Solutions to Problems

Problems in Chap. 1

Problem 1.1

The components of vectors **u**, **v**, and **w** are given by u_i, v_i, w_i. By expanding all the summations, we have:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = u_i v_i. \tag{S.221}$$

Similarly,

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= u_i \mathbf{e}_i \times v_j \mathbf{e}_j \\ &= (u_1 v_2 - u_2 v_1)\, \mathbf{e}_3 + (u_3 v_1 - u_1 v_3)\, \mathbf{e}_2 + (u_2 v_3 - u_3 v_2)\, \mathbf{e}_1 \\ &= \varepsilon_{ijk} \mathbf{e}_i u_j v_k. \end{aligned} \tag{S.222}$$

And, by using (S.221) and (S.222),

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \varepsilon_{ijk} \mathbf{e}_i u_j v_k \cdot w_l \mathbf{e}_l \\ &= \varepsilon_{ijk} u_j v_k w_i \\ &= \varepsilon_{ijk} u_i v_j w_k. \end{aligned} \tag{S.223}$$

From the cyclic properties of $\varepsilon_{ijk} = \varepsilon_{jki}$, and use the last preceding result,

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})\,. \tag{S.224}$$

Next, consider the 4th order tensor $\varepsilon_{nij}\varepsilon_{nkm}$. Clearly, if $i = j$, or $k = m$, this tensor takes on a zero value. If $i = k$, the sum in n will return a zero value, unless $j = m$, and $\varepsilon_{nij} = \varepsilon_{nkm} = \pm 1$, and the product $\varepsilon_{nij}\varepsilon_{nkm} = +1$. Now, if $i = m$ and $j = k$ the product $\varepsilon_{nij}\varepsilon_{nkm}$ will return a zero value unless $\varepsilon_{nij} = \pm 1$, then

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$\varepsilon_{nkm} = \varepsilon_{nji} = \mp 1$, and the product $\varepsilon_{nij}\varepsilon_{nkm} = -1$. These are also the components of the tensor $\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}$. Hence

$$\varepsilon_{nij}\varepsilon_{nkm} = \delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}. \tag{S.225}$$

Using (S.225) in the following:

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} &= \varepsilon_{ijk}\mathbf{e}_i\varepsilon_{jmn}u_m v_n w_k = (\delta_{km}\delta_{in} - \delta_{kn}\delta_{im})\,\mathbf{e}_i u_m v_n w_k \\ &= \mathbf{e}_i u_k v_i w_k - \mathbf{e}_i u_i v_k w_k \\ (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} &= (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\,\mathbf{u}.\end{aligned} \tag{S.226}$$

This could also be derived by expanding components of both sides.

Now, use the result (S.225) in the following

$$\begin{aligned}(\mathbf{u} \times \mathbf{v})^2 &= \varepsilon_{ijk}u_j v_k \varepsilon_{imn}u_m v_n = \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) u_j v_k u_m v_n \\ &= u_j u_j v_k v_k - u_j v_j v_k u_k \\ &= u^2 v^2 - (\mathbf{u} \cdot \mathbf{v})^2 .\end{aligned} \tag{S.227}$$

Problem 1.2

Let $\mathbf{A}$ be a matrix with entries A_{ij},

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

By expanding

$$\begin{aligned}\det[\mathbf{A}] &= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{31}A_{22}A_{13} \\ &\quad - A_{32}A_{23}A_{11} - A_{33}A_{21}A_{12} \\ &= A_{11}(A_{22}A_{33} - A_{23}A_{32}) + A_{12}(A_{23}A_{31} - A_{21}A_{33}) \\ &\quad + A_{13}(A_{21}A_{32} - A_{31}A_{22}) \\ &= \varepsilon_{ijk}A_{1i}A_{2j}A_{3k} \\ &= A_{11}(A_{22}A_{33} - A_{23}A_{32}) + A_{21}(A_{32}A_{13} - A_{12}A_{33}) \\ &\quad + A_{31}(A_{12}A_{23} - A_{22}A_{13}) \\ &= \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}.\end{aligned}$$

First, note that

$$\varepsilon_{123}\det[\mathbf{A}] = \det[\mathbf{A}] = \varepsilon_{123}\varepsilon_{ijk}A_{1i}A_{2j}A_{3k}.$$

If we swap two rows, or two columns, determinant of the resulting matrix will be the negative of det[**A**]. For instance

$$\varepsilon_{ijk}A_{1i}A_{2k}A_{3j} = -\det[\mathbf{A}] = \varepsilon_{132}\det[\mathbf{A}] = \varepsilon_{132}\varepsilon_{ijk}A_{1i}A_{3j}A_{2k}.$$

By inspection,

$$\varepsilon_{lmn}\det[\mathbf{A}] = \varepsilon_{lmn}\varepsilon_{ijk}A_{1i}A_{2j}A_{3k} = \varepsilon_{ijk}A_{li}A_{mj}A_{nk}.$$

Furthermore, since $\varepsilon_{lmn}\varepsilon_{lmn} = 6$,

$$\begin{aligned}\varepsilon_{lmn}\varepsilon_{lmn}\det[\mathbf{A}] = 6\det[\mathbf{A}] &= \varepsilon_{lmn}\varepsilon_{ijk}A_{il}A_{jm}A_{kn},\\ \det[\mathbf{A}] &= \frac{1}{6}\varepsilon_{ijk}\varepsilon_{lmn}A_{il}A_{jm}A_{kn}.\end{aligned} \quad \text{(S.228)}$$

Problem 1.3

The following result has been shown in Problem 1.1:

Given that two matrices of entries

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad [\mathbf{B}] = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix},$$

Then, be expanding the summation: $[\mathbf{C}] = [\mathbf{A}] \cdot [\mathbf{B}], \quad C_{ij} = A_{ik}B_{kj}$.

$[\mathbf{D}] = [\mathbf{A}]^T [\mathbf{B}], \quad D_{ij} = A^T_{ik}B_{kj} = A_{ki}B_{kj}$.

Problem 1.4

Note that the components of $\mathbf{e}'_i$ in frame $\mathcal{F}$ are A_{ij}, i.e., $\mathbf{e}'_i = A_{ij}\mathbf{e}_j$. Thus

$$\left(\mathbf{e}'_1 \times \mathbf{e}'_2\right).\mathbf{e}'_3 = +1 = \varepsilon_{ijk}A_{1i}A_{2j}A_{3k} = \det[\mathbf{A}].$$

Problem 1.5

By expanding, it can be seen that

$$\varepsilon_{ijk}u_iv_jw_k = \det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

Consider a two-tensor W_{ij} and vector defined as $u_i = \varepsilon_{ijk} W_{jk}$. By expanding again, it can be demostrated that, if $\mathbf{W}$ is symmetric, $W_{ij} = W_{ji}$, $\mathbf{u}$ is zero, and if $\mathbf{W}$ is anti-symmetric, $W_{ij} = -W_{ji}$,

$$[\mathbf{W}] = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix}, \qquad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \varepsilon_{1jk} W_{jk} \\ \varepsilon_{2jk} W_{jk} \\ \varepsilon_{3jk} W_{jk} \end{bmatrix} = \begin{bmatrix} 2W_{23} \\ 2W_{31} \\ 2W_{12} \end{bmatrix}.$$

the components of $\mathbf{u}$ are twice those of $\mathbf{W}$. This vector is said to be the axial vector of $\mathbf{W}$. If $\mathbf{W}$ represent the vorticity tensor,

$$[\mathbf{W}] = \frac{1}{2}\left(\nabla \mathbf{u}^T - \nabla \mathbf{u}\right) = \frac{1}{2} \begin{bmatrix} 0 & \dfrac{\partial u_1}{\partial x_2} - \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_1}{\partial x_3} - \dfrac{\partial u_3}{\partial x_1} \\ -\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} & 0 & \dfrac{\partial u_2}{\partial x_3} - \dfrac{\partial u_3}{\partial x_2} \\ -\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1} & -\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2} & 0 \end{bmatrix}$$

Then the axial vector of this vorticity tensor is

$$[\mathbf{w}] = \begin{bmatrix} \dfrac{\partial u_2}{\partial x_3} - \dfrac{\partial u_3}{\partial x_2} \\ \dfrac{\partial u_3}{\partial x_1} - \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_1}{\partial x_2} - \dfrac{\partial u_2}{\partial x_1} \end{bmatrix} = -\varepsilon_{ijk} \nabla_j u_k = [-\nabla \times \mathbf{u}] .$$

Problem 1.6

If $\mathbf{D}$, $\mathbf{S}$, and $\mathbf{W}$ are two-tensors, $\mathbf{D}$ symmetric and $\mathbf{W}$ anti-symmetric,

$$\begin{aligned} \mathbf{D} : \mathbf{S} &= D_{ij} S_{ji} = D_{ji} S_{ji} = D_{ji} S_{ij}^T = \mathbf{D} : \mathbf{S}^T \ \text{(symmetric } \mathbf{D}) \\ &= \tfrac{1}{2} D_{ij} S_{ji} + \tfrac{1}{2} D_{ji} S_{ij}^T = \tfrac{1}{2} D_{ij} \left(S_{ji} + S_{ji}^T\right) = \mathbf{D} : \tfrac{1}{2}\left(\mathbf{S} + \mathbf{S}^T\right). \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathbf{W} : \mathbf{S} &= W_{ij} S_{ji} = -W_{ji} S_{ji} = -W_{ji} S_{ij}^T = -\mathbf{W} : \mathbf{S}^T \ \text{(anti-symmetric } \mathbf{W}) \\ &= \tfrac{1}{2}\left(W_{ij} S_{ji} - W_{ji} S_{ij}^T\right) = \tfrac{1}{2} W_{ij} \left(S_{ji} + S_{ji}^T\right) = \mathbf{W} : \tfrac{1}{2}\left(\mathbf{S} - \mathbf{S}^T\right), \\ \mathbf{D} : \mathbf{W} &= \mathbf{W} : \tfrac{1}{2}\left(\mathbf{D} - \mathbf{D}^T\right) = \mathbf{D} : \tfrac{1}{2}\left(\mathbf{W} + \mathbf{W}^T\right) = 0. \end{aligned}$$

In addition, if

$$\mathbf{T} : \mathbf{S} = 0 \;\; \forall \mathbf{S} \text{ then } \mathbf{T} = 0. \tag{S.229}$$

This is shown by choosing **S** to be unity at any particular entry *ij* and zero elsewhere. The corresponding *ij* component of **T** has to be zero. This implies $\mathbf{T} = 0$.

If

$$\mathbf{T} : \mathbf{S} = 0 \;\; \forall \text{ symmetric } \mathbf{S} \text{ then } \mathbf{S} : \frac{1}{2}\left(\mathbf{T} + \mathbf{T}^T\right) = 0.$$

This leads to $\mathbf{T} + \mathbf{T}^T = 0$, or **T** is anti-symmetric. In a similar manner,

$$\text{if } \mathbf{T} : \mathbf{S} = 0 \;\; \forall \text{ anti-symmetric } \mathbf{S} \text{ then } \mathbf{T} \text{ is symmetric.}$$

Problem 1.7

If **Q** is orthogonal,

$$\begin{aligned}
\mathbf{Q}^{-1} &= \mathbf{Q}^T \\
&\Rightarrow \mathbf{H} + \mathbf{H}^T + \mathbf{H}\mathbf{H}^T = \mathbf{Q} + \mathbf{Q}^T - 2\mathbf{I} + \mathbf{Q}\mathbf{Q}^T - \mathbf{Q} - \mathbf{Q}^T + \mathbf{I} = 0 \\
\mathbf{H}\mathbf{H}^T &= 2\mathbf{I} - \mathbf{Q} - \mathbf{Q}^T = \mathbf{H}^T\mathbf{H}
\end{aligned}$$

Conversely, if

$$\mathbf{H} + \mathbf{H}^T + \mathbf{H}\mathbf{H}^T = 0, \quad \mathbf{H}\mathbf{H}^T = \mathbf{H}^T\mathbf{H},$$

then

$$\begin{aligned}
\mathbf{Q}\mathbf{Q}^T - \mathbf{Q} - \mathbf{Q}^T + \mathbf{I} = \mathbf{Q}^T\mathbf{Q} - \mathbf{Q}^T - \mathbf{Q} + \mathbf{I} \Rightarrow \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q}. \\
\mathbf{Q} + \mathbf{Q}^T - 2\mathbf{I} + \mathbf{Q}\mathbf{Q}^T - \mathbf{Q} - \mathbf{Q}^T + \mathbf{I} = 0 \Rightarrow \mathbf{Q}\mathbf{Q}^T = \mathbf{I} = \mathbf{Q}^T\mathbf{Q},
\end{aligned}$$

and **Q** is orthogonal.

Problem 1.8

To show that *I*, *II*, *III* are invariants, we note that

$$\begin{aligned}
I' &= S'_{ii} = A_{ij}A_{ik}S_{jk} = \delta_{jk}S_{jk} = S_{jj} = I. \\
II' &= S'_{ij}S'_{ji} = A_{ik}A_{jm}S_{km}A_{jn}A_{ir}S_{nr} = \delta_{kr}\delta_{mn}S_{km}S_{nr} \\
&= S_{km}S_{mk} = II.
\end{aligned}$$

$$
\begin{aligned}
III' &= S'_{ij}S'_{jk}S'_{ki} = A_{i\alpha}A_{j\beta}S_{\alpha\beta}A_{j\gamma}A_{k\delta}S_{\gamma\delta}A_{k\varepsilon}A_{i\varphi}S_{\varepsilon\varphi} \\
&= \delta_{\alpha\varphi}\delta_{\beta\gamma}\delta_{\delta\varepsilon}S_{\alpha\beta}S_{\gamma\delta}S_{\varepsilon\varphi} = S_{\alpha\beta}S_{\beta\delta}S_{\delta\alpha} = III.
\end{aligned}
$$

Next,

$$
\begin{aligned}
\det[\mathbf{S} - \omega\mathbf{I}] &= \det\begin{bmatrix} S_{11}-\omega & S_{12} & S_{13} \\ S_{21} & S_{22}-\omega & S_{23} \\ S_{31} & S_{32} & S_{33}-\omega \end{bmatrix} \\
&= (S_{11}-\omega)\left(S_{22}S_{33} - \omega(S_{22}+S_{33}) + \omega^2\right) + S_{12}S_{23}S_{31} \\
&\quad + S_{13}S_{21}S_{32} - S_{31}S_{22}S_{13} + \omega(S_{31}S_{13} + S_{32}S_{23} + S_{21}S_{12}) \\
&\quad - S_{32}S_{23}S_{11} - S_{33}S_{21}S_{12} \\
&= -\omega^3 + \omega^2(S_{11} + S_{22} + S_{33}) - \omega(S_{11}S_{22} + S_{11}S_{33} + S_{22}S_{33} \\
&\quad - S_{31}S_{13} - S_{32}S_{23} - S_{21}S_{12}) + S_{11}S_{22}S_{33} + S_{12}S_{23}S_{31} \\
&\quad + S_{13}S_{21}S_{32} - S_{31}S_{22}S_{13} - S_{32}S_{23}S_{11} - S_{33}S_{21}S_{12} \\
&= -\omega^3 + I_1\omega^2 - I_2\omega + I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= I = tr\mathbf{S}, \\
I_2 &= \tfrac{1}{2}\left(I^2 - II\right) = S_{11}S_{22} + S_{11}S_{33} + S_{22}S_{33} - S_{12}S_{21} - S_{13}S_{31} - S_{23}S_{32} \\
I_3 &= \tfrac{1}{6}\left(I^3 - 3I.II + 2III\right) = \det\mathbf{S}.
\end{aligned}
$$

If $\mathbf{e}$ is an eigenvector of $\mathbf{S}$, with eigenvalue ω then

$$
\mathbf{Se} = \omega\mathbf{e}.
$$

The condition for this to have non-trivial solutions is that

$$
\det[\mathbf{S} - \omega\mathbf{I}] = 0 = -\omega^3 + I_1\omega^2 - I_2\omega + I_3. \tag{S.230}
$$

This is said to be the characteristic equation for $\mathbf{S}$. According to the Cayley-Hamilton theorem, $\mathbf{S}$ satisfies its own characteristic equation.

Problem 1.9

Consider the 2×2 matrix

$$
[\mathbf{C}] = \begin{bmatrix} 1+\gamma^2 & \gamma \\ \gamma & 1 \end{bmatrix}.
$$

Denote $\mathbf{U} = \mathbf{C}^{1/2}$, $\mathbf{U}$ satisfies its own characteristic equation (in 2-D):

$$\mathbf{U}^2 - I_1(\mathbf{U})\,\mathbf{U} + \det[\mathbf{U}]\,\mathbf{I} = 0\mathbf{U} = \frac{1}{I_1(\mathbf{U})}\left(\mathbf{C} + \det(\mathbf{U})\,\mathbf{I}\right).$$

Expressed in eigenspace,

$$\begin{aligned}
&\mathbf{U} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}, \\
&I_1(\mathbf{U}) = \lambda_1 + \lambda_2, \quad I_1(\mathbf{C}) = \lambda_1^2 + \lambda_2^2, \quad I_2(\mathbf{U}) = \lambda_1\lambda_2, \quad I_2(\mathbf{C}) = \lambda_1^2\lambda_2^2, \\
&I_2(\mathbf{U}) = \sqrt{I_2(\mathbf{C})}, \quad I_1(\mathbf{U}) = \sqrt{I_1(\mathbf{C}) + 2\sqrt{I_2(\mathbf{C})}} \\
&I_1(\mathbf{C}) = 2 + \gamma^2, \quad I_2(\mathbf{C}) = \det[\mathbf{C}] = 1 + \gamma^2 - \gamma^2 = 1.
\end{aligned}$$

Thus,

$$\mathbf{U} = \mathbf{C}^{1/2} = \frac{1}{\sqrt{4+\gamma^2}}\begin{bmatrix} 2+\gamma^2 & \gamma \\ \gamma & 2 \end{bmatrix}. \tag{S.231}$$

For the 3×3 matric case:

$$\mathbf{U}^3 - I_1(\mathbf{U})\,\mathbf{U}^2 + I_2(\mathbf{U})\,\mathbf{U} - I_3(\mathbf{U})\,\mathbf{I} = 0 \tag{S.232}$$

leading to

$$\begin{aligned}
&\mathbf{U}^4 + I_2\mathbf{U}^2 - I_3\mathbf{U} = I_1\mathbf{U}^3 = I_1^2\mathbf{U}^2 - I_1I_2\mathbf{U} + I_1I_3\mathbf{I} \\
&\mathbf{U} = \tfrac{1}{I_3 - I_1I_2}\left(\mathbf{C}^2 + \left(I_2 - I_1^2\right)\mathbf{C} - I_1I_3\mathbf{I}\right),
\end{aligned} \tag{S.233}$$

where I_1, I_2, I_3 are the invariants of $\mathbf{U}$, and their dependence on $\mathbf{U}$ has been suppressed for brevity. In terms of the eigenvalues:

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad I_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \quad I_3 = \lambda_1\lambda_2\lambda_3.$$

These need to be expressed in terms of the invariants of $\mathbf{C}$. First

$$I_3(\mathbf{U}) = \sqrt{I_3(\mathbf{C})}. \tag{S.234}$$

Next, in terms of the eigenvalues of $\mathbf{U}$,

$$I_1^2(\mathbf{U}) = (\lambda_1 + \lambda_2 + \lambda_3)^2 = I_1(\mathbf{C}) + 2I_2(\mathbf{U}). \tag{S.235}$$

Next, from (S.232),

$$\begin{aligned}\lambda_1^4+\lambda_2^4+\lambda_3^4 &= (I_3-I_1I_2)(\lambda_1+\lambda_2+\lambda_3)+\left(I_1^2-I_2\right)\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)\\ &\quad +3I_1I_3\\ &= (I_3-I_1I_2)\,I_1+\left(I_1^2-I_2\right)\left(I_1^2-2I_2\right)+3I_1I_3\\ &= 4I_1I_3-4I_1^2I_2+I_1^4+2I_2^2. \end{aligned} \tag{S.236}$$

Left hand of (S.236) can be expressed as

$$I_1^2(\mathbf{C})-2I_2(\mathbf{C})=\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)^2-2\left(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2\right)=\lambda_1^4+\lambda_2^4+\lambda_3^4$$

Thus,

$$\frac{I_1^2(\mathbf{C})-2I_2(\mathbf{C})}{I_1^2(\mathbf{C})}=\frac{1}{I_1^2(\mathbf{C})}\left[4I_1I_3-4I_1^2I_2+I_1^4+2I_2^2\right]$$

With some further algebra,

$$\begin{aligned}2\left[1-\frac{I_1^2(\mathbf{C})-2I_2(\mathbf{C})}{I_1^2(\mathbf{C})}\right]+8\sqrt{\frac{I_3(\mathbf{C})}{I_1^3(\mathbf{C})}}\frac{I_1(\mathbf{U})}{\sqrt{I_1(\mathbf{C})}} &= 2\\ &\quad+\frac{1}{I_1^2(\mathbf{C})}\left[8I_2(\mathbf{U})\,I_1^2(\mathbf{U})-4I_2^2(\mathbf{U})-2I_1^4(\mathbf{U})\right]\\ &=\frac{I_1^4(\mathbf{U})}{I_1^2(\mathbf{C})}-2\frac{I_1^2(\mathbf{U})}{I_1(\mathbf{C})}+1\\ &=\left[\frac{I_1^2(\mathbf{U})}{I_1(\mathbf{C})}-1\right]^2.\end{aligned}$$

This is a quartic equation $\left(x^2-1\right)^2=a+bx$ in x, with

$$\begin{aligned} x&=\frac{I_1(\mathbf{U})}{\sqrt{I_1(\mathbf{C})}},\quad a=2\left[1-\frac{I_1^2(\mathbf{C})-2I_2(\mathbf{C})}{I_1^2(\mathbf{C})}\right]=2\left[1-\frac{tr\mathbf{C}^2}{(tr\mathbf{C})^2}\right],\\ b&=8\sqrt{\frac{I_3(\mathbf{C})}{I_1^3(\mathbf{C})}}=8\sqrt{\frac{\det\mathbf{C}}{(tr\mathbf{C})^3}}.\end{aligned}$$

Once x is obtained, $tr(\mathbf{U})$ is determined in term of $tr(\mathbf{C})$, and from (S.235), the second invariant of **U** is determined in terms of **C**.

Problem 1.10

The components of the strain rate tensor, $\mathbf{D} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$, and the vorticity tensor, $\mathbf{W} = \frac{1}{2}\left(\nabla\mathbf{u}^T - \nabla\mathbf{u}\right)$ are

$$[\mathbf{L}] = [\nabla\mathbf{u}]^T = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix},$$

$$[\mathbf{D}] = \frac{1}{2}\begin{bmatrix} 2\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} & \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} & 2\frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} & \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} & 2\frac{\partial u_3}{\partial x_3} \end{bmatrix},$$

$$[\mathbf{W}] = \frac{1}{2}\begin{bmatrix} 0 & \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} & \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ -\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} & 0 & \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \\ -\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} & -\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} & 0 \end{bmatrix}.$$

Problem 1.11

First, we express **S** in cylindrical coordinates as

$$\mathbf{S} = S_{rr}\mathbf{e}_r\mathbf{e}_r + S_{r\theta}\mathbf{e}_r\mathbf{e}_\theta + S_{rz}\mathbf{e}_r\mathbf{e}_z + S_{\theta r}\mathbf{e}_\theta\mathbf{e}_r \\ + S_{\theta\theta}\mathbf{e}_\theta\mathbf{e}_\theta + S_{\theta z}\mathbf{e}_\theta\mathbf{e}_z + S_{zr}\mathbf{e}_z\mathbf{e}_r + S_{z\theta}\mathbf{e}_z\mathbf{e}_\theta + S_{zz}\mathbf{e}_z\mathbf{e}_z,$$

and taking the gradient operation,

$$\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z},$$

keeping in mind that

$$\frac{\partial}{\partial r}\mathbf{e}_r = 0, \qquad \frac{\partial}{\partial r}\mathbf{e}_\theta = 0, \qquad \frac{\partial}{\partial r}\mathbf{e}_z = 0$$

$$\frac{\partial}{\partial\theta}\mathbf{e}_r = \mathbf{e}_\theta, \quad \frac{\partial}{\partial\theta}\mathbf{e}_\theta = -\mathbf{e}_r, \quad \frac{\partial}{\partial\theta}\mathbf{e}_z = 0$$
$$\frac{\partial}{\partial z}\mathbf{e}_r = 0, \quad \frac{\partial}{\partial z}\mathbf{e}_\theta = 0, \quad \frac{\partial}{\partial z}\mathbf{e}_z = 0$$

to result in

$$\begin{aligned}\nabla\cdot\mathbf{S} = \mathbf{e}_r &\left(\frac{\partial S_{rr}}{\partial r} + \frac{S_{rr}-S_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial S_{\theta r}}{\partial\theta} + \frac{\partial S_{zr}}{\partial z}\right)\\ &+ \mathbf{e}_\theta\left(\frac{\partial S_{r\theta}}{\partial r} + \frac{2S_{r\theta}}{r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial\theta} + \frac{\partial S_{z\theta}}{\partial z} + \frac{S_{\theta r}-S_{r\theta}}{r}\right)\\ &+ \mathbf{e}_z\left(\frac{\partial S_{rz}}{\partial r} + \frac{S_{rz}}{r} + \frac{1}{r}\frac{\partial S_{\theta z}}{\partial\theta} + \frac{\partial S_{zz}}{\partial z}\right).\end{aligned} \tag{S.237}$$

Problem 1.12

In cylindrical coordinates,

$$\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z.$$

Thus

$$\begin{aligned}\nabla\mathbf{u} &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)(u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z)\\ &= \frac{\partial u_r}{\partial r}\mathbf{e}_r\mathbf{e}_r + \frac{\partial u_\theta}{\partial r}\mathbf{e}_r\mathbf{e}_\theta + \frac{\partial u_z}{\partial r}\mathbf{e}_r\mathbf{e}_z\\ &\quad + \left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right)\mathbf{e}_\theta\mathbf{e}_r + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right)\mathbf{e}_\theta\mathbf{e}_\theta + \frac{1}{r}\frac{\partial u_z}{\partial\theta}\mathbf{e}_\theta\mathbf{e}_z\\ &\quad + \frac{\partial u_r}{\partial z}\mathbf{e}_z\mathbf{e}_r + \frac{\partial u_\theta}{\partial z}\mathbf{e}_z\mathbf{e}_\theta + \frac{\partial u_z}{\partial z}\mathbf{e}_z\mathbf{e}_z,\end{aligned}$$

and

$$\begin{aligned}\mathbf{u}\cdot\nabla\mathbf{u} = \mathbf{e}_r&\left[u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta u_\theta}{r}\right]\\ &+\mathbf{e}_\theta\left[u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_\theta u_r}{r}\right]\\ &+\mathbf{e}_z\left[u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta} + u_z\frac{\partial u_z}{\partial z}\right].\end{aligned} \tag{S.238}$$

Additional problem: evaluate $\mathbf{u}\cdot\nabla\mathbf{S}$ in cylindrical coordinates. Now with

$$\mathbf{S} = S_{rr}\mathbf{e}_r\mathbf{e}_r + S_{r\theta}\left(\mathbf{e}_r\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_r\right) + S_{rz}\left(\mathbf{e}_r\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_r\right) + S_{\theta\theta}\mathbf{e}_\theta\mathbf{e}_\theta + S_{\theta z}\left(\mathbf{e}_z\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_z\right) + S_{zz}\mathbf{e}_z\mathbf{e}_z, \qquad \text{(S.239)}$$

$$\begin{aligned}
\nabla\mathbf{S} &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{\partial}{r\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\mathbf{S} \\
&= \mathbf{e}_r\Bigg(\mathbf{e}_r\mathbf{e}_r\frac{\partial S_{rr}}{\partial r} + \left(\mathbf{e}_r\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_r\right)\frac{\partial S_{r\theta}}{\partial r} + \left(\mathbf{e}_r\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_r\right)\frac{\partial S_{rz}}{\partial r} \\
&\quad + \mathbf{e}_\theta\mathbf{e}_\theta\frac{\partial S_{\theta\theta}}{\partial r} + \left(\mathbf{e}_z\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_z\right)\frac{\partial S_{\theta z}}{\partial r} + \mathbf{e}_z\mathbf{e}_z\frac{\partial S_{zz}}{\partial r}\Bigg) \\
&\quad + \mathbf{e}_\theta\Bigg(\mathbf{e}_r\mathbf{e}_r\frac{\partial S_{rr}}{r\partial\theta} + \frac{S_{rr}}{r}\left(\mathbf{e}_\theta\mathbf{e}_r + \mathbf{e}_r\mathbf{e}_\theta\right) + \left(\mathbf{e}_r\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_r\right)\frac{\partial S_{r\theta}}{r\partial\theta} \\
&\quad + 2\frac{S_{r\theta}}{r}\left(\mathbf{e}_\theta\mathbf{e}_\theta - \mathbf{e}_r\mathbf{e}_r\right) + \left(\mathbf{e}_r\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_r\right)\frac{\partial S_{rz}}{r\partial\theta} + \frac{S_{rz}}{r}\left(\mathbf{e}_\theta\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_\theta\right) \\
&\quad + \mathbf{e}_\theta\mathbf{e}_\theta\frac{\partial S_{\theta\theta}}{r\partial\theta} - \frac{S_{\theta\theta}}{r}\left(\mathbf{e}_r\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_r\right) \\
&\quad + \left(\mathbf{e}_z\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_z\right)\frac{\partial S_{\theta z}}{r\partial\theta} - \frac{S_{\theta z}}{r}\left(\mathbf{e}_z\mathbf{e}_r + \mathbf{e}_r\mathbf{e}_z\right) + \mathbf{e}_z\mathbf{e}_z\frac{\partial S_{zz}}{r\partial\theta}\Bigg) \\
&\quad + \mathbf{e}_z\Bigg(\mathbf{e}_r\mathbf{e}_r\frac{\partial S_{rr}}{\partial z} + \left(\mathbf{e}_r\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_r\right)\frac{\partial S_{r\theta}}{\partial z} + \left(\mathbf{e}_r\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_r\right)\frac{\partial S_{rz}}{\partial z} \\
&\quad + \mathbf{e}_\theta\mathbf{e}_\theta\frac{\partial S_{\theta\theta}}{\partial z} + \left(\mathbf{e}_z\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_z\right)\frac{\partial S_{\theta z}}{\partial z} + \mathbf{e}_z\mathbf{e}_z\frac{\partial S_{zz}}{\partial z}\Bigg).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{u}\cdot\nabla\mathbf{S} &= u_r\Bigg(\mathbf{e}_r\mathbf{e}_r\frac{\partial S_{rr}}{\partial r} + \left(\mathbf{e}_r\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_r\right)\frac{\partial S_{r\theta}}{\partial r} + \left(\mathbf{e}_r\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_r\right)\frac{\partial S_{rz}}{\partial r} \\
&\quad + \mathbf{e}_\theta\mathbf{e}_\theta\frac{\partial S_{\theta\theta}}{\partial r} + \left(\mathbf{e}_z\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_z\right)\frac{\partial S_{\theta z}}{\partial r} + \mathbf{e}_z\mathbf{e}_z\frac{\partial S_{zz}}{\partial r}\Bigg) \\
&\quad + u_\theta\Bigg(\mathbf{e}_r\mathbf{e}_r\frac{\partial S_{rr}}{r\partial\theta} + \frac{S_{rr}}{r}\left(\mathbf{e}_\theta\mathbf{e}_r + \mathbf{e}_r\mathbf{e}_\theta\right) + \left(\mathbf{e}_r\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_r\right)\frac{\partial S_{r\theta}}{r\partial\theta} \\
&\quad + 2\frac{S_{r\theta}}{r}\left(\mathbf{e}_\theta\mathbf{e}_\theta - \mathbf{e}_r\mathbf{e}_r\right) + \left(\mathbf{e}_r\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_r\right)\frac{\partial S_{rz}}{r\partial\theta} + \frac{S_{rz}}{r}\left(\mathbf{e}_\theta\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_\theta\right) \\
&\quad + \mathbf{e}_\theta\mathbf{e}_\theta\frac{\partial S_{\theta\theta}}{r\partial\theta} - \frac{S_{\theta\theta}}{r}\left(\mathbf{e}_r\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_r\right) \\
&\quad + \left(\mathbf{e}_z\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_z\right)\frac{\partial S_{\theta z}}{r\partial\theta} - \frac{S_{\theta z}}{r}\left(\mathbf{e}_z\mathbf{e}_r + \mathbf{e}_r\mathbf{e}_z\right) + \mathbf{e}_z\mathbf{e}_z\frac{\partial S_{zz}}{r\partial\theta}\Bigg)
\end{aligned}$$

$$+ u_z \left(\mathbf{e}_r\mathbf{e}_r \frac{\partial S_{rr}}{\partial z} + (\mathbf{e}_r\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_r) \frac{\partial S_{r\theta}}{\partial z} + (\mathbf{e}_r\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_r) \frac{\partial S_{rz}}{\partial z} + \mathbf{e}_\theta\mathbf{e}_\theta \frac{\partial S_{\theta\theta}}{\partial z} + (\mathbf{e}_z\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_z) \frac{\partial S_{\theta z}}{\partial z} + \mathbf{e}_z\mathbf{e}_z \frac{\partial S_{zz}}{\partial z} \right).$$

$$\begin{aligned}
\mathbf{u}\cdot\nabla\mathbf{S} = {} & \mathbf{e}_r\mathbf{e}_r \left(u_r \frac{\partial S_{rr}}{\partial r} + u_\theta \frac{\partial S_{rr}}{r\partial\theta} + u_z \frac{\partial S_{rr}}{\partial z} - 2u_\theta \frac{S_{r\theta}}{r} \right) \\
& + (\mathbf{e}_r\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_r) \left(u_r \frac{\partial S_{r\theta}}{\partial r} + u_\theta \frac{\partial S_{r\theta}}{r\partial\theta} + u_z \frac{\partial S_{r\theta}}{\partial z} + u_\theta \frac{S_{rr}}{r} - u_\theta \frac{S_{\theta\theta}}{r} \right) \\
& + (\mathbf{e}_r\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_r) \left(u_r \frac{\partial S_{rz}}{\partial r} + u_\theta \frac{\partial S_{rz}}{r\partial\theta} + u_z \frac{\partial S_{rz}}{\partial z} - u_\theta \frac{S_{\theta z}}{r} \right) \\
& + \mathbf{e}_\theta\mathbf{e}_\theta \left(u_r \frac{\partial S_{\theta\theta}}{\partial r} + u_\theta \frac{\partial S_{\theta\theta}}{r\partial\theta} + u_z \frac{\partial S_{\theta\theta}}{\partial z} + 2u_\theta \frac{S_{r\theta}}{r} \right) \\
& + (\mathbf{e}_z\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_z) \left(u_r \frac{\partial S_{\theta z}}{\partial r} + u_\theta \frac{\partial S_{\theta z}}{r\partial\theta} + u_z \frac{\partial S_{\theta z}}{\partial z} + u_\theta \frac{S_{rz}}{r} \right) \\
& + \mathbf{e}_z\mathbf{e}_z \left(u_r \frac{\partial S_{zz}}{\partial r} + u_\theta \frac{\partial S_{zz}}{r\partial\theta} + u_z \frac{\partial S_{zz}}{\partial z} \right).
\end{aligned}$$

Problem 1.13

The stress tensor in a material satisfies $\nabla \cdot \mathbf{S} = \mathbf{0}$. Thus

$$\nabla_j \left(x_k S_{ij} \right) = S_{ik}.$$

and

$$\begin{aligned}
\langle S_{ik} \rangle &= \tfrac{1}{V} \int_V S_{ik} dV = \tfrac{1}{V} \int_V \nabla_j \left(x_k S_{ij} \right) dV = \tfrac{1}{V} \int_S x_k S_{ij} n_j dA \\
&= \tfrac{1}{2V} \int_S \left(x_k t_i + x_i t_k \right) dA
\end{aligned}$$

after a symmetrisation.

Problem 1.14

We can regard

$$\langle \mathbf{nn} \rangle = \frac{1}{S} \int_S \mathbf{nn} dS$$

and

$$\langle \mathbf{nnnn} \rangle = \frac{1}{S} \int_S \mathbf{nnnn} dS$$

as averages of various moments of the normal unit vector, uniformly distributed in space.

We further note that < **nn** > and < **nnnn** > are isotropic tensors (unchanged with coordinate rotation) and thus their general forms are given by

$$\langle \mathbf{nn} \rangle = \alpha \mathbf{I} = \frac{1}{S} \int_S \mathbf{nn} dS,$$
$$\langle n_i n_j n_k n_l \rangle = \beta \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right) = \frac{1}{S} \int_S n_i n_j n_k n_l dS.$$

The scalars α and β are found by contracting indices and performing some simple integrations:

$$3\alpha = \frac{1}{S} \int_S dS = 1 \ \alpha = \frac{1}{3}.$$

$$15\beta = \frac{1}{S} \int_S dS = 1 \ \beta = \frac{1}{15}.$$

Another approach (here a is the radius of the sphere): for $\langle \mathbf{nn} \rangle$:

$$\begin{aligned} \frac{1}{S} \int n_i n_j dS &= \frac{1}{aS} \int_S x_i n_j dS \\ &= \frac{1}{aS} \int_V \frac{\partial}{\partial x_j} (x_i)\, dV \\ &= \frac{1}{aS} \delta_{ij} \int_V dV = \frac{V}{aS} \delta_{ij} \\ &= \frac{1}{3} \delta_{ij} \end{aligned}$$

And for $\langle n_i n_j n_k n_l \rangle$:

$$\begin{aligned} \frac{1}{S} \int n_i n_j n_k n_l dS &= \frac{1}{a^3 S} \int_S x_i x_j x_k n_l dS \\ &= \frac{1}{a^3 S} \int_V \frac{\partial}{\partial x_l} \left(x_i x_j x_k \right) dV \\ &= \frac{1}{a^3 S} \int_V \left(\delta_{il} x_j x_k + \delta_{jl} x_i x_k + \delta_{kl} x_i x_j \right) dV \end{aligned}$$

Also, from

$$\frac{\partial}{\partial x_k}\left(x_i x_j x_k\right) = \delta_{ik} x_j x_k + \delta_{jk} x_i x_k + \delta_{kk} x_i x_j = 5 x_i x_j$$

Thus

$$\begin{aligned}\frac{1}{a^3 S}\int_V \delta_{il} x_j x_k dV &= \frac{1}{5a^3 S}\int_V \delta_{il}\frac{\partial}{\partial x_m}\left(x_j x_k x_m\right) dV \\ &= \frac{1}{5a^3 S}\delta_{il}\int_S x_j x_k x_m n_m dS \\ &= \frac{a^3}{5a^3 S}\delta_{il}\int_S n_j n_k dS \\ &= \frac{1}{15}\delta_{il}\delta_{jk}\end{aligned}$$

Similarly

$$\frac{1}{a^3 S}\int_V \delta_{jl} x_i x_k dV = \frac{1}{15}\delta_{jl}\delta_{ik},\ \ \frac{1}{a^3 S}\int_V \delta_{kl} x_i x_j dV = \frac{1}{15}\delta_{kl}\delta_{ij}$$

and

$$\frac{1}{S}\int n_i n_j n_k n_l dS = \frac{1}{15}\left(\delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik} + \delta_{kl}\delta_{ij}\right).$$

Problems in Chap. 3

Problem 3.1

From $\mathbf{F}\mathbf{F}^{-1} = \mathbf{I}$,

$$\mathbf{F}\frac{d}{dt}\mathbf{F}^{-1} + \left(\frac{d}{dt}\mathbf{F}\right)\mathbf{F}^{-1} = \mathbf{0}\ \mathbf{F}\frac{d}{dt}\mathbf{F}^{-1} = -\left(\frac{d}{dt}\mathbf{F}\right)\mathbf{F}^{-1} = -\mathbf{L}\mathbf{F}\mathbf{F}^{-1}$$

yielding

$$\frac{d}{dt}\mathbf{F}^{-1} = -\mathbf{F}^{-1}\mathbf{L}, \quad \mathbf{F}(0) = \mathbf{I} = \mathbf{F}^{-1}(0).$$

Problem 3.2

For a simple shear flow, where the velocity field takes the form

$$u = \dot{\gamma} y, \quad v = 0, \quad w = 0,$$

the velocity gradient is

$$[\mathbf{L}] = [\nabla \mathbf{u}]^T = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The deformation gradient obeys

$$\frac{d\mathbf{F}}{dt} = \mathbf{LF}, \quad \mathbf{F}(0) = \mathbf{I},$$

which has the solution for a constant $\mathbf{L}$,

$$\mathbf{F}(t) = \exp(t\mathbf{L}) = \mathbf{I} + \sum_{n=1} \frac{1}{n!} (t\mathbf{L})^n = \mathbf{I} + t\mathbf{L} \text{ for } \mathbf{L}^n = 0, n \geq 2$$

The particle X is carried to $\mathbf{x}$ at time t according to

$$\mathbf{x}(t) = \mathbf{F}(t)\mathbf{X} \text{ or } \mathbf{X} = \mathbf{F}^{-1}(t)\mathbf{x}$$

At time τ if the particle is at point $\boldsymbol{\zeta}$, then

$$\begin{aligned} \boldsymbol{\zeta}(\tau) &= \mathbf{F}(\tau)\mathbf{X} = \mathbf{F}(\tau)\mathbf{F}^{-1}(t)\mathbf{x} = \exp((\tau - t)\mathbf{L})\mathbf{x} \\ &= \mathbf{x} + (\tau - t)\mathbf{Lx}. \end{aligned} \tag{S.240}$$

This shows the linearly stretching nature of a fluid filament.

Problem 3.3

For an elongational flow, the velocity field is

$$u = ax, \quad v = by, \quad w = cz, \quad a + b + c = 0 \text{ (for incompressibility)}. \tag{S.241}$$

the velocity gradient is

$$[\mathbf{L}] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \quad \left[e^{t\mathbf{L}}\right] = \begin{bmatrix} e^{at} & 0 & 0 \\ 0 & e^{bt} & 0 \\ 0 & 0 & e^{ct} \end{bmatrix}.$$

The path lines are given by

$$[\boldsymbol{\xi}(\tau)] = \begin{bmatrix} \xi \\ \psi \\ \zeta \end{bmatrix} = \begin{bmatrix} e^{a(\tau-t)} & 0 & 0 \\ 0 & e^{b(\tau-t)} & 0 \\ 0 & 0 & e^{c(\tau-t)} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{S.242}$$

and thus exponential flow can stretch the fluid element exponentially fast.

Problem 3.4

Consider a super-imposed oscillatory shear flow:

$$u = \dot{\gamma}_m y, \quad v = 0, \quad w = \omega\gamma_a y \cos\omega t. \tag{S.243}$$

The path lines are defined by

$$\frac{d}{dt}(x, y, z) = (\dot{\gamma}_m y, 0, \omega\gamma_a y \cos\omega t), \quad x(0) = X, \; y(0) = Y, \; z(0) = Z$$

This has the solution

$$x(t) = X + t\dot{\gamma}_m Y, \; y(t) = Y, \; z(t) = Z + \gamma_a Y \sin\omega t$$

The path line, $(\xi(\tau), \psi(\tau), \zeta(\tau))$,

$$\begin{aligned} (\xi(\tau), \psi(\tau), \zeta(\tau)) &= (X + \tau\dot{\gamma}_m Y, Y, Z + \gamma_a Y \sin\omega\tau) \\ &= (x(t) + (\tau - t)\dot{\gamma}_m y, y, z(t) + \gamma_a y (\cos\omega\tau - \cos\omega t)). \end{aligned}$$

Problem 3.5

We want to calculate Rivlin–Ericksen tensor for an elongation flow (S.241) where the velocity gradient is

$$[\mathbf{L}] = \mathrm{diag}(a, b, c).$$

The first Rivlin–Ericksen tensor is

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T = 2\mathbf{L}.$$

The subsequent two Rivlin–Ericksen tensors are, because of **L** being diagonal,

$$\mathbf{A}_2 = \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1 = \mathbf{A}_1^2, \quad \mathbf{A}_3 = \mathbf{A}_2\mathbf{L} + \mathbf{L}^T\mathbf{A}_2 = \mathbf{A}_1^3.$$

By induction,

$$\mathbf{A}_n = \mathbf{A}_1^n = \mathrm{diag}\left((2a)^n, (2b)^n, (2c)^n\right).$$

Problem 3.6

For the velocity field of (S.243) takes the form

$$u = \dot{\gamma}_m y, \quad v = 0, \quad w = \omega\gamma_a y \cos\omega t.$$

The velocity gradient tensor is

$$[\mathbf{L}] = \begin{bmatrix} 0 & \dot{\gamma}_m & 0 \\ 0 & 0 & 0 \\ 0 & \omega\gamma_a \cos\omega t & 0 \end{bmatrix}.$$

The first Rivlin–Ericksen tensor is

$$[\mathbf{A}_1] = \left[\mathbf{L} + \mathbf{L}^T\right] = \begin{bmatrix} 0 & \dot{\gamma}_m & 0 \\ \dot{\gamma}_m & 0 & \omega\gamma_a \cos\omega t \\ 0 & \omega\gamma_a \cos\omega t & 0 \end{bmatrix}.$$

The second Rivlin–Ericksen tensor is

$$\begin{aligned}
[\mathbf{A}_2] = \left[\tfrac{d}{dt}\mathbf{A}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1\right] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega^2\gamma_a \sin\omega t \\ 0 & -\omega^2\gamma_a \sin\omega t & 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & \dot{\gamma}_m & 0 \\ \dot{\gamma}_m & 0 & \omega\gamma_a \cos\omega t \\ 0 & \omega\gamma_a \cos\omega t & 0 \end{bmatrix} \begin{bmatrix} 0 & \dot{\gamma}_m & 0 \\ 0 & 0 & 0 \\ 0 & \omega\gamma_a \cos\omega t & 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & 0 \\ \dot{\gamma}_m & 0 & \omega\gamma_a \cos\omega t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \dot{\gamma}_m & 0 \\ \dot{\gamma}_m & 0 & \omega\gamma_a \cos\omega t \\ 0 & \omega\gamma_a \cos\omega t & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\left(\dot{\gamma}_m^2 + \omega^2\gamma_a^2 \cos^2\omega t\right) & -\omega^2\gamma_a \sin\omega t \\ 0 & -\omega^2\gamma_a \sin\omega t & 0 \end{bmatrix}.
\end{aligned}$$

The third Rivlin–Ericksen tensor is

$$[\mathbf{A}_3] = \left[\tfrac{d}{dt}\mathbf{A}_2 + \mathbf{A}_2\mathbf{L} + \mathbf{L}^T\mathbf{A}_2\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\omega^3\gamma_a^2 \sin 2\omega t & -\omega^3\gamma_a \cos\omega t \\ 0 & -\omega^3\gamma_a \cos\omega t & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\left(\dot\gamma_m^2 + \omega^2\gamma_a^2 \cos^2\omega t\right) & -\omega^2\gamma_a \sin\omega t \\ 0 & -\omega^2\gamma_a \sin\omega t & 0 \end{bmatrix} \begin{bmatrix} 0 & \dot\gamma_m & 0 \\ 0 & 0 & 0 \\ 0 & \omega\gamma_a \cos\omega t & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ \dot\gamma_m & 0 & \omega\gamma_a \cos\omega t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\left(\dot\gamma_m^2 + \omega^2\gamma_a^2 \cos^2\omega t\right) & -\omega^2\gamma_a \sin\omega t \\ 0 & -\omega^2\gamma_a \sin\omega t & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3\omega^3\gamma_a^2 \sin 2\omega t & -\omega^3\gamma_a \cos\omega t \\ 0 & -\omega^3\gamma_a \cos\omega t & 0 \end{bmatrix}.$$

The 4th Rivlin–Ericksen tensor:

$$[\mathbf{A}_4] = \left[\tfrac{d}{dt}\mathbf{A}_3 + \mathbf{A}_3\mathbf{L} + \mathbf{L}^T\mathbf{A}_3\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6\omega^4\gamma_a^2 \cos 2\omega t & \omega^4\gamma_a \sin\omega t \\ 0 & \omega^4\gamma_a \sin\omega t & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3\omega^3\gamma_a^2 \sin 2\omega t & -\omega^3\gamma_a \cos\omega t \\ 0 & -\omega^3\gamma_a \cos\omega t & 0 \end{bmatrix} \begin{bmatrix} 0 & \dot\gamma_m & 0 \\ 0 & 0 & 0 \\ 0 & \omega\gamma_a \cos\omega t & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ \dot\gamma_m & 0 & \omega\gamma_a \cos\omega t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3\omega^3\gamma_a^2 \sin 2\omega t & -\omega^3\gamma_a \cos\omega t \\ 0 & -\omega^3\gamma_a \cos\omega t & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\omega^4\gamma_a^2\left(3 - 7\cos^2\omega t\right) & \omega^4\gamma_a \sin\omega t \\ 0 & \omega^4\gamma_a \sin\omega t & 0 \end{bmatrix}.$$

The 5th Rivlin–Ericksen tensor:

$$[\mathbf{A}_5] = \left[\tfrac{d}{dt}\mathbf{A}_4 + \mathbf{A}_4\mathbf{L} + \mathbf{L}^T\mathbf{A}_4\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 14\omega^5\gamma_a^2 \sin 2\omega t & \omega^5\gamma_a \cos\omega t \\ 0 & \omega^5\gamma_a \cos\omega t & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\omega^4\gamma_a^2\left(3 - 7\cos^2\omega t\right) & \omega^4\gamma_a \sin\omega t \\ 0 & \omega^4\gamma_a \sin\omega t & 0 \end{bmatrix} \begin{bmatrix} 0 & \dot\gamma_m & 0 \\ 0 & 0 & 0 \\ 0 & \omega\gamma_a \cos\omega t & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ \dot\gamma_m & 0 & \omega\gamma_a \cos\omega t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\omega^4\gamma_a^2\left(3 - 7\cos^2\omega t\right) & \omega^4\gamma_a \sin\omega t \\ 0 & \omega^4\gamma_a \sin\omega t & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 15\omega^5\gamma_a^2 \sin 2\omega t & \omega^5\gamma_a \cos\omega t \\ 0 & \omega^5\gamma_a \cos\omega t & 0 \end{bmatrix}.$$

The 6th Rivlin–Ericksen tensor is

$$[\mathbf{A}_6] = \left[\frac{d}{dt}\mathbf{A}_5 + \mathbf{A}_5\mathbf{L} + \mathbf{L}^T\mathbf{A}_5\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 30\omega^6\gamma_a^2\cos 2\omega t & -\omega^6\gamma_a\sin\omega t \\ 0 & -\omega^6\gamma_a\sin\omega t & 0 \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 \\ 0 & 15\omega^5\gamma_a^2\sin 2\omega t & \omega^5\gamma_a\cos\omega t \\ 0 & \omega^5\gamma_a\cos\omega t & 0 \end{bmatrix}\begin{bmatrix} 0 & \dot{\gamma}_m & 0 \\ 0 & 0 & 0 \\ 0 & \omega\gamma_a\cos\omega t & 0 \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 \\ \dot{\gamma}_m & 0 & \omega\gamma_a\cos\omega t \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 15\omega^5\gamma_a^2\sin 2\omega t & \omega^5\gamma_a\cos\omega t \\ 0 & \omega^5\gamma_a\cos\omega t & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 30\omega^6\gamma_a^2\cos 2\omega t & -\omega^6\gamma_a\sin\omega t \\ 0 & -\omega^6\gamma_a\sin\omega t & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\omega^6\gamma_a^2\cos^2\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\omega^6\gamma_a^2\left(15 - 31\cos^2\omega t\right) & \omega^6\gamma_a\sin\omega t \\ 0 & \omega^6\gamma_a\sin\omega t & 0 \end{bmatrix}.$$

By induction, the general form for Rivlin–Ericksen tensors is ($n \geq 1$)

$$\left[\mathbf{A}_{2n+1}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (-1)^n\,\alpha\omega^{2n+1}\gamma_a^2\sin 2\omega t & (-1)^n\,\omega^{2n+1}\gamma_a\cos\omega t \\ 0 & (-1)^n\,\omega^{2n+1}\gamma_a\cos\omega t & 0 \end{bmatrix}$$

$$\left[\mathbf{A}_{2n+2}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\,(-1)^{n+1}\,\omega^{2n+2}\gamma_a^2\left(\alpha - \beta\cos^2\omega t\right) & (-1)^{n+1}\,\omega^{2n+2}\gamma_a\sin\omega t \\ 0 & (-1)^{n+1}\,\omega^{2n+2}\gamma_a\sin\omega t & 0 \end{bmatrix}$$

$\alpha = 3.5\ldots(2n+1), \quad \beta = 2\alpha + 1.$

Problem 3.7

Cylindrical Coordinates. Conservation of mass: $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho u_\theta) + \frac{\partial}{\partial z}(\rho u_z) = 0. \qquad \text{(S.244)}$$

Conservation of linear momentum $\rho\mathbf{a} = \rho\mathbf{b} + (\nabla\cdot\mathbf{S})^T$ where

$$\mathbf{a} = \left(\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right)$$

is the fluid acceleration, **b** is the body force and **S** is the (symmetric) stress tensor:

$$\begin{aligned}
a_r &= \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z \frac{\partial u_r}{\partial z}, \\
a_\theta &= \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta u_r}{r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z}, \\
a_z &= \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z},
\end{aligned} \tag{S.245}$$

$$\begin{aligned}
\rho a_r &= \rho b_r + \frac{1}{r}\frac{\partial}{\partial r}(r S_{rr}) - \frac{S_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial}{\partial \theta} S_{r\theta} + \frac{\partial}{\partial z} S_{rz}, \\
\rho a_\theta &= \rho b_\theta + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 S_{\theta r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta} S_{\theta\theta} + \frac{\partial}{\partial z} S_{\theta z}, \\
\rho a_z &= \rho b_z + \frac{1}{r}\frac{\partial}{\partial r}(r S_{zr}) + \frac{1}{r}\frac{\partial}{\partial \theta} S_{z\theta} + \frac{\partial}{\partial z} S_{zz}.
\end{aligned} \tag{S.246}$$

Spherical Coordinates. Conservation of mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho u_r\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho u_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}\left(\rho u_\phi\right) = 0. \tag{S.247}$$

Conservation of linear momentum $\rho \mathbf{a} = \rho \mathbf{b} + (\nabla \cdot \mathbf{S})^T$ where

$$\mathbf{a} = \left(\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}\right)$$

is the fluid acceleration, **b** is the body force and **S** is the (symmetric) stress tensor:

$$\begin{aligned}
a_r &= \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_\phi \left(\frac{1}{r \sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r}\right), \\
a_\theta &= \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta u_r}{r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} \\
&\quad + u_\phi \left(\frac{1}{r \sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi}{r}\cot\theta\right), \\
a_\phi &= \frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + u_\phi \left(\frac{1}{r \sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta\right),
\end{aligned} \tag{S.248}$$

$$\rho a_r = \rho b_r + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 S_{rr}\right) - \frac{S_{\theta\theta} + S_{\phi\phi}}{r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(S_{r\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}S_{r\phi},$$

$$\rho a_\theta = \rho b_\theta + \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3 S_{\theta r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(S_{\theta\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}S_{\theta\phi} - \frac{S_{\phi\phi}}{r}\cot\theta, \quad \text{(S.249)}$$

$$\rho a_\phi = \rho b_\phi + \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3 S_{\phi r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(S_{\phi\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}S_{\phi\phi} + \frac{S_{\theta\phi}}{r}\cot\theta.$$

Problems in Chap. 4

Problem 4.1

In a simple shear deformation of a linearly elastic body (4.7), in which the displacement field takes the form

$$v_1 = \gamma y, \quad v_2 = 0, \quad v_3 = 0, \quad \text{(S.250)}$$

where γ is the amount of shear, the infinitesimal strain tensor is

$$[\varepsilon] = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{(S.251)}$$

corresponding to the stress tensor

$$[\mathbf{T}] = \mu\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad \text{(S.252)}$$

The only non-trivial stress component is the shear stress, $T_{12} = \mu\gamma$. Thus μ is called the shear modulus.

Problem 4.2

In a uni-axial extension of a linearly elastic material (4.7), in which the displacement field is given by

$$v_1 = \varepsilon x, \quad v_2 = -\nu \varepsilon y, \quad v_3 = -\nu \varepsilon z, \tag{S.253}$$

where ε is the elongational strain, and ν is the amount of lateral contraction due to the axial elongation, called Poisson's ratio. The infinitesimal strain tensor is

$$[\boldsymbol{\varepsilon}] = diag\left[\varepsilon, -\nu\varepsilon, -\nu\varepsilon\right], \tag{S.254}$$

leading to the stress tensor

$$[\mathbf{T}] = \lambda\left(1 - 2\nu\right)\varepsilon\mathbf{I} + 2\mu\, diag\left[\varepsilon, -\nu\varepsilon, -\nu\varepsilon\right], \tag{S.255}$$

or

$$\begin{aligned} T_{xx} &= \left[\lambda\left(1 - 2\nu\right) + 2\mu\right]\varepsilon, \\ T_{yy} &= \left[\lambda\left(1 - 2\nu\right) - 2\mu\nu\right]\varepsilon, \\ T_{zz} &= \left[\lambda\left(1 - 2\nu\right) - 2\mu\nu\right]\varepsilon. \end{aligned} \tag{S.256}$$

All the other components of $\mathbf{T}$ are zero. If the lateral stresses are zero, then

$$\nu = \frac{\lambda}{2\left(\lambda + \mu\right)}. \tag{S.257}$$

In addition, if we set

$$\lambda\left(1 - 2\nu\right) + 2\mu = 2\mu\left(1 + \nu\right) = E, \tag{S.258}$$

then

$$T_{xx} = E\varepsilon. \tag{S.259}$$

E is called the Young's modulus of the material.

Problem 4.3

Consider a deformation characterised by

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}},$$

$\mathbf{F}$ takes an element $d\mathbf{X}$ into $d\mathbf{x} = \mathbf{F}d\mathbf{X}$. Thus, if $d\mathbf{X} = dX\mathbf{P}$, where $\mathbf{P}$ is a unit vector,

$$\begin{aligned} dx^2 &= F_{ik}dX_k F_{il}dX_l = F_{ik}F_{il}P_k P_l dX^2 \\ &= C_{kl}P_k P_l dX^2. \end{aligned} \tag{S.260}$$

If $\mathbf{P}$ is distributed randomly in space, then on the average, $\langle P_k P_l \rangle = \delta_{kl}/3$ and one has

$$\left\langle dx^2 \right\rangle = \frac{1}{3} dX^2 tr\ \mathbf{C}, \tag{S.261}$$

and a measure of the amount of stretch (Weissenberg number) could be defined as

$$Wi = \frac{1}{3} tr\ \mathbf{C}. \tag{S.262}$$

Problem 4.4

Let $\mathbf{f}$ be a vector-valued, isotropic polynomial of a symmetric tensor $\mathbf{S}$ and a vector $\mathbf{v}$. By definition, it satisfies, $\forall$ orthogonal $\mathbf{Q}$,

$$\mathbf{Qf}(\mathbf{S}, \mathbf{v}) = \mathbf{f}\left(\mathbf{QSQ}^T, \mathbf{Qv}\right). \tag{S.263}$$

Now, define a scalar function $g = g(\mathbf{u}, \mathbf{S}, \mathbf{v})$ as

$$g(\mathbf{u}, \mathbf{S}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{f}(\mathbf{S}, \mathbf{v}). \tag{S.264}$$

Then,

$$g\left(\mathbf{Qu}, \mathbf{QSQ}^T, \mathbf{Qv}\right) = \mathbf{Qu} \cdot \mathbf{f}\left(\mathbf{QSQ}^T, \mathbf{Qv}\right), \tag{S.265}$$

and from the properties of $\mathbf{f}$, (S.263),

$$\begin{aligned} g\left(\mathbf{Qu}, \mathbf{QSQ}^T, \mathbf{Qv}\right) &= \mathbf{Qu} \cdot \mathbf{Qf}(\mathbf{S}, \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{f}(\mathbf{S}, \mathbf{v}) = g(\mathbf{u}, \mathbf{S}, \mathbf{v}) \end{aligned} \tag{S.266}$$

Thus $g(\mathbf{u}, \mathbf{S}, \mathbf{v})$ is an isotropic scalar function in all of its arguments. If $\mathbf{f}$ is a polynomial in its arguments then g is also a polynomial in its arguments - its integrity basis has been listed in (4.36):

$$\begin{aligned} &\mathrm{tr}\,\mathbf{S},\ \mathrm{tr}\,\mathbf{S}^2,\ \mathrm{tr}\,\mathbf{S}^3,\ \mathbf{u}\cdot\mathbf{u},\ \mathbf{u}\cdot\mathbf{v},\ \mathbf{v}\cdot\mathbf{v},\ \mathbf{u}\cdot\mathbf{Su},\ \mathbf{u}\cdot\mathbf{S}^2\mathbf{u}, \\ &\mathbf{v}\cdot\mathbf{Sv},\ \mathbf{v}\cdot\mathbf{S}^2\mathbf{v},\ \mathbf{u}\cdot\mathbf{Sv},\ \mathbf{u}\cdot\mathbf{S}^2\mathbf{v}. \end{aligned} \tag{S.267}$$

But g is linear in $\mathbf{u}$ and cannot depend on any non-linear manner on $\mathbf{u}$. Consequently,

$$\begin{aligned} g &= \mathbf{u} \cdot \mathbf{f}\left(\operatorname{tr} \mathbf{S},\ \operatorname{tr} \mathbf{S}^2,\ \operatorname{tr} \mathbf{S}^3, \mathbf{v} \cdot \mathbf{v}, \mathbf{v} \cdot \mathbf{S}\mathbf{v},\ \mathbf{v} \cdot \mathbf{S}^2\mathbf{v}\right) \\ &= \mathbf{u} \cdot [f_0 \mathbf{1} + f_1 \mathbf{S} + f_2 \mathbf{S}^2]\mathbf{v}. \end{aligned} \tag{S.268}$$

This choice is dictated by the fact the term multiplied with $\mathbf{u}$ is a vector. This leads to the form

$$\mathbf{f}(\mathbf{S}, \mathbf{v}) = [f_0 \mathbf{1} + f_1 \mathbf{S} + f_2 \mathbf{S}^2]\mathbf{v}, \tag{S.269}$$

where the scalar valued coefficients are polynomials in the six invariants involving only $\mathbf{S}$ and $\mathbf{v}$ in the list (S.268).

Problem 4.5

Consider a simple shear deformation of a rubber-like material (4.55), where

$$x = X + \gamma Y, \quad y = Y, \quad z = Z. \tag{S.270}$$

The deformation gradient is

$$[\mathbf{F}] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

leading to the Finger tensor

$$\begin{aligned} [\mathbf{B}] = \left[\mathbf{F}\mathbf{F}^T\right] &= \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{S.271}$$

$$\left[\mathbf{B}^{-1}\right] = \begin{bmatrix} 1 & -\gamma & 0 \\ -\gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{S.272}$$

Consequently, the stress tensor is given by

$$[\mathbf{T}] = -P\,[\mathbf{I}] + \begin{bmatrix} \beta_1\left(1+\gamma^2\right) - \beta_2 & (\beta_1+\beta_2)\,\gamma & 0 \\ (\beta_1+\beta_2)\,\gamma & \beta_1 - \beta_2\left(1+\gamma^2\right) & 0 \\ 0 & 0 & \beta_1 - \beta_2 \end{bmatrix}, \tag{S.273}$$

where P is the hydrostatic pressure. The shear stress and the normal stress differences are

$$S = (\beta_1 + \beta_2)\,\gamma, \quad N_1 = (\beta_1 + \beta_2)\,\gamma^2, \quad N_2 = -\beta_2\gamma^2. \tag{S.274}$$

Deduce that the linear shear modulus of elasticity is

$$G = \lim_{\gamma\to 0} (\beta_1 + \beta_2)\,. \tag{S.275}$$

The ratio

$$\frac{N_1}{S} = \gamma \tag{S.276}$$

is independent of the material properties. Such a relation is called *universal*.

Problem 4.6

In a uniaxial elongational deformation of a rubber-like material (4.55), where (in cylindrical coordinates)

$$R = \lambda^{1/2} r, \quad \Theta = \theta, \quad Z = \lambda^{-1} z, \tag{S.277}$$

show that the inverse deformation gradient is

$$\left[\mathbf{F}^{-1}\right] = \begin{bmatrix} \dfrac{\partial R}{\partial r} & \dfrac{1}{r}\dfrac{\partial R}{\partial \theta} & \dfrac{\partial R}{\partial z} \\ 0 & \dfrac{R}{r} & 0 \\ \dfrac{\partial Z}{\partial r} & \dfrac{1}{r}\dfrac{\partial Z}{\partial \theta} & \dfrac{\partial Z}{\partial z} \end{bmatrix} = \begin{bmatrix} \lambda^{1/2} & 0 & 0 \\ 0 & \lambda^{1/2} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}. \tag{S.278}$$

Consequently, the strains are

$$\left[\mathbf{B}^{-1}\right] = \left[\mathbf{F}^{-T}\mathbf{F}^{-1}\right] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}, \quad [\mathbf{B}] = \begin{bmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{2} \end{bmatrix}. \tag{S.279}$$

Thus the total stress tensor for a rubber-like material (4.55) is

$$[\mathbf{T}] = -P\,[\mathbf{I}] + \begin{bmatrix} \beta_1\lambda^{-1} - \beta_2\lambda & 0 & 0 \\ 0 & \beta_1\lambda^{-1} - \beta_2\lambda & 0 \\ 0 & 0 & \beta_1\lambda^2 - \beta_2\lambda^{-2} \end{bmatrix}. \tag{S.280}$$

Under the condition that the lateral tractions are zero, i.e., $T_{rr} = 0$, the pressure can be found

$$P = \beta_1 \lambda^{-1} - \beta_2 \lambda, \tag{S.281}$$

and thus the tensile stress is

$$\begin{aligned} T_{zz} &= -P + \beta_1 \lambda^2 - \beta_2 \lambda^{-2} = \beta_1 \lambda^2 - \beta_2 \lambda^{-2} - \beta_1 \lambda^{-1} + \beta_2 \lambda \\ &= \left(\lambda^2 - \lambda^{-1}\right)\left(\beta_1 + \beta_2 \lambda^{-1}\right). \end{aligned} \tag{S.282}$$

This tensile stress is the force per unit area in the deformed configuration. As $r = \lambda^{1/2} R$, the corresponding force per unit area in the undeformed configuration is

$$T_{ZZ} = T_{zz} \lambda^{-1} = \left(\lambda - \lambda^{-2}\right)\left(\beta_1 + \beta_2 \lambda^{-1}\right). \tag{S.283}$$

Problem 4.7

Note that

$$\mathbf{C}_t(\tau) = \mathbf{F}_t(\tau)^T \mathbf{F}_t(\tau), \tag{S.284}$$

but (use subscript notation for clarity)

$$\mathbf{F}_t(\tau) = \frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{x}(t)} = \frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{X}} \frac{\partial \mathbf{X}}{\partial \mathbf{x}(t)} = \mathbf{F}(\tau)\, \mathbf{F}(t)^{-1}$$

and therefore

$$\mathbf{C}_t(\tau) = \mathbf{F}(t)^{-T} \mathbf{F}(\tau)^T \mathbf{F}(\tau)\, \mathbf{F}(t)^{-1} = \mathbf{F}(t)^{-T} \mathbf{C}(\tau)\, \mathbf{F}(t)^{-1}$$

leading to the desired result

$$\mathbf{C}(\tau) = \mathbf{F}(t)^T \mathbf{C}_t(\tau)\, \mathbf{F}(t). \tag{S.285}$$

Problem 4.8

Consider a simple shear flow

$$u = \dot{\gamma} y, \quad v = 0, \quad w = 0. \tag{S.286}$$

The first and second Rivlin–Ericksen tensors are

$$[\mathbf{A}_1] = \begin{bmatrix} 0 & \dot\gamma & 0 \\ \dot\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \left[\mathbf{A}_1^2\right] = \begin{bmatrix} \dot\gamma^2 & 0 & 0 \\ 0 & \dot\gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{A}_2] = \left[\mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\dot\gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the stress tensor in the second-order model is given by

$$[\mathbf{S}] = \eta_0 \begin{bmatrix} 0 & \dot\gamma & 0 \\ \dot\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (\nu_1 + \nu_2) \begin{bmatrix} \dot\gamma^2 & 0 & 0 \\ 0 & \dot\gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{\nu_1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\dot\gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \text{(S.287)}$$

and the three viscometric functions are

$$S = \eta_0\dot\gamma, \quad N_1 = S_{11} - S_{22} = \nu_1\dot\gamma^2, \quad N_2 = S_{22} - S_{33} = \nu_2\dot\gamma^2. \qquad \text{(S.288)}$$

Problem 4.9

In an elongational flow

$$u = \dot\varepsilon x, \quad v = -\frac{\dot\varepsilon}{2}y, \quad w = -\frac{\dot\varepsilon}{2}z, \qquad \text{(S.289)}$$

the first and second Rivlin–Ericksen tensors are

$$[\mathbf{A}_1] = \begin{bmatrix} 2\dot\varepsilon & 0 & 0 \\ 0 & -\dot\varepsilon & 0 \\ 0 & 0 & -\dot\varepsilon \end{bmatrix}, \ \left[\mathbf{A}_1^2\right] = \begin{bmatrix} 4\dot\varepsilon^2 & 0 & 0 \\ 0 & \dot\varepsilon^2 & 0 \\ 0 & 0 & \dot\varepsilon^2 \end{bmatrix},$$

$$[\mathbf{A}_2] = \left[\mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1\right] = \begin{bmatrix} 4\dot\varepsilon^2 & 0 & 0 \\ 0 & \dot\varepsilon^2 & 0 \\ 0 & 0 & \dot\varepsilon^2 \end{bmatrix}.$$

Thus the stress is given by

$$[\mathbf{S}] = \eta_0 \begin{bmatrix} 2\dot{\varepsilon} & 0 & 0 \\ 0 & -\dot{\varepsilon} & 0 \\ 0 & 0 & -\dot{\varepsilon} \end{bmatrix} + (\nu_1 + \nu_2) \begin{bmatrix} 4\dot{\varepsilon}^2 & 0 & 0 \\ 0 & \dot{\varepsilon}^2 & 0 \\ 0 & 0 & \dot{\varepsilon}^2 \end{bmatrix} - \frac{\nu_1}{2} \begin{bmatrix} 4\dot{\varepsilon}^2 & 0 & 0 \\ 0 & \dot{\varepsilon}^2 & 0 \\ 0 & 0 & \dot{\varepsilon}^2 \end{bmatrix}. \tag{S.290}$$

Consequently the elongational viscosity is given by

$$\eta_E = \frac{S_{xx} - S_{yy}}{\dot{\varepsilon}} = 3\eta_0 + 3\left(\frac{\nu_1}{2} + \nu_2\right)\dot{\varepsilon}. \tag{S.291}$$

Problem 4.10

For potential flows, the velocity field is a gradient of a potential:

$$\mathbf{u} = \nabla\phi, \tag{S.292}$$

from which, incompressibility demands

$$\nabla \cdot \mathbf{u} = \nabla^2\phi = 0. \tag{S.293}$$

Now, since

$$L_{ij} = \frac{\partial u_i}{\partial x_j} = \phi_{,ij}, \quad A_{1ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = 2\phi_{,ij} \tag{S.294}$$

where the comma denotes a spatial derivative, it follows that

$$\nabla \cdot \mathbf{A}_1 = 0. \tag{S.295}$$

Next,

$$\begin{aligned} \left(\nabla \cdot \mathbf{A}_1^2\right)_i &= \left(A_{1ik}A_{1kj}\right)_{,j} = 4\left(\phi_{,ik}\phi_{,kj}\right)_{,j} = 4\phi_{,kj}\phi_{,ikj} \\ &= \frac{1}{2}\left(4\phi_{,kj}\phi_{,kj}\right)_{,i} \\ \nabla \cdot \mathbf{A}_1^2 &= \frac{1}{2}\left(\nabla(tr\ \mathbf{A}_1^2)\right). \end{aligned} \tag{S.296}$$

Furthermore,

$$A_{2ij} = u_k A_{1ij,k} + A_{1ik} L_{kj} + L_{ki} A_{1kj}$$
$$= 2\phi_{,k}\phi_{,ijk} + 4\phi_{,ik}\phi_{,kj}$$

$$A_{2ij,j} = 2\phi_{,kj}\phi_{,ijk} + 4\phi_{,kj}\phi_{,ikj} = 6\phi_{,kj}\phi_{,ijk}$$
$$= 3\left(\phi_{,kj}\phi_{,kj}\right)_i$$
$$= \frac{3}{4}\left(4\phi_{,kj}\phi_{,kj}\right)_i$$
$$\nabla \cdot \mathbf{A}_2 = \frac{3}{4}\nabla(\text{tr } \mathbf{A}_1^2). \tag{S.297}$$

In a second-order fluid model, the stress is given by

$$\mathbf{T} = -P\mathbf{I} + \eta_0 \mathbf{A}_1 + (\nu_1 + \nu_2)\,\mathbf{A}_1^2 - \frac{\nu_1}{2}\mathbf{A}_2,$$

and conservation of momentum yields

$$\nabla P = \nabla \cdot \left[\eta_0 \mathbf{A}_1 + (\nu_1 + \nu_2)\,\mathbf{A}_1^2 - \frac{\nu_1}{2}\mathbf{A}_2\right] - \rho\mathbf{a}, \tag{S.298}$$

where **a** is the acceleration field. If the same flow, of the same kinematics occurs in a Newtonian fluid of viscosity η_0, then we must have

$$\nabla P_N = \eta_0 \nabla \cdot \mathbf{A}_1 - \rho\mathbf{a}, \tag{S.299}$$

where P_N is the Newtonian pressure field. Thus, for this to occur, one must have an "extra" pressure field

$$\nabla P_E = \nabla \cdot \left[(\nu_1 + \nu_2)\,\mathbf{A}_1^2 - \frac{\nu_1}{2}\mathbf{A}_2\right]. \tag{S.300}$$

From the results (S.296) and (S.297), this is possible, and thus in potential flows, the velocity fields for a Newtonian and a second-order fluid are identical, with the extra pressure given by

$$\nabla P_E = \nabla \cdot \left[(\nu_1 + \nu_2)\,\mathbf{A}_1^2 - \frac{\nu_1}{2}\mathbf{A}_2\right]$$
$$= \left[\frac{1}{2}(\nu_1 + \nu_2) - \frac{3\nu_1}{8}\right]\nabla\left(tr\mathbf{A}_1^2\right),$$

or that

$$P_E = \frac{1}{8}(\nu_1 + 4\nu_2)\, tr\mathbf{A}_1^2, \tag{S.301}$$

to within a constant which can be absorbed in P_N.

Problem 4.11

For steady two-dimensional incompressible flows, a stream function $\psi = \psi(x, y)$ can be defined such that the velocity components u and v can be expressed as

$$u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x}. \tag{S.302}$$

If the fluid is incompressible Newtonian, then for a flow at zero Reynolds number, the Newtonian stresses are

$$[\mathbf{S}_N] = \eta \begin{bmatrix} 2\psi_{,xy} & \psi_{,yy} - \psi_{,xx} \\ \psi_{,yy} - \psi_{,xx} & -2\psi_{,xy} \end{bmatrix}$$

and the balance of momentum requires

$$\begin{aligned} \frac{\partial P}{\partial x} &= 2\eta\frac{\partial^2 u}{\partial x^2} + \eta\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) \\ &= \eta\left[2\psi_{,yxx} + \psi_{,yyy} - \psi_{,xxy}\right], \\ \frac{\partial P}{\partial y} &= \eta\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + 2\eta\frac{\partial^2 v}{\partial y^2} \\ &= \eta\left[\psi_{,yyx} - \psi_{,xxx} - 2\psi_{,xyy}\right]. \end{aligned}$$

For compatibility, the stream function must satisfy a bi-harmonic equation:

$$\begin{aligned} \psi_{,xxy}y + \psi_{,yyyy} &= -\psi_{,xxxx} - \psi_{,xxyy} \\ \nabla^2\nabla^2\psi &= \Delta^2\psi = 0. \end{aligned} \tag{S.303}$$

Now,

$$[\mathbf{L}] = \begin{bmatrix} \psi_{,xy} & \psi_{,yy} \\ -\psi_{,xx} & -\psi_{xy} \end{bmatrix}$$

$$[\mathbf{A}_1] = \begin{bmatrix} 2\psi_{,xy} & \psi_{,yy} - \psi_{,xx} \\ \psi_{,yy} - \psi_{,xx} & -2\psi_{,xy} \end{bmatrix}, \tag{S.304}$$

$$\left[\mathbf{A}_1^2\right] = \left(4\psi_{,xy}^2 + \psi_{,yy}^2 - 2\psi_{,xx}\psi_{,yy} + \psi_{,xx}^2\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{S.305}$$

$$\begin{aligned}
[\mathbf{A}_2] &= [\mathbf{u}\cdot\nabla\mathbf{A}_1] + [\mathbf{A}_1\mathbf{L}] + \left[\mathbf{L}^T\mathbf{A}_1\right] \qquad \text{(S.306)}\\
&= \left(\psi_y\frac{\partial}{\partial x} - \psi_x\frac{\partial}{\partial y}\right)\begin{bmatrix} 2\psi_{,xy} & \psi_{,yy}-\psi_{,xx} \\ \psi_{,yy}-\psi_{,xx} & -2\psi_{,xy} \end{bmatrix}\\
&+ \begin{bmatrix} 2\psi_{,xy}^2 - \psi_{,xx}\left(\psi_{,yy}-\psi_{,xx}\right) & 2\psi_{,xy}\psi_{,yy} - \psi_{,xy}\left(\psi_{,yy}-\psi_{,xx}\right) \\ 2\psi_{,xy}\psi_{,xx} + \psi_{,xy}\left(\psi_{,yy}-\psi_{,xx}\right) & 2\psi_{,xy}^2 + \psi_{,yy}\left(\psi_{,yy}-\psi_{,xx}\right) \end{bmatrix}\\
&+ \begin{bmatrix} 2\psi_{,xy}^2 - \psi_{,xx}\left(\psi_{,yy}-\psi_{,xx}\right) & 2\psi_{,xy}\psi_{,xx} + \psi_{,xy}\left(\psi_{,yy}-\psi_{,xx}\right) \\ 2\psi_{,xy}\psi_{,yy} - \psi_{,xy}\left(\psi_{,yy}-\psi_{,xx}\right) & 2\psi_{,xy}^2 + \psi_{,yy}\left(\psi_{,yy}-\psi_{,xx}\right) \end{bmatrix},
\end{aligned}$$

from which the second-order fluid stresses (see 4.71) can be determined as

$$\begin{aligned}
S_{xx} &= 2\eta\psi_{,xy} + (\nu_1+\nu_2)\left(4\psi_{,xy}^2 + \psi_{,yy}^2 - 2\psi_{,xx}\psi_{,yy} + \psi_{,xx}^2\right)\\
&\quad -\nu_1\left[\left(\psi_y\frac{\partial}{\partial x} - \psi_x\frac{\partial}{\partial y}\right)\psi_{,xy} + 2\psi_{,xy}^2 - \psi_{,xx}\left(\psi_{,yy}-\psi_{,xx}\right)\right]
\end{aligned} \tag{S.307}$$

$$\begin{aligned}
S_{xy} &= \eta\left(\psi_{,yy}-\psi_{,xx}\right) - \frac{\nu_1}{2}[\left(\psi_y\frac{\partial}{\partial x} - \psi_x\frac{\partial}{\partial y}\right)\left(\psi_{,yy}-\psi_{,xx}\right)\\
&\quad +2\psi_{,xy}\left(\psi_{,yy}+\psi_{,xx}\right)]
\end{aligned} \tag{S.308}$$

$$\begin{aligned}
S_{yy} &= -2\eta\psi_{,xy} + (\nu_1+\nu_2)\left(4\psi_{,xy}^2 + \psi_{,yy}^2 - 2\psi_{,xx}\psi_{,yy} + \psi_{,xx}^2\right)\\
&\quad -\nu_1\left[\left(\psi_x\frac{\partial}{\partial y} - \psi_y\frac{\partial}{\partial x}\right)\psi_{,xy} + 2\psi_{,xy}^2 + \psi_{,yy}\left(\psi_{,yy}-\psi_{,xx}\right)\right].
\end{aligned} \tag{S.309}$$

The balance of momentum requires

$$\begin{aligned}
\frac{\partial P}{\partial x} &= \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y},\\
\frac{\partial P}{\partial y} &= \frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y}.
\end{aligned}$$

Compatibility requires

$$\frac{\partial^2}{\partial x\partial y}\left(S_{xx}-S_{yy}\right) + \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)S_{xy} = 0. \tag{S.310}$$

The second-order fluid stresses can be substituted in the preceding results to yield

$$0 = \frac{\partial^2}{\partial x \partial y}\left[4\eta\psi_{,xy} - \nu_1\left(2\left(\psi_y\frac{\partial}{\partial x} - \psi_x\frac{\partial}{\partial y}\right)\psi_{,xy} - \left(\psi_{,yy}^2 - \psi_{,xx}^2\right)\right)\right]$$
$$+\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)[\eta\left(\psi_{,yy} - \psi_{,xx}\right) - \frac{\nu_1}{2}[\left(\psi_y\frac{\partial}{\partial x} - \psi_x\frac{\partial}{\partial y}\right)\left(\psi_{,yy} - \psi_{,xx}\right)$$
$$+2\psi_{,xy}\left(\psi_{,yy} + \psi_{,xx}\right)]].$$

This is simplified to

$$\eta\Delta^2\psi - \frac{\nu_1}{2}\mathbf{u}\cdot\nabla\left(\triangle^2\psi\right) = 0. \tag{S.311}$$

This shows that a Newtonian velocity field ($\Delta^2\psi = 0$) is also a velocity field for the second-order fluid.

Problem 4.12

In a simple shear flow,

$$u = \dot{\gamma}(t)\,y, \quad v = 0, \quad w = 0,$$

the path lines **x** satisfy

$$\frac{d}{dt}(x, y, z) = (\dot{\gamma}(t)\,y, 0, 0), \quad \mathbf{x}(0) = \mathbf{X}.$$

That is,

$$x(t) = X + Y\int_0^t \dot{\gamma}\left(t'\right)dt', \;\; y = Y, \;\; z = Z.$$

Thus the path lines ¸$(\tau) = (\xi, \psi, \zeta)$

$$\begin{aligned}\xi(\tau) &= X + Y\int_0^\tau \dot{\gamma}\left(t'\right)dt' \\ &= x + y\int_t^\tau \dot{\gamma}\left(t'\right)dt', \\ \psi &= y, \\ \zeta &= z,\end{aligned}$$

or

$$\xi(\tau) = x + y\gamma(t,\tau), \quad \psi(\tau) = y, \quad \zeta(\tau) = z, \tag{S.312}$$

where

$$\gamma(t,\tau) = \int_t^\tau \dot{\gamma}(s)\, ds. \tag{S.313}$$

The relative deformation gradient:

$$\mathbf{F}_t(\tau) = (\nabla_{\mathbf{x}}\xi)^T = \begin{bmatrix} 1 & \gamma(t,\tau) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{C}_t(\tau) = \mathbf{F}_t(\tau)^T \mathbf{F}_t(\tau) = \begin{bmatrix} 1 & \gamma(t,\tau) & 0 \\ \gamma(t,\tau) & 1+\gamma^2(t,\tau) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{aligned} \mathbf{S}(t) &= \int_0^\infty \mu(s)\left(\mathbf{C}_t(t-s) - \mathbf{I}\right) ds \\ &= \int_0^\infty \mu(s) \begin{bmatrix} 0 & \gamma(t,t-s) & 0 \\ \gamma(t,t-s) & \gamma^2(t,t-s) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

That the shear stress and the normal stress differences are given by

$$S_{12}(t) = \int_0^\infty \mu(s)\gamma(t,t-s)\, ds, \tag{S.314}$$

$$N_1(t) = -\int_0^\infty \mu(s)\left[\gamma(t,t-s)\right]^2 ds = -N_2. \tag{S.315}$$

For a constant shear rate $\dot{\gamma}$,

$$\gamma(t,\tau) = \dot{\gamma}(\tau - t),$$

and with the memory function $\mu(s) = -\frac{G}{\lambda}e^{-s/\lambda}$,

$$\begin{aligned} S_{12}(t) &= \int_0^\infty \mu(s)\gamma(t,t-s)\, ds \\ &= -\int_0^\infty \mu(s)\dot{\gamma}s\, ds \\ &= \frac{G}{\lambda}\dot{\gamma}\int_0^\infty e^{-s/\lambda} s\, ds \\ &= G\dot{\gamma}. \end{aligned} \tag{S.316}$$

$$N_1(t) = \frac{G}{\lambda}\dot{\gamma}^2 \int_0^\infty e^{-s/\lambda} s^2 ds$$
$$= 2G(\lambda\dot{\gamma})^2. \tag{S.317}$$

For a sinusoidal shear rate $\dot{\gamma} = \dot{\gamma}_0 \cos(\omega t)$

$$\gamma(t,\tau) = \int_t^\tau \dot{\gamma}_0 \cos\omega t dt = \frac{\dot{\gamma}_0}{\omega}(\sin\omega\tau - \sin\omega t).$$

$$\begin{aligned} S_{12}(t) &= \int_0^\infty \mu(s)\gamma(t, t-s) ds \\ &= -\frac{G\dot{\gamma}_0}{\lambda\omega}\int_0^\infty e^{-s/\lambda}(\sin\omega(t-s) - \sin\omega t) ds \\ &= \frac{G\dot{\gamma}_0}{\lambda\omega}\int_0^\infty e^{-s/\lambda}(\sin\omega t - \sin\omega t\cos\omega s + \cos\omega t\sin\omega s) ds \\ &= \frac{G\dot{\gamma}_0}{\omega}\left[\sin\omega t - \frac{\sin\omega t}{1+\lambda^2\omega^2} + \frac{\omega\lambda\cos\omega t}{1+\lambda^2\omega^2}\right] \\ &= G\dot{\gamma}_0\lambda\frac{\lambda\omega\sin\omega t + \cos\omega t}{1+\lambda^2\omega^2}. \end{aligned} \tag{S.318}$$

$$\begin{aligned} N_1 &= \frac{G\dot{\gamma}_0^2}{\lambda\omega^2}\int_0^\infty e^{-s/\lambda}(\sin\omega t - \sin\omega t\cos\omega s + \cos\omega t\sin\omega s)^2 ds \\ &= \frac{G\dot{\gamma}_0^2}{\lambda\omega^2}\int_0^\infty e^{-s/\lambda}[\sin^2\omega t - 2\sin^2\omega t\cos\omega s + \sin 2\omega t\sin\omega s \\ &\quad + \frac{1}{2}\sin 2\omega t\sin 2\omega s + \sin^2\omega t\cos^2\omega s + \cos^2\omega t\sin^2\omega s] ds \\ &= \frac{G\dot{\gamma}_0^2}{\omega^2}\left[\sin^2\omega t - \frac{2\sin^2\omega t}{1+\lambda^2\omega^2} + \frac{\omega\lambda\sin 2\omega t}{1+\lambda^2\omega^2}\right. \\ &\quad \left. + \frac{\omega\lambda\sin 2\omega t}{1+4\lambda^2\omega^2} + \frac{\sin^2\omega t\left(1+2\lambda^2\omega^2\right)}{1+4\lambda^2\omega^2} + \frac{2\lambda^2\omega^2\cos^2\omega t}{1+4\lambda^2\omega^2}\right]. \end{aligned} \tag{S.319}$$

Problems in Chap. 5

Problem 5.1

Assume the relaxation modulus function (5.13)

$$G(t) = \sum_{j=1}^N G_j e^{-t/\lambda_j},$$

the relation (5.11) is

$$\mathbf{S}(t) = 2\int_{-\infty}^{t} \sum_{j=1}^{N} G_j e^{-\left(t-t'\right)/\lambda_j} \mathbf{D}\left(t'\right) dt'$$
$$= \sum_{j=1}^{N} \mathbf{S}^{(j)}$$
$$\mathbf{S}^{(j)} = 2\int_{-\infty}^{t} G_j e^{-\left(t-t'\right)/\lambda_j} \mathbf{D}\left(t'\right) dt'.$$

Each component $\mathbf{S}^{(j)}$ satisfies, by direct differentiation

$$\dot{\mathbf{S}}^{(j)} = 2G_j \mathbf{D}(t) - 2\frac{G_j}{\lambda_j}\int_{-\infty}^{t} e^{-\left(t-t'\right)/\lambda_j} \mathbf{D}\left(t'\right) dt',$$

or that

$$\mathbf{S}^{(j)} + \lambda_j \dot{\mathbf{S}}^{(j)} = 2\eta_j \mathbf{D}, \quad \eta_j = G_j \lambda_j.$$

This relation is called the linear Maxwell equation.

Problem 5.2

In an oscillatory flow where the shear rate and the shear strain are

$$\dot{\gamma} = \dot{\gamma}_0 \cos \omega t, \quad \gamma = \gamma_0 \sin \omega t, \quad \dot{\gamma}_0 = \omega \gamma_0 \tag{S.320}$$

the only non-zero component of the stress is

$$\begin{aligned} S_{12} &= \int_{-\infty}^{t} G\left(t-t'\right) \dot{\gamma}_0 \cos \omega t' dt', \\ &= \int_{0}^{\infty} \dot{\gamma}_0 G(s) \cos \omega (t-s)\, ds, \\ &= \int_{0}^{\infty} \dot{\gamma}_0 G(s) \left[\cos \omega t \cos \omega s + \sin \omega t \sin \omega s\right] ds, \\ &= G'(\omega)\, \gamma_0 \sin \omega t + \eta'(\omega)\, \dot{\gamma}_0 \cos \omega t, \end{aligned} \tag{S.321}$$

where the coefficients in the strain is the *storage modulus*,

$$G'(\omega) = \int_{0}^{\infty} \omega G(s) \sin \omega s ds, \tag{S.322}$$

and in the strain rate, the *dynamic viscosity*,

$$\eta'(\omega) = \int_0^\infty G(s)\cos\omega s ds.$$

The shear stress can be written as

$$S_{12} = G'(\omega)\gamma_0 \sin\omega t + G''(\omega)\gamma_0\cos\omega t, \tag{S.323}$$

where the *loss modulus* G'' is defined as

$$G''(\omega) = \omega\eta'(\omega). \tag{S.324}$$

Re-write the shear stress as

$$\begin{aligned} S_{12} &= S_{12}^0 \sin(\omega t + \phi) \\ &= S_{12}^0 (\sin\omega t\cos\phi + \sin\phi\cos\omega t). \end{aligned}$$

Identify this with the previous result for S_{12} :

$$S_{12}^0\cos\phi = \gamma_0 G'(\omega), \;\; S_{12}^0\sin\phi = \gamma_0 G''(\omega),$$

or that

$$S_{12}^0 = \gamma_0\sqrt{G'^2(\omega) + G''^2(\omega)}, \;\; \tan\phi = \frac{G''(\omega)}{G'(\omega)}, \tag{S.325}$$

where $\tan\phi$ is called the loss tangent. Sometimes it is more convenient to work with complex numbers, and the complex modulus G^* and the complex viscosity η^* are thus defined as

$$G^*(\omega) = G'(\omega) + iG''(\omega), \quad \eta^*(\omega) = \eta'(\omega) - i\eta''(\omega). \tag{S.326}$$

One can define the complex shear strain as

$$\gamma^* = \gamma_0 e^{-i\omega t}, \tag{S.327}$$

then the shear stress is

$$\begin{aligned} S_{12} &= \mathrm{Re}\left(G^*\gamma^*\right) = \gamma_0\,\mathrm{Re}\left[\left(G'(\omega) + iG''(\omega)\right)(\sin\omega t - i\cos\omega t)\right] \\ &= \gamma_0\left(G'(\omega)\sin\omega t + G''(\omega)\cos\omega t\right). \end{aligned} \tag{S.328}$$

Problem 5.3

Recall the definition of the spectrum $H(\lambda)$:

$$G'(\omega) = \int_{-\infty}^{\infty} \frac{\omega^2\lambda^2}{1+\omega^2\lambda^2} H(\lambda)\, d\ln\lambda, \tag{S.329}$$

or, equivalently,

$$\eta'(\omega) = \int_0^{\infty} \frac{H(\lambda)}{1+\omega^2\lambda^2} d\lambda. \tag{S.330}$$

For the spectrum of the form

$$\begin{aligned} H(\lambda) &= \cos^2(n\lambda), \\ \eta'(\omega) &= \int_0^{\infty} \frac{H(\omega)}{1+\lambda^2\omega^2} d\omega \\ &= \frac{\pi}{4\omega}\left(1 - e^{-2n/\omega}\right). \end{aligned} \tag{S.331}$$

At large n, the data η' is smooth ($e^{-2n/\omega}$ goes to zero), but the spectrum is highly oscillatory. Thus, the inverse problem of finding $H(\lambda)$, given the data η' in the chosen form is ill-conditioned – that is, a small variation in the data (in the exponentially small term) may lead to a large variation in the solution.

Problem 5.4

For the Maxwell discrete relaxation spectrum (5.13),

$$G(t) = \sum_{j=1}^{N} G_j e^{-t/\lambda_j}$$

the storage modulus and the dynamic viscosity are given by (5.17):

$$\begin{aligned} G'(\omega) &= \int_0^{\infty} \omega G(s) \sin\omega s ds \\ &= \sum_{j=1}^{N} \int_0^{\infty} \omega G_j e^{-s/\lambda_j} \sin\omega s ds \end{aligned}$$

$$= \sum_{j=1}^{N} \frac{\omega^2 G_j \lambda_j^2}{1 + \omega^2 \lambda_j^2}, \tag{S.332}$$

$$\begin{aligned} \eta'(\omega) &= \int_0^\infty G(s) \cos \omega s ds \\ &= \sum_{j=1}^{N} \int_0^\infty G_j e^{-s/\lambda_j} \cos \omega s ds \\ &= \sum_{j=1}^{N} \frac{G_j \lambda_j}{1 + \omega^2 \lambda_j^2}. \end{aligned} \tag{S.333}$$

In particular, with one relaxation mode $\lambda = \lambda_1$,

$$G'(\omega) = \frac{\omega^2 G_1 \lambda_1^2}{1 + \omega^2 \lambda_1^2}, \quad \eta'(\omega) = \frac{G_1 \lambda_1}{1 + \omega^2 \lambda_1^2}, \quad \tan \phi = \frac{\omega \eta'(\omega)}{G'(\omega)} = \frac{1}{\omega \lambda_1} \tag{S.334}$$

As as $\omega\lambda_1 = 0$, $\eta'(\omega) = G_1\lambda_1$, $G'(\omega) = 0$ (Newtonian fluid-like response), and when $\omega\lambda_1 \to \infty$, $\eta' \to 0$, $G' \to G_1$ corresponds to a solid-like response.

Problem 5.5

Suppose we have a Maxwell material with one relaxation time,

$$G(t) = \frac{\eta_0}{\lambda} e^{-t/\lambda}.$$

and Ω_i = constant. For the circular Couette flow problem, the torque on the inner cylinder is $\Gamma = M(R_o - R_i)$, where

$$\begin{aligned} M(t) &= \frac{4\pi R_o^2}{1 - R_o^2/R_i^2} \int_0^t G\left(t - t'\right) \Omega_i \left(t'\right) dt' \\ &= \frac{4\pi R_o^2 \eta_0 \Omega_i}{\left(1 - R_o^2/R_i^2\right) \lambda} \int_0^t e^{-\left(t - t'\right)/\lambda} dt' \\ &= \frac{4\pi R_o^2 \eta_0 \Omega_i}{\left(1 - R_o^2/R_i^2\right)} \left(1 - e^{-t/\lambda}\right). \end{aligned}$$

With the Newtonian result (corresponds to $\lambda \to 0$),

$$M_N = \frac{4\pi R_o^2 \eta_0 \Omega_i}{\left(1 - R_o^2/R_i^2\right)},$$

we thus have

$$\frac{M(t)}{M_N} = 1 - e^{-t/\lambda}. \tag{S.335}$$

Problem 5.6

Working in Laplace transform domain (s is the Laplace transform variable, and the overbar denotes the Laplace transform function), and denote the displacement across the spring G_1 as x_1, across the Kelvin-Voigt element (G_2, η_2) as $x_2 - x_1$, and across the dashpot η_1 as $x - x_2$, the relationship between force and various displacements so defined are

- across the spring G_1:

$$\frac{\bar{F}}{G_1} = \bar{x}_1,$$

- across the dashpot of η_1:

$$\frac{\bar{F}}{s\eta_1} = \bar{x} - \bar{x}_2$$

- across the Kelvin–Voigt–Meyer element (G_2, η_2):

$$\frac{\bar{F}}{G_2 + s\eta_2} = \bar{x}_2 - \bar{x}_1.$$

These may be summed up to yield

$$\frac{\bar{F}}{G_1} + \frac{\bar{F}}{G_2 + s\eta_2} + \frac{\bar{F}}{s\eta_1} = \bar{x}$$

$$\left[s\eta_1 (G_2 + s\eta_2) + s\eta_1 G_1 + G_1 (G_2 + s\eta_2)\right] \bar{F} = s\eta_1 G_1 (G_2 + s\eta_2)\, \bar{x}$$

leading to

$$\left[1 + a_1 s + a_2 s^2\right] \bar{F} = (b_1 + b_2 s)\, s\bar{x},$$

where

$$a_1 = \frac{\eta_1}{G_2}\left(1 + \frac{G_2}{G_1} + \frac{\eta_2}{\eta_1}\right), \quad a_2 = \frac{\eta_1\eta_2}{G_1 G_2}, \quad b_1 = \eta_1, \quad b_2 = \frac{\eta_1\eta_2}{G_2}.$$

In time domain, this is equivalent to

$$F + a_1 \dot{F} + a_2 \ddot{F} = b_1 \dot{x} + b_2 \ddot{x},$$

which corresponds to the following stress-strain relation for the four-element model

$$S_{ij} + a_1 \dot{S}_{ij} + a_2 \ddot{S}_{ij} = b_1 \dot{\gamma}_{ij} + b_2 \ddot{\gamma}_{ij}. \tag{S.336}$$

Problems in Chap. 6

Problem 6.1

The velocity field for (6.3) is

$$\mathbf{u} = \dot{\gamma} \left(\mathbf{b} \cdot \mathbf{x} \right) \mathbf{a}, \tag{S.337}$$

where **a**, **b**, **c** are a set of orthonormal vectors. The velocity gradient is

$$\begin{aligned} L_{ij} &= \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\dot{\gamma} b_k x_k a_i \right) \\ &= \dot{\gamma} b_j a_i , \end{aligned}$$

leading to the stated result

$$\mathbf{L} = \dot{\gamma} \mathbf{ab}. \tag{S.338}$$

Since

$$tr\ \mathbf{L} = 0,$$

all the flows represented by (6.3) are isochoric (volume-conserving). Now

$$\begin{aligned} 4\mathbf{D}^2 &= \dot{\gamma}^2 \left(\mathbf{ab} + \mathbf{ba} \right) \cdot \left(\mathbf{ab} + \mathbf{ba} \right) \\ &= \dot{\gamma}^2 \left(\mathbf{aa} + \mathbf{bb} \right), \end{aligned}$$

the shear rate is

$$\sqrt{2tr\ \mathbf{D}^2} = \sqrt{\dot{\gamma}^2} = \left| \dot{\gamma} \right|. \tag{S.339}$$

Problem 6.2

For the helicoidal flow (6.14), the velocity field is

$$\mathbf{u} = (r\mathbf{e}_\theta + c\mathbf{e}_z)\,\omega\,(r, z - c\theta)\,, \tag{S.340}$$

where c is a constant. The velocity gradient is

$$\begin{aligned}
(\nabla\mathbf{u})^T &= \left[\nabla\,(r\omega\mathbf{e}_\theta + c\omega\mathbf{e}_z)\right]^T \\
&= \left[\nabla\omega\,(r\mathbf{e}_\theta + c\mathbf{e}_z) + \omega\nabla\,(r\mathbf{e}_\theta + c\mathbf{e}_z)\right]^T \\
&= \left[\nabla\omega\,(r\mathbf{e}_\theta + c\mathbf{e}_z) + \omega\,(\mathbf{e}_r\mathbf{e}_\theta - \mathbf{e}_\theta\mathbf{e}_r)\right]^T \\
&= (r\mathbf{e}_\theta + c\mathbf{e}_z)\,\nabla\omega + \omega\,(\mathbf{e}_\theta\mathbf{e}_r - \mathbf{e}_r\mathbf{e}_\theta) \\
&= \dot{\gamma}\mathbf{ab} = \mathbf{L}
\end{aligned} \tag{S.341}$$

We first note that

$$\begin{aligned}
2\mathbf{D} = &\,(r\mathbf{e}_\theta + c\mathbf{e}_z)\,\nabla\omega + \nabla\omega\,(r\mathbf{e}_\theta + c\mathbf{e}_z) \\
&+\omega\,(\mathbf{e}_\theta\mathbf{e}_r - \mathbf{e}_r\mathbf{e}_\theta) + \omega\,(-\mathbf{e}_\theta\mathbf{e}_r + \mathbf{e}_r\mathbf{e}_\theta) \\
= &\,(r\mathbf{e}_\theta + c\mathbf{e}_z)\,\nabla\omega + \nabla\omega\,(r\mathbf{e}_\theta + c\mathbf{e}_z)\,.
\end{aligned} \tag{S.342}$$

Thus

$$\begin{aligned}
4\mathbf{D}^2 &= (r\mathbf{e}_\theta + c\mathbf{e}_z)\,\nabla\omega \cdot (r\mathbf{e}_\theta + c\mathbf{e}_z)\,\nabla\omega \\
&\quad +2\nabla\omega\,(r\mathbf{e}_\theta + c\mathbf{e}_z) \cdot (r\mathbf{e}_\theta + c\mathbf{e}_z)\,\nabla\omega \\
&\quad +\nabla\omega\,(r\mathbf{e}_\theta + c\mathbf{e}_z) \cdot \nabla\omega\,(r\mathbf{e}_\theta + c\mathbf{e}_z) \\
&= (r\mathbf{e}_\theta + c\mathbf{e}_z)\left(\frac{\partial\omega}{\partial\theta} + c\frac{\partial\omega}{\partial z}\right)\nabla\omega \\
&\quad +2\left(r^2 + c^2\right)\nabla\omega\nabla\omega \\
&\quad +\nabla\omega\left(\frac{\partial\omega}{\partial\theta} + c\frac{\partial\omega}{\partial z}\right)(r\mathbf{e}_\theta + c\mathbf{e}_z)\,.
\end{aligned}$$

Since $\omega = \omega\,(r, z - c\theta)$ is a function of r and $z - c\theta = Z$,

$$\frac{\partial\omega}{\partial\theta} + c\frac{\partial\omega}{\partial z} = -c\frac{\partial\omega}{\partial Z} + c\frac{\partial\omega}{\partial Z} = 0,$$

and the shear rate squared given by

$$\dot{\gamma}^2 = \left(r^2 + c^2\right)\nabla\omega \cdot \nabla\omega. \tag{S.343}$$

The vectors **a** and **b** are identified as

$$\dot{\gamma}\mathbf{a} = \mathbf{L}\mathbf{L}^T, \quad \dot{\gamma}\mathbf{b} = \mathbf{L}^T\mathbf{L}, \tag{S.344}$$

neither is constant.

Problem 6.3

This flow is one of the fan flows - the flow is uniaxial and by inspection of the boundary conditions we find that the velocity field is

$$\mathbf{u} = U\frac{\theta}{\theta_0}\mathbf{e}_z. \tag{S.345}$$

This velocity field may be written as

$$\mathbf{u} = \dot{\gamma}\,(\mathbf{b}\cdot\mathbf{x})\,\mathbf{a}, \tag{S.346}$$

where $\mathbf{a} = \mathbf{e}_z$. To identify $\mathbf{b}$ we note

$$\begin{aligned}\mathbf{L} = \nabla\mathbf{u}^T &= \left(\mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{u}\right)^T \\ &= \mathbf{e}_z\mathbf{e}_\theta\frac{U}{r\theta_0} = \dot{\gamma}\mathbf{a}\mathbf{b}.\end{aligned}$$

Clearly

$$\dot{\gamma} = \frac{U}{r\theta_0}, \quad \mathbf{b} = \mathbf{e}_\theta.$$

The stress is given by (see equation preceding 6.20)

$$\mathbf{T} = -P\mathbf{I} + \dot{\gamma}\eta\,(\mathbf{ab} + \mathbf{ba}) + (N_1 + N_2)\,(\mathbf{aa} + \mathbf{bb}) - N_1\mathbf{bb},$$

or

$$\mathbf{T} = -P\mathbf{I} + \eta\dot{\gamma}\,(\mathbf{e}_z\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_z) + (N_1 + N_2)\,\mathbf{e}_z\mathbf{e}_z + N_2\mathbf{e}_\theta\mathbf{e}_\theta. \tag{S.347}$$

In full, the non-trivial components for the stress are

$$T_{rr} = -P, \quad T_{\theta\theta} = -P + N_2, \quad T_{zz} = -P + N_1 + N_2, \quad T_{z\theta} = T_{\theta z} = \eta\dot{\gamma}.$$

To find the pressure field, we note that the velocity field is uni-directional, with zero inertia force $\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{0}$. Thus, the non-trivial component of the conservation of momentum equation is

$$0 = \frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{\partial T_{zr}}{\partial z}$$

$$\frac{\partial P}{\partial r} = -\frac{N_2}{r} = -\frac{\dot{\gamma}^2 \nu_2 (\dot{\gamma})}{r}$$

Noting that

$$\dot{\gamma} = \frac{U}{r\theta_0} = \frac{r_0 \dot{\gamma}_0}{r}, \quad \dot{\gamma}_0 = \frac{U}{r_0 \theta_0}$$

and $d\dot{\gamma}/\dot{\gamma} = -dr/r$, which can be used in integrating for the pressure:

$$\begin{aligned} P &= P(r_0) - \int_{r_0}^{r} \frac{\dot{\gamma}^2 \nu_2 (\dot{\gamma})}{r} dr \\ &= P(r_0) + \int_{\dot{\gamma}_0}^{\dot{\gamma}} \dot{\gamma} \nu_2 (\dot{\gamma}) . \end{aligned}$$

Alternatively,

$$P = P(r_0) + I_2(\dot{\gamma}) - I_2(\dot{\gamma}_0), \quad I_2(\dot{\gamma}) = \int_0^{\dot{\gamma}} \dot{\gamma} \nu_2 d\dot{\gamma}. \tag{S.348}$$

Suppose we have a pressure measurement $P = P(\dot{\gamma})$ at different shear rates. Then according to the preceding

$$\frac{dP}{d\dot{\gamma}} = \frac{dI_2}{d\dot{\gamma}} = \dot{\gamma} \nu_2 (\dot{\gamma}) .$$

This may be used to deduce ν_2, or $N_2 = \dot{\gamma}^2 \nu_2 (\dot{\gamma})$.

Problem 6.4

In the flow between two parallel, coaxial disks, or the *torsional flow*, the flow is a sub-class of the helicoidal flow (6.14), and by the boundary conditions on the plates, the velocity may be shown that (another way would be to show that the kinematics below satisfy conservation of mass and momentum)

$$\mathbf{u} = \Omega r \frac{z}{h} \mathbf{e}_\theta = \dot{\gamma} z \mathbf{e}_\theta, \quad \dot{\gamma} = \Omega \frac{r}{h}. \tag{S.349}$$

This velocity is already in the form (6.3)

$$\mathbf{u} = \dot{\gamma}\,(\mathbf{b}\cdot\mathbf{x})\,\mathbf{a}, \tag{S.350}$$

where

$$\mathbf{a} = \mathbf{e}_\theta, \tag{S.351}$$

and

$$(\mathbf{b}\cdot\mathbf{x}) = z, \quad \mathbf{b} = \mathbf{e}_z. \tag{S.352}$$

The stress is

$$\mathbf{T} = -P\mathbf{I} + \mathbf{S} = -P\mathbf{I} + \eta\dot{\gamma}\,(\mathbf{ab}+\mathbf{ba}) + \dot{\gamma}^2\,(\nu_1+\nu_2)\,\mathbf{aa} + \dot{\gamma}^2\nu_2\mathbf{bb}, \tag{S.353}$$

with the non-trivial components

$$T_{rr} = -P, \;\; T_{\theta\theta} = -P + N_1 + N_2, \;\; T_{zz} = -P + N_2, \;\; S_{z\theta} = S_{\theta z} = \eta\dot{\gamma}. \tag{S.354}$$

The conservation of momentum required, neglecting inertia terms,

$$\frac{\partial P}{\partial r} = \frac{\partial S_{rr}}{\partial r} + \frac{S_{rr} - S_{\theta\theta}}{r} = -\frac{N_1 + N_2}{r},$$

$$\frac{\partial P}{r\partial\theta} = 0,$$
$$\frac{\partial P}{\partial z} = 0.$$

Consequently, P is a function of r only. It is given by

$$P\,(r) = C - \int_0^r \frac{N_1\,(\zeta) + N_2\,(\zeta)}{\zeta}d\zeta \tag{S.355}$$

where C is a constant of integration. This is found by the assumption that at the free surface at $r = R$, the radial stress is zero:

$$T_{rr}\,(R) = -P\,(R) = 0. \tag{S.356}$$

This implies

$$C = \int_0^R \frac{N_1\,(\zeta) + N_2\,(\zeta)}{\zeta}d\zeta,$$

or that

$$P\left(r\right)=\int_r^R\frac{N_1\left(\zeta\right)+N_2\left(\zeta\right)}{\zeta}d\zeta$$
$$=\int_{\dot{\gamma}(r)}^{\dot{\gamma}_R}\frac{N_1\left(\dot{\gamma}\right)+N_2\left(\dot{\gamma}\right)}{\dot{\gamma}}d\dot{\gamma}$$
$$=\int_{\dot{\gamma}}^{\dot{\gamma}_R}\dot{\gamma}\left(\nu_1+\nu_2\right)d\dot{\gamma}, \quad \text{(S.357)}$$

where $\dot{\gamma}_R=\Omega R/h$ is the shear rate at the rim $r=R$.

The torque required to turn the top disk is $\dot{\gamma}=\Omega r/h=\dot{\gamma}_R r/R$

$$M=\int_0^R 2\pi r^2 S_{z\theta}dr$$
$$=2\pi\int_0^R\dot{\gamma}\eta\left(\dot{\gamma}\right)r^2dr$$
$$=2\pi\int_0^{\dot{\gamma}_R}\eta r^3 d\dot{\gamma}$$
$$=2\pi R^3\dot{\gamma}_R^{-3}\int_0^{\dot{\gamma}_R}\dot{\gamma}^3\eta\left(\dot{\gamma}\right)d\dot{\gamma}. \quad \text{(S.358)}$$

From the axial stress and the result for the pressure,

$$T_{zz}=-P+N_2$$
$$=N_2+\int_R^r\frac{N_1\left(\zeta\right)+N_2\left(\zeta\right)}{\zeta}d\zeta, \quad \text{(S.359)}$$

and therefore the normal force on the top disk is, by an integration by parts, keeping in mind that normal stresses are zero at the free surface $r=R$,

$$F=-2\pi\int_0^R T_{zz}r\,dr$$
$$=\pi\int_0^R T_{zz}'r^2dr$$
$$=\pi\int_0^R\left(-2rN_2+r\left(N_1+N_2\right)\right)dr$$
$$=\pi\int_0^R r\left(N_1-N_2\right)dr \quad \text{(S.360)}$$

Convert this into an integral with respect to the shear rate, the normal force is given by

$$F = \pi R^2 \dot{\gamma}_R^{-2} \int_0^{\gamma_R} \dot{\gamma} \left(N_1 - N_2\right) d\dot{\gamma}. \tag{S.361}$$

By normalizing the torque and the force as

$$m = \frac{M}{2\pi R^3}, \quad f = \frac{F}{\pi R^2}, \tag{S.362}$$

we have

$$m = \dot{\gamma}_R^{-3} \int_0^{\dot{\gamma}_R} \dot{\gamma}^3 \eta\left(\dot{\gamma}\right) d\dot{\gamma}.$$

Thus

$$\frac{dm}{d\dot{\gamma}_R} = -3\frac{m}{\dot{\gamma}_R} + \eta\left(\dot{\gamma}_R\right)$$

or

$$\begin{aligned} \eta\left(\dot{\gamma}_R\right) &= \frac{m}{\dot{\gamma}_R}\left(3 + \frac{\dot{\gamma}_R}{m}\frac{dm}{d\dot{\gamma}_R}\right) \\ &= \frac{m}{\dot{\gamma}_R}\left(3 + \frac{d\ln m}{d\ln \dot{\gamma}_R}\right), \end{aligned} \tag{S.363}$$

as required to show.

In addition

$$f = \dot{\gamma}_R^{-2} \int_0^{\gamma_R} \dot{\gamma} \left(N_1 - N_2\right) d\dot{\gamma}.$$

And thus

$$\frac{df}{d\dot{\gamma}_R} = -2\frac{f}{\dot{\gamma}_R} + N_1\left(\dot{\gamma}_R\right) - N_2\left(\dot{\gamma}_R\right),$$

or

$$N_1\left(\dot{\gamma}_R\right) - N_2\left(\dot{\gamma}_R\right) = f\left(2 + \frac{d\ln f}{d\ln \dot{\gamma}_R}\right). \tag{S.364}$$

Relations (S.363) and (S.364) are the basis for the operation of the parallel-disk viscometer.

Problem 6.5

Pipe flow is a special case for helical flow, where the velocity field is (uniaxial flow)

$$\mathbf{u} = u(r)\,\mathbf{e}_z. \tag{S.365}$$

This velocity is already in the viscometric form (6.3), and the non-trivial stresses are functions of r. The only non-trivial momentum equation (in the axial z direction) is

$$\frac{\partial P}{\partial z} = \frac{\partial S_{rz}}{\partial r} + \frac{S_{rz}}{r} = \frac{1}{r}\frac{d}{dr}\left(r S_{rz}\right), \tag{S.366}$$

where

$$S_{rz} = \eta \frac{du}{dr}.$$

The right side of (S.366) is a function of r alone, thus

$$\begin{aligned} P &= \left(\frac{\partial S_{rz}}{\partial r} + \frac{S_{rz}}{r}\right) z + P_0 \\ &= -\frac{\Delta P}{L} z + P_0, \end{aligned} \tag{S.367}$$

where $\Delta P/L$ is the pressure drop per unit length. This leads to

$$\begin{aligned} \frac{1}{r}\frac{d}{dr}\left(r S_{rz}\right) &= -\frac{\Delta P}{L} \\ S_{rz} &= -\frac{\Delta P}{2L} r. \end{aligned} \tag{S.368}$$

noting the boundedness of the stress in r. Thus the shear rate is

$$\frac{du}{dr} = -\frac{\Delta P}{2L} r \eta^{-1}. \tag{S.369}$$

The flow rate is

$$\begin{aligned} Q &= 2\pi \int_0^R r u(r)\, dr \\ &= -\pi \int_0^R r^2 \frac{du}{dr} dr \\ &= \frac{\pi \Delta P}{2L} \int_0^R r^3 \eta^{-1} dr \end{aligned}$$

Since $r = -2L\tau/\Delta P$, where $\tau = S_{rz}$, a change in variable yields

$$Q = 8\pi \left(\frac{L}{\Delta P}\right)^3 \int_0^{\tau_w} \tau^3 \eta^{-1} d\tau \tag{S.370}$$

where $\tau_w = -\Delta P R/(2L)$ is the shear stress at the wall. In terms of the reduced discharge rate,

$$q = \frac{Q}{\pi R^3} = \tau_w^{-3} \int_0^{\tau_w} \tau^3 \eta^{-1} d\tau, \tag{S.371}$$

and therefore

$$\frac{dq}{d\tau_w} = \frac{1}{\eta(\tau_w)} - \frac{3q}{\tau_w}, \tag{S.372}$$

or

$$\eta^{-1}(\tau_w) = \frac{q}{\tau_w}\left(3 + \frac{d\ln q}{d\ln \tau_w}\right). \tag{S.373}$$

Since $\eta(\tau_w)\dot{\gamma}_w = \tau_w$, or $\eta^{-1}(\tau_w) = \dot{\gamma}_w/\tau_w$

$$\dot{\gamma}_w = q(\tau_w)\left[3 + \frac{d\ln q}{d\ln \tau_w}\right]. \tag{S.374}$$

The relation (S.374) is due to Rabinowitch and is the basis for capillary viscometry.

Problems in Chap. 7

Problem 7.1

The solution to (7.22) is given by (7.24), reproduced here

$$\dot{\mathbf{x}}(t) = \int_0^t \exp\left\{\mathbf{m}^{-1}\boldsymbol{\zeta}\left(t' - t\right)\right\} \mathbf{m}^{-1}\mathbf{F}^{(b)}\left(t'\right) dt'. \tag{S.375}$$

Now, the diffusivity is defined

$$\mathbf{D} = \lim_{t\to\infty} \frac{1}{2}\int_0^t \langle \dot{\mathbf{x}}(t)\dot{\mathbf{x}}(t-\tau) + \dot{\mathbf{x}}(t-\tau)\dot{\mathbf{x}}(t)\rangle \, d\tau. \tag{S.376}$$

First, we form the correlation function

$$
\begin{aligned}
\mathbf{R}(\tau) &= \langle \dot{\mathbf{x}}(t+\tau)\ \dot{\mathbf{x}}(t)\rangle \\
&= \int_0^t \int_0^t \exp\left\{\mathbf{m}^{-1}\boldsymbol{\zeta}\left(t'-t-\tau\right)\right\}\mathbf{m}^{-1}\left\langle \mathbf{F}^{(b)}\left(t'\right)\ \mathbf{F}^{(b)}\left(t''\right)\right\rangle \\
&\qquad\qquad \mathbf{m}^{-1}\exp\left\{\mathbf{m}^{-1}\boldsymbol{\zeta}\left(t''-t\right)\right\}dt'dt'' \\
&= \int_0^t \exp\left\{\mathbf{m}^{-1}\boldsymbol{\zeta}\left(t'-t-\tau\right)\right\}\mathbf{m}^{-1}2\mathbf{f}\mathbf{m}^{-1}\exp\left\{\mathbf{m}^{-1}\boldsymbol{\zeta}\left(t'-t\right)\right\}dt' \\
&= e^{-\mathbf{m}^{-1}\boldsymbol{\zeta}\tau}\langle \dot{\mathbf{x}}(t)\ \dot{\mathbf{x}}(t)\rangle\,.
\end{aligned} \tag{S.377}
$$

Thus

$$
\begin{aligned}
\mathbf{D} &= \lim_{t\to\infty}\int_0^t e^{-\mathbf{m}\boldsymbol{\zeta}\tau}\left\langle \dot{\mathbf{x}}\left(t'\right)\dot{\mathbf{x}}\left(t'\right)\right\rangle d\tau \\
&= \boldsymbol{\zeta}^{-1}\mathbf{m}kT\mathbf{m}^{-1} \\
&= kT\boldsymbol{\zeta}^{-1}.
\end{aligned} \tag{S.378}
$$

This is the *Stokes–Einstein relation*, relating the diffusivity to the mobility of a Brownian particle.

Problem 7.2

In the limit $\mathbf{m}\to\mathbf{0}$, the Langevin equation (7.20) becomes (note that all the material matrices are symmetric)

$$
\dot{\mathbf{x}} = -\boldsymbol{\zeta}^{-1}\cdot\mathbf{K}\mathbf{x} + \boldsymbol{\zeta}^{-1}\mathbf{F}^{(b)}(t)\,, \tag{S.379}
$$

which has the solution

$$
\Delta\mathbf{x}(t) = -\boldsymbol{\zeta}^{-1}\cdot\mathbf{K}\mathbf{x}\Delta t + \int_t^{t+\Delta t}\boldsymbol{\zeta}^{-1}\mathbf{F}^{(b)}\left(t'\right)dt'. \tag{S.380}
$$

From this,

$$
\begin{aligned}
\langle \Delta\mathbf{x}(t)\rangle &= -\boldsymbol{\zeta}^{-1}\cdot\mathbf{K}\mathbf{x}\Delta t + \int_t^{t+\Delta t}\boldsymbol{\zeta}^{-1}\left\langle \mathbf{F}^{(b)}\left(t'\right)\right\rangle dt' \\
&= -\boldsymbol{\zeta}^{-1}\cdot\mathbf{K}\mathbf{x}.
\end{aligned} \tag{S.381}
$$

In addition,

$$\begin{aligned}\langle \Delta \mathbf{x}(t)\, \Delta \mathbf{x}(t) \rangle &= O\left(Deltat^2\right) \\ &\quad + \int_t^{t+\Delta t} \int_t^{t+\Delta t} dt'' \zeta^{-1} \left\langle \mathbf{F}^{(b)}\left(t'\right) \mathbf{F}^{(b)}\left(t''\right) \right\rangle \zeta^{-1} dt' \\ &= 2 \int_t^{t+\Delta t} dt' \zeta^{-1} \mathbf{f}\, \zeta^{-1} \\ &= 2kT\zeta^{-1}\Delta t. \end{aligned} \tag{S.382}$$

Thus the Fokker–Planck equation is

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial \mathbf{x}} \cdot \left[\frac{\langle \Delta \mathbf{x} \Delta \mathbf{x} \rangle}{2\Delta t} \frac{\partial \phi}{\partial \mathbf{x}} + \frac{\langle \Delta \mathbf{x} \rangle}{\Delta t} \phi \right]$$

is simply

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial \mathbf{x}} \cdot \left[kT\zeta^{-1} \frac{\partial \phi}{\partial \mathbf{x}} + \zeta^{-1} \cdot \mathbf{K}\mathbf{x}\phi \right]. \tag{S.383}$$

Problem 7.3

In a non-homogeneous flow and using the dumbell model, the particle centre of gravity will migrate from the streamline according to

$$\left\langle \dot{\mathbf{R}}^{(c)} - \mathbf{u}^{(c)} \right\rangle = \frac{1}{8} \langle \mathbf{R}\mathbf{R} \rangle : \nabla\nabla \mathbf{u}^{(c)}. \tag{S.384}$$

In the planar Poiseuille flow,

$$\mathbf{u}^{(c)} = u(y)\,\mathbf{i}, \quad u(y) = U\left(1 - y^2/h^2\right)$$

$$\begin{aligned}\left\langle \dot{\mathbf{R}}^{(c)} - \mathbf{u}^{(c)} \right\rangle &= \frac{1}{8} \left[\left\langle R_1^2 \right\rangle \frac{\partial^2}{\partial x^2} + 2\langle R_1 R_2 \rangle \frac{\partial^2}{\partial x \partial y} + 2 \langle R_1 R_3 \rangle \frac{\partial^2}{\partial x \partial z} \right. \\ &\quad \left. + \left\langle R_2^2 \right\rangle \frac{\partial^2}{\partial y^2} + 2\langle R_2 R_3 \rangle \frac{\partial^2}{\partial y \partial z} + \left\langle R_3^2 \right\rangle \frac{\partial^2}{\partial z^2} \right] \mathbf{u}^{(c)} \\ &= \frac{1}{8} \left\langle R_2^2 \right\rangle \frac{\partial^2}{\partial y^2} \mathbf{u}^{(c)} \\ &= -\frac{U}{4h^2} \left\langle R_2^2 \right\rangle \mathbf{i}. \end{aligned} \tag{S.385}$$

This must be solved together with the constitutive equation for $\mathbf{R}$ to obtain the migration velocity. In fact, putting $\mathbf{A} = \langle \mathbf{RR} \rangle$, and using the elastic dumbbell model, we have

$$\begin{aligned} A_{11} + \lambda \left(\dot{A}_{11} - 2u' A_{12} \right) &= \frac{1}{3}, \\ A_{12} + \lambda \left(\dot{A}_{12} - u' A_{22} \right) &= 0, \\ A_{22} + \lambda \dot{A}_{22} &= \frac{1}{3}. \end{aligned}$$

Thus the migration is along a streamline, and of the amount

$$\left\langle \dot{\mathbf{R}}^{(c)} - \mathbf{u}^{(c)} \right\rangle = -\frac{U}{12h^2}\mathbf{i}. \tag{S.386}$$

Of course, this is a simplistic model, to have cross-streamline migration, a better model is required, see for example, Goh et al., J Chem Phys 81 (1985) 6259–6265 and some of the references cited thereon.

Problem 7.4

The average end-to-end vector of a linear dumbbell evolves in time according to

$$\left\langle \dot{\mathbf{R}} \right\rangle = \mathbf{L} \left\langle \mathbf{R} \right\rangle - 2H\zeta^{-1} \left\langle \mathbf{R} \right\rangle , \tag{S.387}$$

This has the integrating factor $e^{t/2\lambda} e^{\mathbf{L}t}$:

$$\begin{aligned} \frac{d}{dt} \left(e^{t/2\lambda} e^{-\mathbf{L}t} \left\langle \mathbf{R} \right\rangle \right) &= e^{t/2\lambda} e^{-\mathbf{L}t} \left\langle \dot{\mathbf{R}} + \frac{1}{2\lambda}\mathbf{R} - \mathbf{LR} \right\rangle \\ &= 0, \end{aligned}$$

where $\lambda = \zeta / (4H)$ is the relaxation time. This has the solution

$$\left\langle \mathbf{R}\,(t) \right\rangle = e^{-t/2\lambda} e^{\mathbf{L}t} \mathbf{R}_0. \tag{S.388}$$

Whether or not $\langle \mathbf{R} \rangle$ decays to zero depends on the eigenvalues of $\mathbf{L} - \mathbf{I}/2\lambda$, if this is positive, $\langle \mathbf{R} \rangle$ is a run-away process, and if this is negative, then $\langle \mathbf{R} \rangle$ will decay to zero. Thus a strong flow will result if

$$\text{eigen } (\mathbf{L}) \geq 1/2\lambda, \tag{S.389}$$

where eigen $(\mathbf{L})$ is the maximum eigenvalue of $\mathbf{L}$. Otherwise we have a weak flow.

Problem 7.5

The upper-convected Maxwell model is written as

$$\mathbf{S}^{(p)} + \lambda \left\{ \frac{d}{dt}\mathbf{S}^{(p)} - \mathbf{L}\mathbf{S}^{(p)} - \mathbf{S}^{(p)}\mathbf{L}^T \right\} = G\mathbf{I}, \tag{S.390}$$

Now, consider the following integral model:

$$\mathbf{S}^{(p)}(t) = \frac{G}{\lambda} \int_{-\infty}^{t} e^{(s-t)/\lambda} \mathbf{C}_t(s)^{-1} \, ds, \tag{S.391}$$

one has

$$\dot{\mathbf{S}}^{(p)}(t) = \frac{G}{\lambda}\mathbf{I} + \frac{G}{\lambda}\int_{-\infty}^{t} -\frac{1}{\lambda} e^{(s-t)/\lambda}\mathbf{C}_t(s)^{-1}\, ds$$
$$+ \frac{G}{\lambda}\int_{-\infty}^{t} e^{(s-t)/\lambda}\left[\mathbf{L}(t)\,\mathbf{C}_t(s)^{-1} + \mathbf{C}_t(s)^{-1}\mathbf{L}^T(t)\right] ds,$$

or that

$$\mathbf{S}^{(p)} + \lambda \left\{ \frac{d}{dt}\mathbf{S}^{(p)} - \mathbf{L}\mathbf{S}^{(p)} - \mathbf{S}^{(p)}\mathbf{L}^T \right\} = G\mathbf{I}.$$

We conclude that (S.391) indeed solves (S.390).

Problems in Chap. 8

Problem 8.1

In the shear reversal experiments of Gadala-Maria and Acrivos (1980), it was found that if shearing is stopped after a steady state has been reached in a Couette device, the torque is reduced to zero instantaneously. This is due to the insignificant inertia of the suspended particles, and the micromechanics are governed by the Stokes equation (8.3), only the present boundary conditions matter. When the flow stops, all the forces, including the torque, go to zero instantaneously.

If shearing is resumed in the same direction after a period of rest, then the torque would attain its final value that corresponds to the resumed shear rate almost instantaneously. This is due to the equilibrated configuration of the suspended particles has been achieved and frozen in place after the flow stops. If the flow starts in the same direction, the particles are happy to stay in their previously preferred configuration, and the forces and torques instantaneously assume their previous values.

However, if shearing is resumed in the opposite direction, then the particles find a different preferred configuration corresponding to the reversed shear, and the forces and torques go through a period of adjustment to their steady state values. Zero, fading or infinite memory is a convenient description - in this suspension case, it is irrelevant to think of fading memory - the whole rheology is what matters.

Problem 8.2

Jeffery's solution for a unit vector $\mathbf{p}$ directed along the major axis of a spheroidal supsended particle obeys

$$\begin{aligned}\dot{\mathbf{p}} &= \mathbf{W}\cdot\mathbf{p} + \frac{R^2-1}{R^2+1}\left(\mathbf{D}\cdot\mathbf{p} - \mathbf{D}:\mathbf{ppp}\right) \\ &= \mathbf{L}\cdot\mathbf{p} - \frac{2}{R^2+1}\mathbf{D}\cdot\mathbf{p} - \frac{R^2-1}{R^2+1}\mathbf{D}:\mathbf{ppp},\end{aligned} \tag{S.392}$$

where R is the aspect ratio of the particle (major to minor diameter ratio), $\mathbf{L}$ is the velocity gradient, $\mathbf{D} = \left(\mathbf{L}+\mathbf{L}^T\right)/2$ is the strain rate, and $\mathbf{W} = \left(\mathbf{L}-\mathbf{L}^T\right)/2$ is the vorticity tensor. Denote the effective velocity gradient as

$$\mathcal{L} = \mathbf{L} - \frac{2}{R^2+1}\mathbf{D} = \mathbf{L} - \zeta\mathbf{D}, \tag{S.393}$$

then (S.392) can be simplified to

$$\dot{\mathbf{p}} = \mathcal{L}\cdot\mathbf{p} - \mathcal{L}:\mathbf{ppp}. \tag{S.394}$$

Now, consider the linear system

$$\dot{\mathbf{Q}} = \mathcal{L}\cdot\mathbf{Q}. \tag{S.395}$$

If we denote $\mathbf{Q} = Q\mathbf{p}$, where Q is the magnitude of $\mathbf{Q}$, and $\mathbf{p}$ is a unit vector, then

$$\dot{\mathbf{Q}} = \dot{Q}\mathbf{p} + Q\dot{\mathbf{p}} = Q\mathcal{L}\cdot\mathbf{p}.$$

Since $\mathbf{p}\cdot\dot{\mathbf{p}} = 0$,

$$\dot{Q} = Q\mathcal{L}:\mathbf{pp},$$

and thus

$$\dot{\mathbf{p}} = \mathcal{L}\cdot\mathbf{p} - \mathcal{L}:\mathbf{ppp}.$$

We conclude that (S.395) solves (S.394) and therefore solves (S.392). The parameter $\zeta = 2/(R^2+1)$ is a 'non-affine' parameter, representing the straining inefficiency of the flow.

Problem 8.3

In the start-up of a simple shear flow, the velocity gradient is

$$[\mathbf{L}] = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathcal{L}] = \begin{bmatrix} 0 & \left(1-\frac{\zeta}{2}\right)\dot{\gamma} & 0 \\ -\frac{\zeta}{2}\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where the shear rate is $\dot{\gamma}$, the process $\mathbf{Q}$ obeys

$$\begin{aligned} \dot{Q}_1 &= \left(1-\frac{\zeta}{2}\right)\dot{\gamma}Q_2, \\ \dot{Q}_2 &= -\frac{\zeta}{2}\dot{\gamma}Q_1, \\ \dot{Q}_3 &= 0. \end{aligned} \tag{S.396}$$

This implies $Q_3 = Q_{30}$, a constant, and

$$\begin{aligned} \ddot{Q}_1 + \frac{\zeta}{2}\left(1-\frac{\zeta}{2}\right)\dot{\gamma}^2 Q_1 &= 0, \\ \ddot{Q}_2 + \frac{\zeta}{2}\left(1-\frac{\zeta}{2}\right)\dot{\gamma}^2 Q_2 &= 0. \end{aligned}$$

The solutions are

$$\begin{aligned} Q_1 &= Q_{10}\cos\omega t + \sqrt{\frac{2-\zeta}{\zeta}}\,Q_{20}\sin\omega t, \\ Q_2 &= Q_{20}\cos\omega t - \sqrt{\frac{\zeta}{2-\zeta}}\,Q_{10}\sin\omega t, \end{aligned}$$

where $\{Q_{10}, Q_{20}, Q_{30}\}$ are the initial components of $\mathbf{Q}$, and the frequency of the oscillation is

$$\omega = \frac{1}{2}\dot{\gamma}\sqrt{\zeta(2-\zeta)} = \frac{\dot{\gamma}R}{R^2+1}.$$

From these results, the result for $\mathbf{p}$ can be obtained.

The stress is

$$\text{œ} = 2\eta_s \mathbf{D} + 2\eta_s \phi\{A\mathbf{D} : \mathbf{pppp} + B\left(\mathbf{D}\cdot\mathbf{pp} + \mathbf{pp}\cdot\mathbf{D}\right) + C\mathbf{D}\}, \tag{S.397}$$

In full we have the reduced viscosity:

$$\frac{\langle\sigma_{12}\rangle - \eta_s\dot{\gamma}}{\eta_s\dot{\gamma}\phi} = 2Ap_1^2p_2^2 + B\left(p_1^2 + p_2^2\right) + C, \tag{S.398}$$

the reduced first normal stress difference:

$$\frac{N_1}{\eta_s\dot{\gamma}\phi} = 2Ap_1p_2\left(p_1^2 - p_2^2\right), \tag{S.399}$$

and the reduced second normal stress difference:

$$\frac{N_2}{\eta_s\dot{\gamma}\phi} = 2p_1p_2\left(Ap_2^2 + B\right). \tag{S.400}$$

Thus, the particles tumble along with the flow, with a period of $T = 2\pi(R^2+1)/\dot{\gamma}R$, spending most of their time aligned with the flow.

Problem 8.4

In the start-up of an elongational flow with a positive elongational rate $\dot{\gamma}$, the velocity gradient is

$$[\mathbf{L}] = \begin{bmatrix} \dot{\gamma} & 0 & 0 \\ 0 & -\dot{\gamma}/2 & 0 \\ 0 & 0 & -\dot{\gamma}/2 \end{bmatrix}, \quad [\mathcal{L}] = (1-\zeta)\begin{bmatrix} \dot{\gamma} & 0 & 0 \\ 0 & -\dot{\gamma}/2 & 0 \\ 0 & 0 & -\dot{\gamma}/2 \end{bmatrix},$$

$$\begin{aligned} \dot{Q}_1 &= (1-\zeta)\,\dot{\gamma}Q_1, \\ \dot{Q}_2 &= -\frac{1}{2}(1-\zeta)\,\dot{\gamma}Q_2, \\ \dot{Q}_3 &= -\frac{1}{2}(1-\zeta)\,\dot{\gamma}Q_3, \end{aligned}$$

which have the solutions

$$\begin{aligned} Q_1 &= Q_{10}\exp\{(1-\zeta)\dot{\gamma}t\}, \\ Q_2 &= Q_{20}\exp\left\{-\frac{1}{2}(1-\zeta)\dot{\gamma}t\right\}, \end{aligned}$$

$$Q_3 = Q_{30} \exp\left\{-\frac{1}{2}(1-\zeta)\dot{\gamma}t\right\},$$

so that the particle is quickly aligned with the flow in a time scale $O(\dot{\gamma}^{-1})$, $\mathbf{p} \to (1, 0, 0)$. The stress is

$$\text{œ} = 2\eta_s \mathbf{D} + 2\eta_s \phi\{A\mathbf{D} : \mathbf{pppp} + B\left(\mathbf{D}\cdot\mathbf{pp} + \mathbf{pp}\cdot\mathbf{D}\right) + C\mathbf{D}\}, \tag{S.401}$$

In full we have:

$$\begin{aligned}
\frac{\sigma_{11}}{\eta_s\dot{\gamma}} &= 2 + 2\phi\left[A\left(p_1^2 - \frac{1}{2}p_2^2 - \frac{1}{2}p_3^2\right)p_1^2 + 2Bp_1^2 + C\right] \\
\frac{\sigma_{22}}{\eta_s\dot{\gamma}} &= -1 + 2\phi\left[A\left(p_1^2 - \frac{1}{2}p_2^2 - \frac{1}{2}p_3^2\right)p_2^2 - 2Bp_2^2 - \frac{1}{2}C\right] \\
\frac{\sigma_{33}}{\eta_s\dot{\gamma}} &= -1 + 2\phi\left[A\left(p_1^2 - \frac{1}{2}p_2^2 - \frac{1}{2}p_3^2\right)p_3^2 - 2Bp_3^2 - \frac{1}{2}C\right]
\end{aligned}$$

This yields the reduced elongational viscosity

$$\frac{N_1 - 3\eta_s\dot{\gamma}}{\eta_s\dot{\gamma}\phi} = 2\left[A\left(p_1^2 - \frac{1}{2}p_2^2 - \frac{1}{2}p_3^2\right)\left(p_1^2 - p_2^2\right) + 2B\left(p_1^2 + p_2^2\right) + \frac{3}{2}C\right]$$

At a steady state, show that the reduced elongational viscosity is given by

$$\frac{N_1 - 3\eta_s\dot{\gamma}}{\eta_s\dot{\gamma}\phi} = 2A + 4B + 3C \approx 2A = \frac{R^2}{\ln 2R - 1.5}. \tag{S.402}$$

This elongational viscosity could be several order of magnitudes greater than the shear viscosity, due to the term $O\left(R^2\right)$.

Problems in Chap. 9

Problem 9.1

We start with the 1-D system

$$\begin{aligned}
&\frac{dr}{dt} = v, \quad m\frac{dv}{dt} = F_c - \gamma w_D v + \sigma w_R \theta(t), \; r(0) = r_0, \; v(0) = v_0, \\
&\langle\theta(t)\rangle = 0, \quad \langle\theta(t)\,\theta(t+\tau)\rangle = \delta(\tau).
\end{aligned} \tag{S.403}$$

In the inertial time scale, the displacement, the force F_c *and* the wighting functions may be regarded as constant. We may re-define the velocity as

$$v = u + F_c/\gamma w_D,$$

thus eliminating F_c in the governing equation altogether, thus only deal with a linear system

$$m\frac{dv}{dt} = -\gamma w_D v + \sigma w_R \theta(t)\,. \tag{S.404}$$

This system has the integrating constant $e^{m^{-1}\gamma w_D t}$

$$\begin{aligned}\frac{d}{dt}\left(e^{m^{-1}\gamma w_D t} v\right) &= e^{m^{-1}\gamma w_D t}\left(\dot{v} + m^{-1}\gamma w_D v\right)\\ &= e^{m^{-1}\gamma w_D t} m^{-1}\sigma w_R \theta(t)\,,\end{aligned}$$

which can be integrated to yield

$$v(t) = \int_0^t e^{m^{-1}\gamma w_D (t'-t)} m^{-1}\sigma w_R \theta\left(t'\right) dt'. \tag{S.405}$$

Its mean square velocity is

$$\begin{aligned}\langle v(t)\, v(t)\rangle &= \frac{\sigma^2}{m^2}\int_0^t\int_0^t e^{m^{-1}\gamma w_D (t'-t)} w_R^2 \left\langle\theta\left(t'\right)\theta\left(t''\right)\right\rangle e^{m^{-1}\gamma w_D (t''-t)} dt' dt''\\ &= \frac{\sigma^2}{m^2}\int_0^t e^{m^{-1}\gamma w_D (t'-t)} w_R e^{m^{-1}\gamma w_D (t'-t)} w_R dt',\\ &= \frac{1}{2} m^{-1}\sigma^2 w_R^2 \gamma^{-1} w_D^{-1}.\end{aligned} \tag{S.406}$$

Assuming the equi-partition principle

$$\frac{1}{2} m\left\langle v^2(t)\right\rangle = \frac{1}{2} k_B T, \tag{S.407}$$

leads directly to

$$\sigma^2 w_R^2 \gamma^{-1} w_D^{-1} = 2 k_B T. \tag{S.408}$$

Problem 9.2

In the case of small inertia, our main stochastic system becomes

$$\frac{dr}{dt} = \gamma^{-1} w_D^{-1} F_c + \gamma^{-1} w_D^{-1}\sigma w_R \theta(t)\,. \tag{S.409}$$

This may be solved in one time step Δt

$$\Delta r = \gamma^{-1} w_D^{-1} F_c \Delta t + \int_t^{t+\Delta t} \gamma^{-1} w_D^{-1} \sigma w_R \theta \left(t'\right) dt'. \tag{S.410}$$

From this, and the properties of white noise, the drift velocity is given by

$$\frac{\langle \Delta r \rangle}{\Delta t} = \gamma^{-1} w_D^{-1} F_c. \tag{S.411}$$

In addition

$$\begin{aligned} \langle \Delta r \Delta r \rangle \quad &= O\left(\Delta t^2\right) + \gamma^{-2} w_D^{-2} \sigma^2 w_R^2 \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \left\langle \theta\left(t'\right) \theta\left(t''\right) \right\rangle \\ &= O\left(\Delta t^2\right) + \gamma^{-2} w_D^{-2} \sigma^2 w_R^2 \Delta t \\ &= O\left(\Delta t^2\right) + 2 k_B T \gamma^{-1} w_D^{-1} \Delta t. \end{aligned}$$

In the last step, the equipartition principle has been used. Thus

$$\frac{\langle \Delta r \Delta r \rangle}{2 \Delta t} = O\left(\Delta t\right) + k_B T \gamma^{-1} w_D^{-1}, \tag{S.412}$$

and the Fokker–Planck equation is

$$\frac{\partial}{\partial t} W\left(r, t\right) = \lim_{\Delta t \to 0} \frac{\partial}{\partial r} \left[\left(\frac{\langle \Delta r \Delta r \rangle}{2 \Delta t} \frac{\partial}{\partial r} - \frac{\langle \Delta r \rangle}{\Delta t} \right) W\left(r, t\right) \right], \tag{S.413}$$

or

$$\frac{\partial}{\partial t} W\left(r, t\right) = \frac{\partial}{\partial r} \left[\left(\frac{k_B T}{\gamma w_D} \frac{\partial}{\partial r} - \frac{F_c}{\gamma w_D} \right) W\left(r, t\right) \right]. \tag{S.414}$$

Problem 9.3

In order to focus on events on the time scale τ_I we can regard the restoring force F_c as constant, so that it can be absorbed in a re-definition of the system state system:

$$\frac{dv}{dt} = -m^{-1} \gamma w_D v + m^{-1} \sigma w_R \theta\left(t\right). \tag{S.415}$$

Noting that

$$2vr = \frac{d}{dt}\left(r^2\right), \; 2\left(\dot{v} r + v^2\right) = \frac{d^2}{dt^2}\left(r^2\right),$$

we find, by multiplying (S.415) with r,

$$m\left(\dot{v}r + v^2\right) - mv^2 - \gamma w_D vr = \sigma w_R \theta(t)\, r,$$

or,

$$\frac{1}{2}m\frac{d}{dt}\left(\frac{d}{dt}\left\langle r^2\right\rangle\right) - m\left\langle v^2\right\rangle + \frac{1}{2}\gamma w_D \frac{d}{dt}\left\langle r^2\right\rangle = \sigma w_R \left\langle \theta(t)\, r\right\rangle. \tag{S.416}$$

We now define

$$e = d\left\langle r^2\right\rangle/dt.$$

From the temperature definition, $m\left\langle v^2\right\rangle = k_B T$; furthermore, $\left\langle \theta(t)\, r\right\rangle = 0$ due to different time scales of $\theta(t)$ and r. Thus

$$\dot{e} + m^{-1}\gamma w_D e = 2k_B T m^{-1}, \quad e(0) = 0.$$

This has the integration factor

$$\frac{d}{dt}\left(e\exp\left(m^{-1}\gamma w_D t\right)\right) = 2k_B T m^{-1}\exp\left(m^{-1}\gamma w_D t\right),$$

which has the solution, for the assumed initial condition,

$$e = \frac{d}{dt}\left\langle r^2\right\rangle = 2k_B T\gamma^{-1}w_D^{-1}\left[1 - e^{-m^{-1}\gamma w_D t}\right]. \tag{S.417}$$

Consequently, if $\Delta t \gg \tau_I = O\left(m^{-1}\gamma w_D\right)$, writing $\Delta r = r(\Delta t)$,

$$\frac{\left\langle \Delta r\,\Delta r\right\rangle}{2\Delta t} = k_B T\gamma^{-1}w_D^{-1}. \tag{S.418}$$

Problem 9.4

Define the velocity correlation as

$$R(\tau) = \lim_{t\to\infty}\left\langle v(t+\tau)\, v(t)\right\rangle, \tag{S.419}$$

where the limit refers to large time compared to the inertial time scale, but yet small compared to the relaxation time scale. A formal solution of (S.415) is

$$v(t) = \int_0^t e^{m^{-1}\gamma w_D\left(t'-t\right)} m^{-1}\sigma w_R \theta\left(t'\right) dt'. \tag{S.420}$$

From this solution, and for $\tau > 0$

$$R(\tau) = \lim_{t\to\infty} \int_0^{t+\tau} dt' \int_0^t e^{m^{-1}\gamma w_D(t'-t-\tau)} \left(m^{-1}\sigma w_R\right)^2$$
$$\left\langle\theta\left(t'\right)\theta\left(t''\right)\right\rangle e^{m^{-1}\gamma w_D(t''-t)} dt'' \tag{S.421}$$
$$= e^{-m^{-1}\gamma w_D\tau} \lim_{t\to\infty} \langle v(t)\, v(t)\rangle = e^{-m^{-1}\gamma w_D\tau} R(0)$$
$$= k_B T m^{-1} e^{-m^{-1}\gamma w_D \tau}. \tag{S.422}$$

That is, the velocity correlation decays after an inertial time scale, after which the velocity is independent to its previous state.

Next, the diffusivity can also be defined as

$$D = \lim_{t\to\infty} \frac{1}{2} \langle v(t)\, r(t) + r(t)\, v(t)\rangle .$$

This is equivalent to

$$D = \lim_{t\to\infty} \int_0^t \langle v(t)\, v(t+\tau)\rangle\, d\tau = \int_0^\infty R(\tau)\, d\tau$$
$$= k_B T \gamma^{-1} w_D^{-1},$$

consistent with previous results.

Problem 9.5

Now, consider the Langevin system

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \quad m\frac{d\mathbf{v}_i}{dt} = \sum_j \mathbf{F}_{ij}. \tag{S.423}$$

Here, $\mathbf{F}_{ij}$ is the pairwise additive interparticle force by particle j on particle i; this force consists of three parts, a conservative force $\mathbf{F}_{ij}^C$, a dissipative force $\mathbf{F}_{ij}^D$, and a random force $\mathbf{F}_{ij}^R$:

$$\mathbf{F}_{ij} = \mathbf{F}_{ij}^C + \mathbf{F}_{ij}^D + \mathbf{F}_{ij}^R. \tag{S.424}$$

From (S.423), the drift velocity of the process is

$$\left\langle \frac{\Delta \mathbf{v}_i}{\Delta t} \right\rangle = -m^{-1} \sum_j \left(\mathbf{F}_{ij}^C + \mathbf{F}_{ij}^D\right) = -\, m^{-1} \sum_j \left(\mathbf{F}_{ij}^C - \gamma w_{ij}^D \mathbf{e}_{ij}\mathbf{e}_{ij} \cdot \mathbf{v}_{ij}\right). \tag{S.425}$$

Furthermore,

$$\left\langle \Delta\mathbf{v}_\alpha \Delta\mathbf{v}_\beta \right\rangle = O\left(\Delta t^2\right) + \sum_{k,m} \int^{\Delta t} dt' \int^{\Delta t} dt'' \frac{\sigma^2}{m^2} w^R_{\alpha k} w^R_{\beta m} \left(\delta_{\alpha\beta}\delta_{km} + \delta_{\alpha m}\delta_{\beta k}\right) \delta\left(t' - t''\right) \mathbf{e}_{\alpha k}\mathbf{e}_{\beta m},$$

or

$$\left\langle \Delta\mathbf{v}_\alpha \Delta\mathbf{v}_\beta \right\rangle = O\left(\Delta t^2\right) + \Delta t \frac{\sigma^2}{m^2} \left[\delta_{\alpha\beta} \sum_k \left(w^R_{\alpha k}\right)^2 \mathbf{e}_{\alpha k}\mathbf{e}_{\beta k} + \left(w^R_{\alpha\beta}\right)^2 \mathbf{e}_{\alpha\beta}\mathbf{e}_{\beta\alpha} \right],$$

Using the fluctuation-dissipation theorem,

$$\left\langle \Delta\mathbf{v}_\alpha \Delta\mathbf{v}_\beta \right\rangle = O\left(\Delta t^2\right) + 2\frac{k_B T \gamma}{m^2} \Delta t \left[\delta_{\alpha\beta} \sum_k w^D_{\alpha k} \mathbf{e}_{\alpha k}\mathbf{e}_{\beta k} - w^D_{\alpha\beta} \mathbf{e}_{\alpha\beta}\mathbf{e}_{\alpha\beta} \right],$$

$$\left\langle \frac{\Delta\mathbf{v}_\alpha \Delta\mathbf{v}_\beta}{2\Delta t} \right\rangle = \frac{\gamma k_B T}{m^2} \left[\delta_{\alpha\beta} \sum_k w^D_{\alpha k} \mathbf{e}_{\alpha k}\mathbf{e}_{\beta k} - w^D_{\alpha\beta} \mathbf{e}_{\alpha\beta}\mathbf{e}_{\alpha\beta} \right]. \tag{S.426}$$

The Fokker–Planck equation for the process is

$$\frac{\partial f}{\partial t} + \sum_i \frac{\partial}{\partial \mathbf{r}_i} \cdot (\mathbf{v}_i f) + \sum_i \frac{\partial}{\partial \mathbf{v}_i} \cdot \left(\left\langle \frac{\Delta\mathbf{v}_i}{\Delta t} \right\rangle f \right) = \sum_{i,j} \frac{\partial}{\partial \mathbf{v}_i} \cdot \left(\left\langle \frac{\Delta\mathbf{v}_i \Delta\mathbf{v}_j}{2\Delta t} \right\rangle \cdot \frac{\partial f}{\partial \mathbf{v}_j} \right), \tag{S.427}$$

where the limit $\Delta t \to 0$ is implied. Using the results obtained above for the drift and the diffusivity, we finally obtain the Fokker–Planck equation for the process

$$\frac{\partial f}{\partial t} + \sum_i \frac{\partial}{\partial \mathbf{r}_i} \cdot (\mathbf{v}_i f) + \sum_{i,j} \frac{\partial}{\partial \mathbf{p}_i} \cdot \left(\mathbf{F}^C_{ij} f\right) = \gamma \sum_{i,j} w^D_{ij} \mathbf{e}_{ij} \frac{\partial}{\partial \mathbf{p}_i} \cdot \left(\mathbf{e}_{ij} \cdot \mathbf{v}_{ij} f\right) + \gamma k_B T \sum_{i,j} w^D_{ij} \mathbf{e}_{ij} \cdot \frac{\partial}{\partial \mathbf{p}_i} \cdot \left(\mathbf{e}_{ij} \cdot \left(\frac{\partial f}{\partial \mathbf{p}_i} - \frac{\partial f}{\partial \mathbf{p}_j} \right) \right), \tag{S.428}$$

where $\mathbf{p}_i = m\mathbf{v}_i$ is the linear momentum of particle i.

Problem 9.6

We note that the Hamiltonian of the associate system is

$$\mathcal{H}=\sum_i \frac{\mathbf{p}_i \cdot \mathbf{p}_i}{2m} + \frac{1}{2}\sum_{i,j}\varphi\left(r_{ij}\right), \tag{S.429}$$

noting

$$\frac{\partial \mathcal{H}}{\partial \mathbf{p}_\alpha} = \frac{\mathbf{p}_\alpha}{m} = \mathbf{v}_\alpha, \quad \frac{\partial \mathcal{H}}{\partial \mathbf{r}_{\alpha\beta}} = -\mathbf{F}^C_{\alpha\beta}.$$

Thus, the equilibrium distribution of the associate system to is

$$\begin{aligned} f_{eq}\left(\chi, t\right) &= \frac{1}{Z}\exp\left[-\frac{1}{k_B T}\left(\sum_i \frac{\mathbf{p}_i \cdot \mathbf{p}_i}{2m} + \frac{1}{2}\sum_{i,j}\varphi\left(r_{ij}\right)\right)\right] \\ &= \frac{1}{Z}\exp\left[-\frac{\mathcal{H}}{k_B T}\right], \end{aligned} \tag{S.430}$$

where Z is a normalizing constant.

We further note that,

$$\sum_i \mathbf{v}_i \cdot \frac{\partial f_{eq}}{\partial \mathbf{r}_i} = \frac{1}{k_B T}\sum_{i,j}\mathbf{v}_i \cdot \mathbf{F}^C_{ij} f_{eq}, \quad \sum_{i,j}\mathbf{F}^C_{ij} \cdot \frac{\partial f_{eq}}{\partial \mathbf{p}_i} = -\frac{1}{k_B T}\sum_{i,j}\mathbf{F}^C_{ij} \cdot \mathbf{v}_i f_{eq},$$

$$\gamma \sum_{i,j} w^D_{ij}\mathbf{e}_{ij}\frac{\partial}{\partial \mathbf{p}_i} \cdot \left(\mathbf{e}_{ij} \cdot \mathbf{v}_{ij} f_{eq}\right) = \gamma \sum_{i,j} w^D_{ij}\left(-\frac{1}{k_B T}\mathbf{e}_{ij} \cdot \mathbf{v}_i \mathbf{e}_{ij} \cdot \mathbf{v}_{ij} + \frac{1}{m}\right) f_{eq},$$

$$\begin{aligned} &\gamma k_B T \sum_{i,j} w^D_{ij}\mathbf{e}_{ij} \cdot \frac{\partial}{\partial \mathbf{p}_i} \cdot \left(\mathbf{e}_{ij} \cdot \left(\frac{\partial}{\partial \mathbf{p}_i} - \frac{\partial}{\partial \mathbf{p}_j}\right) f_{eq}\right) \\ &= \gamma \sum_{i,j} w^D_{ij}\left(-\frac{1}{m} + \frac{1}{k_B T}\mathbf{e}_{ij} \cdot \mathbf{v}_i \mathbf{e}_{ij}\mathbf{v}_{ij}\right), \end{aligned}$$

and conclude that f_{eq} is also a stationary solution (i.e., solution that is independent of time) of the Fokker–Planck equation (S.428).

Problem 9.7

The stress contributed from the conservative forces,

$$\mathbf{S}_C(\mathbf{r},t) = -\frac{1}{2}\int d\mathbf{R}\mathbf{F}^C\mathbf{R}\left\{1-\frac{1}{2}\mathbf{R}\cdot\nabla+\cdots\right\}\bar{f}_2(\mathbf{r}+\mathbf{R},\mathbf{r},t)\,. \qquad \text{(S.431)}$$

Marsh used a different technique in deriving the stresses, which involves expressing the delta function as an integral. From this, the stresses from the conservative forces is expressed as,

$$\mathbf{S}_C(\mathbf{r},t) = -\frac{1}{2}\int d\mathbf{v}'\int d\mathbf{v}''\int \mathbf{F}^C\mathbf{R}\bar{W}_2\left(\chi',\chi'',t;\mathbf{r}\right)d\mathbf{R}, \qquad \text{(S.432)}$$

where

$$\bar{W}_2\left(\mathbf{r}',\mathbf{v}',\mathbf{r}'',\mathbf{v}'',t;\mathbf{r}\right) = \int_0^1 f_2\left(\mathbf{r}+\lambda\mathbf{R},\mathbf{r}-(1-\lambda)\mathbf{R},\mathbf{v}',\mathbf{v}'',t\right)d\lambda. \qquad \text{(S.433)}$$

Expressing $f_2\left(\mathbf{r}+\lambda\mathbf{R},\mathbf{r}-(1-\lambda)\mathbf{R},\mathbf{v}',\mathbf{v}'',t\right)$ as a function of $(\mathbf{r}-\varepsilon\mathbf{R})$, where $\varepsilon = 1-\lambda$,

$$f_2 = f_2\left(\mathbf{r}+\mathbf{R}-\varepsilon\mathbf{R},\mathbf{r}-\varepsilon\mathbf{R},\mathbf{v}',\mathbf{v}'',t\right),$$

and taking a Taylor's series in ε,

$$f_2 = f_2\left(\mathbf{r}+\mathbf{R},\mathbf{r},\mathbf{v}',\mathbf{v}'',t\right) - \varepsilon\mathbf{R}\cdot\nabla f_2\left(\mathbf{r}+\mathbf{R},\mathbf{r},\mathbf{v}',\mathbf{v}'',t\right)+\cdots$$

Integrating

$$\begin{aligned}\bar{W}_2\left(\mathbf{r}',\mathbf{v}',\mathbf{r}'',\mathbf{v}'',t;\mathbf{r}\right) &= \int_0^1 f_2\left(\mathbf{r}+\lambda\mathbf{R},\mathbf{r}-(1-\lambda)\mathbf{R},\mathbf{v}',\mathbf{v}'',t\right)d\varepsilon \\ &= \left\{1-\frac{1}{2}\mathbf{R}\cdot\nabla+\cdots\right\}f_2\left(\mathbf{r}+\mathbf{R},\mathbf{r},\mathbf{v}',\mathbf{v}'',t\right).\end{aligned}$$

This, substituted in (S.432) and the velocity spaces are integrated out, leads to (S.431). The remaining part of the question is demonstrated similarly.

Problem 9.8

The "heat flux" has been shown to be

$$\mathbf{q}_C = \left\langle\sum_{i,j}\frac{1}{4}\mathbf{F}^C_{ij}\cdot\left(\mathbf{v}_i+\mathbf{v}_j\right)\mathbf{r}_{ij}\left(1-\frac{1}{2}\mathbf{r}_{ij}\cdot\nabla+\cdots\right)\delta\left(\mathbf{r}-\mathbf{r}_j\right)\right\rangle. \qquad \text{(S.434)}$$

This can be demonstrated, in an almost verbatim manner to the treatment of the stress, to be

$$\mathbf{q}_C(\mathbf{r},t)=\frac{1}{4}\int d\mathbf{R}\int d\mathbf{v}\int d\mathbf{v}'\mathbf{F}^C(\mathbf{R})\cdot\left(\mathbf{v}+\mathbf{v}'\right)\mathbf{R}\left\{1-\frac{1}{2}\mathbf{R}\cdot\nabla+\cdots\right\}$$
$$\cdot f_2\left(\mathbf{r}+\mathbf{R},\mathbf{r},\mathbf{v},\mathbf{v}',t\right). \tag{S.435}$$

Problem 9.9

The stress contributed from the damping forces is,

$$\mathbf{S}_D(\mathbf{r},t)=-\frac{1}{2}\int d\mathbf{R}\int d\mathbf{v}\int d\mathbf{v}'\gamma w^D(R)\,\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot\left(\mathbf{v}-\mathbf{v}'\right)\mathbf{R}\left\{1+O(R)\right\}$$
$$\cdot f_2\left(\mathbf{r}+\mathbf{R},\mathbf{r},\mathbf{v},\mathbf{v}',t\right).$$

For a homogeneously shear flow, the velocity gradient is constant. By taking the particles' velocities as the fluid's velocities at the particles' locations: $\mathbf{v}-\mathbf{v}'=\mathbf{LR}$, where $\mathbf{L}$ is the velocity gradient, together with Groot and Warren's approximation, $f_2=n^2\left(1+O(R)\right)$, the stress contributed by the damping forces is

$$S_{D,\alpha\beta}(\mathbf{r},t)=-\frac{\gamma n^2}{2}\left\langle\hat{R}_\alpha\hat{R}_\beta\hat{R}_i\hat{R}_jL_{ij}\int R^2w^D(R)\,4\pi R^2dR\right\rangle$$
$$=-\frac{2\pi\gamma n^2}{15}\left(\delta_{\alpha\beta}\delta_{ij}+\delta_{\alpha i}\delta_{\beta j}+\delta_{\alpha j}\delta_{\beta i}\right)L_{ij}\int R^4w^D(R)\,dR.$$

Here we have assume a completely isotropic distribution for the structure tensor:

$$\left\langle\hat{R}_\alpha\hat{R}_\beta\hat{R}_i\hat{R}_j\right\rangle=\frac{1}{15}\left(\delta_{\alpha\beta}\delta_{ij}+\delta_{\alpha i}\delta_{\beta j}+\delta_{\alpha j}\delta_{\beta i}\right).$$

For the standard weighting function (9.88) adopted in DPD, one has

$$S_{D,\alpha\beta}(\mathbf{r},t)=-\frac{2\pi\gamma n^2}{15}\left(L_{\alpha\beta}+L_{\beta\alpha}+L_{ii}\delta_{\alpha\beta}\right)\int_0^{r_c}R^4\left(1-R/r_c\right)^2dR$$
$$=\frac{2\pi\gamma n^2r_c^5}{1575}\left(L_{\alpha\beta}+L_{\beta\alpha}+L_{ii}\delta_{\alpha\beta}\right), \tag{S.436}$$

and consequently the damping-contributed viscosities are given by

$$\eta_D=\frac{2\pi\gamma n^2r_c^5}{1575},\quad \zeta_D=\frac{5}{3}\eta_D=\frac{2\pi\gamma n^2r_c^5}{945}. \tag{S.437}$$

For the modified weighting function (9.125) with $s = 1/2$,

$$S_{D,\alpha\beta}(\mathbf{r}, t) = \frac{512\pi\gamma n^2 r_c^5}{51975}\left(L_{\alpha\beta} + L_{\beta\alpha} + L_{ii}\delta_{\alpha\beta}\right) \qquad \text{(S.438)}$$

and consequently

$$\eta_D = \frac{512\pi\gamma n^2 r_c^5}{51975}, \quad \zeta_D = \frac{5}{3}\eta_D = \frac{512\pi\gamma n^2 r_c^5}{31185}. \qquad \text{(S.439)}$$

Problem 9.10

By using the kinetic theory or by considering the case of a homogeneously shear flow (as shown above), one can obtain an estimate for the viscosity and diffusivity

$$\eta = \eta_K + \eta_D = \frac{\rho D}{2} + \frac{\gamma n^2 \left[R^2 w^D\right]_R}{30},$$
$$D = \frac{3k_B T}{n\gamma \left[w^D\right]_R},$$

where

$$\left[w^D\right]_R = \int_0^{r_c} 4\pi R^2 w^D(R)\, dR, \quad \left[R^2 w^D\right]_R = \int_0^{r_c} 4\pi R^4 w^D(R)\, dR\ .$$

With the standard weighting function,

$$\left[w^D\right]_R = \frac{2\pi}{15} r_c^3, \quad \left[R^2 w^D\right]_R = \frac{4\pi}{105} r_c^5,$$

which leads to

$$\eta = \frac{45mkT}{4\pi\gamma r_c^3} + \frac{2\pi\gamma n^2 r_c^5}{1575}, \quad D = \frac{45k_B T}{2\pi\gamma n r_c^3}.$$

Problem 9.11

With the modified weighting function ($s = 1/2$),

$$\left[w^D\right]_R = \frac{64\pi}{105} r_c^3, \quad \left[R^2 w^D\right]_R = \frac{1024\pi}{3465} r_c^5,$$

which leads to

$$\eta = \frac{315mk_BT}{128\pi\gamma r_c^3} + \frac{512\pi\gamma n^2 r_c^5}{51975}, \quad D = \frac{315k_BT}{64\pi\gamma n r_c^3}.$$

Problem 9.12

```
%%%%%%%%%%%%%%%%%% INITILISATION %%%%%%%%%%%%%%%%
% Array to store the number of times that the particles cross boundaries
ncc=zeros(2,nFreeAtom);
% Array to store the mean square displacements against time
MSDs=zeros(stepSample,2);

%%%%%%%%%%% COMPUTATION %%%%%%%%%%%%%%%%%%%%%%%%
    for i=1:nFreeAtom
        if r(1,i) > 0.5*Lx
            r(1,i)=r(1,i)-Lx; ncc(1,i) = ncc(1,i)+1;
        elseif r(1,i) < -0.5*Lx
            r(1,i)=r(1,i)+Lx; ncc(1,i) = ncc(1,i)-1;
        end
        if r(2,i) > 0.5*Ly
            r(2,i)=r(2,i)-Ly; ncc(2,i) = ncc(2,i)+1;
        elseif r(2,i) < -0.5*Ly
            r(2,i)=r(2,i)+Ly; ncc(2,i) = ncc(2,i)-1;
        end
    end

    if stepCount == stepEquil
        % Actual positions of particles at the thermal equilibrium
        r0=r+[Lx*ncc(1,:);Ly*ncc(2,:)];
    end

    if stepCount > stepEquil
        timeNow=timeNow+deltaT;
        % Actual positions of particles at the current time t
        rt = r+[Lx*ncc(1,:);Ly*ncc(2,:)];
        % Relative positions
        DeltaR = rt-r0;
        % Mean Square Displacement
        MSD = sum(sum(DeltaR.^2,1))/nFreeAtom;
        MSDs(round(timeNow/deltaT),:) = [timeNow,MSD];
    end

%%%%%%%%%%%%%%%% OUTPUT %%%%%%%%%%%%%%%%%%
% Plot MSD and Diffusivity of DPD particles
figure; plot(MSDs(:,1),MSDs(:,2),'b-')
xlabel('Time'); ylabel('MSD')
title('Mean Square Displacements of DPD particles');
figure; plot(MSDs(:,1),(1/4)*(MSDs(:,2)./MSDs(:,1)),'b-');
xlabel('Time'); ylabel('D');
title('Self-diffusion coefficient of DPD particles');
```

Problem 9.13

```
%%%%%%%%%%%%%%%%%%%% INPUT %%%%%%%%%%%%%%%%%%%%%%
% Number of annular elements and the domain size
nRDF=100; rRDF=1;

%%%%%%%%%%%%%%%%%% INITILISATION %%%%%%%%%%%%%%%%
% Array to store RDF values against time
RDFs=zeros(stepSample,nRDF);

%%%%%%%%%%% COMPUTATION %%%%%%%%%%%%%%%%%%%%%%%%
    if stepCount > stepEquil
        timeNow=timeNow+deltaT;

        % Pick up particles in the core region for computing their RDF
        id = find(abs(r(1,:))<Lx/2-rRDF & abs(r(2,:))<Ly/2-rRDF);
        rs = r(:,id); ns=size(rs,2); RDF=zeros(ns,nRDF);
        for i=1:ns
            DeltaR = [r(1,:)-rs(1,i);r(2,:)-rs(2,i)];
            DistR = sqrt(sum(DeltaR.^2,1));
            for j=1:nRDF
                Rmin=(j-1)*(rRDF/nRDF); Rmax=j*(rRDF/nRDF);
                id=find(DistR>Rmin & DistR<=Rmax); RDF(i,j)=length(id);
            end
        end
        RDFs(round(timeNow/deltaT),:) = mean(RDF,1);
    end

%%%%%%%%%%%%%%%% OUTPUT %%%%%%%%%%%%%%%%%%
% Plot RDF of the DPD particles
R = mean(RDFs,1); dq=(rRDF/nRDF); q=dq*(1:nRDF);
R(1)=R(1)/(numDen*pi*dq^2);
R(2:end)=R(2:end)./(numDen*2*pi*(q(2:end)-dq)*dq);
figure; plot(q,R,'b-o')
xlabel('Distance'),ylabel('RDF')
title('Radial distribution function for DPD particles')
```

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Index

N. Phan-Thien and N. Mai-Duy, *Understanding Viscoelasticity*,
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