). \tag{H.2}$$

A direct integration yields $\Gamma(1) = 1$. Therefore, when n is a positive integer,

$$\Gamma(n) = \int_0^\infty y^{n-1} e^{-y} dy = (n-1)!\,, \quad n \text{ is positive integer.} \tag{H.3}$$

Changing the integration variable by substituting $y = z^2$ in (H.1), observing that the integrand is an even function, and using (A.2), it is found that

$$\Gamma(1/2) = \sqrt{\pi}. \tag{H.4}$$

The *incomplete gamma functions* are defined as

$$\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt, \quad Re\,(a) > 0 \tag{H.5}$$

and

$$\gamma(a, x) = \int_a^x e^{-t} t^{a-1} dt, \quad Re\,(a) > 0. \tag{H.6}$$

D. Torrieri, *Principles of Spread-Spectrum Communication Systems*,
https://doi.org/10.1007/978-3-319-70569-9

Therefore,

$$\Gamma(a) = \Gamma(a, x) + \gamma(a, x). \tag{H.7}$$

When $a = n$ is a positive integer, the integration of $\Gamma(n, x)$ by parts $n - 1$ times yields

$$\Gamma(n, x) = (n-1)! e^{-x} \sum_{i=0}^{n-1} \frac{x^i}{i!} \tag{H.8}$$

and hence

$$\gamma(n, x) = (n-1)! \left(1 - e^{-x} \sum_{i=0}^{n-1} \frac{x^i}{i!} \right). \tag{H.9}$$

H.2 Beta Function

The *beta function* is defined as

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt, \quad x > 0, \quad y > 0. \tag{H.10}$$

The identity

$$B(x, y) = B(y, x) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{H.11}$$

is proved by substituting $y = z^2$ in the integrand of (H.1), expressing the product $\Gamma(a)\Gamma(b)$ as a double integral, changing to polar coordinates, integrating over the radius to obtain a result proportional to $\Gamma(a + b)$, and then changing the variable in the remaining integral to obtain $B(a, b)\Gamma(a + b)$.

In (H.10), set $x = (k + 1)/2$ and $y = 1/2$, and then apply (H.11) . The change of variable $t = cos^2\theta$, $0 \leq \theta \leq \pi/2$, in the integrand of (H.10), and similarly the change of variable $t = \sin^2 \theta$, $0 \leq \theta \leq \pi/2$, yield

$$\int_0^{\pi/2} \cos^k \theta \, d\theta = \int_0^{\pi/2} \sin^k \theta \, d\theta = \frac{\sqrt{\pi}\,\Gamma\left(\frac{k+1}{2}\right)}{2\Gamma\left(\frac{k+2}{2}\right)}, \quad k \geq 0. \tag{H.12}$$

H.3 Bessel Functions of the First Kind

The *modified Bessel function* of the first kind and order n is defined as

$$I_n(x) = \sum_{i=0}^{\infty} \frac{(x/2)^{n+2i}}{i!\Gamma(n+i+1)} \tag{H.13}$$

where the gamma function may be replaced by a factorial if n is an integer. Therefore, the modified Bessel function of the first kind and order zero is defined as

$$I_0(x) = \sum_{i=0}^{\infty} \frac{1}{i!i!} \left(\frac{x}{2}\right)^{2i}. \tag{H.14}$$

A substitution of the series expansion of the exponential function and a term-by-term integration using (H.12) verifies the representation

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos u} du. \tag{H.15}$$

Since the cosine is a periodic function and the integration is over the same period, we may replace $\cos u$ with $\cos(u+\theta)$ for any θ in (H.15). A trigonometric expansion with $x_1 = |x| \cos\theta$ and $x_2 = |x| \sin\theta$ then yields

$$\begin{aligned} I_0(|x|) &= \frac{1}{2\pi} \int_0^{2\pi} \exp\{\mathrm{Re}[|x|\, e^{j(u+\theta)}]\} du \\ &= \frac{1}{2\pi} \int_0^{2\pi} \exp\left(x_1 \cos u - x_2 \sin u\right) du, \quad |x| = \sqrt{x_1^2 + x_2^2}\,. \end{aligned} \tag{H.16}$$

A term-by-term differentiation of (H.14) yields

$$I_1(x) = \frac{d}{dx} I_0(x). \tag{H.17}$$

The *Bessel function* of the first kind and order n is defined as

$$J_n(x) = \sum_{i=0}^{\infty} \frac{(-1)^i\, (x/2)^{n+2i}}{i!\, \Gamma(n+i+1)}\,. \tag{H.18}$$

A substitution of the series expansion of the exponential function and a term-by-term integration using (H.12) verifies the representation

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{jx \sin u} du. \tag{H.19}$$

H.4 Q-Functions

The Gaussian *Q-function* is defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right) \tag{H.20}$$

where erfc$(\cdot)$ is the *complementary error function*. The results in Appendix A.1 imply that $Q(-\infty) = 1$, $Q(0) = 1/2$, and $Q(\infty) = 0$.

The Gaussian Q-function can be recast into a form in which the limits of the integral are not only finite but also independent of the argument of the function. This form facilitates computation and enables simplified analyses. To derive this form, we use (H.20) and $Q(-\infty) = 1$. For $x \geq 0$,

$$\begin{aligned} Q(x) &= \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2}\right) dy \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left(-\frac{z^2}{2}\right) dz\right] \\ &= \frac{1}{2\pi} \iint\limits_{y \geq x \geq 0,\, z > -\infty} \exp\left(-\frac{y^2 + z^2}{2}\right) dydz \\ &= \frac{1}{2\pi} \int_0^\pi \int_{x/\sin\theta}^\infty \exp\left(-\frac{r^2}{2}\right) rdrd\theta. \end{aligned} \tag{H.21}$$

In the second equality, Fubini's theorem justifies expressing the successive integrations as a double integral over a region in the plane. The third equality is the result of changing the Cartesian coordinates to polar coordinates. After integrating over the radius and using the periodic character of the angular coordinate, we obtain the desired form:

$$Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta, \; x \geq 0. \tag{H.22}$$

If $x, y \geq 0$, then

$$Q(x+y) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2 + y^2 + 2xy}{2\sin^2\theta}\right) d\theta. \tag{H.23}$$

Since $\left(y^2 + 2xy\right) / \sin^2\theta \geq y^2$ when $\theta \in [0, \pi/2]$,

$$Q(x+y) \leq \exp\left(-\frac{y^2}{2}\right) Q(x), \; x, y \geq 0. \tag{H.24}$$

This inequality provides a generalization of the Chernoff bound on the standard Gaussian Q-function. Similarly,

$$Q(\sqrt{x+y}) \leq \exp\left(-\frac{y}{2}\right) Q(\sqrt{x}), \; x, y \geq 0. \tag{H.25}$$

The *generalized Marcum Q-function* is defined as

$$Q_m(\alpha, \beta) = \int_\beta^\infty x \left(\frac{x}{\alpha}\right)^{m-1} \exp\left(-\frac{x^2+\alpha^2}{2}\right) I_{m-1}(\alpha x)\, dx \tag{H.26}$$

and m is an integer. Since $Q_m(\alpha, 0) - Q_m(\alpha, \beta)$=$1 - Q_m(\alpha, \beta)$ is an integral over a finite interval, it can be numerically integrated, and hence $Q_m(\alpha, \beta)$ numerically evaluated.

H.5 Hypergeometric Function

The *confluent hypergeometric function* is defined as

$${}_1F_1(\alpha, \beta; x) = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha+i)\Gamma(\beta)x^i}{\Gamma(\alpha)\Gamma(\beta+i)i!}, \quad \beta \neq 0, -1, -2, \ldots \tag{H.27}$$

and the series converges for all finite x.

References

1. F. Adachi, D. Garg, S. Takaoka, and K. Takeda, "Broadband CDMA Techniques," İ *IEEE Wireless Commun.*, vol. 44, pp. 8–18, April 2005.
2. F. Adachi and K. Takeda, "Bit Error Rate Analysis of DS-CDMA with Joint Frequency-Domain Equalization and Antenna Diversity Combining," *IEICE Trans. Commun.*, vol. E87-B, pp. 2291–3001, Oct. 2004.
3. F. Adachi, M. Sawahashi, and K. Okawa, "Tree-structured Generation of Orthogonal Spreading Codes with Different Lengths for Forward Link of DS-CDMA Mobile Radio," *IEE Electronics Letters*, vol. 33, pp. 27–28, Jan. 1997.
4. S. Ahmed, L.-L. Yang, and L. Hanzo, "Erasure Insertion in RS-Coded SFH FSK Subjected to Tone Jamming and Rayleigh Fading," *IEEE Trans. Commun.*, vol. 56, pp. 3563–3571, Nov. 2007.
5. R. B. Ash and W. P. Novinger, *Complex Variables, 2nd ed.* Dover, 2004.
6. R. B. Ash and C. A. Doleans-Dade, *Probability and Measure Theory, 2nd ed.,* Academic Press, 2000.
7. J. R. Barry, E. A. Lee, and D. G. Messerschmitt, *Digital Communication, 3rd ed.,* Kluwer Academic, 2004.
8. N. C. Beaulieu, W. L. Hopkins, and P. J. McLane, "Interception of Frequency-Hopped Spread-Spectrum Signals," *IEEE. J. Select. Areas Commun.*, vol. 8, pp. 853–870, June 1990.
9. P. Billingsley, *Probability and Measure, 3rd ed., Wiley,* 1995.
10. N. Bonello, S. Chen, and L. Hanzo, "Low-Density Parity-Check Codes and Their Rateless Relatives," *IEEE Commun. Surveys Tut.*, vol. 13, pp. 3–26, first quarter, 2011.
11. W. R. Braun, "PN Acquisition and Tracking Performance in DS/CDMA Systems with Symbol-Length Spreading Sequence," *IEEE Trans. Commun.*, vol. 45, pp. 1595–1601, Dec. 1997.
12. H. Cai, Y. Yang, Z. Zhou, and X. Tang, "Strictly Optimal Frequency-Hopping Sequence Sets With Optimal Family Sizes," *IEEE Trans. Inf. Theory, vol. 62,* pp. 1087–1093, Feb. 2016.
13. G. Caire, G. Taricco, and E. Biglieri, "Bit-interleaved coded modulation," *IEEE Trans. Inform. Theory*, vol. 44, pp. 927–946, May 1998.
14. J.-J. Chang, D.-J. Hwang, and M.-C. Lin, "Some Extended Results on the Search for Good Convolutional Codes," *IEEE Trans. Inform. Theory*, vol. 43, pp. 1682–1697, Sept. 1997.
15. H. Chen, R. G. Maunder, and L. Hanzo, "A Survey and Tutorial on Low-Complexity Turbo Coding Techniques and a Holistic Hybrid ARQ Design Example," *IEEE Commun. Surveys Tut.*, vol. 15, pp. 1546–1566, fourth quarter, 2013.

D. Torrieri, *Principles of Spread-Spectrum Communication Systems*,
https://doi.org/10.1007/978-3-319-70569-9

16. A. Chockalingam, P. Dietrich, L. B. Milstein, and R. R. Rao, "Performance of Closed-Loop Power Control in DS-CDMA Cellular Systems," *IEEE Trans. Veh. Technol.*, vol. 47, pp. 774–789, Aug. 1998.
17. J. Choi, *Adaptive and Iterative Signal Processing in Communications,* Cambridge Univ. Press, 2006.
18. K. Choi, K. Cheun, and T. Jung, "Adaptive PN Code Acquisition Using Instantaneous Power-Scaled Detection Threshold under Rayleigh Fading and Pulsed Gaussian Noise Jamming," *IEEE Trans. Commun.*, vol. 50, pp. 1232–1235, Aug. 2002.
19. E. K. P. Chong and S. H. Zak, *An Introduction to Optimization, 4th ed.*, Wiley, 2013.
20. T. M. Cover and J. M. Thomas, *Elements of Information Theory, 2nd ed.*, Wiley, 2006.
21. F. F. Digham, M. S. Alouini, and M. K. Simon, "On the Energy Detection of Unknown Signals over Fading Channels," *IEEE Trans. Commun.*, vol. 55, pp. 21–24, Jan. 2007.
22. P. S. R. Diniz, *Adaptive Filtering: Algorithms and Practical Implementation, 4th ed.,* Springer, 2012.
23. S. Emami, "UWB Communication Systems: Conventional and 60 GHz: Principles, Design, and Standards," Springer, 2013.
24. Y. Fang, G. Bi, Y. L. Guan, and F. C. M. Lau, "A Survey on Protograph LDPC Codes and Their Applications," *IEEE Commun. Surveys Tut.,* vol. 17, pp. 1989–2016, fourth quarter, 2015.
25. Y. Fang, G. Han, P. Chen, F. C. M. Lau, G. Chen, and L. Wang, "A Survey on DCSK-based Communication Systems and Their Application to UWB Scenarios," *IEEE Commun. Surveys Tut.,* vol. 18, pp. 1804–1837, fourth quarter, 2016.
26. B. Farhang-Boroujeny, *Adaptive Filters: Theory and Applications, 2nd ed.,* Wiley, 2013.
27. S. Gezici, "Mean Acquisition Time Analysis of Fixed-Step Serial Search Algorithms," *IEEE Trans. Commun.*, vol. 8, pp. 1096–1101, March 2009.
28. M. Goresky and A. Klapper, *Algebraic Shift Register Sequences*, Cambridge Univ. Press, 2012.
29. T. T. Ha, *Theory and Design of Digital Communication Systems*, Cambridge Univ. Press, 2011.
30. L. Hanzo, T. H. Liew, B. L. Yeap, R. Y. S. Lee, and S. X. Ng, *Turbo Coding, Turbo Equalisation and Space-Time Coding, 2nd ed.,* Wiley, 2011.
31. A. R. Hammons and P. V. Kumar, "On a Recent 4-Phase Sequence Design for CDMA," *IEICE Trans. Commun.*, vol. E76-B, pp. 804–813, Aug. 1993.
32. S. Haykin, *Adaptive Filter Theory, 5th ed.,* Prentice-Hall, 2013.
33. C. W. Helstrom, *Elements of Signal Detection and Estimation*, Prentice Hall, 1995.
34. K. Higuchi et al., "Experimental Evaluation of Combined Effect of Coherent Rake Combining and SIR-Based Fast Transmit Power Control for Reverse Link of DS-CDMA Mobile Radio," *IEEE J. Select. Areas Commun.*, vol. 18, pp. 1526–1535, Aug. 2000.
35. D. A. Hill and E. B. Felstead, "Laboratory Performance of Spread Spectrum Detectors," *IEE Proc.-Commun.*, vol. 142, pp. 243–249, Aug. 1995.
36. D. Horgan and C. C. Murphy, "Fast and Accurate Approximations for the Analysis of Energy Detection in Nakagami-m Channels," *IEEE Commun. Lett.*, vol. 17, pp. 83–86, Jan. 2013.
37. D. Horgan and C. C. Murphy, "On the Convergence of the Chi Square and Noncentral Chi Square Distributions to the Normal Distribution," *IEEE Commun. Lett.*, vol. 17, pp. 2233–2236, Dec. 2013.
38. V. M. Jovanovic, "Analysis of Strategies for Serial-Search Spread-Spectrum Code Acquisition—Direct Approach," *IEEE Trans. Commun.*, vol. 36, pp. 1208–1220, Nov. 1988.
39. E. Kreyszig, *Advanced Engineering Mathematics, 10th ed.,* Wiley, 2011.
40. V. Kuhn, *Wireless Communications over MIMO Channels*, Wiley, 2006.
41. Y. M. Lam and P. H. Wittke, "Frequency-Hopped Spread-Spectrum Transmission with Band-Efficient Modulations and Simplified Noncoherent Sequence Estimation," *IEEE Trans. Commun.*, vol. 38, pp. 2184–2196, Dec. 1990.
42. K. Lange, *Optimization*, 2nd ed., Springer, 2013.

43. E. G. Larsson and P. Stoica, *Space-Time Block Coding for Wireless Communications*, Cambridge Univ. Press, 2003.
44. J. S. Lee, L. E. Miller, and Y. K. Kim, “Probability of Error Analyses of a BFSK Frequency-Hopping System with Diversity under Partial-Band Jamming Interference—Part II: Performance of Square-Law Nonlinear Combining Soft Decision Receivers,” *IEEE Trans. Commun.*, vol. 32, pp. 1243–1250, Dec. 1984.
45. S. J. Leon, *Linear Algebra with Applications, 9th ed.,* Pearson, 2014.
46. B. K. Levitt, U. Cheng, A. Polydoros, and M. K. Simon, “optimal Detection of Slow Frequency-Hopped Signals,” *IEEE Trans. Commun.*, vol. 42, pp. 1990–2000, Feb./March/April 1994.
47. B. Levy, *Principles of Signal Detection and Parameter Estimation*, Springer, 2008.
48. X. Li, A. Chindapol, and J. A. Ritcey, “Bit-interleaved Coded Modulation with Iterative Decoding and 8PSK Modulation,” *IEEE Trans. Commun.*, vol. 50, pp. 1250–1257, Aug. 2002.
49. C-F. Li, Y-S Chu, J-S Ho, and W-H Sheen, “Cell Search in WCDMA Under Large-Frequency and Clock Errors: Algorithms to Hardware Implementation,” *IEEE Trans. Circuits and Syst.*, vol. 55, pp. 659–671, March 2008.
50. T. G. Macdonald and M. B. Pursley, “The Performance of Direct-Sequence Spread Spectrum with Complex Processing and Quaternary Data Modulation,” *IEEE J. Select. Areas Commun.*, vol. 18, pp. 1408–1417, Aug. 2000.
51. A. Mariani, A. Giorgetti, and M. Chiani, “Effects of Noise Power Estimation on Energy Detection for Cognitive Radio Applications,” *IEEE Trans. Commun.*, vol. 59, pp. 3410–3420, Dec. 2011.
52. M. Medley, G. Saulnier, and P. Das, “The Application of Wavelet-Domain Adaptive Filtering to Spread-Spectrum Communications,” *Proc. SPIE Wavelet Applications for Dual-Use*, vol. 2491, pp. 233–247, April 1995.
53. H. Meyr and G. Polzer, “Performance Analysis for General PN Spread-spectrum Acquisition Techniques,” *IEEE Trans. Commun.*, vol. 31, pp. 1317–1319, Dec. 1983.
54. L. E. Miller, J. S. Lee, R. H. French, and D. J. Torrieri, “Analysis of an Anti-jam FH Acquisition Scheme,” *IEEE Trans. Commun.*, vol. 40, pp. 160–170, Jan. 1992.
55. L. E. Miller, J. S. Lee, and D. J. Torrieri, “Frequency-Hopping Signal Detection Using Partial Band Coverage,” *IEEE Trans. Aerosp. Electron. Syst.*, vol. 29, pp. 540–553, April 1993.
56. L. B. Milstein, “Interference Rejection Techniques in Spread Spectrum Communications,” *Proc. IEEE*, vol. 76, pp. 657–671, June 1988.
57. T. K. Moon, *Error Correction Coding*, Wiley, 2005.
58. D. Morgan, *Surface Acoustic Wave Filters with Applications to Electronic Communications and Signal Processing, 2nd Ed.*, Academic Press, 2007.
59. H. Mukhtar, A. Al-Dweik, and A. Shami, “Turbo Product Codes: Applications, Challenges and Future Directions,” *IEEE Trans. Commun. Surveys Tut.*, vol. 18, pp. 3052–3069, fourth quarter, 2016.
60. J. T. Oden and L. Demkowicz, *Applied Functional Analysis, 2nd. ed.*, CRC Press, 2010.
61. S.-M. Pan, D. E. Dodds, and S. Kumar, “Acquisition Time Distribution for Spread-Spectrum Receivers,” *IEEE J. Select. Areas Commun.*, vol. 8, pp. 800–808, June 1990.
62. D. R. Pauluzzi and N. C. Beaulieu, “A Comparison of SNR Estimation Techniques for the AWGN Channel,” *IEEE Trans. Commun.*, vol. 48, pp. 1681–1691, Oct. 2000.
63. D. Peng and P. Fan, “Lower bounds on the Hamming auto- and cross correlations of frequency-hopping sequences,” *IEEE Trans. Inf.* Theory, vol. 50, pp. 2149–2154, Sept. 2004.
64. R. L. Peterson, R. E. Ziemer, and D. E. Borth, *Introduction to Spread Spectrum Communications,* Prentice Hall, 1995.
65. C. Phillips, D. Sicker, and D. Grunwald, “A Survey of Wireless Path Loss Prediction and Coverage Mapping Methods,” *IEEE Trans. Commun. Surveys Tut.*, vol. 15, pp. 255–270, first quarter, 2013.
66. A. Polydoros and C. L. Weber, “A Unified Approach to Serial-Search Spread Spectrum Code Acquisition,” *IEEE Trans. Commun.*, vol. 32, pp. 542–560, May 1984.

67. A. Polydoros and C. L. Weber, "Detection Performance Considerations for Direct-Sequence and Time-Hopping LPI Waveforms," *IEEE J. Select. Areas Commun.*, vol. 3, pp. 727–744, Sept. 1985.
68. B. Porat, *A Course in Digital Signal Processing*, Wiley, 1997.
69. J. G. Proakis and M. Salehi, *Digital Communications, 5th ed.,* McGraw-Hill, 2008.
70. M. B. Pursley, "Spread-Spectrum Multiple-Access Communications," in *Multi-User Communications Systems*, G. Longo, ed., Springer-Verlag, 1981.
71. M. B. Pursley, D. V. Sarwate, and W. E. Stark, "Error Probability for Direct-Sequence Spread-Spectrum Multiple-Access Communications—Part 1: Upper and Lower Bounds," *IEEE Trans. Commun.*, vol. 30, pp. 975–984, May 1982.
72. H. Puska, H. Saarnisaari, J. Iinatti, and P. Lilja, "Serial Search Code Acquisition Using Smart Antennas with Single Correlator or Matched Filter," *IEEE Trans. Commun.*, vol. 56, pp. 299–308, Feb. 2008.
73. C. A. Putman, S. S. Rappaport, and D. L. Schilling, "Comparison of Strategies for Serial Acquisition of Frequency-Hopped Spread-Spectrum Signals," *IEE Proc.*, vol. 133, pt. F, pp. 129–137, April 1986.
74. C. A. Putman, S. S. Rappaport, and D. L. Schilling, "Tracking of Frequency-Hopped Spread-Spectrum Signals in Adverse Environments," *IEEE Trans. Commun.*, vol. 31, pp. 955–963, Aug. 1983.
75. R. Pyndiah, "Near-optimal decoding of product codes: block turbo codes," *IEEE Trans. Commun.*, vol. 46, pp. 1003–1010, Aug. 1998.
76. Y. Rahmatallah and S. Mohan, Member, "Peak-To-Average Power Ratio Reduction in OFDM Systems: A Survey And Taxonomy," *IEEE Commun. Surveys Tut.*, vol. 15, pp. 1567–1592, fourth quarter, 2013.
77. B. Razavi, *RF Microelectronics, 2nd ed.,* Pearson Prentice Hall, 2012.
78. M. Rice, *Digital Communications: A Discrete-Time Approach,* Pearson Prentice Hall, 2009.
79. U. L. Rohde and M. Rudolph, *RF/Microwave Circuit Design for Wireless Applications, 2nd ed.,* Wiley, 2013.
80. W. E. Ryan and S. Lin, *Channel Codes: Classical and Modern,* Cambridge Univ. Press, 2009.
81. D. S. Saini and M. Upadhyay, "Multiple Rake Combiners and Performance Improvement in 3G and Beyond WCDMA Systems," *IEEE Trans. Veh. Technol.*, vol. 58, pp. 3361–3370, Sept. 2009.
82. M. Sawahashi, K. Higuchi, H. Andoh, and F. Adachi, "Experiments on Pilot Symbol-Assisted Coherent Multistage Interference canceler for DS-CDMA Mobile Radio," *IEEE J. Select. Areas Commun.*, vol. 20, pp. 433–449, Feb. 2002.
83. A. H. Sayed, *Adaptive Filters,* Wiley, 2008.
84. C. Schlegel and A. Grant, *Coordinated Multiuser Communications.* Springer, 2006.
85. M. K. Simon and M.-S. Alouini, *Digital Communication over Fading Channels, 2nd ed.,* Wiley, 2004.
86. J. R. Smith, *Modern Communication Circuits, 2nd ed.*, McGraw-Hill, 1998.
87. R. A. Soni and R. M. Buehrer, "On the Performance of Open-Loop Transmit Diversity Techniques for IS-2000 Systems: A Comparative Study," *IEEE Trans. Wireless Commun.*, vol. 3, pp. 1602–1615, Sept. 2004.
88. G. Stuber, *Principles of Mobile Communication, 4th ed.,* Springer, 2017.
89. S-Y. Sun, H.-H. Chen, and W.-X. Meng, "A Survey on Complementary-Coded MIMO CDMA Wireless Communications," *IEEE Commun. Surveys Tut.*, vol. 17, pp. 52–69, first quarter, 2015.
90. W. Suwansantisuk and M. Z. Win, "Multipath Aided Rapid Acquisition: Optimal Search Strategies," I*EEE Trans. Inform. Theory*, vol. 53, pp. 174–193, Jan. 2007.
91. S. Talarico, M. C. Valenti, and D. Torrieri, "Optimization of an Adaptive Frequency-Hopping Network," *IEEE Military Commun. Conf.*, Oct. 2015.
92. W. M. Tam, F. C. M. Lau, and C. K. Tse, *Digital Communications with Chaos: Multiple Access Techniques and Performance Evaluation,* Oxford, U. K.: Elsevier, 2007.

93. B. S. Tan, K. H. Li, and K. C. Teh, “Transmit Antenna Selection Systems,” *IEEE Vehicular Technol. Mag.*, vol. 8, pp. 104–112, Sept. 2013.
94. X. Tan and J. M. Shea, “An EM Approach to Multiple-Access interference Mitigation in Asynchronous Slow FHSS Systems,” *IEEE Trans. Wireless Commun.*, vol. 7, pp. 2661–2670, July 2008.
95. S. Tanaka, A. Harada, and F. Adachi, “Experiments on Coherent Adaptive Antenna Array Diversity for Wideband DS-CDMA Mobile Radio,” *IEEE J. Select. Areas Commun.*, vol. 18, pp. 1495–1504, Aug. 2000.
96. S. Tantaratana, A. W. Lam, and P. J. Vincent, “Noncoherent Sequential Acquisition of PN Sequences for DS/SS Communications with/without Channel Fading,” *IEEE Trans. Commun.*, vol. 43, pp. 1738–1745, Feb./March/April 1995.
97. D. J. Torrieri, “The Performance of Five Different Metrics Against Pulsed Jamming,” *IEEE Trans. Commun.*, vol. 34, pp. 200–207, Feb. 1986.
98. D. J. Torrieri, “Fundamental Limitations on Repeater Jamming of Frequency-Hopping Communications,” *IEEE J. Select. Areas Commun.*, vol. 7, pp. 569–578, May 1989.
99. D. Torrieri, *Principles of Secure Communication Systems*, 2nd ed., Artech House, 1992.
100. D. Torrieri, “Performance of Direct-Sequence Systems with Long Pseudonoise Sequences,” *IEEE J. Select. Areas Commun.*, vol. 10, pp. 770–781, May 1992.
101. D. Torrieri, “The Radiometer and Its Practical Implementation,” *IEEE Military Commun. Conf.*, Oct. 2010.
102. D. Torrieri and K. Bakhru, “Anticipative Maximin Adaptive-Array Algorithm for Frequency-Hopping Systems,” *IEEE Military Commun. Conf.*, Nov. 2006.
103. D. Torrieri and K. Bakhru, “The Maximin Adaptive-Array Algorithm for Direct-Sequence Systems,” *IEEE Trans. Signal Processing*, vol. 55, pp.1853–1861, May 2007.
104. D. Torrieri and K. Bakhru, “Adaptive-Array Algorithm for Interference Suppression Prior to Acquisition of Direct-Sequence Signal,” *IEEE Trans. Wireless Commun.*, vol. 7, pp. 3341–3346, Sept. 2008.
105. D. Torrieri, S. Cheng, and M. C. Valenti, “Robust Frequency Hopping for Interference and Fading Channels,” IEEE Trans. Communications, vol. 56, pp. 1343–1351, Aug. 2008.
106. D. Torrieri, A. Mukherjee, and H. M. Kwon, “Coded DS-CDMA systems with Iterative Channel Estimation and No Pilot Symbols,” *IEEE Trans. Wireless Commun.*, vol. 9, pp. 2012–2021, June 2010.
107. D. Torrieri and M. C. Valenti, “The Outage Probability of a Finite Ad Hoc Network in Nakagami Fading,” *IEEE Trans. Commun.*, vol. 60, pp. 3509–3518, Nov. 2012.
108. D. Torrieri and M. C. Valenti, “Exclusion and Guard Zones in DS-CDMA Ad Hoc Networks,” *IEEE Trans. Commun.*, vol. 61, pp. 2468–2476, June 2013.
109. D. Torrieri, M. C. Valenti, and S. Talarico, “An Analysis of the DS-CDMA Cellular Uplink for Arbitrary and Constrained Topologies,” *IEEE Trans. Commun.*, vol. 61, pp. 3318–3326, Aug. 2013.
110. M. C. Valenti, S. Cheng, and D. Torrieri, “Iterative Multisymbol Noncoherent Reception of Coded CPFSK,” *IEEE Trans. Commun.*, vol. 58, pp. 2046–2054, July 2010.
111. M. C. Valenti and S. Cheng, “Iterative Demodulation and Decoding of Turbo Coded M-ary Noncoherent Orthogonal Modulation,” *IEEE J. Select. Areas Commun.*, vol. 23, pp. 1738–1747, Sept. 2005.
112. M.C. Valenti and M. Fanaei, “The interplay between modulation and channel coding,” Chapter 5 of *Transmission Techniques for Digital Communications*, Elsevier, 2016.
113. M. C. Valenti, D. Torrieri, and S. Talarico, “A New Analysis of the DS-CDMA Cellular Downlink Under Spatial Constraints,” *Intern. Conf. Computing, Networking, Commun.*, Jan. 2013.
114. S. S. Venkatesh, *The Theory of Probability*, Cambridge Univ. Press, 2013.
115. S. Verdu, *Multiuser Detection,* Cambridge Univ. Press, 1998.
116. X. Wang and H. V. Poor, *Wireless Communication Systems*, Prentice Hall, 2004.

117. S. P. Weber, J. G. Andrews, X. Yang, and G. de Veciana, "Transmission Capacity of Wireless Ad Hoc Networks with Successive Interference Cancellation," *IEEE Trans. Inf. Theory, vol. 53,* pp. 2799–2812, Aug. 2007.
118. P. H. Wittke, Y. M. Lam, and M. J. Schefter, "The Performance of Trellis-Coded Nonorthogonal Noncoherent FSK in Noise and Jamming," *IEEE Trans. Commun.*, vol. 43, pp. 635–645, Feb./March/April 1995.
119. S. Won and L. Hanzo, "Initial Synchronisation of Wideband and UWB Direct Sequence Systems: Single- and Multiple-Antenna Aided Solutions," *IEEE Commun. Surveys Tut.*, vol. 14, pp. 87–108, first quarter, 2012.
120. S. Xie and S. Rahardja, "Performance Evaluation for Quaternary DS-SSMA Communications with Complex Signature Sequences over Rayleigh-fading Channels," *IEEE Trans. Wireless Commun.*, vol. 4, pp. 266–77, Jan. 2005.
121. L. L. Yang and L. Hanzo, "Serial Acquisition Performance of Single-Carrier and Multicarrier DS-CDMA over Nakagami-m Fading Channels," *IEEE Trans. Wireless Commun.,* vol. 1, pp. 692–702, Oct. 2002.
122. Z. Zhou, X. Tang, X. Niu, and U. Parampalli, "New Classes of Frequency-Hopping Sequences With Optimal Partial Correlation," *IEEE Trans. Inf. Theory,* , vol. 58, pp.453–458, Jan. 2012.

Index

D. Torrieri, *Principles of Spread-Spectrum Communication Systems*,
https://doi.org/10.1007/978-3-319-70569-9

D