Appendix A
Uncertainty Inequalities

For a set of observables $\{\hat{X}_i\}_{i=1,\ldots,N}$ of a quantum mechanical system, the fact that their expectation values are determined by an underlying quantum state, determine a set of fundamental bounds to be satisfied by their outcome statistics.

The experiment where an identical quantum state of the system, $\hat{\rho}$, is independently prepared and any of the observables is measured once per preparation, determines a distribution of outcomes for each of the observables. Denoting by $x_i \in \mathbb{R}$ the continuous eigenvalues of the observable $\hat{X}_i$ corresponding to the eigenstate $|x_i\rangle$, the probability distribution of the outcomes of $\hat{X}_i$ is given by [1–3],

$$\Pr[x_i] = \mathrm{Tr}\left[|x_i\rangle\langle x_i|\,\hat{\rho}\right]. \tag{A.0.1}$$

Broadly, uncertainty relations are general statements that describe the constraints satisfied by the set of these probability distributions.

In the case of a large number of experimental trials, each of these distributions tend to a gaussian distribution, in which case, a convenient measure of measurement uncertainty is the deviation from the mean outcome, represented by the operators,

$$\delta\hat{X}_i := \hat{X}_i - \left\langle \hat{X}_i \right\rangle. \tag{A.0.2}$$

The uncertainty in each observable may be characterized as the variance of the distribution,

$$\mathrm{Var}\left[\hat{X}_i\right] := \left\langle \delta\hat{X}_i^{\,2} \right\rangle, \tag{A.0.3}$$

while the mutual correlations between the observables described by the covariance,

$$\mathrm{Cov}\left[\hat{X}_i, \hat{X}_j\right] := \left\langle \frac{1}{2}\left\{\delta\hat{X}_i, \delta\hat{X}_j\right\} \right\rangle, \tag{A.0.4}$$

where the possible non-commutativity of observables necessitates the symmetrization. Note that, $\mathrm{Var}\left[\hat{X}_i\right] = \mathrm{Cov}\left[\hat{X}_i, \hat{X}_i\right]$. In a general setting, the observables may

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commute amongst themselves according to,

$$\left[\hat{X}_j, \hat{X}_k\right] = i\hat{C}_{jk}, \tag{A.0.5}$$

where $\hat{C}_{jk}$ are some operators encoding the commutation structure. Note that $\hat{C}_{jk}$ are necessarily hermitian, and satisfy, $\hat{C}_{jk} = -\hat{C}_{kj}$.

Theorem 1 (Uncertainty principle) *In the setting described herein, the covariance matrix satisfies the matrix inequality,*

$$\mathrm{Cov}\left[\hat{X}_j, \hat{X}_k\right] + \frac{i}{2}\left\langle\hat{C}_{jk}\right\rangle \geq 0. \tag{A.0.6}$$

Proof Note that a general operator, defined by,

$$\hat{M} := \sum_j \alpha_j \, \delta\hat{X}_j,$$

for some arbitrary complex numbers α_i, satisfies the identity, $\mathrm{Tr}\left[\hat{M}^\dagger\hat{M}\hat{\rho}\right] \geq 0$, for any state $\hat{\rho}$ (see Lemma 2.1). Working out the trace explicitly,

$$\begin{aligned}
\mathrm{Tr}\left[\hat{M}^\dagger\hat{M}\hat{\rho}\right] &= \sum_{j,k} \alpha_j^*\alpha_k \, \mathrm{Tr}[\delta\hat{X}_j\delta\hat{X}_k\,\hat{\rho}] \\
&= \sum_{j,k} \alpha_j^*\alpha_k \, \mathrm{Tr}\left[\left(\frac{1}{2}\left\{\delta\hat{X}_j, \delta\hat{X}_k\right\} + \frac{1}{2}\left[\delta\hat{X}_j, \delta\hat{X}_k\right]\right)\hat{\rho}\right] \\
&= \sum_{j,k} \alpha_j^*\alpha_k \left(\mathrm{Cov}\left[\hat{X}_i, \hat{X}_j\right] + \frac{i}{2}\left\langle\hat{C}_{jk}\right\rangle\right) \\
&= \boldsymbol{\alpha}^H\mathbf{M}\boldsymbol{\alpha},
\end{aligned}$$

where, $\boldsymbol{\alpha} := [\alpha_1, \ldots, \alpha_N]^T$, is the vector of the arbitrary complex numbers α_i, $\boldsymbol{\alpha}^H = (\alpha^*)^T$ is its hermitian conjugate, and $\mathbf{M}$ is a complex matrix whose elements are given by,

$$M_{jk} := \mathrm{Cov}\left[\hat{X}_i, \hat{X}_j\right] + \frac{i}{2}\left\langle\hat{C}_{jk}\right\rangle.$$

The identity $\mathrm{Tr}\left[\hat{M}^\dagger\hat{M}\hat{\rho}\right] \geq 0$ implies that the quadratic form,

$$\boldsymbol{\alpha}^H\mathbf{M}\boldsymbol{\alpha} \geq 0, \quad \text{for any } \alpha_i.$$

This implies that the matrix $\mathbf{M}$ must itself be positive, giving the desired result. □

Corollary 1 (Robertson-Schrodinger [4, 5]) *For the case of two observables,* $\hat{X}_1$, $\hat{X}_2$,

$$\text{Var}[\hat{X}_1]\text{Var}[\hat{X}_2] \geq \frac{1}{4}\left|\left\langle\left\{\delta\hat{X}_1, \delta\hat{X}_2\right\}\right\rangle\right|^2 + \frac{1}{4}\left|\left\langle\left[\delta\hat{X}_1, \delta\hat{X}_2\right]\right\rangle\right|^2. \tag{A.0.7}$$

Proof The $N = 2$ case of Eq. (A.0.6) gives,

$$\begin{pmatrix} \text{Var}\left[\hat{X}_1\right] & \text{Cov}\left[\hat{X}_1, \hat{X}_2\right] + \frac{i}{2}\left\langle\left[\hat{X}_1, \hat{X}_2\right]\right\rangle \\ \text{Cov}\left[\hat{X}_1, \hat{X}_2\right] - \frac{i}{2}\left\langle\left[\hat{X}_1, \hat{X}_2\right]\right\rangle & \text{Var}\left[\hat{X}_2\right] \end{pmatrix} \geq 0.$$

The sufficient condition for this to be true is that its lowest eigenvalue be positive, i.e.

$$\left(\text{Var}\left[\hat{X}_1\right] + \text{Var}\left[\hat{X}_2\right]\right) - \sqrt{\left(\text{Var}\left[\hat{X}_1\right] - \text{Var}\left[\hat{X}_2\right]\right)^2 + 4\text{Cov}\left[\hat{X}_1, \hat{X}_2\right]^2 + \left\langle\left[\hat{X}_1, \hat{X}_2\right]\right\rangle^2} \geq 0;$$

simplifying this gives the required result. □

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Appendix B
Miscellanea on Elastodynamics

B.1 Principle of Least Action

Following from Sect. 3.1, the action for the elastodynamic field is,

$$\mathscr{S}[\![u_i]\!] = \int \mathrm{d}t \int_D \mathrm{d}^3 r\, \mathscr{L}(u_i, \dot{u}_i, \partial_j u_i), \tag{B.1.1}$$

for the set of independent displacement fields $u_i(\mathbf{r}, t)$. Note that for the sake of generality, we here retain a possible functional dependence of the Lagrangian on u_i, even though the actual Lagrangian of interest (Eq. (3.1.15)),

$$\mathscr{L} = \frac{\rho}{2}\dot{u}_i \dot{u}_i - \frac{1}{2} t_{ij} u_{ij}^{(1)}, \tag{B.1.2}$$

depends only on the derivatives of u_i. Note the constitute relation for the stress in terms of the strain (Eq. (3.1.16))

$$t_{ij} = \alpha_{ijkl} u_{kl}^{(1)}, \tag{B.1.3}$$

with the Hooke tensor given by (Eq. (3.1.17)),

$$\alpha_{ijkl} = \mu_1\, \delta_{ij}\delta_{kl} + \mu_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{B.1.4}$$

The principle of least action dictates that the field configuration $u_i(\mathbf{r}, t)$ that is realized is the one that renders the action minimum. Note that the action Eq. (B.1.1) is an example of a *functional*, i.e., a map that associates to a set of functions (here u_i), a real number (here, the value of the definite integral in Eq. (B.1.1)). Thus, it is reasonable to compare values of the action for different field configurations and determine one for which the action attains a minimum. In order to determine such a point, we are led to consider a space of test functions, so as to be able to explore

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the neighbourhood of each element of this functional space in a systematic fashion. Variational calculus [1–3] provides the machinery to accomplish this task.[1]

We consider the *variation* of the functions u_i,

$$u_i(\mathbf{r}, t) \to u_i(\mathbf{r}, t) + \mathrm{D}u_i(\mathbf{r}, t),$$

where the symbol D denotes a functional variation, signifying the fact that these changes are simply a device to enable exploration of the functional neighbourhood of u_i. Since the fields u_i are independent, they maybe varied independently, and so the corresponding variations $\mathrm{D}u_i$ are also independent. The resulting variation in the action is,

$$\mathrm{D}\mathscr{S} = \int \mathrm{d}t\, \mathrm{d}^3r \left[\frac{\partial \mathscr{L}}{\partial u_i}\, \mathrm{D}u_i + \frac{\partial \mathscr{L}}{\partial \dot{u}_i}\, \mathrm{D}\dot{u}_i + \frac{\partial \mathscr{L}}{\partial(\partial_j u_i)}\, \mathrm{D}(\partial_j u_i)\right].$$

The second and third terms of the integrand can be re-expressed as,

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial \dot{u}_i}\, \mathrm{D}\dot{u}_i &= \frac{\partial \mathscr{L}}{\partial \dot{u}_i}\, \partial_t(\mathrm{D}u_i) = \partial_t\left(\frac{\partial \mathscr{L}}{\partial \dot{u}_i}\, \mathrm{D}u_i\right) - \partial_t\left(\frac{\partial \mathscr{L}}{\partial \dot{u}_i}\right)\mathrm{D}u_i \\ \frac{\partial \mathscr{L}}{\partial(\partial_j u_i)}\, \mathrm{D}(\partial_j u_i) &= \frac{\partial \mathscr{L}}{\partial(\partial_j u_i)}\, \partial_j(\mathrm{D}u_i) = \partial_j\left(\frac{\partial \mathscr{L}}{\partial(\partial_j u_i)}\, \mathrm{D}u_i\right) - \partial_j\left(\frac{\partial \mathscr{L}}{\partial(\partial_j u_i)}\right)\mathrm{D}u_i \end{aligned} \tag{B.1.5}$$

and re-inserted back. Thus we arrive at,

$$\begin{aligned} \mathrm{D}\mathscr{S} = &\int \mathrm{d}t \int_D \mathrm{d}^3r \left[\frac{\partial \mathscr{L}}{\partial u_i} - \partial_t\left(\frac{\partial \mathscr{L}}{\partial \dot{u}_i}\right) - \partial_j\left(\frac{\partial \mathscr{L}}{\partial(\partial_j u_i)}\right)\right]\mathrm{D}u_i \\ &+ \int_D \mathrm{d}^3r \left[\frac{\partial \mathscr{L}}{\partial \dot{u}_i}\, \mathrm{D}u_i\right]_{t=0}^{t=\infty} + \int \mathrm{d}t \oint_{\partial D} \mathrm{d}A_j\, \frac{\partial \mathscr{L}}{\partial(\partial_j u_i)}\, \mathrm{D}u_i. \end{aligned} \tag{B.1.6}$$

Here, the second and third integrals arise from integrating the total derivatives in Eq. (B.1.5). In the third integral, this is performed through an application of the divergence theorem, resulting in an integral over the boundary ∂D of the domain D.

For the principle of least action to be implemented in the form,

$$\frac{\mathrm{D}\mathscr{S}}{\mathrm{D}u_i} = 0,$$

it is therefore required that each of the integrals in Eq. (B.1.6) vanish separately. Since the variations $\mathrm{D}u_i$ are arbitrary, this is tantamount to each of the integrands vanishing independently. This results in three conditions:

1. the Euler-Lagrange equations,

[1] Incidentally, note that the principle of least action is really a principle of stationary action [1].

$$\frac{\partial \mathscr{L}}{\partial u_i} - \partial_t \left(\frac{\partial \mathscr{L}}{\partial \dot{u}_i} \right) - \partial_j \left(\frac{\partial \mathscr{L}}{\partial(\partial_j u_i)} \right) = 0 \tag{B.1.7}$$

2. fulfilment of initial and/or final conditions,

$$\left[\frac{\partial \mathscr{L}}{\partial \dot{u}_i} \, \mathrm{D}u_i \right]_{t=0}^{t=\infty} = 0 \tag{B.1.8}$$

3. fulfilment of boundary conditions,

$$\oint_{\partial D} \mathrm{d}A_j \, \frac{\partial \mathscr{L}}{\partial(\partial_j u_i)} \, \mathrm{D}u_i = 0. \tag{B.1.9}$$

Note that the principle of least action not only furnishes the dynamical equation Eq. (B.1.7) to be satisfied by the true field configuration, but also provides a consistent set of *natural* boundary conditions Eqs. (B.1.8) and (B.1.9).

B.1.1 Equations of Motion

To implement the Euler-Lagrange equation Eq. (B.1.7), we compute the various terms in it, for the Lagrangian Eq. (B.1.2):

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial u_i} &= 0 \\ \frac{\partial \mathscr{L}}{\partial \dot{u}_i} &= \frac{\partial}{\partial \dot{u}_i} \left(\frac{\rho}{2} \dot{u}_a \dot{u}_a \right) = \frac{\rho}{2} (2 \dot{u}_a \delta_{ia}) = \rho \dot{u}_i \\ \frac{\partial \mathscr{L}}{\partial(\partial_j u_i)} &= \frac{\partial}{\partial(\partial_j u_i)} \left(-\frac{\alpha_{abcd}}{2} (\partial_b u_a)(\partial_d u_c) \right) = -\alpha_{ijcd} (\partial_d u_c). \end{aligned} \tag{B.1.10}$$

Inserting these in Eq. (B.1.7) gives,

$$\rho \ddot{u}_i - \alpha_{ijkl} \partial_j \partial_l u_k = 0. \tag{B.1.11}$$

Finally using the explicit form of the Hooke tensor (Eq. (B.1.3)) gives the Navier equations,

$$\begin{aligned} \rho \, \ddot{u}_i &= (\mu_1 + \mu_2) \partial_i \partial_j u_j + \mu_2 \, \partial_j \partial_j u_i \\ \text{or,} \quad \rho \ddot{\mathbf{u}} &= (\mu_1 + \mu_2) \nabla (\nabla \cdot \mathbf{u}) + \mu_2 \nabla^2 \mathbf{u} \\ &= (\mu_1 + 2\mu_2) \nabla (\nabla \cdot \mathbf{u}) - \mu_2 \nabla \times (\nabla \times \mathbf{u}), \end{aligned} \tag{B.1.12}$$

where the last two forms are expressed using vector operators appropriate for 3D domains. The third form is obtained by using the generally valid vector identity,

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}. \tag{B.1.13}$$

B.1.2 Boundary Conditions

The natural boundary condition Eq. (B.1.9), applied to the Lagrangian Eq. (B.1.2) results in,

$$\oint_{\partial D} \mathrm{d}A_j \, t_{ij} \, \mathrm{D}u_i = 0,$$

where, $t_{ij} := \alpha_{ijkl} \partial_l u_k$, is the stress tensor according to Hooke's law. Using the fact that the force F_i (along the i direction) is given in terms of the stress t_{ij} acting on the area element $\mathrm{d}A_j$ (normal to the direction j), $t_{ij}\,\mathrm{d}A_j = \mathrm{d}F_i$, the boundary condition reads,

$$\oint_{\partial D} \mathrm{d}F_i \, \mathrm{D}u_i = 0.$$

Since the variation $\mathrm{D}u_i$ are independent of the force on the boundary, this is equivalent to two conditions, viz.

$$\begin{aligned} \mathrm{d}F_i|_{\partial D} &= t_{ij}\mathrm{d}A_j|_{\partial D} = 0 \\ \mathrm{D}u_i|_{\partial D} &= 0. \end{aligned} \tag{B.1.14}$$

Physically, the first is appropriate for a free boundary, on which no force impinges, whereas the second is appropriate for a fixed boundary, whose displacement is prescribed.

B.2 Transverse and Longitudinal Elastic Waves

The Navier equations Eq. (B.1.12) expressed as a single vector equation,

$$\ddot{\mathbf{u}} = \left(\frac{\mu_1 + 2\mu_2}{\rho}\right) \nabla(\nabla \cdot \mathbf{u}) - \left(\frac{\mu_2}{\rho}\right) \nabla \times (\nabla \times \mathbf{u}), \tag{B.2.1}$$

makes explicit the two kinds of excitations referred to in Sect. 3.1. In order to exhibit this claim, we make use of the fact that any vector field, here $\mathbf{u}$, in a simply connected domain D, maybe expressed uniquely in terms of *potentials*, $\phi(\mathbf{r}, t)$ and $\mathbf{\Phi}(\mathbf{r}, t)$:

$$\mathbf{u} = \nabla \phi + \nabla \times \mathbf{\Phi}. \tag{B.2.2}$$

Identifying these two terms as $\mathbf{u}_L$ and $\mathbf{u}_T$ respectively, standard vector identities imply $\nabla \cdot \mathbf{u}_T = 0$ and $\nabla \times \mathbf{u}_L = 0$; $\mathbf{u}_T$ ($\mathbf{u}_L$) is the transverse (longitudinal) component of $\mathbf{u}$. Substituting this decomposition into Eq. (B.2.1), and realizing that the transverse and longitudinal components are independent, results in two wave equations,

$$\ddot{\mathbf{u}}_L = \left(\frac{\mu_1 + 2\mu_2}{\rho}\right) \nabla^2 \mathbf{u}_L$$
$$\ddot{\mathbf{u}}_T = \left(\frac{\mu_2}{\rho}\right) \nabla^2 \mathbf{u}_T. \tag{B.2.3}$$

The phase velocities of the two elastic waves can be immediately identified, viz.,

$$c_L := \sqrt{\frac{\mu_1 + 2\mu_2}{\rho}}, \quad c_T := \sqrt{\frac{\mu_2}{\rho}}. \tag{B.2.4}$$

B.3 Hermiticity of the Elastic Operator

The elasticity operator $\hat{\mathbf{L}}$, defined in Eq. (3.1.20) viz.

$$\hat{L}_{ik} = \frac{\alpha_{ijkl}}{\rho} \partial_j \partial_l, \tag{B.3.1}$$

acts on vector functions $\mathbf{u}$ defined on some finite domain D. Corresponding to some such function $\mathbf{v}$, we define a linear functional $\langle \mathbf{v}, \cdot \rangle$ that acts as,

$$\langle \mathbf{v}, \mathbf{u} \rangle := \frac{1}{\mathrm{Vol}(D)} \int_D v_i^*(\mathbf{r}) u_i(\mathbf{r})\, \mathrm{d}^3 r. \tag{B.3.2}$$

We now restrict attention to functions $\mathbf{u}$ for which $\langle \mathbf{u}, \mathbf{u} \rangle < \infty$, and satisfies one of the boundary conditions in Eq. (B.1.14) viz.

$$\text{Type 1:} \quad \mathrm{d}F_i|_{\partial D} = t_{ij} \mathrm{d}A_j|_{\partial D} = \alpha_{ijkl} (\partial_j u_i)(\partial_l u_k)|_{\partial D} = 0$$
$$\text{Type 2:} \quad u_i|_{\partial D} = 0, \tag{B.3.3}$$

where we have assumed (without loss of generality) that in the case of a fixed boundary condition, the boundary displacement is zero.

Each set of such functions—bounded and satisfying boundary condition of Type s ($s = 1, 2$)—forms a Hilbert space[2] $\mathscr{H}_s$ under the inner product $\langle \cdot, \cdot \rangle$. For every $\mathbf{u} \in \mathscr{H}_s$, there is a functional $\langle \mathbf{u}, \cdot \rangle \in \mathrm{Dual}(\mathscr{H}_s)$ in the dual of $\mathscr{H}_s$ [4].

Having identified the two distinct Hilbert spaces at play, the proof of the hermiticity of $\hat{\mathbf{L}}$ is straightforward. Using the definition of $\hat{\mathbf{L}}$ (Eq. (B.3.1)),

$$\langle \mathbf{v}, \hat{\mathbf{L}}\mathbf{u} \rangle = \frac{\rho^{-1}}{\mathrm{Vol}(D)} \int_D v_i^*(\mathbf{r})\, \alpha_{ijkl} \partial_j \partial_l u_k(\mathbf{r})\, \mathrm{d}^3 r.$$

[2] Physically the two spaces $\mathscr{H}_{1,2}$ describe the displacement fields for the physically incompatible boundary conditions of each type; mathematically, this incompatibility manifests as the fact that a function satisfying one type of boundary condition does not form a superposition with that satisfying a different boundary condition, such that the superposed function satisfies any well-defined boundary condition. Closure under superposition is necessary for a Hilbert space.

Manipulating the integral, and freely using the symmetries of the Hooke tensor (Eq. (3.1.10)) $\alpha_{ijkl} = \alpha_{jikl} = \alpha_{ijlk} = \alpha_{klij}$,

$$\begin{aligned}\int_D v_i^* \,\alpha_{ijkl}\partial_j\partial_l u_k \,\mathrm{d}^3r &= \int_D v_i^* \,\alpha_{ijkl}\partial_j\partial_k u_l \,\mathrm{d}^3r \\ &= \int_D \partial_j(v_i^* \,\alpha_{ijkl}\,\partial_k u_l)\,\mathrm{d}^3r - \int_D (\partial_j v_i^*)\,\alpha_{ijkl}(\partial_k u_l)\,\mathrm{d}^3r \\ &= \int_{\partial D} v_i^* \underbrace{\alpha_{ijkl}\,\partial_k u_l}_{t_{ij}}\,\mathrm{d}A_j - \int_D (\partial_j v_i^*)\,\alpha_{ijkl}(\partial_k u_l)\,\mathrm{d}^3r;\end{aligned}$$

the second equality follows by partial integration, while the third follows from Gauss' Theorem. Finally, either type of boundary condition ensures that the first term in the last line is zero. Treating the remaining integral similarly,

$$\begin{aligned}\int_D v_i^* \,\alpha_{ijkl}\partial_j\partial_l u_k \,\mathrm{d}^3r &= -\int_D (\partial_j v_i^*)\,\alpha_{ijkl}(\partial_k u_l)\,\mathrm{d}^3r \\ &= -\int_{\partial D} (\partial_j v_i^*)\alpha_{ijkl}u_l\,\mathrm{d}A_k + \int_D (\partial_k\partial_j v_i^*)\alpha_{ijkl}u_l\,\mathrm{d}^3r \\ &= -\int_{\partial D} u_i \underbrace{\alpha_{ijkl}\partial_k v_l^*}_{t_{ij}^*}\,\mathrm{d}A_k + \int_D (\alpha_{ijkl}\partial_j\partial_l v_k^*)u_i\,\mathrm{d}^3r \\ &= \int_D (\alpha_{ijkl}\partial_j\partial_l v_k^*)u_i\,\mathrm{d}^3r,\end{aligned}$$

i.e., the differential operator $\partial_j\partial_l$ can be freely commuted within the integral as long as the functions satisfy one of the boundary conditions (Eq. (B.3.3)), and the Hooke tensor is symmetric. In particular, this means that the inner product satisfies,

$$\langle \mathbf{v}, \hat{\mathbf{L}}\mathbf{u}\rangle = \langle \hat{\mathbf{L}}\mathbf{v}, \mathbf{u}\rangle, \tag{B.3.4}$$

i.e. $\hat{\mathbf{L}}$ is hermitian in either Hilbert space $\mathscr{H}_{1,2}$.

B.4 Eigensolution of the Doubly-Clamped Stressed Elastic Beam

The normalized mode functions $v_n(\zeta)$, of a 1D stressed elastic beam are given by the Euler-Bernoulli equations with stress Eq. (5.1.9), viz.

$$\epsilon\frac{\partial^4 v_n}{\partial\zeta^4} - \frac{\partial^2 v_n}{\partial\zeta^2} = \left(\frac{\Omega_n}{\Omega_0}\right)^2 v_n, \tag{B.4.1}$$

where, $\epsilon = KM/T\ell_z^2$ is the (dimensionless) ratio of bending to tensile energy, $\Omega_0 = (T/\rho\mathscr{A}\ell_z^2)^{1/2}$ is the frequency determined by the ratio of tensile energy to inertia. The equation is well-posed for the case where the beam is clamped on both ends, described by the boundary conditions,

$$v(0) = v(1) = 0, \quad \partial_\zeta v(0) = \partial_\zeta v(1) = 0. \tag{B.4.2}$$

The fourth order differential operator forming the right-hand side of Eq. (B.4.1) has four eigenfunctions, viz. $e^{\pm k_n^+ \zeta}$, $e^{\pm i k_n^- \zeta}$, where,

$$k_n^{\pm} := \left(\frac{1}{2\epsilon}\right)^{1/2} \left(\pm 1 + \sqrt{1 + 4\epsilon(\Omega_n/\Omega_0)^2}\right)^{1/2}, \tag{B.4.3}$$

are the normalized wave vectors of the vibration at frequency Ω_n. Note that this relation indicates a nonlinear dispersion for waves excited on the stressed beam. Indeed, the small$-\epsilon$ approximation,

$$\begin{aligned} k_n^+ &\approx \frac{1}{\sqrt{\epsilon}}\left[1 + \frac{(\Omega_n/\Omega_0)^2}{2}\epsilon + \mathscr{O}(\epsilon^3)\right] \\ k_n^- &\approx \frac{\Omega_n}{\Omega_0}\left[1 - \frac{(\Omega_n/\Omega_0)^2}{2}\epsilon + \mathscr{O}(\epsilon^2)\right] \end{aligned} \tag{B.4.4}$$

seems to suggest that the k_n^- branch describes excitations with linear dispersion—familiar from the case of the purely tensile string ($\epsilon = 0$), while the k_n^+ branch arises from corrections due to the bending term—leading to deviations from a sinusoidal mode that occupy a spatial scale approximated by $\ell_z/k_n^+ \approx \ell_z\sqrt{\epsilon}$.

In the following, exact shapes of the mode functions, and their small$-\epsilon$ approximation—describing the afore-mentioned deviations—will be presented. The general mode $v_n(\zeta)$ is that superposition of the four exponential eigenfunctions that satisfies the double-clamped boundary conditions in Eq. (B.4.2), viz. (see also [5])

$$v_n(\zeta) \propto \frac{k_n^+ \sin k_n^- \zeta - k_n^- \sinh k_n^+ \zeta}{k_n^+ \sin k_n^+ - k_n^- \sinh k_n^-} - \frac{\cos k_n^- \zeta - \cosh k_n^+ \zeta}{\cos k_n^- - \cosh k_n^+}. \tag{B.4.5}$$

Here, the proportionality indicates that an overall constant— fixed by the normalization of the mode function—is omitted. In order for the boundary conditions to be satisfied consistently, it is required that,

$$\frac{(k_n^+)^2 - (k_n^-)^2}{2k_n^- k_n^+} = \frac{\cosh k_n^+ \cos k_n^- - 1}{\sinh k_n^+ \sin k_n^-}; \tag{B.4.6}$$

an algebraic equation that, expressed in terms of Ω_n (via Eq. (B.4.3)), determines the eigenfrequencies of the beam.

Convenient approximations, relevant for the case $\epsilon \ll 1$, can be derived from noting that in this regime, $k_n^+ \gg 1$, and, $k_n^+ \gg k_n^-$. Applied to the characteristic Eq. (B.4.6),

$$k_n^+/k_n^- \approx 2\coth k_n^+ \cot k_n^- \approx 2\cot k_n^-,$$

where the second approximation follows from, $\coth k_n^+ \to 1$, for $k_n^+ \approx \Omega_n/\Omega_0 \gtrsim 1$ (and improving for higher order modes). Thus the approximate characteristic equation,

$$k_n^- \cot k_n^- \approx 2k_n^+,$$

holds. Since $k_n^+ \gg 1$, the solutions of this equation are well approximated by those values of k_n^- that make, $\cot k_n^-$, singular; i.e., $k_n^- \approx n\pi$, for $n \in \mathbb{Z}$. Finally using Eq. (B.4.3), the approximate eigenfrequencies are given by,

$$\Omega_n \approx n\pi\Omega_0\sqrt{1+(n\pi)^2\epsilon}. \tag{B.4.7}$$

For the mode functions, a similar approach may be followed, noting that for $\epsilon \ll 1$, $\sinh k_n^+ \approx \cosh k_n^+ \gg 1$. Applying these crude estimates in Eq. (B.4.5), for the case $\zeta < 1$, gives the approximate mode function, $f_n(\zeta) := v_n(0 \le \zeta \lesssim \frac{1}{2})|_{\epsilon\ll 1}$, viz.

$$\begin{aligned}
f_n(\zeta) &\approx \frac{k_n^+ \sin k_n^-\zeta - k_n^- \sinh k_n^+\zeta}{-k_n^- \sinh k_n^+} - \frac{\cosh k_n^+\zeta - \cos k_n^-\zeta}{\cosh k_n^+} \\
&\propto \sin k_n^-\zeta - \frac{k_n^-}{k_n^+}\sinh k_n^+\zeta + \frac{k_n^-}{k_n^+}\tanh k_n^+\left(\cosh k_n^+\zeta - \cos k_n^-\zeta\right) \\
&\approx \sin k_n^-\zeta + \frac{k_n^-}{k_n^+}\left(\cosh k_n^+\zeta - \sinh k_n^+\zeta - \cos k_n^-\zeta\right) \\
&= \sin k_n^-\zeta + \frac{k_n^-}{k_n^+}\left(e^{-k_n^+\zeta} - \cos k_n^-\zeta\right).
\end{aligned}$$

This approximate form indicates that the mode functions deviate slightly from the sinusoidal modes of a tensile string, by a factor proportional to $\sqrt{\epsilon}$, and the form of the deviation is an exponential correction at the boundary. Indeed the mode function, $v_n(\zeta)$, over the full domain can be approximated by the piecewise smooth function (used, for example in [6]),

$$v_n(\zeta) \approx \begin{cases} f_n(\zeta), & 0 \le \zeta \le \frac{1}{2} \\ (-1)^{n+1} f_n(1-\zeta), & \frac{1}{2} < \zeta \le 1 \end{cases}. \tag{B.4.8}$$

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Appendix C
Response of an Imbalanced Interferometer

Following the discussion in Sect. 3.2.2, assume that the amplitude flux $\hat{a}(t)$ of a coherent source of mean amplitude $\bar{a}$ undergoes classical amplitude and phase fluctuations, so that in the rotating frame (the ansatz in Eq. (3.2.6)),

$$a_{\text{in}}(t) = (\bar{a} + \delta\alpha(t))e^{i\delta\phi(t)}, \tag{C.1}$$

where $\delta\alpha(t)$ and $\delta\phi(t)$ are real-valued stochastic processes. Note that since we are interested in classical noise in the amplitude $\delta\alpha(t)$ and in phase $\delta\phi(t)$, all vacuum contributions will be ignored here.

Figure C.1 shows such a field passing through an interferometer. When the input field is split at a beam splitter of transmissivity η_1 at the input of the interferometer, each arm is fed with the fields $a_{1,\text{in}}(t)$ and $a_{2,\text{in}}(t)$, given by,

$$a_{1,\text{in}}(t) = \sqrt{\eta_1}\, a_{\text{in}}(t), \qquad a_{2,\text{in}}(t) = i\sqrt{1-\eta_1}\, a_{\text{in}}(t). \tag{C.2}$$

The first field propagates through a path containing a frequency-shifting element (for example, AOM) implementing a radio frequency shift $\Omega_{\text{IF}} \ll \Omega_{\text{det}} \ll \omega_\ell$ (where Ω_{det} is the final detection span), while the other field propagates through a relative delay (for example using a long path length) of duration τ. The two fields emerging at the end of these paths are,

$$a_{1,\text{out}}(t) = a_{1,\text{in}}(t)e^{-i\Omega_{\text{IF}}t}, \qquad a_{2,\text{out}}(t) = a_{2,\text{in}}(t-\tau). \tag{C.3}$$

Finally, the beams are combined at a beam-splitter of transmissivity η_2 and one of the outputs,

$$\begin{aligned} a_{\text{out}}(t) &= \sqrt{\eta_2}\, a_{1,\text{out}}(t) + i\sqrt{1-\eta_2}\, a_{2,\text{out}}(t) \\ &= \sqrt{\eta_1\eta_2}\, a_{\text{in}}(t)e^{-i\Omega_{\text{IF}}t} - \sqrt{(1-\eta_1)(1-\eta_2)}\, a_{\text{in}}(t-\tau), \end{aligned}$$

V. Sudhir, *Quantum Limits on Measurement and Control of a Mechanical Oscillator*, Springer Theses, https://doi.org/10.1007/978-3-319-69431-3

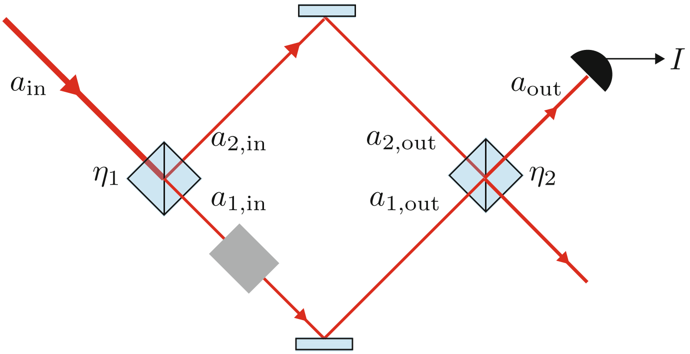

Fig. C.1 Schematic of an imbalanced interferometer. An interferometer in Mach-Zehnder configuration with a noisy input field that is possibly frequency-shifted in one of the arms, and phase delayed in the other

is photodetected. The resulting photocurrent $I(t) \propto |a_{out}(t)|^2$ is given by,

$$I(t) = \eta_1\eta_2\ |a_{in}(t)|^2 + (1-\eta_1)(1-\eta_2)\ |a_{in}(t-\tau)|^2 + 2\sqrt{\eta_1\eta_2(1-\eta_1)(1-\eta_2)}\,\mathrm{Re}\, a_{in}^*(t-\tau)a_{in}(t)e^{-i\Omega_{IF}t}.$$

The last (interference) term describes fluctuations in the photocurrent,

$$\delta I(t) := \mathrm{Re}\, a_{in}^*(t-\tau)a_{in}(t)e^{-i\Omega_{IF}t} = \bar{a}^2\left(1+\frac{\delta\alpha(t-\tau)}{\bar{a}}\right)\left(1+\frac{\delta\alpha(t)}{\bar{a}}\right)\cos\left[\delta\phi(t)-\delta\phi(t-\tau)-\Omega_{IF}t\right],$$

that carry traces of the amplitude and phase fluctuations of the field at the input of the interferometer. Introducing the cumulative relative amplitude fluctuations,

$$\delta \mathrm{A}(t) := \left(\delta\alpha(t)+\delta\alpha(t-\tau)\right)/\bar{a}, \tag{C.4}$$

and the differential phase fluctuations,

$$\delta\Phi(t) := \delta\phi(t)-\delta\phi(t-\tau), \tag{C.5}$$

the photocurrent fluctuations can be approximated as,

$$\delta I(t) \approx (1+\delta \mathrm{A}(t))\cos\left[\delta\Phi(t)-\Omega_{IF}t\right]. \tag{C.6}$$

Henceforth, we assume that the amplitude ($\delta\alpha(t)$) and phase ($\delta\phi(t)$) fluctuations are stationary gaussian processes with zero mean; a property that is inherited by $\delta\mathrm{A}(t)$, and, $\delta\Phi(t)$. However, due to the nonlinear transformation relating the phase to the photocurrent, the latter is not gaussian.

Despite this fact, useful information about the amplitude and phase fluctuations can be garnered from the lowest order correlation function of the fluctuating photocurrent. Indeed, assuming that the amplitude and phase fluctuations are uncorrelated (see in Chap. 7 footnote 4, on page 183), the two-time correlation of the photocurrent fluctuations take the form,

$$\begin{aligned}\langle \delta I(t)\delta I(0)\rangle &= \bar{a}^2\Big\langle\Big(1+\delta\mathrm{A}(t)\Big)\Big(1+\delta\mathrm{A}(0)\Big)\,\cos\big[\delta\Phi(t)-\Omega_{\mathrm{IF}}t\big]\cos\big[\delta\Phi(0)\big]\Big\rangle\\ &= \bar{a}^2\Big(1+\langle\delta\mathrm{A}(t)\delta\mathrm{A}(0)\rangle\Big)\Big(\big\langle\cos[\delta\Phi(t)]\cos[\delta\Phi(0)]\big\rangle\cos\Omega_{\mathrm{IF}}t\\ &\qquad+\big\langle\sin[\delta\Phi(t)]\cos[\delta\Phi(0)]\big\rangle\sin\Omega_{\mathrm{IF}}t\Big).\end{aligned} \tag{C.7}$$

Using standard techniques,[3] the expectation values of the product of the cosine/sine phase terms can be shown to be equal, and given by,

$$\begin{aligned}\langle\cos[\delta\Phi(t)]\cos[\delta\Phi(0)]\rangle &= \langle\sin[\delta\Phi(t)]\cos[\delta\Phi(0)]\rangle\\ &= \tfrac{1}{2}+\tfrac{1}{2}\exp\big[-\langle\delta\Phi(t)\delta\Phi(0)\rangle-\langle\delta\Phi(0)^2\big]\rangle.\end{aligned} \tag{C.8}$$

Finally using the Fourier representation of $\delta\Phi$, and then using its definition given in Eq. (C.5),

$$\begin{aligned}\langle\delta\Phi(t)\delta\Phi(0)\rangle &= \int\frac{\mathrm{d}\Omega\,\mathrm{d}\Omega'}{(2\pi)^2}e^{-i\Omega t}\,\big\langle\delta\Phi[\Omega]\delta\Phi[\Omega']\big\rangle\\ &= \int\frac{\mathrm{d}\Omega\,\mathrm{d}\Omega'}{(2\pi)^2}e^{-i\Omega t}\,\Big\langle\delta\phi[\Omega](1-e^{i\Omega\tau})\,\delta\phi[\Omega'](1-e^{i\Omega'\tau})\Big\rangle\\ &= \int\frac{\mathrm{d}\Omega\,\mathrm{d}\Omega'}{(2\pi)^2}e^{-i\Omega t}\,(1-e^{i\Omega\tau})(1-e^{i\Omega'\tau})\cdot 2\pi\,S_{\phi\phi}[\Omega]\,\delta[\Omega-\Omega']\\ &= -4\int\frac{\mathrm{d}\Omega}{2\pi}e^{-i\Omega(t-\tau)}\,\sin^2\left(\frac{\Omega\tau}{2}\right)S_{\phi\phi}[\Omega];\end{aligned} \tag{C.9}$$

thus, the two-time correlators in the exponent of Eq. (C.8) can be expressed in terms of the spectrum of phase fluctuations. Similarly, the two-time correlator, $\langle\delta\mathrm{A}(t)\delta\mathrm{A}(0)\rangle$ in Eq. (C.7), can be expressed in terms of the spectrum of amplitude fluctuations, viz.

[3] Re-writing the trigonometric functions as exponentials, multiplying them out, and then using the identity $\big\langle\exp[i\delta X(t)]\big\rangle=\exp\big[-\frac{1}{2}\langle\delta X(t)\delta X(0)\rangle\big]$, on each exponential term; here δX denotes the relevant random process.

$$\begin{aligned}\langle \delta \mathrm{A}(t)\delta \mathrm{A}(0)\rangle &= \int \frac{\mathrm{d}\Omega\,\mathrm{d}\Omega'}{(2\pi)^2} e^{-i\Omega t} \left\langle \delta \mathrm{A}[\Omega]\delta \mathrm{A}[\Omega']\right\rangle \\ &= \int \frac{\mathrm{d}\Omega\,\mathrm{d}\Omega'}{(2\pi)^2} e^{-i\Omega t} \left\langle \delta \mathrm{A}[\Omega](1+e^{i\Omega\tau})\, \delta \mathrm{A}[\Omega'](1+e^{i\Omega'\tau})\right\rangle \\ &= 4\int \frac{\mathrm{d}\Omega}{2\pi} e^{-i\Omega(t-\tau)} \cos^2\left(\frac{\Omega\tau}{2}\right) \frac{S_{\alpha\alpha}[\Omega]}{\bar{a}^2}.\end{aligned} \tag{C.10}$$

Inserting Eq. (C.9) in Eq. (C.8) and subsequently in Eq. (C.7), and inserting Eq. (C.10) in Eq. (C.7), taking the limit where $S_{\phi\phi} \ll 1$, and dropping irrelevant constant factors, the photocurrent correlation takes the approximate form,

$$\begin{aligned}\langle \delta I(t)\delta I(0)\rangle \propto \sin\left(\Omega_{\mathrm{IF}} t + \frac{\pi}{4}\right) \Bigg[1 &+ 4\int \frac{\mathrm{d}\Omega}{2\pi} e^{-i\Omega(t-\tau)} \cos^2\left(\frac{\Omega\tau}{2}\right) \frac{S_{\alpha\alpha}[\Omega]}{\bar{a}^2} \\ &+ 4\int \frac{\mathrm{d}\Omega}{2\pi} e^{-i\Omega(t-\tau)} \sin^2\left(\frac{\Omega\tau}{2}\right) S_{\phi\phi}[\Omega] \Bigg].\end{aligned} \tag{C.11}$$

The (symmetrised) spectrum of photocurrent fluctuations recorded by a spectrum analyser is the cosine transform of this quantity. Shifted by the heterodyne beat frequency, the photocurrent spectrum is,

$$\bar{S}_{II}[\Omega - \Omega_{\mathrm{IF}}] \propto \delta[\Omega - \Omega_{\mathrm{IF}}] + \frac{2}{\pi}\left(\cos^2\left(\frac{\Omega\tau}{2}\right) \frac{\bar{S}_{\alpha\alpha}[\Omega]}{\bar{a}^2} + \sin^2\left(\frac{\Omega\tau}{2}\right) \bar{S}_{\phi\phi}[\Omega] \right), \tag{C.12}$$

a result consistent with earlier treatments of phase fluctuations alone [1, 2].

Thus, an imbalanced interferometer transduces input phase and relative amplitude fluctuations, onto the output photocurrent, depending on the time delay τ between the two arms. Typically, by operating a laser far above threshold with a large photon flux $\bar{a}$, the input relative intensity noise can be made arbitrarily small, so that an imbalanced Mach-Zehnder interferometer can be used to measure input phase noise.

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