Conclusion

We have considered the processes of stochastic structure formation in the two-dimensional geophysical fluid dynamics based on statistical analysis of Gaussian random fields, as well as stochastic structure formation in dynamic systems with parametric excitation of positive random fields $f(\mathbf{r}, t)$ described by partial differential equations. We also considered two examples of stochastic structure formation in dynamic systems with parametric excitation in the presence of the Gaussian pumping. Such structure formation in dynamic systems with parametric excitation in space and time either happens or not! However, if it occurs in space, then this almost always happens (exponentially fast) in individual realizations, i.e., with a unit probability, and for the spatially homogeneous statistical case consists in the following:

(1) the field decays at almost all points in space with time (clearly, with fluctuations superimposed);

(2) the small regions where this field is concentrated (clustered) develop in space, and stochastic structure formation is caused by diffusion of random field $f(\mathbf{r}, t)$ in its phase space $\{f\}$.

In the case considered, clustering of the field $f(\mathbf{r}, t)$ of any nature is a general feature of dynamic fields, and one may claim that structure formation is the *Law of Nature* for arbitrary random fields of such type.

In this study, we clarified conditions under which such structure formation takes place. It is worth noting that these conditions have a transparent physical-mathematical sense and are described at a sufficiently elementary mathematical level by resorting to the ideas of statistical topography.

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Appendix
Elements of Mathematical Tools for Describing Coherent Phenomena

A. Solution Dependence on Problem Type, Medium Parameters, and Initial Data

Earlier, in Sect. 3 we considered a number of dynamic systems described by both ordinary and partial differential equations. Many applications concerning research of statistical characteristics of the solutions to these equations require the knowledge of the solution dependence (generally, in the functional form) on the medium parameters appeared in the equations as coefficients and initial values. Some properties appear common of all such dependencies, and two of them are of special interest in the context of statistical descriptions. We illustrate these dependencies by the example of the simplest problem, namely, the system of ordinary differential equations (3.1) that describes particle dynamics under random velocity field and which we reformulate in the form of the nonlinear integral equation

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^{t} d\tau \mathbf{U}(\mathbf{r}(\tau), \tau). \tag{A.1}$$

The solution to Eq. (A.1) functionally depends on vector field $\mathbf{U}(\mathbf{r}', \tau)$ and initial values $\mathbf{r}_0,\ t_0$.

A.1 Functional Representation of Problem Solution

A.1.1 Variational (Functional) Derivatives

Recall first the general definition of a functional. One says that a functional is given if a rule is fixed that associates a number to every function from certain function family. Below, we give some examples of functionals:

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$$\textbf{(a)} \qquad F[\varphi(\tau)] = \int_{t_1}^{t_2} d\tau a(\tau)\varphi(\tau),$$

where $a(t)$ is the given (fixed) function and limits t_1 and t_2 can be both finite and infinite. This is the linear functional.

$$\textbf{(b)} \qquad F[\varphi(\tau)] = \int_{t_1}^{t_2}\int_{t_1}^{t_2} d\tau_1 d\tau_2 B(\tau_1, \tau_2)\varphi(\tau_1)\varphi(\tau_2),$$

where $B(t_1, t_2)$ is the given (fixed) function. This is the quadratic functional.

$$\textbf{(c)} \qquad F[\varphi(\tau)] = f\left(\Phi[\varphi(\tau)]\right),$$

where $f(x)$ is the given function and quantity $\Phi[\varphi(\tau)]$ is the functional.

Estimate the difference between the values of a functional calculated for functions $\varphi(\tau)$ and $\varphi(\tau) + \delta\varphi(\tau)$ for $t - \dfrac{\Delta\tau}{2} < \tau < t + \dfrac{\Delta\tau}{2}$ (see Fig. A.1).

The variation of a functional is defined as the linear (in $\delta\varphi(\tau)$) portion of the difference

$$\delta F[\varphi(\tau)] = \{F\left[\varphi(\tau) + \delta\varphi(\tau)\right] - F[\varphi(\tau)]\}.$$

The limit

$$\frac{\delta F[\varphi(\tau)]}{\delta\varphi(t)dt} = \lim_{\Delta\tau \to 0} \frac{\delta F[\varphi(\tau)]}{\int_{\Delta\tau} d\tau \delta\varphi(\tau)} \tag{A.2}$$

is called the *variational* (or *functional*) *derivative* (see, e.g., [41]).

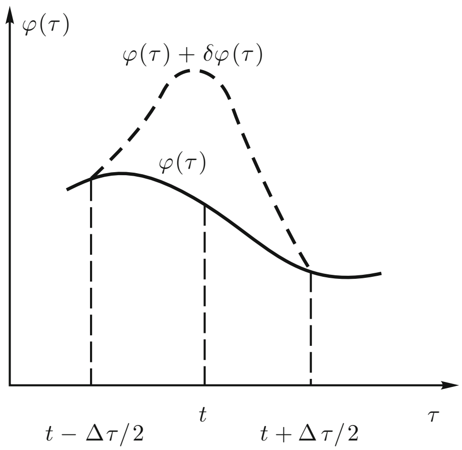

Fig. A.1 To definition of variational derivative

For short, we will use notation $\dfrac{\delta F[\varphi(\tau)]}{\delta\varphi(t)}$ instead of $\dfrac{\delta F[\varphi(\tau)]}{\delta\varphi(t)dt}$.

Note that, if we use function $\delta\varphi(\tau) = \alpha\delta(\tau)$, where $\delta(\tau)$ is the Dirac delta function, then Eq. (A.2) can be represented in the form of the ordinary derivative

$$\frac{\delta F[\varphi(\tau)]}{\delta\varphi(t)} = \lim_{\alpha\to 0}\frac{d}{d\alpha}F[\varphi(\tau)+\alpha\delta(\tau-t)].$$

The variational derivative of functional $F[\varphi(\tau)]$ is again the functional of $\varphi(\tau)$, which depends additionally on point t as a parameter. As a result, this variational derivative will have two types of derivatives; one can differentiate it in the ordinary sense with respect to parameter t and in the functional sense with respect to $\varphi(\tau)$ at point $\tau = t'$ thus obtaining the second variational derivative of the initial functional

$$\frac{\delta^2 F[\varphi(\tau)]}{\delta\varphi(t')\delta\varphi(t)} = \frac{\delta}{\delta\varphi(t')}\left[\frac{\delta F[\varphi(\tau)]}{\delta\varphi(t)}\right].$$

The second variational derivative will now be the functional of $\varphi(\tau)$ dependent on two points t and t', and so forth.

Determine the variational derivatives of functionals (**a**), (**b**), and (**c**).

In the case (**a**), we have

$$\delta F[\varphi(\tau)] = F[\varphi(\tau)+\delta\varphi(\tau)] - F[\varphi(\tau)] = \int\limits_{t-\frac{\Delta\tau}{2}}^{t+\frac{\Delta\tau}{2}} d\tau a(\tau)\delta\varphi(\tau).$$

If function $a(t)$ is continuous on segment $\Delta\tau$, then, by the average theorem,

$$\delta F[\varphi(\tau)] = a(t')\int\limits_{\Delta\tau} d\tau\delta\varphi(\tau),$$

where point t' belongs to segment $\left[t-\dfrac{\Delta\tau}{2}, t+\dfrac{\Delta\tau}{2}\right]$. Consequently,

$$\frac{\delta F[\varphi(\tau)]}{\delta\varphi(t)} = \lim_{\Delta\tau\to 0} a(t') = a(t). \tag{A.3}$$

In the case (**b**), we obtain similarly

$$\frac{\delta F[\varphi(\tau)]}{\delta\varphi(t)} = \int\limits_{t_1}^{t_2} d\tau\,[B(\tau,t)+B(t,\tau)]\,\varphi(\tau) \quad (t_1 < t < t_2).$$

Note that function $B(\tau_1, \tau_2)$ can always be assumed a symmetric function of its arguments here.

In the case (**c**), we have

$$F[\varphi(\tau) + \delta\varphi(\tau)] = f\,(\Phi[\varphi(\tau)]) + \frac{\partial f(\Phi[\varphi(\tau)])}{\partial \Phi}\delta\Phi[\varphi(\tau)] + \cdots$$
$$= F[\varphi(\tau)] + \frac{\partial f(\Phi[\varphi(\tau)])}{\partial \Phi}\delta\Phi[\varphi(\tau)] + \cdots$$

and, consequently,

$$\frac{\delta}{\delta\varphi(t)} f\,(\Phi[\varphi(\tau)]) = \frac{\partial f(\Phi[\varphi(\tau)])}{\partial \Phi}\frac{\delta}{\delta\varphi(t)}\Phi[\varphi(\tau)]. \tag{A.4}$$

Consider now functional $\Phi[\varphi(\tau)] = F_1[\varphi(\tau)]F_2[\varphi(\tau)]$. We have

$$\delta\Phi[\varphi(\tau)] = F_1[\varphi(\tau) + \delta\varphi(\tau)]F_2[\varphi(\tau) + \delta\varphi(\tau)] - F_1[\varphi(\tau)]F_2[\varphi(\tau)]$$
$$= F_1[\varphi(\tau)]\delta F_2[\varphi(\tau)] + F_2[\varphi(\tau)]\delta F_1[\varphi(\tau)]$$

and, consequently,

$$\frac{\delta}{\delta\varphi(t)} F_1[\varphi(\tau)]F_2[\varphi(\tau)] = F_1[\varphi(\tau)]\frac{\delta}{\delta\varphi(t)}F_2[\varphi(\tau)] + F_2[\varphi(\tau)]\frac{\delta}{\delta\varphi(t)}F_1[\varphi(\tau)]. \tag{A.5}$$

We can define the expression for the variational derivative of functional $\varphi(\tau_0)$ with respect to function $\varphi(t)$ by the formal relationship

$$\frac{\delta\varphi(\tau_0)}{\delta\varphi(t)} = \delta(\tau_0 - t). \tag{A.6}$$

Formula (A.6) can be proved, for example, by considering the linear functional of the form

$$F[\varphi(\tau)] = \frac{1}{\sqrt{2\pi}\sigma}\int\limits_{-\infty}^{\infty} d\tau\varphi(\tau)\exp\left\{-\frac{(\tau - \tau_0)^2}{2\sigma^2}\right\}. \tag{A.7}$$

According to Eq. (A.3), the variational derivative of this functional has the form

$$\frac{\delta}{\delta\varphi(t)}F[\varphi(\tau)] = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(t - \tau_0)^2}{2\sigma^2}\right\}. \tag{A.8}$$

Performing now formal limit process $\sigma \to 0$ in Eqs. (A.7) and (A.8), we obtain the desired formula (A.6). Moreover,

$$\frac{\delta F[\varphi(\tau)]}{\delta\varphi(t)} = \frac{\partial F[\varphi(\tau)]}{\partial\varphi(\tau)}\frac{\delta\varphi(\tau)}{\delta\varphi(t)} = \frac{\partial F[\varphi(\tau)]}{\partial\varphi(\tau)}\delta(\tau - t).$$

Formula (A.6) is very convenient for functional differentiation of functionals explicitly dependent on $\varphi(\tau)$. Indeed, for the quadratic functional (**b**), we have

$$\frac{\delta}{\delta\varphi(t)}\int_{t_1}^{t_2}\int_{t_1}^{t_2} d\tau_1 d\tau_2 B(\tau_1,\tau_2)\varphi(\tau_1)\varphi(\tau_2)$$

$$\overset{(250)}{=}\int_{t_1}^{t_2}\int_{t_1}^{t_2} d\tau_1 d\tau_2 B(\tau_1,\tau_2)\left[\frac{\delta\varphi(\tau_1)}{\delta\varphi(t)}\varphi(\tau_2)+\varphi(\tau_1)\frac{\delta\varphi(\tau_2)}{\delta\varphi(t)}\right]$$

$$\overset{(251)}{=}\int_{t_1}^{t_2} d\tau\left[B(t,\tau)+B(\tau,t)\right]\varphi(\tau)\quad(t_1<t<t_2).$$

Consider the functional

$$F[\varphi(\tau)]=\int_{t_1}^{t_2} d\tau L\left(\tau,\varphi(\tau),\frac{d\varphi(\tau)}{d\tau}\right)$$

as another example. In this case,

$$\frac{\delta}{\delta\varphi(t)}F[\varphi(\tau)]\overset{(249)}{=}\int_{t_1}^{t_2} d\tau\left[\frac{\partial L\left(\tau,\varphi(\tau),\frac{d\varphi(\tau)}{d\tau}\right)}{\partial\varphi(\tau)}+\frac{\partial L\left(\tau,\varphi(\tau),\frac{d\varphi(\tau)}{d\tau}\right)}{\partial\dot{\varphi}(\tau)}\frac{d}{d\tau}\right]\frac{\delta\varphi(\tau)}{\delta\varphi(t)}$$

$$\overset{(251)}{=}\int_{t_1}^{t_2} d\tau\left[\frac{\partial L\left(\tau,\varphi(\tau),\frac{d\varphi(\tau)}{d\tau}\right)}{\partial\varphi(\tau)}+\frac{\partial L\left(\tau,\varphi(\tau),\frac{d\varphi(\tau)}{d\tau}\right)}{\partial\dot{\varphi}(\tau)}\frac{d}{d\tau}\right]\delta(\tau-t)$$

$$=\left(-\frac{d}{dt}\frac{\partial}{\partial\dot{\varphi}(t)}+\frac{\partial}{\partial\varphi(t)}\right)L\left(t,\varphi(t),\frac{d\varphi(t)}{dt}\right),$$

where $\dot{\varphi}(t)=\frac{d}{dt}\varphi(t)$ if point t belongs to interval (t_1,t_2).

Just as a function can be expanded in the Taylor series, a functional $F[\varphi(\tau)+\eta(\tau)]$ can be expanded in the functional Taylor series in function $\eta(\tau)$

$$F[\varphi(\tau)+\eta(\tau)]=F[\varphi(\tau)]+\int_{-\infty}^{\infty} dt\frac{\delta F[\varphi(\tau)]}{\delta\varphi(t)}\eta(t)$$
$$+\frac{1}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2\frac{\delta^2 F[\varphi(\tau)]}{\delta\varphi(t_1)\delta\varphi(t_2)}\eta(t_1)\eta(t_2)+\cdots\,. \qquad \text{(A.9)}$$

Note that the operator expression

$$1+\int_{-\infty}^{\infty} dt\eta(t)\frac{\delta}{\delta\varphi(t)}+\frac{1}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1 dt_2\eta(t_1)\eta(t_2)\frac{\delta^2}{\delta\varphi(t_1)\delta\varphi(t_2)}+\cdots$$
$$=1+\int_{-\infty}^{\infty} dt\eta(t)\frac{\delta}{\delta\varphi(t)}+\frac{1}{2!}\left[\int_{-\infty}^{\infty} dt\eta(t)\frac{\delta}{\delta\varphi(t)}\right]^2+\cdots \tag{A.10}$$

can be written shortly as the operator

$$\exp\left\{\int_{-\infty}^{\infty} dt\eta(t)\frac{\delta}{\delta\varphi(t)}\right\}, \tag{A.11}$$

whose action should be treated precisely in the sense of expansion (A.10). Using this operator, we can rewrite Eq. (A.9) in the form

$$F[\varphi(\tau)+\eta(\tau)]=e^{\int_{-\infty}^{\infty} dt\eta(t)\frac{\delta}{\delta\varphi(t)}}F[\varphi(\tau)], \tag{A.12}$$

which enables us to interpret operator (A.11) as the functional shift operator.

Consider now functional $F[t;\varphi(\tau)]$ dependent on parameter t. We can differentiate this functional with respect to t and determine its variational derivative with respect to $\varphi(t')$, as well. One can easily see that these operations commute, i.e., the equality

$$\frac{\partial}{\partial t}\frac{\delta F[t;\varphi(\tau)]}{\delta\varphi(t')}=\frac{\delta}{\delta\varphi(t')}\frac{\partial F[t;\varphi(\tau)]}{\partial t} \tag{A.13}$$

holds. If the domain of τ is independent of t, the validity of Eq. (A.13) is obvious. Otherwise, for example, for functionals $F[t;\varphi(\tau)]$ with $0\le\tau\le t$, the validity of Eq. (A.13) can be checked on by expanding functional $F[t;\varphi(\tau)]$ in the functional Taylor series.

A.1.2 Principle of Dynamic Causality

Vary Eq. (A.1) with respect to field $\mathbf{U}(\mathbf{r},t)$. Assuming that the initial position $\mathbf{r}_0$ is independent of field $\mathbf{U}$, we obtain the equation linear in variational derivative (the linear variational differential equation)

$$\frac{\delta r_i(t)}{\delta U_j(\mathbf{r},t')}=\delta_{ij}\delta\left(\mathbf{r}-\mathbf{r}\left(t'\right)\right)\theta\left(t'-t_0\right)\theta\left(t-t'\right)+\int_{t_0}^{t} d\tau\frac{\partial U_i(\mathbf{r}(\tau),\tau)}{\partial r_k}\frac{\delta r_k(\tau)}{\delta U_j(\mathbf{r},t')}, \tag{A.14}$$

where $\delta\left(\mathbf{r}-\mathbf{r}'\right)$ is the Dirac delta function, and $\theta(z)$ is the Heaviside step function. From Eq. (A.14) follows that

$$\frac{\delta r_i(t)}{\delta U_j(\mathbf{r},t')}=0 \quad \text{for } t'>t \text{ or } t'<t_0, \tag{A.15}$$

which means that solution $\mathbf{r}(t)$ to the dynamic problem (A.1) as a functional of field $\mathbf{U}\left(\mathbf{r},t'\right)$ depends on $\mathbf{U}\left(\mathbf{r},t'\right)$ only for $t_0<t'<t$. Consequently, function $\mathbf{r}(t)$ will remain unchanged if field $\mathbf{U}\left(\mathbf{r},t'\right)$ varies outside the interval (t_0,t), i.e., for $t'<t_0$ or $t'>t$. We will call condition (A.15) the *dynamic causality condition*. Taking this condition into account, we can rewrite Eq. (A.14) in the form

$$\frac{\delta r_i(t)}{\delta U_j(\mathbf{r},t')}=\delta_{ij}\delta\left(\mathbf{r}-\mathbf{r}\left(t'\right)\right)\theta\left(t'-t_0\right)\theta\left(t-t'\right)+\int\limits_{t'}^{t}d\tau\frac{\partial U_i(\mathbf{r}(\tau),\tau)}{\partial r_k}\frac{\delta r_k(\tau)}{\delta U_j(\mathbf{r},t')}. \tag{A.16}$$

As a consequence, limit $t\to t'+0$ yields the equality

$$\left.\frac{\delta r_i(t)}{\delta U_j(\mathbf{r},t')}\right|_{t=t'+0}=\delta_{ij}\delta\left(\mathbf{r}-\mathbf{r}\left(t'\right)\right). \tag{A.17}$$

Integral equation (A.16) in variational derivative is obviously equivalent to the linear differential equation with the initial value

$$\begin{gathered}\frac{\partial}{\partial t}\left(\frac{\delta r_i(t)}{\delta U_j(\mathbf{r},t')}\right)=\frac{\partial U_i(\mathbf{r}(t),t)}{\partial r_k}\left(\frac{\delta r_k(t)}{\delta U_j(\mathbf{r},t')}\right),\\ \left.\frac{\delta r_i(t)}{\delta U_j(\mathbf{r},t')}\right|_{t=t'}=\delta_{ij}\delta\left(\mathbf{r}-\mathbf{r}\left(t'\right)\right).\end{gathered} \tag{A.18}$$

The dynamic causality condition is the general property of problems described by differential equations with initial values. The boundary-value problems possess no such property. Indeed, in the case of problem (3.12), (3.13) that describes propagation of a plane wave in a layer of inhomogeneous medium, wavefield $u(x)$ at point x and reflection and transmission coefficients depend functionally on function $\varepsilon(x)$ for all x of layer (L_0,L). However, using the imbedding method, we can convert this problem into the initial-value problem with respect to an auxiliary parameter L and make use the causality property in terms of the equations of the imbedding method.

A.2 Solution Dependence on Problem's Parameters

A.2.1 Solution Dependence on Initial Data

Here, we will use the vertical bar symbol to isolate the dependence of solution $\mathbf{r}(t)$ to Eq. (A.1) on the initial parameters $\mathbf{r}_0$ and t_0:

$$\mathbf{r}(t) = \mathbf{r}(t|\mathbf{r}_0, t_0), \quad \mathbf{r}_0 = \mathbf{r}(t_0|\mathbf{r}_0, t_0).$$

Let us differentiate Eq. (A.1) with respect to parameters r_{0k} and t_0. As a result, we obtain linear equations for Jacobi's matrix $\dfrac{\partial}{\partial r_{0k}} r_i(t|\mathbf{r}_0, t_0)$ and quantity $\dfrac{\partial}{\partial t_0} r_i(t|\mathbf{r}_0, t_0)$

$$\begin{aligned} \frac{\partial r_i(t|\mathbf{r}_0, t_0)}{\partial r_{0k}} &= \delta_{ik} + \int_{t_0}^{t} d\tau \frac{\partial U_i(\mathbf{r}(\tau), \tau)}{\partial r_j} \frac{\partial r_j(\tau|\mathbf{r}_0, t_0)}{\partial r_{0k}}, \\ \frac{\partial r_i(t|\mathbf{r}_0, t_0)}{\partial t_0} &= -U_i(\mathbf{r}_0(t_0), t_0) + \int_{t_0}^{t} d\tau \frac{\partial U_i(\mathbf{r}(\tau), \tau)}{\partial r_j} \frac{\partial r_j(\tau|\mathbf{r}_0, t_0)}{\partial t_0}. \end{aligned} \tag{A.19}$$

Multiplying now the first of these equations by $U_k(\mathbf{r}_0(t), t)$, summing over index k, adding the result to the second equation, and introducing the vector function

$$F_i(t|\mathbf{r}_0, t_0) = \left(\frac{\partial}{\partial t_0} + \mathbf{U}(\mathbf{r}_0, t_0)\frac{\partial}{\partial \mathbf{r}_0}\right) r_i(t|\mathbf{r}_0, t_0),$$

we obtain that this function satisfies the linear homogeneous equation

$$F_i(t|\mathbf{r}_0, t_0) = \int_{t_0}^{t} d\tau \frac{\partial U_i(\mathbf{r}(\tau), \tau)}{\partial r_k} F_k(\tau|\mathbf{r}_0, t_0). \tag{A.20}$$

Differentiating this equation with respect to time, we arrive at the ordinary differential equation

$$\frac{\partial}{\partial t} F_i(t|\mathbf{r}_0, t_0) = \frac{\partial U_i(\mathbf{r}(t), t)}{\partial r_k} F_k(t|\mathbf{r}_0, t_0)$$

with the initial condition $F_i(t_0|\mathbf{r}_0, t_0) = 0$ at $t = t_0$, which follows from Eq. (A.20); as a consequence, we have $F_i(t|\mathbf{r}_0, t_0) \equiv 0$. Therefore, we obtain the equality

$$\left(\frac{\partial}{\partial t_0} + \mathbf{U}(\mathbf{r}_0, t_0)\frac{\partial}{\partial \mathbf{r}_0}\right) r_i(t|\mathbf{r}_0, t_0) = 0, \tag{A.21}$$

which can be considered as the linear partial differential equation with the derivatives with respect to variables $\mathbf{r}_0$, t_0 and the initial value at $t_0 = t$

$$\mathbf{r}\,(t|\mathbf{r}_0, t) = \mathbf{r}_0. \tag{A.22}$$

The variable t appears now in problem (A.21), (A.22) as a parameter.

Equation (A.21) is solved using the time direction inverse to that used in solving problem (3.1); for this reason, we will call it the *backward equation*.

Equation (A.21) with the initial condition (A.22) can be rewritten as the integral equation

$$\mathbf{r}(t|\mathbf{r}_0, t_0) = \mathbf{r}_0 + \int_{t_0}^{t} d\tau \left(\mathbf{U}\,(\mathbf{r}_0, \tau)\,\frac{\partial}{\partial \mathbf{r}_0}\right) \mathbf{r}(t|\mathbf{r}_0, \tau). \tag{A.23}$$

Varying now Eq. (A.23) with respect to function $U_j(\mathbf{r}', t')$, we obtain the integral equation

$$\begin{aligned}\frac{\delta r_i(t|\mathbf{r}_0, t_0)}{\delta U_j(\mathbf{r}', t')} &= \delta(\mathbf{r}_0 - \mathbf{r}')\theta(t' - t_0)\theta(t - t')\frac{\partial r_i(t|\mathbf{r}_0, t')}{\partial r_{j0}} \\ &\quad + \int_{t_0}^{t} d\tau \left(\mathbf{U}\,(\mathbf{r}_0, \tau)\,\frac{\partial}{\partial \mathbf{r}_0}\right) \frac{\delta r_i(t|\mathbf{r}_0, \tau)}{\delta U_j(\mathbf{r}', t')},\end{aligned} \tag{A.24}$$

from which follows that

$$\frac{\delta r_i\,(t|\mathbf{r}_0, t_0)}{\delta U_j\,(\mathbf{r}', t')} = 0, \quad \text{if } t' > t \quad \text{or} \quad t' < t_0,$$

which means that function $\mathbf{r}(t|\mathbf{r}_0, t_0)$ also possesses the property of dynamic causality with respect to parameter t_0 and Eq. (A.24) can be rewritten in the form (for $t_0 < t' < t$)

$$\frac{\delta r_i(t|\mathbf{r}_0, t_0)}{\delta U_j(\mathbf{r}', t')} = \delta(\mathbf{r}_0 - \mathbf{r}')\frac{\partial r_i(t|\mathbf{r}_0, t')}{\partial r_{j0}} + \int_{t_0}^{t'} d\tau \left(\mathbf{U}\,(\mathbf{r}_0, \tau)\,\frac{\partial}{\partial \mathbf{r}_0}\right) \frac{\delta r_i(t|\mathbf{r}_0, \tau)}{\delta U_j(\mathbf{r}', t')}. \tag{A.25}$$

Setting now $t' \to t_0 + 0$, we obtain the equality

$$\left.\frac{\delta r_i\,(t|\mathbf{r}_0, t_0)}{\delta U_j\,(\mathbf{r}, t')}\right|_{t'=t_0+0} = \delta\,(\mathbf{r}_0 - \mathbf{r})\,\frac{\partial r_i\,(t|\mathbf{r}_0, t_0)}{\partial r_{0j}}. \tag{A.26}$$

A.2.2 Imbedding Method for Boundary-Value Problems

Consider first boundary-value problems formulated in terms of ordinary differential equations. The *imbedding method* (or *invariant imbedding method*, as it is usually called in mathematical literature) offers a possibility of reducing boundary-value problems at hand to the evolution-type initial-value problems possessing the property of dynamic causality with respect to an auxiliary parameter.

The idea of this method was first suggested by V.A. Ambartsumyan (the so-called *Ambartsumyan invariance principle*) for solving the equations of linear theory of radiative transfer. Further, mathematicians grasped this idea and used it to convert boundary-value (nonlinear, in the general case) problems into evolution-type initial-value problems that are more convenient for simulations. Several monographs (see, e.g., [28, 29]) deal with this method and consider both physical and computational aspects.

Consider the dynamic system described in terms of the system of ordinary differential equations

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{F}\left(t, \mathbf{x}(t)\right), \tag{A.27}$$

defined on segment $t \in [0, T]$ with the boundary conditions

$$g\mathbf{x}(0) + h\mathbf{x}(T) = \mathbf{v}, \tag{A.28}$$

where g and h are the constant matrixes.

Dynamic problem (A.27), (A.28) possesses no dynamic causality property, which means that the solution $\mathbf{x}(t)$ to this problem at instant t functionally depends on external forces $\mathbf{F}\left(\tau, \mathbf{x}(\tau)\right)$ for all $0 \leq \tau \leq T$. Moreover, even boundary values $\mathbf{x}(0)$ and $\mathbf{x}(T)$ are functionals of field $\mathbf{F}\left(\tau, \mathbf{x}(\tau)\right)$. The absence of dynamic causality in problem (A.27), (A.28) prevents us from using the known statistical methods of analyzing statistical characteristics of the solution to Eq. (A.27) if external force functional $\mathbf{F}\left(t, \mathbf{x}\right)$ is the random space- and time-domain field. Introducing the one-time probability density $P(t; \mathbf{x})$ of the solution to Eq. (A.27), we can easily see that condition (A.28) is insufficient for determining the value of this probability at any point. The boundary condition imposes only certain functional restriction.

Note that the solution to problem (A.27), (A.28) parametrically depends on T and $\mathbf{v}$, i.e., $\mathbf{x}(t) = \mathbf{x}(t; T, \mathbf{v})$. We introduce functions

$$\mathbf{R}(T, \mathbf{v}) = \mathbf{x}(T; T, \mathbf{v}), \quad \mathbf{S}(T, \mathbf{v}) = \mathbf{x}(0; T, \mathbf{v})$$

that describe the boundary values of the solution to Eq. (A.27).

Differentiate Eq. (A.27) with respect to T and $\mathbf{v}$. We obtain two linear equations in the corresponding derivatives

$$\frac{d}{dt}\frac{\partial x_i(t;T,\mathbf{v})}{\partial T} = \frac{\partial F_i(t,\mathbf{x})}{\partial x_l}\frac{\partial x_l(t;T,\mathbf{v})}{\partial T},$$
$$\frac{d}{dt}\frac{\partial x_i(t;T,\mathbf{v})}{\partial v_k} = \frac{\partial F_i(t,\mathbf{x})}{\partial x_l}\frac{\partial x_l(t;T,\mathbf{v})}{\partial v_k}. \quad \text{(A.29)}$$

These equations are identical in form; consequently, we can expect that their solutions are related by the linear expression

$$\frac{\partial x_i(t;T,\mathbf{v})}{\partial T} = \lambda_k(T,\mathbf{v})\frac{\partial x_i(t;T,\mathbf{v})}{\partial v_k} \quad \text{(A.30)}$$

if vector quantity $\lambda(T,\mathbf{v})$ is such that boundary conditions (A.28) are satisfied and the solution is unique. To determine vector quantity $\lambda(T,\mathbf{v})$, we first set $t = 0$ in Eq. (A.30) and multiply the result by matrix g; then, we set $t = T$ and multiply the result by matrix h; and, finally, we combine the obtained expressions. Taking into account Eq. (A.28), we obtain

$$g\frac{\partial \mathbf{x}(0;T,\mathbf{v})}{\partial T} + h\left.\frac{\partial \mathbf{x}(t;T,\mathbf{v})}{\partial T}\right|_{t=T} = \boldsymbol{\lambda}(T,\mathbf{v}).$$

In view of the fact that

$$\left.\frac{\partial \mathbf{x}(t;T,\mathbf{v})}{\partial T}\right|_{t=T} = \frac{\partial \mathbf{x}(T;T,\mathbf{v})}{\partial T} - \left.\frac{\partial \mathbf{x}(t;T,\mathbf{v})}{\partial t}\right|_{t=T} = \frac{\partial \mathbf{R}(T,\mathbf{v})}{\partial T} - \mathbf{F}\left(T,\mathbf{R}(T,\mathbf{v})\right)$$

(with allowance for Eq. (A.27)), we obtain the desired expression for quantity $\lambda(T,\mathbf{v})$,

$$\lambda(T,\mathbf{v}) = -h\mathbf{F}\left(T,\mathbf{R}(T,\mathbf{v})\right). \quad \text{(A.31)}$$

Expression (A.30) with parameter $\lambda(T,\mathbf{v})$ defined by Eq. (A.31), i.e., the expression

$$\frac{\partial x_i(t;T,\mathbf{v})}{\partial T} = -h_{kl}F_l\left(T,\mathbf{R}(T,\mathbf{v})\right)\frac{\partial x_i(t;T,\mathbf{v})}{\partial v_k}, \quad \text{(A.32)}$$

can be considered as the linear differential equation; one needs only to supplement it with the corresponding initial condition

$$\mathbf{x}(t;T,\mathbf{v})|_{T=t} = \mathbf{R}(t,\mathbf{v})$$

assuming that function $\mathbf{R}(T,\mathbf{v})$ is known.

The equation for this function can be obtained from the equality

$$\frac{\partial \mathbf{R}(T,\mathbf{v})}{\partial T} = \left.\frac{\partial \mathbf{x}(t;T,\mathbf{v})}{\partial t}\right|_{t=T} + \left.\frac{\partial \mathbf{x}(t;T,\mathbf{v})}{\partial T}\right|_{t=T}. \quad \text{(A.33)}$$

The right-hand side of Eq. (A.33) is the sum of the right-hand sides of Eq. (A.27) and (A.30) at $t = T$. As a result, we obtain the closed nonlinear (quasilinear) equation

$$\frac{\partial \mathbf{R}(T,\mathbf{v})}{\partial T} = -h_{kl}F_l\left(T,\mathbf{R}(T,\mathbf{v})\right)\frac{\partial \mathbf{R}(T,\mathbf{v})}{\partial v_k} + \mathbf{F}\left(T,\mathbf{R}(T,\mathbf{v})\right). \tag{A.34}$$

The initial condition for Eq. (A.34) follows from Eq. (A.28) for $T \to 0$

$$\mathbf{R}(T,\mathbf{v})|_{T=0} = (g+h)^{-1}\mathbf{v}. \tag{A.35}$$

Setting now $t = 0$ in Eq. (A.29), we obtain for the secondary boundary quantity $\mathbf{S}(T,\mathbf{v}) = \mathbf{x}(0;T,\mathbf{v})$ the equation

$$\frac{\partial \mathbf{S}(T,\mathbf{v})}{\partial T} = -h_{kl}F_l\left(T,\mathbf{R}(T,\mathbf{v})\right)\frac{\partial \mathbf{S}(T,\mathbf{v})}{\partial v_k} \tag{A.36}$$

with the initial condition

$$\mathbf{S}(T,\mathbf{v})|_{T=0} = (g+h)^{-1}\mathbf{v}$$

following from Eq. (A.35).

Thus, the problem reduces to the closed quasilinear equation (A.34) with initial value (A.35) and linear equation (A.30) whose coefficients and initial value are determined by the solution of Eq. (A.34).

In the problem under consideration, input 0 and output T are symmetric. For this reason, one can solve it not only from T to 0, but also from 0 to T. In the latter case, functions $\mathbf{R}(T,\mathbf{v})$ and $\mathbf{S}(T,\mathbf{v})$ switch the places.

An important point consists in the fact that, despite the initial problem (A.27) is nonlinear, Eq. (A.30) is the linear equation, because it is essentially the equation in variations. It is Eq. (A.34) that is responsible for nonlinearity.

Note that the above technique of deriving imbedding equations for Eq. (A.27) can be easily extended to the boundary condition of the form

$$\mathbf{g}\left(\mathbf{x}(0)\right) + \mathbf{h}\left(\mathbf{x}(T)\right) + \int\limits_0^T d\tau \mathbf{K}\left(\tau,\mathbf{x}(\tau)\right) = \mathbf{v},$$

where $\mathbf{g}\left(\mathbf{x}\right)$, $\mathbf{h}\left(\mathbf{x}\right)$ and $\mathbf{K}\left(T,\mathbf{x}\right)$ are arbitrary given vector functions.

If function $\mathbf{F}\left(t,\mathbf{x}\right)$ is linear in $\mathbf{x}$, $F_i\left(t,\mathbf{x}\right) = A_{ij}(t)x_j(t)$, then boundary-value problem (A.27), (A.28) assumes the simpler form

$$\frac{d}{dt}\mathbf{x}(t) = A\left(t\right)\mathbf{x}(t), \quad g\mathbf{x}(0) + h\mathbf{x}(T) = \mathbf{v},$$

and the solution of Eq. (A.30), (A.34) and (A.36) will be the function linear in $\mathbf{v}$

$$\mathbf{x}(t; T, \mathbf{v}) = X(t; T)\mathbf{v}. \tag{A.37}$$

As a result, we arrive at the closed matrix Riccati equation for matrix $R(T) = X(T; T)$

$$\frac{d}{dT} R(T) = A(T)R(T) - R(T)hA(T)R(T), \quad R(0) = (g + h)^{-1}. \tag{A.38}$$

As regards matrix $X(t, T)$, it satisfies the linear matrix equation with the initial condition

$$\frac{\partial}{\partial T} X(t; T) = -X(t; T)hA(T)R(T), \quad X(t; T)_{T=t} = R(t). \tag{A.39}$$

Note that, for particular linear boundary-value problems of wave propagation, the direct derivation of imbedding equations from the specific problem statement is usually more convenient.

B. Statistical Characteristics of Random Processes, and Fields

B.1 General Remarks

If we deal with random function (random process) $z(t)$, then all statistical characteristics of this function at any fixed instant t are exhaustively described in terms of the one-time probability density

$$P(z, t) = \langle \delta\left(z(t) - z\right)\rangle \tag{B.1}$$

dependent parametrically on time t by the following relationship

$$\langle f(z(t))\rangle = \int\limits_{-\infty}^{\infty} dz f(z) P(z, t).$$

Note that the singular Dirac delta function

$$\varphi(z, t) = \delta\left(z(t) - z\right)$$

appeared in Eq. (B.1) in angle brackets of averaging is called the *indicator function*.

The integral distribution function for this process, i.e. the probability of the event that process $z(t) < Z$ at instant t, is calculated by the formula

$$F(t, Z) = \mathsf{P}(z(t) < Z) = \int_{-\infty}^{Z} dz P(z, t)$$

from which follows that

$$F(t, Z) = \langle \theta(Z - z(t)) \rangle , \quad F(t, \infty) = 1, \tag{B.2}$$

where $\theta(z)$ is the Heaviside step function equal to zero for $z < 0$ and unity for $z > 0$.

Similar definitions hold for the two-time probability density

$$P(z_1, t_1; z_2, t_2) = \langle \varphi(z_1, t_1; z_2, t_2) \rangle$$

and for the general case of the n-time probability density

$$P(z_1, t_1; \cdots ; z_n, t_n) = \langle \varphi(z_1, t_1; \cdots ; z_n, t_n) \rangle ,$$

where

$$\varphi(z_1, t_1; \cdots ; z_n, t_n) = \delta(z(t_1) - z_1) \cdots \delta(z(t_n) - z_n).$$

is the n-time indicator function.

Process $z(t)$ is called *stationary* if all its statistical characteristics are invariant with respect to arbitrary temporal shift, i.e., if

$$P(z_1, t_1 + \tau; \cdots ; z_n, t_n + \tau) = P(z_1, t_1; \cdots ; z_n, t_n). \tag{B.3}$$

In particular, the one-time probability density of stationary process is at all independent of time, and the correlation function depends only on difference of times,

$$B_z(t_1, t_2) = \langle z(t_1) z(t_2) \rangle = B_z(t_1 - t_2).$$

Temporal scale τ_0 characteristic of correlation function $B_z(t)$ is called the temporal correlation radius of process $z(t)$. We can determine this scale, say, by the equality

$$\int_0^{\infty} \langle z(t + \tau) z(t) \rangle \, d\tau = \tau_0 \left\langle z^2(t) \right\rangle . \tag{B.4}$$

Note that the Fourier transform of stationary process correlation function

$$\Phi_z(\omega) = \int_{-\infty}^{\infty} dt B_z(t) e^{i\omega t}$$

is called the *temporal spectral function* (or simply *temporal spectrum*).

For random field $f(\mathbf{x}, t)$, the one- and n-point probability densities are defined similarly to those for random processes

$$P(\mathbf{x}, t; f) = \langle \varphi(\mathbf{x}, t; f) \rangle , \tag{B.5}$$

$$P(\mathbf{x}_1, t_1, f_1; \cdots ; \mathbf{x}_n, t_n, f_n) = \langle \varphi(\mathbf{x}_1, t_1, f_1; \cdots ; \mathbf{x}_n, t_n, f_n) \rangle , \tag{B.6}$$

where the indicator functions are defined as follows:

$$\begin{gathered} \varphi(\mathbf{x}, t; f) = \delta(f(\mathbf{x}, t) - f), \\ \varphi(\mathbf{x}_1, t_1, f_1; \cdots ; \mathbf{x}_n, t_n, f_n) = \delta\left(f(\mathbf{x}_1, t_1) - f_1\right) \cdots \delta\left(f(\mathbf{x}_n, t_n) - f_n\right) . \end{gathered} \tag{B.7}$$

For clarity, we use here variables $\mathbf{x}$ and t as spatial and temporal coordinates; however, in many physical problems, some preferred spatial coordinate can play the role of the temporal coordinate.

Random field $f(\mathbf{x}, t)$ is called the spatially homogeneous field if all its statistical characteristics are invariant relative to spatial translations by arbitrary vector $\mathbf{a}$, i.e., if

$$P(\mathbf{x}_1 + \mathbf{a}, t_1, f_1; \cdots ; \mathbf{x}_n + \mathbf{a}, t_n, f_n) = P(\mathbf{x}_1, t_1, f_1; \cdots ; \mathbf{x}_n, t_n, f_n).$$

In this case, the one-point probability density $P(\mathbf{x}, t; f) = P(t; f)$ is independent of $\mathbf{x}$, and the spatial correlation function $B_f(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2)$ depends on the difference $\mathbf{x}_1 - \mathbf{x}_2$:

$$B_f(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2) = \langle f(\mathbf{x}_1, t_1) f(\mathbf{x}_2, t_2) \rangle = B_f(\mathbf{x}_1 - \mathbf{x}_2; t_1, t_2).$$

If random field $f(\mathbf{x}, t)$ is additionally invariant with respect to rotation of all vectors $\mathbf{x}_i$ by arbitrary angle, i.e., with respect to rotations of the reference system, then field $f(\mathbf{x}, t)$ is called the homogeneous isotropic random field. In this case, the correlation function depends only on length $|\mathbf{x}_1 - \mathbf{x}_2|$:

$$B_f(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2) = \langle f(\mathbf{x}_1, t_1) f(\mathbf{x}_2, t_2) \rangle = B_f(|\mathbf{x}_1 - \mathbf{x}_2|; t_1, t_2).$$

The corresponding Fourier transform of the correlation function with respect to the spatial variable defines the spatial spectral function (called also the angular spectrum)

$$\Phi_f(\mathbf{k}, t) = \int d\mathbf{x} B_f(\mathbf{x}, t) e^{i\mathbf{k}\mathbf{x}},$$

and the Fourier transform of the correlation function of random field $f(\mathbf{x}, t)$ stationary in time and homogeneous in space defines the space–time spectrum

$$\Phi_f(\mathbf{k}, \omega) = \int d\mathbf{x} \int_{-\infty}^{\infty} dt B_f(\mathbf{x}, t) e^{i(\mathbf{k}\mathbf{x} + \omega t)}.$$

In the case of isotropic random field $f(\mathbf{x}, t)$, the space–time spectrum appears isotropic in the $\mathbf{k}$-space:

$$\Phi_f(\mathbf{k}, \omega) = \Phi_f(k, \omega).$$

An exhaustive description of random function $z(t)$ can be given in terms of the characteristic functional

$$\Phi[v(\tau)] = \left\langle \exp\left\{ i \int\limits_{-\infty}^{\infty} d\tau v(\tau) z(\tau) \right\} \right\rangle,$$

where $v(t)$ is arbitrary (but sufficiently smooth) function. Functional $\Phi[v(\tau)]$ being known, one can determine such characteristics of random function $z(t)$ as mean value $\langle z(t) \rangle$, correlation function $\langle z(t_1) z(t_2) \rangle$, n-time moment function $\langle z(t_1) \cdots z(t_n) \rangle$, etc.

Indeed, expanding functional $\Phi[v(\tau)]$ in the functional Taylor series, we obtain the representation of characteristic functional in terms of the moment functions of process $z(t)$:

$$\Phi[v(\tau)] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int\limits_{-\infty}^{\infty} dt_1 \cdots \int\limits_{-\infty}^{\infty} dt_n M_n(t_1, \ldots, t_n) v(t_1) \cdots v(t_n),$$

$$M_n(t_1, \ldots, t_n) = \langle z(t_1) \cdots z(t_n) \rangle = \frac{1}{i^n} \frac{\delta^n}{\delta v(t_1) \cdots \delta v(t_n)} \Phi[v(\tau)] \bigg|_{v=0}.$$

Consequently, the moment functions of random process $z(t)$ are expressed in terms of variational derivatives of the characteristic functional.

Represent functional $\Phi[v(\tau)]$ in the form $\Phi[v(\tau)] = \exp\{\Theta[v(\tau)]\}$. Functional $\Theta[v(\tau)]$ also can be expanded in the functional Taylor series

$$\Theta[v(\tau)] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int\limits_{-\infty}^{\infty} dt_1 \cdots \int\limits_{-\infty}^{\infty} dt_n K_n(t_1, \ldots, t_n) v(t_1) \cdots v(t_n), \tag{B.8}$$

where function

$$K_n(t_1, \ldots, t_n) = \frac{1}{i^n} \frac{\delta^n}{\delta v(t_1) \cdots \delta v(t_n)} \Theta[v(\tau)] \bigg|_{v=0}$$

is called the n-th order *cumulant function* of random process $z(t)$.

The characteristic functional and the n-th order cumulant functions of scalar random field $f(\mathbf{x}, t)$ are defined similarly

$$\Phi[v(\mathbf{x}',\tau)] = \left\langle \exp\left\{ i\int d\mathbf{x}\int\limits_{-\infty}^{\infty} dt v(\mathbf{x},t) f(\mathbf{x},t)\right\}\right\rangle = \exp\left\{\Theta[v(\mathbf{x}',\tau)]\right\},$$

$$M_n(\mathbf{x}_1,t_1,\ldots,\mathbf{x}_n,t_n) = \frac{1}{i^n}\frac{\delta^n}{\delta v(\mathbf{x}_1,t_1)\cdots\delta v(\mathbf{x}_n,t_n)}\Phi[v(\mathbf{x}',\tau)]\bigg|_{v=0},$$

$$K_n(\mathbf{x}_1,t_1,\ldots,\mathbf{x}_n,t_n) = \frac{1}{i^n}\frac{\delta^n}{\delta v(\mathbf{x}_1,t_1)\cdots\delta v(\mathbf{x}_n,t_n)}\Theta[v(\mathbf{x}',\tau)]\bigg|_{v=0}.$$

In the case of vector random field $\mathbf{f}(\mathbf{x},t)$, we must assume that $\mathbf{v}(\mathbf{x},t)$ is the vector function.

As was mentioned, characteristic functionals ensure a complete description of random processes and fields. However, even one-point probability densities of random processes and fields give certain data on random process evolution in the entire interval of times and on the spatial structure of random fields, as well. These data can be obtained on the basis of ideas of statistical topography of random processes and fields.

B.2 Gaussian Random Process

We start the discussion with the continuous processes; namely, we consider the Gaussian random process $z(t)$ with zero-valued mean ($\langle z(t)\rangle = 0$) and correlation function $B(t_1,t_2) = \langle z(t_1)z(t_2)\rangle$. The corresponding characteristic functional assumes the form

$$\Phi[v(\tau)] = \exp\left\{-\frac{1}{2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} dt_1 dt_2 B(t_1,t_2)v(t_1)v(t_2)\right\}. \tag{B.9}$$

Only one cumulant function (the correlation function)

$$K_2(t_1,t_2) = B(t_1,t_2)$$

is different from zero for this process, so that

$$\Theta[v(\tau)] = -\frac{1}{2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} dt_1 dt_2 B(t_1,t_2)v(t_1)v(t_2). \tag{B.10}$$

Consider the nth-order variational derivative of functional $\Phi[v(\tau)]$. It satisfies the following line of equalities:

$$\frac{\delta^n}{\delta v(t_1)\cdots\delta v(t_n)}\Phi[v(\tau)] = \frac{\delta^{n-1}}{\delta v(t_2)\cdots\delta v(t_n)}\frac{\delta\Theta[v(\tau)]}{\delta v(t_1)}\Phi[v(\tau)]$$
$$= \frac{\delta^2\Theta[v(\tau)]}{\delta v(t_1)\delta v(t_2)}\frac{\delta^{n-2}}{\delta v(t_3)\cdots\delta v(t_n)}\Phi[v(\tau)] + \frac{\delta^{n-2}}{\delta v(t_3)\cdots\delta v(t_n)}\frac{\delta\Theta[v(\tau)]}{\delta v(t_1)}\frac{\delta\Phi[v(\tau)]}{\delta v(t_2)}.$$

Setting now $v = 0$, we obtain that moment functions of the Gaussian process $z(t)$ satisfy the recurrence formula

$$M_n(t_1, \ldots, t_n) = \sum_{k=2}^{n} B(t_1, t_2)M_{n-2}(t_2, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n). \tag{B.11}$$

From this formula follows that, for the Gaussian process with zero-valued mean, all moment functions of odd orders are identically equal to zero and the moment functions of even orders are represented as sums of terms which are the products of averages of all possible pairs $z(t_i)z(t_k)$.

If we assume that function $v(\tau)$ in Eq. (B.10) is different from zero only in interval $0 < \tau < t$, the characteristic functional

$$\Phi[t; v(\tau)] = \left\langle \exp\left(i\int_0^t d\tau z(\tau)v(\tau)\right)\right\rangle = \exp\left\{-\int_0^t d\tau_1\int_0^{\tau_1} d\tau_2 B(\tau_1, \tau_2)v(\tau_1)v(\tau_2)\right\} \tag{B.12}$$

becomes a function of time t and satisfies the ordinary differential equation

$$\frac{d}{dt}\Phi[t; v(\tau)] = -v(t)\int_0^t dt_1 B(t, t_1)v(t_1)\Phi[t; v(\tau)], \quad \Phi[0; v(\tau)] = 1. \tag{B.13}$$

To obtain the one-time characteristic function of the Gaussian random process at instant t, we specify function $v(\tau)$ in Eq. (B.9) in the form

$$v(\tau) = v\delta(\tau - t).$$

Then, we obtain

$$\Phi(v, t) = \left\langle e^{ivz(t)}\right\rangle = \int_{-\infty}^{\infty} dz\, P(z, t)e^{ivz} = \exp\left\{-\frac{1}{2}\sigma^2(t)v^2\right\}, \tag{B.14}$$

where $\sigma^2(t) = B(t, t)$. Using the inverse Fourier transform of (B.14), we obtain the one-time probability density of the Gaussian random process

$$P(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dv\, \Phi(v,t)e^{-ivz} = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp\left\{-\frac{z^2}{2\sigma^2(t)}\right\}. \quad \text{(B.15)}$$

Note that, in the case of stationary random process $z(t)$, variance $\sigma^2(t)$ is independent of time t, i.e., $\sigma^2(t) = \sigma^2 = \text{const}$.

Density $P(z,t)$ as a function of z is symmetric relative to point $z = 0$,

$$P(z,t) = P(-z,t).$$

If mean value of the Gaussian random process is different from zero, then we can consider process $z(t) - \langle z(t)\rangle$ to obtain instead of Eq. (B.15) the expression

$$P(z,t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp\left\{-\frac{(z - \langle z(t)\rangle)^2}{2\sigma^2(t)}\right\} \quad \text{(B.16)}$$

and the corresponding integral distribution function assumes the form

$$F(z,t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}}\int_{-\infty}^{z} dz\, \exp\left\{-\frac{(z - \langle z(t)\rangle)^2}{2\sigma^2(t)}\right\} = \Pr\left(\frac{z - \langle z(t)\rangle}{\sigma(t)}\right),$$

where probability integral $\Pr(z)$ is defined by Eq. (5.4).

B.3 Correlation Splitting for Random Processes and Fields

For simplicity, we content ourselves here with the one-dimensional random processes (extensions to multidimensional cases are obvious). We need the ability of calculating correlation $\langle z(t)]R[z(\tau)]\rangle$, where $R[z(\tau)]$ is the functional that can depend on process $z(t)$ both explicitly and implicitly.

To calculate this average, we consider auxiliary functional $R[z(\tau) + \eta(\tau)]$, where $\eta(t)$ is arbitrary deterministic function, and calculate the correlation

$$\langle z(t) R[z(\tau) + \eta(\tau)]\rangle\,. \quad \text{(B.17)}$$

The correlation of interest will be obtained by setting $\eta(\tau) = 0$ in the final result.

We can expand the above auxiliary functional $R[z(\tau) + \eta(\tau)]$ in the functional Taylor series with respect to $z(\tau)$. The result can be represented in the form

$$R\left[z(\tau) + \eta(\tau)\right] = \exp\left\{\int_{-\infty}^{\infty} d\tau z(\tau) \frac{\delta}{\delta\eta(\tau)}\right\} R\left[\eta(\tau)\right],$$

where we introduced the functional shift operator. With this representation, we can obtain the following expression for correlation (B.17)

$$\langle z(t)R[z(\tau)+\eta(\tau)]\rangle = \Omega\left[t;\,\frac{\delta}{i\delta\eta(\tau)}\right]\langle R[z(\tau)+\eta(\tau)]\rangle\,, \tag{B.18}$$

where functional

$$\Omega[t;\,v(\tau)] = \frac{\left\langle z(t)\exp\left\{i\int\limits_{-\infty}^{\infty}d\tau z(\tau)v(\tau)\right\}\right\rangle}{\left\langle \exp\left\{i\int\limits_{-\infty}^{\infty}d\tau z(\tau)v(\tau)\right\}\right\rangle} = \frac{1}{\Phi[v(\tau)]}\frac{\delta}{i\delta v(t)}\Phi[v(\tau)] = \frac{\delta}{i\delta v(t)}\Theta[v(\tau)]. \tag{B.19}$$

Here $\Theta[v(\tau)] = \ln\Phi[v(\tau)]$ and $\Phi[v(\tau)]$ is the characteristic functional of random process $z(t)$.

Replacing now differentiation with respect to $\eta(\tau)$ by differentiation with respect to $z(\tau)$ and setting $\eta(\tau)=0$, we obtain the expression

$$\langle z(t)R[z(\tau)\rangle = \left\langle \Omega\left[t;\,\frac{\delta}{i\delta z(\tau)}\right]R[z(\tau)]\right\rangle. \tag{B.20}$$

If we expand functional $\Theta[v(\tau)]$ in the functional Taylor series (B.8) and differentiate the result with respect to $v(t)$, we obtain

$$\Omega[t;\,v(\tau)] = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int\limits_{-\infty}^{\infty}dt_1\cdots\int\limits_{-\infty}^{\infty}dt_n K_{n+1}(t,t_1,\ldots,t_n)v(t_1)\cdots v(t_n)$$

and expression (B.20) will assume the form

$$\langle z(t)R[z(\tau)]\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{-\infty}^{\infty}dt_1\cdots\int\limits_{-\infty}^{\infty}dt_n K_{n+1}(t,t_1,\ldots,t_n)\left\langle\frac{\delta^n R[z(\tau)]}{\delta z(t_1)\cdots\delta z(t_n)}\right\rangle. \tag{B.21}$$

In physical problems satisfying the condition of dynamic causality in time, statistical characteristics of the solution at instant t depend on the statistical characteristics of process $z(\tau)$ for $0\le\tau\le t$, which are completely described by the characteristic functional

$$\Phi[t;\,v(\tau)] = \exp\{\Theta[t;\,v(\tau)]\} = \left\langle\exp\left\{i\int\limits_0^t d\tau z(\tau)v(\tau)\right\}\right\rangle.$$

In this case, the obtained formulas hold also for calculating statistical averages $\left\langle z(t')R[t; z(\tau)]\right\rangle$ for $t' < t, \quad \tau \le t$, i.e., we have the equality

$$\left\langle z(t')R[t; z(\tau)]\right\rangle = \left\langle \Omega\left[t', t; \frac{\delta}{i\delta z(\tau)}\right] R[t; z(\tau)]\right\rangle \quad (0 < t' < t), \tag{B.22}$$

where

$$\begin{aligned}\Omega[t', t; v(\tau)] &= \frac{\delta}{i\delta v(t')]}\Theta[t; v(\tau)] \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_0^t dt_1 \cdots \int_0^t dt_n K_{n+1}(t', t_1, \ldots, t_n) v(t_1) \cdots v(t_n).\end{aligned} \tag{B.23}$$

For $t' = t - 0$, formula (B.22) holds as before, i.e.

$$\langle z(t)R[t; z(\tau)]\rangle = \left\langle \Omega\left[t, t; \frac{\delta}{i\delta z(\tau)}\right] R[t; z(\tau)]\right\rangle. \tag{B.24}$$

However, expansion (B.23) not always gives the correct result in the limit $t' \to t - 0$ (which means that the limiting process and the procedure of expansion in the functional Taylor series can be non-commutable). In this case,

$$\Omega[t, t; v(\tau)] = \frac{\left\langle z(t) \exp\left\{ i \int_0^t d\tau z(\tau) v(\tau) \right\} \right\rangle}{\left\langle \exp\left\{ i \int_0^t d\tau z(\tau) v(\tau) \right\} \right\rangle} = \frac{d}{iv(t)dt}\Theta[t; v(\tau)], \tag{B.25}$$

and statistical averages in Eqs. (B.22) and (B.24) can be discontinuous at $t' = t - 0$.

B.3.1 Correlation Splitting for Random Gaussian Processes and Fields (Furutzu–Novikov Formula)

In the case of the Gaussian random process $z(t)$, all formulas obtained in the previous section become significantly simpler. In this case, the logarithm of characteristic functional $\Phi[v(\tau)]$ is given by Eq. (B.10) (we assume that the mean value of process $z(t)$ is zero), and functional $\Theta[t, v(\tau)]$ assumes the form

$$\Theta[t, v(\tau)] = -\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 B(\tau_1, \tau_2) v(\tau_1) v(\tau_2).$$

As a consequence, functional $\Omega[t; v(\tau)]$ (B.19) is the linear functional

$$\Omega[t; v(\tau)] = i \int_{-\infty}^{\infty} d\tau_1 B(t, \tau_1) v(\tau_1), \tag{B.26}$$

and Eq. (B.18) assumes the form

$$\langle z(t) R[z(\tau) + \eta(\tau)] \rangle = \int_{-\infty}^{\infty} d\tau_1 B(t, \tau_1) \frac{\delta}{\delta \eta(\tau_1)} \langle R[z(\tau) + \eta(\tau)] \rangle . \tag{B.27}$$

Replacing differentiation with respect to $\eta(\tau)$ by differentiation with respect to $z(\tau)$ and setting $\eta(\tau) = 0$, we obtain the equality

$$\langle z(t) R[z(\tau)] \rangle = \int_{-\infty}^{\infty} d\tau_1 B(t, \tau_1) \left\langle \frac{\delta}{\delta z(\tau_1)} R[z(\tau)] \right\rangle \tag{B.28}$$

commonly known in physics as the *Furutsu–Novikov formula*.

One can easily obtain the multi-dimensional extension of Eq. (B.28); it can be written in the form

$$\left\langle z_{i_1,\ldots,i_n}(\mathbf{r}) R[\mathbf{z}] \right\rangle = \int d\mathbf{r}' \left\langle z_{i_1,\ldots,i_n}(\mathbf{r}) z_{j_1,\ldots,j_n}(\mathbf{r}') \right\rangle \left\langle \frac{\delta R[\mathbf{z}]}{\delta z_{j_1,\ldots,j_n}(\mathbf{r}')} \right\rangle, \tag{B.29}$$

where $\mathbf{r}$ stands for all continuous arguments of random vector field $\mathbf{z}(\mathbf{r})$ and $i_1, \ldots, i_n$ are the discrete (index) arguments. Repeated index arguments in the right-hand side of Eq. (B.29) assume summation.

If random process $z(\tau)$ is defined only on time interval $[0, t]$, then functional $\Theta[t, v(\tau)]$ assumes the form

$$\Theta[t, v(\tau)] = -\frac{1}{2} \int_0^t \int_0^t d\tau_1 d\tau_2 B(\tau_1, \tau_2) v(\tau_1) v(\tau_2), \tag{B.30}$$

and functionals $\Omega[t', t; v(\tau)]$ and $\Omega[t, t; v(\tau)]$ are the linear functionals

$$\begin{aligned} \Omega[t', t; v(\tau)] &= \frac{\delta}{i\delta v(t')} \Theta[t, v(\tau)] = i \int_0^t d\tau B(t', \tau) v(\tau), \\ \Omega[t, t; v(\tau)] &= \frac{d}{iv(t)dt} \Theta[t, v(\tau)] = i \int_0^t d\tau B(t, \tau) v(\tau). \end{aligned} \tag{B.31}$$

As a consequence, Eqs. (B.22), (B.24) assume the form

$$\left\langle z(t')R[t, z(\tau)]\right\rangle = \int_0^t d\tau B(t', \tau)\left\langle \frac{\delta R[z(\tau)]}{\delta z(\tau)}\right\rangle \quad (t' \leqslant t) \tag{B.32}$$

that coincides with Eq. (B.28) if the condition

$$\frac{\delta R[t; z(\tau)]}{\delta z(\tau)} = 0 \quad \text{for} \quad \tau < 0, \quad \tau > t \tag{B.33}$$

holds.

C. Approximation of Gaussian Random Field Delta-Correlated in Time

C.1 The Fokker–Planck Equation

Let vector function $\mathbf{x}(t) = \{x_1(t),\ x_2(t), \ldots,\ x_n(t)\}$ satisfies the dynamic equation

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{v}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{C.1}$$

where $v_i(\mathbf{x}, t)$ $(i = 1, \ldots, n)$ are the deterministic functions and $f_i(\mathbf{x}, t)$ are the random functions of $(n+1)$ variable that have the following properties:

(a) $f_i(\mathbf{x}, t)$ is the Gaussian random field in the $(n+1)$-dimensional space $(\mathbf{x}, t)$;
(b) $\langle f_i(\mathbf{x}, t)\rangle = 0$.

For definiteness, we assume that t is the temporal variable and $\mathbf{x}$ is the spatial variable.

Statistical characteristics of field $f_i(\mathbf{x}, t)$ are completely described by correlation tensor

$$B_{ij}(\mathbf{x}, t; \mathbf{x}', t') = \left\langle f_i(\mathbf{x}, t)f_j(\mathbf{x}', t')\right\rangle.$$

Because Eq. (C.1) is the first-order equation with the initial value, its solution satisfies the dynamic causality condition

$$\frac{\delta}{\delta f_j(\mathbf{x}', t')}x_i(t) = 0 \quad \text{for} \quad t' < t_0 \quad \text{and} \quad t' > t, \tag{C.2}$$

which means that solution $\mathbf{x}(t)$ depends only on values of function $f_j(\mathbf{x}, t')$ for times t' preceding time t, i.e., $t_0 \le t' \le t$. In addition, we have the following equality for the variational derivative

$$\frac{\delta}{\delta f_j(\mathbf{x}', t-0)} x_i(t) = \delta_{ij}\delta(\mathbf{x}(t) - \mathbf{x}'). \tag{C.3}$$

Nevertheless, the statistical relationship between $\mathbf{x}(t)$ and function values $f_j(\mathbf{x}, t'')$ for consequent times $t'' > t$ can exist, because such function values $f_j(\mathbf{x}, t'')$ are correlated with values $f_j(\mathbf{x}, t')$ for $t' \le t$. It is obvious that the correlation between function $\mathbf{x}(t)$ and consequent values $f_j(\mathbf{x}, t'')$ is appreciable only for $t'' - t \le \tau_0$, where τ_0 is the correlation radius of field $\mathbf{f}(\mathbf{x}, t)$ with respect to variable t.

For many actual physical processes, characteristic temporal scale T of function $\mathbf{x}(t)$ significantly exceeds correlation radius τ_0 ($T \gg \tau_0$); in this case, the problem has small parameter τ_0/T that can be used to construct an approximate solution.

In the first approximation with respect to this small parameter, one can consider the asymptotic solution for $\tau_0 \to 0$. In this case values of function $\mathbf{x}(t')$ for $t' < t$ will be independent of values $\mathbf{f}(\mathbf{x}, t'')$ for $t'' > t$ not only functionally, but also statistically. This approximation is equivalent to the replacement of correlation tensor B_{ij} with the effective tensor

$$B_{ij}^{\text{eff}}(\mathbf{x}, t; \mathbf{x}', t') = 2\delta(t - t')F_{ij}(\mathbf{x}, \mathbf{x}'; t). \tag{C.4}$$

Here, quantity $F_{ij}(\mathbf{x}, \mathbf{x}', t)$ is determined from the condition that integrals of $B_{ij}(\mathbf{x}, t; \mathbf{x}', t')$ and $B_{ij}^{\text{eff}}(\mathbf{x}, t; \mathbf{x}', t')$ over t' coincide

$$F_{ij}(\mathbf{x}, \mathbf{x}', t) = \frac{1}{2}\int_{-\infty}^{\infty} dt' B_{ij}(\mathbf{x}, t; \mathbf{x}', t'),$$

which just corresponds to the passage to the Gaussian random field delta-correlated in time t.

Introduce the indicator function

$$\varphi(\mathbf{x}, t) = \delta(\mathbf{x}(t) - \mathbf{x}), \tag{C.5}$$

where $\mathbf{x}(t)$ is the solution to Eq. (C.1), which satisfies the Liouville equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\mathbf{v}(\mathbf{x}, t)\right)\varphi(\mathbf{x}, t) = -\frac{\partial}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}, t)\varphi(\mathbf{x}, t) \tag{C.6}$$

and the equality

$$\frac{\delta}{\delta f_j(\mathbf{x}', t-0)}\varphi(\mathbf{x}, t) = -\frac{\partial}{\partial x_j}\left\{\delta(\mathbf{x} - \mathbf{x}')\varphi(\mathbf{x}, t)\right\}. \tag{C.7}$$

The equation for the probability density of the solution to Eq. (C.1)

$$P(\mathbf{x}, t) = \langle\varphi(\mathbf{x}, t)\rangle = \langle\delta(\mathbf{x}(t) - \mathbf{x})\rangle$$

can be obtained by averaging Eq. (C.6) over an ensemble of realizations of field $\mathbf{f}(\mathbf{x}, t)$,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\mathbf{v}(\mathbf{x}, t)\right) P(\mathbf{x}, t) = -\frac{\partial}{\partial \mathbf{x}} \langle \mathbf{f}(\mathbf{x}, t)\varphi(\mathbf{x}, t)\rangle . \tag{C.8}$$

We rewrite Eq. (C.8) in the form

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\mathbf{v}(\mathbf{x}, t)\right)P(\mathbf{x}, t) = -\frac{\partial}{\partial x_i} \int d\mathbf{x}' \int_{t_0}^{t} dt' B_{ij}(\mathbf{x}, t; \mathbf{x}', t') \left\langle \frac{\delta\varphi(\mathbf{x}, t)}{\delta f_j(\mathbf{x}', t')} \right\rangle, \tag{C.9}$$

where we used the Furutsu–Novikov formula (B.29)

$$\langle f_k(\mathbf{x}, t)R[t; \mathbf{f}(\mathbf{y}, \tau)]\rangle = \int d\mathbf{x}' \int dt' B_{kl}(\mathbf{x}, t; \mathbf{x}', t') \left\langle \frac{\delta R[t; \mathbf{f}(\mathbf{y}, \tau)]}{\delta f_l(\mathbf{x}', t')} \right\rangle \tag{C.10}$$

for the correlator of the Gaussian random field $\mathbf{f}(\mathbf{x}, t)$ with arbitrary functional $R[t; \mathbf{f}(\mathbf{y}, \tau)]$ of it and the dynamic causality condition (C.2).

Equation (C.9) shows that the one-time probability density of solution $\mathbf{x}(t)$ at instant t is governed by functional dependence of solution $\mathbf{x}(t)$ on field $\mathbf{f}(\mathbf{x}', t)$ for all times in the interval (t_0, t).

In the general case, there is no closed equation for the probability density $P(\mathbf{x}, t)$. However, if we use approximation (C.4) for the correlation function of field $\mathbf{f}(\mathbf{x}, t)$, there appear terms related to variational derivatives $\delta\varphi[\mathbf{x}, t; \mathbf{f}(\mathbf{y}, \tau)]/\delta f_j(\mathbf{x}', t')$ at coincident temporal arguments $t' = t - 0$,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\mathbf{v}(\mathbf{x}, t)\right) P(\mathbf{x}, t) = -\frac{\partial}{\partial x_i} \int d\mathbf{x}' F_{ij}(\mathbf{x}, \mathbf{x}', t) \left\langle \frac{\delta\varphi(\mathbf{x}, t)}{\delta f_j(\mathbf{x}', t-0)} \right\rangle.$$

According to Eq. (C.7), these variational derivatives can be expressed immediately in terms of quantity $\varphi[\mathbf{x}, t; \mathbf{f}(\mathbf{y}, \tau)]$. Thus, we obtain the closed Fokker–Planck equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_k} \left[v_k(\mathbf{x}, t) + A_k(\mathbf{x}, t)\right]\right) P(\mathbf{x}, t) = \frac{\partial^2}{\partial x_k \partial x_l} \left[F_{kl}(\mathbf{x}, \mathbf{x}, t)P(\mathbf{x}, t)\right], \tag{C.11}$$

where

$$A_k(\mathbf{x}, t) = \left. \frac{\partial}{\partial x_l'} F_{kl}(\mathbf{x}, \mathbf{x}', t)\right|_{\mathbf{x}'=\mathbf{x}} .$$

Equation (C.11) should be solved with the initial condition

$$P(\mathbf{x}, t_0) = \delta(\mathbf{x} - \mathbf{x}_0),$$

or with a more general initial condition $P(\mathbf{x}, t_0) = W_0(\mathbf{x})$ if the initial conditions are also random, but statistically independent of field $\mathbf{f}(\mathbf{x}, t)$.

The Fokker–Planck equation (C.11) is a partial differential equation and its further analysis essentially depends on boundary conditions with respect to $\mathbf{x}$ whose form can vary depending on the problem at hand.

Consider the quantities appeared in Eq. (C.11). In this equation, the terms containing $A_k(\mathbf{x}, t)$ and $F_{kl}(\mathbf{x}, \mathbf{x}', t)$ are stipulated by fluctuations of field $\mathbf{f}(\mathbf{x}, t)$. If field $\mathbf{f}(\mathbf{x}, t)$ is stationary in time, quantities $A_k(\mathbf{x})$ and $F_{kl}(\mathbf{x}, \mathbf{x}')$ are independent of time. If field $\mathbf{f}(\mathbf{x}, t)$ is additionally homogeneous and isotropic in all spatial coordinates, then

$$F_{kl}(\mathbf{x}, \mathbf{x}, t) = \text{const},$$

which corresponds to the constant tensor of diffusion coefficients, and $A_k(\mathbf{x}, t) = 0$ (note however that quantities $F_{kl}(\mathbf{x}, \mathbf{x}', t)$ and $A_k(\mathbf{x}, t)$ can depend on $\mathbf{x}$ because of the use of a curvilinear coordinate systems).

C.2 Transition Probability Distributions

Turn back to dynamic system (C.1) and consider the m-time probability density

$$P_m(\mathbf{x}_1, t_1; \cdots; \mathbf{x}_m, t_m) = \langle \delta(\mathbf{x}(t_1) - \mathbf{x}_1) \cdots \delta(\mathbf{x}(t_m) - \mathbf{x}_m) \rangle \tag{C.12}$$

for m different instants $t_1 < t_2 < \cdots < t_m$. Differentiating Eq. (C.12) with respect to time t_m and using then dynamic equation (C.1), dynamic causality condition (C.2), definition of function $F_{kl}(\mathbf{x}, \mathbf{x}', t)$, and the Furutsu–Novikov formula (C.10), one can obtain the equation similar to the Fokker–Planck equation (C.11),

$$\begin{aligned}
&\frac{\partial}{\partial t_m} P_m(\mathbf{x}_1, t_1; \cdots; \mathbf{x}_m, t_m) \\
&\qquad + \sum_{k=1}^{n} \frac{\partial}{\partial x_{mk}} \left[v_k(\mathbf{x}_m, t_m) + A_k(\mathbf{x}_m, t_m) \right] P_m(\mathbf{x}_1, t_1; \cdots; \mathbf{x}_m, t_m) \\
&\qquad = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial^2}{\partial x_{mk} \partial x_{ml}} \left[F_{kl}(\mathbf{x}_m, \mathbf{x}_m, t_m) P_m(\mathbf{x}_1, t_1; \cdots; \mathbf{x}_m, t_m)) \right].
\end{aligned} \tag{C.13}$$

No summation over index m is performed here. The initial value to Eq. (C.13) can be determined from Eq. (C.12). Setting $t_m = t_{m-1}$ in (C.12), we obtain

$$\begin{aligned}
&P_m(\mathbf{x}_1, t_1; \cdots; \mathbf{x}_m, t_{m-1}) \\
&\qquad = \delta(\mathbf{x}_m - \mathbf{x}_{m-1}) P_{m-1}(\mathbf{x}_1, t_1; \cdots; \mathbf{x}_{m-1}, t_{m-1}).
\end{aligned} \tag{C.14}$$

Equation (C.13) assumes the solution in the form

$$P_m(\mathbf{x}_1, t_1; \cdots; \mathbf{x}_m, t_m) = p(\mathbf{x}_m, t_m|\mathbf{x}_{m-1}, t_{m-1}) P_{m-1}(\mathbf{x}_1, t_1; \cdots; \mathbf{x}_{m-1}, t_{m-1}). \quad \text{(C.15)}$$

Because all differential operations in Eq. (C.13) concern only t_m and $\mathbf{x}_m$, we can find the equation for the *transitional probability density* by substituting Eq. (C.15) in (C.13) and (C.14):

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_k}\left[v_k(\mathbf{x}, t) + A_k(\mathbf{x}, t)\right]\right) p(\mathbf{x}, t|\mathbf{x}_0, t_0) = \frac{\partial^2}{\partial x_k \partial x_l}\left[F_{kl}(\mathbf{x}, \mathbf{x}, t) p(\mathbf{x}, t|\mathbf{x}_0, t_0)\right] \quad \text{(C.16)}$$

with initial condition

$$p(\mathbf{x}, t|\mathbf{x}_0, t_0)|_{t \to t_0} = \delta(\mathbf{x} - \mathbf{x}_0),$$

where

$$p(\mathbf{x}, t|\mathbf{x}_0, t_0) = \langle \delta(\mathbf{x}(t) - \mathbf{x}) | \mathbf{x}(t_0) = \mathbf{x}_0 \rangle .$$

In Eq. (C.16) we denoted variables $\mathbf{x}_m$ and t_m as $\mathbf{x}$ and t, and variables $\mathbf{x}_{m-1}$ and t_{m-1} as $\mathbf{x}_0$ and t_0.

Using formula (C.15) $(m - 1)$ times, we obtain the relationship

$$P_m(\mathbf{x}_1, t_1; \cdots; \mathbf{x}_m, t_m) = p(\mathbf{x}_m, t_m|\mathbf{x}_{m-1}, t_{m-1}) \cdots p(\mathbf{x}_2, t_2|\mathbf{x}_1, t_1) P(\mathbf{x}_1, t_1), \quad \text{(C.17)}$$

where $P(\mathbf{x}_1, t_1)$ is the one-time probability density governed by Eq. (C.11). Equality (C.17) expresses the multi-time probability density as the product of transitional probability densities, which means that random process $\mathbf{x}(t)$ is the Markovian process.

Equation (C.11) is usually called the *forward Fokker–Planck equation*. The *backward Fokker–Planck equation* (it describes the transitional probability density as a function of the initial parameters t_0 and $\mathbf{x}_0$) can also be easily derived (see, for example, monographs [28, 29]).

C.3 Applicability Range of the Fokker–Planck Equation

To estimate the applicability range of the Fokker–Planck equation, we must include into consideration the finite-valued correlation radius τ_0 of field $\mathbf{f}(\mathbf{x}, t)$ with respect to time. Thus, smallness of parameter τ_0/T is the necessary but generally not sufficient condition in order that one can describe statistical characteristics of the solution to Eq. (C.1) using the approximation of the delta-correlated random field of which a consequence is the Fokker–Planck equation. Every particular problem requires more detailed investigations. Below, we give a more physical method called the *diffusion approximation*. This method also leads to the Markovian property of the solution to Eq. (C.1); however, it considers to some extent the finite value of the temporal correlation radius.

Here, we emphasize that the approximation of the delta-correlated random field does not reduce to the formal replacement of random field $\mathbf{f}(\mathbf{x}, t)$ in Eq. (C.1) with the random field with correlation function (C.4). This approximation corresponds to the construction of an asymptotic expansion in temporal correlation radius τ_0 of filed $\mathbf{f}(\mathbf{x}, t)$ for $\tau_0 \to 0$. It is in such limit process that exact average quantities like

$$\left\langle \mathbf{f}(\mathbf{x}, t) R[t; \mathbf{f}(\mathbf{x}', \tau)] \right\rangle$$

grade into the expressions obtained by the formal replacement of the correlation tensor of field $\mathbf{f}(\mathbf{x}, t)$ with the effective tensor (C.4).

C.3.1 Langevin Equation

We illustrate the above speculation by the example of the *Langevin equation* that allows an exhaustive statistical analysis. This equation has the form

$$\frac{d}{dt} x(t) = -\lambda x(t) + f(t), \quad x(t_0) = 0 \tag{C.18}$$

and assumes that the sufficiently fine smooth function $f(t)$ is the stationary Gaussian process with zero-valued mean and correlation function

$$\left\langle f(t) f(t') \right\rangle = B_f(t - t').$$

For any individual realization of random force $f(t)$, the solution to Eq. (C.18) has the form

$$x(t) = \int_{t_0}^{t} d\tau f(\tau) e^{-\lambda(t-\tau)}.$$

Consequently, this solution $x(t)$ is also the Gaussian process with the parameters

$$\langle x(t) \rangle = 0, \quad \left\langle x(t) x(t') \right\rangle = \int_{t_0}^{t} d\tau_1 \int_{t_0}^{t'} d\tau_2 B_f(\tau_1 - \tau_2) e^{-\lambda(t + t' - \tau_1 - \tau_2)}.$$

In addition, we have, for example,

$$\langle f(t) x(t) \rangle = \int_{0}^{t - t_0} d\tau B_f(\tau) e^{-\lambda \tau}.$$

Note that the one-point probability density $P(x, t) = \langle \delta(x(t) - x) \rangle$ of the solution to Eq. (C.18) satisfies the exact equation

$$\left(\frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial x} x \right) P(x, t) = \int\limits_0^{t-t_0} d\tau B_f(\tau) e^{-\lambda\tau} \frac{\partial^2}{\partial x^2} P(x, t), \quad P(x, t_0) = \delta(x),$$

which rigorously follows from Eq. (C.18). As a consequence, we obtain

$$\frac{d}{dt} \left\langle x^2(t) \right\rangle = -2\lambda \left\langle x^2(t) \right\rangle + 2 \int\limits_0^{t-t_0} d\tau B_f(\tau) e^{-\lambda\tau}.$$

For $t_0 \to -\infty$, process $x(t)$ grades into the stationary Gaussian process with the following one-time statistical parameters ($\langle x(t) \rangle = 0$)

$$\sigma_x^2 = \left\langle x^2(t) \right\rangle = \frac{1}{\lambda} \int\limits_0^{\infty} d\tau B_f(\tau) e^{-\lambda\tau}, \quad \langle f(t) x(t) \rangle = \int\limits_0^{\infty} d\tau B_f(\tau) e^{-\lambda\tau}.$$

In particular, for exponential correlation function $B_f(t)$,

$$B_f(t) = \sigma_f^2 e^{-|\tau|/\tau_0},$$

we obtain the expressions

$$\langle x(t) \rangle = 0, \quad \left\langle x^2(t) \right\rangle = \frac{\sigma_f^2 \tau_0}{\lambda(1 + \lambda\tau_0)}, \quad \langle f(t) x(t) \rangle = \frac{\sigma_f^2 \tau_0}{1 + \lambda\tau_0}, \tag{C.19}$$

which grade into the asymptotic expressions

$$\left\langle x^2(t) \right\rangle = \frac{\sigma_f^2 \tau_0}{\lambda}, \quad \langle f(t) x(t) \rangle = \sigma_f^2 \tau_0 \tag{C.20}$$

for $\tau_0 \to 0$.

Multiply now Eq. (C.18) by $x(t)$. Assuming that function $x(t)$ is sufficiently fine function, we obtain the equality

$$x(t) \frac{d}{dt} x(t) = \frac{1}{2} \frac{d}{dt} x^2(t) = -\lambda x^2(t) + f(t) x(t).$$

Averaging this equation over an ensemble of realizations of function $f(t)$, we obtain the equation

$$\frac{1}{2} \frac{d}{dt} \left\langle x^2(t) \right\rangle = -\lambda \left\langle x^2(t) \right\rangle + \langle f(t) x(t) \rangle, \tag{C.21}$$

whose steady-state solution (it corresponds to the limit process $t_0 \to -\infty$ and $\tau_0 \to 0$)

$$\left\langle x^2(t) \right\rangle = \frac{1}{\lambda} \left\langle f(t)x(t) \right\rangle$$

coincides with Eqs. (C.19) and (C.20).

Taking into account the fact that $\delta x(t)/\delta f(t-0) = 1$, we obtain the same result for correlation $\langle f(t)x(t) \rangle$ by using the formula

$$\langle f(t)x(t) \rangle = \int\limits_{-\infty}^{t} d\tau B_f(t-\tau) \left\langle \frac{\delta}{\delta f(\tau)} x(t) \right\rangle \tag{C.22}$$

with the effective correlation function

$$B_f^{\text{eff}}(t) = 2\sigma_f^2 \tau_0 \delta(t).$$

Earlier, we mentioned that statistical characteristics of solutions to dynamic problems in the approximation of the delta-correlated random process (field) coincide with the statistical characteristics of the Markovian processes. However, one should clearly understand that this is the case only for statistical averages and equations for these averages. In particular, realizations of process $x(t)$ satisfying the Langevin equation (C.18) drastically differ from realizations of the corresponding Markovian process. The latter satisfies Eq. (C.18) in which function $f(t)$ in the right-hand side is the ideal white noise with the correlation function $B_f(t) = 2\sigma_f^2 \tau_0 \delta(t)$; moreover, this equation must be treated in the sense of generalized functions, because the Markovian processes are not differentiable in the ordinary sense. At the same time, process $x(t)$ — whose statistical characteristics coincide with the characteristics of the Markovian process — behaves as sufficiently fine function and is differentiable in the ordinary sense. For example,

$$x(t) \frac{d}{dt} x(t) = \frac{1}{2} \frac{d}{dt} x^2(t),$$

and we have for $t_0 \to -\infty$ in particular

$$\left\langle x(t) \frac{d}{dt} x(t) \right\rangle = 0. \tag{C.23}$$

On the other hand, in the case of the ideal Markovian process $x(t)$ satisfying (in the sense of generalized functions) the Langevin equation (C.18) with the white noise in the right-hand side, Eq. (C.23) makes no sense at all, and the meaning of the relationship

$$\left\langle x(t) \frac{d}{dt} x(t) \right\rangle = -\lambda \left\langle x^2(t) \right\rangle + \langle f(t)x(t) \rangle \tag{C.24}$$

depends on the definition of averages. Indeed, if we will treat Eq. (C.24) as the limit of the equality

$$\left\langle x(t+\Delta)\frac{d}{dt}x(t)\right\rangle = -\lambda\left\langle x(t)x(t+\Delta)\right\rangle + \left\langle f(t)x(t+\Delta)\right\rangle \tag{C.25}$$

for $\Delta \to 0$, the result will be essentially different depending on whether we use limit processes $\Delta \to +0$, or $\Delta \to -0$. For limit process $\Delta \to +0$, we have

$$\lim_{\Delta\to+0}\left\langle f(t)x(t+\Delta)\right\rangle = 2\sigma_f^2\tau_0,$$

and, taking into account Eq. (C.22), we can rewrite Eq. (C.25) in the form

$$\left\langle x(t+0)\frac{d}{dt}x(t)\right\rangle = \sigma_f^2\tau_0. \tag{C.26}$$

On the contrary, for limit process $\Delta \to -0$, we have

$$\left\langle f(t)x(t-0)\right\rangle = 0$$

because of the dynamic causality condition, and Eq. (C.25) assumes the form

$$\left\langle x(t-0)\frac{d}{dt}x(t)\right\rangle = -\sigma_f^2\tau_0. \tag{C.27}$$

Comparing Eq. (C.23) with (C.26) and (C.27), we see that, for the ideal Markovian process described by the solution to the Langevin equation with the white noise in the right-hand side and commonly called the *Ohrnstein–Ulenbeck process*, we have

$$\left\langle x(t+0)\frac{d}{dt}x(t)\right\rangle \neq \left\langle x(t-0)\frac{d}{dt}x(t)\right\rangle \neq \frac{1}{2}\frac{d}{dt}\left\langle x^2(t)\right\rangle.$$

Note that equalities (C.26) and (C.27) can also be obtained from the correlation function

$$\left\langle x(t)x(t+\tau)\right\rangle = \frac{\sigma_f^2\tau_0}{\lambda}e^{-\lambda|\tau|}$$

of process $x(t)$.

To conclude with the discussion of the approximation of the delta-correlated random process (field), we emphasize that, in all further examples, we will treat the sentence like 'dynamic system (equation) with the delta-correlated parameter fluctuations' as the asymptotic limit in which these parameters have temporal correlation radii small in comparison with all characteristic temporal scales of the problem under consideration.

C.3.2 Diffusion Approximation

Applicability of the approximation of the delta-correlated random field $\mathbf{f}(\mathbf{x}, t)$ (i.e., applicability of the Fokker–Planck equation) is restricted by the smallness of the temporal correlation radius τ_0 of random field $\mathbf{f}(\mathbf{x}, t)$ with respect to all temporal scales of the problem under consideration. The effect of the finite-valued temporal correlation radius of random field $\mathbf{f}(\mathbf{x}, t)$ can be considered within the framework of the diffusion approximation. The diffusion approximation appears to be more obvious and physical than the formal mathematical derivation of the approximation of the delta-correlated random field. This approximation also holds for sufficiently weak parameter fluctuations of the stochastic dynamic system and allows describing new physical effects caused by the finite-valued temporal correlation radius of random parameters, rather than only obtaining the applicability range of the delta-correlated approximation. The diffusion approximation assumes that the effect of random actions is insignificant during temporal scales about τ_0, i.e., the system behaves during these time intervals as the free system.

Again, let vector function $\mathbf{x}(t)$ satisfies the dynamic equation (C.1)

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{v}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{C.28}$$

where $\mathbf{v}(\mathbf{x}, t)$ is the deterministic vector function and $\mathbf{f}(\mathbf{x}, t)$ is the random statistically homogeneous and stationary Gaussian vector field with the statistical characteristics

$$\langle f(\mathbf{x}, t)\rangle = 0, \quad B_{ij}(\mathbf{x}, t; \mathbf{x}', t') = B_{ij}(\mathbf{x} - \mathbf{x}', t - t') = \left\langle f_i(\mathbf{x}, t) f_j(\mathbf{x}', t')\right\rangle .$$

Introduce the indicator function

$$\varphi(\mathbf{x}, t) = \delta(\mathbf{x}(t) - \mathbf{x}), \tag{C.29}$$

where $\mathbf{x}(t)$ is the solution to Eq. (C.28) satisfying the Liouville equation (C.6)

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\mathbf{v}(\mathbf{x}, t)\right)\varphi(\mathbf{x}, t) = -\frac{\partial}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}, t)\varphi(\mathbf{x}, t). \tag{C.30}$$

As earlier, we obtain the equation for the probability density of the solution to Eq. (C.28)

$$P(\mathbf{x}, t) = \langle \varphi(\mathbf{x}, t)\rangle = \langle \delta(\mathbf{x}(t) - \mathbf{x})\rangle$$

by averaging Eq. (C.30) over an ensemble of realizations of field $\mathbf{f}(\mathbf{x}, t)$

$$\begin{gathered}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\mathbf{v}(\mathbf{x}, t)\right)P(\mathbf{x}, t) = -\frac{\partial}{\partial \mathbf{x}}\left\langle \mathbf{f}(\mathbf{x}, t)\varphi(\mathbf{x}, t)\right\rangle , \\ P(\mathbf{x}, t_0) = \delta(\mathbf{x} - \mathbf{x}_0).\end{gathered} \tag{C.31}$$

Using the Furutsu–Novikov formula (C.10)

$$\langle f_k(\mathbf{x},t)R[t;\mathbf{f}(\mathbf{y},\tau)]\rangle = \int d\mathbf{x}' \int dt' B_{kl}(\mathbf{x},t;\mathbf{x}',t') \left\langle \frac{\delta}{\delta f_l(\mathbf{x}',t')} R[t;\mathbf{f}(\mathbf{y},\tau)] \right\rangle$$

valid for the correlation between the Gaussian random field $\mathbf{f}(\mathbf{x},t)$ and arbitrary functional $R[t;\mathbf{f}(\mathbf{y},\tau)]$ of this field, we can rewrite Eq. (C.31) in the form

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\mathbf{v}(\mathbf{x},t)\right) P(\mathbf{x},t) = -\frac{\partial}{\partial x_i} \int d\mathbf{x}' \int_{t_0}^{t} dt' B_{ij}(\mathbf{x},t;\mathbf{x}',t') \left\langle \frac{\delta}{\delta f_j(\mathbf{x}',t')} \varphi(\mathbf{x},t) \right\rangle. \qquad \text{(C.32)}$$

In the diffusion approximation, Eq. (C.32) is the exact equation, and the variational derivative and indicator function satisfy, within temporal intervals of about temporal correlation radius τ_0 of random field $\mathbf{f}(\mathbf{x},t)$, the system of dynamic equations

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\delta\varphi(\mathbf{x},t)}{\delta f_i(\mathbf{x}',t')} &= -\frac{\partial}{\partial \mathbf{x}}\left\{\mathbf{v}(\mathbf{x},t)\frac{\delta\varphi(\mathbf{x},t)}{\delta f_i(\mathbf{x}',t')}\right\}, \\ \left.\frac{\delta\varphi(\mathbf{x},t)}{\delta f_i(\mathbf{x}',t')}\right|_{t=t'} &= -\frac{\partial}{\partial x_i}\left\{\delta(\mathbf{x}-\mathbf{x}')\varphi(\mathbf{x},t')\right\}, \\ \frac{\partial}{\partial t}\varphi(\mathbf{x},t) = -\frac{\partial}{\partial \mathbf{x}}&\left\{\mathbf{v}(\mathbf{x},t)\varphi(\mathbf{x},t)\right\}, \quad \varphi(\mathbf{x},t)|_{t=t'} = \varphi(\mathbf{x},t'). \end{aligned} \qquad \text{(C.33)}$$

The solution to problem (C.32), (C.33) holds for all times t. In this case, the solution $\mathbf{x}(t)$ to problem (C.28) cannot be considered as the Markovian vector random process because its multi-time probability density cannot be factorized in terms of the transition probability density. However, in asymptotic limit $t \gg \tau_0$, the diffusion-approximation solution to the initial dynamic system (C.28) will be the Markovian random process, and the corresponding conditions of applicability are formulated as smallness of all statistical effects within temporal intervals of about temporal correlation radius τ_0.

C.4 The Simplest Markovian Random Processes

There are only few Fokker–Planck equations that allow an exact solution. First of all, among them are the Fokker–Planck equations corresponding to the stochastic equations that are themselves solvable in the analytic form. Such problems often allow determination of not only the one-point and transitional probability densities, but also the characteristic functional and other statistical characteristics important for practice.

The simplest special case of Eq. (C.11) is the equation that defines the *Wiener random process*. In view of the significant role that such processes plays in physics (for example, they describe the *Brownian motion of particles*), we consider the Wiener process in detail.

C.4.1 Wiener Random Process

The Wiener random process is defined as the solution to the stochastic equation

$$\frac{d}{dt}w(t) = z(t), \quad w(0) = 0,$$

where $z(t)$ is the Gaussian process delta-correlated in time and described by the parameters

$$\langle z(t) \rangle = 0, \quad \left\langle z(t)z(t') \right\rangle = 2D\delta(t - t').$$

The solution to this equation

$$w(t) = \int_0^t d\tau z(\tau) \tag{C.34}$$

is the continuous Gaussian nonstationary random process with the parameters

$$\langle w(t) \rangle = 0, \quad \left\langle w(t)w(t') \right\rangle = 2D \min(t, t').$$

Figure C.1 shows a realization of the Wiener process (C.34) simulated numerically.

C.4.2 Wiener Random Process with Shear

Consider a more general process that includes additionally the drift dependent on parameter α

$$w(t; \alpha) = -\alpha t + w(t), \quad \alpha > 0.$$

Process $w(t; \alpha)$ is the Markovian process, and its probability density

$$P(w, t; \alpha) = \langle \delta(w(t; \alpha) - w) \rangle$$

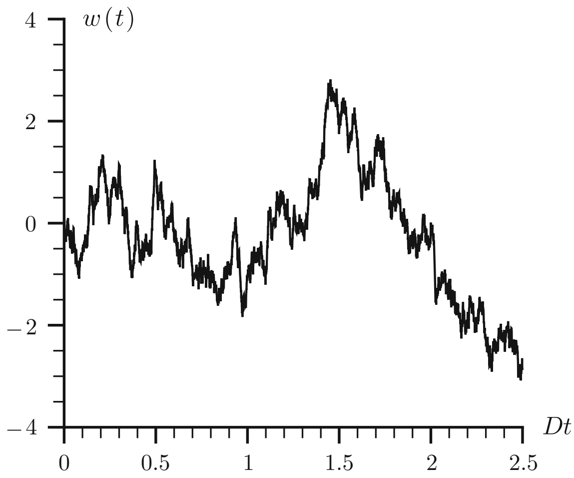

Fig. C.1 Realization of the Wiener process (C.34)

satisfies the Fokker–Planck equation

$$\left(\frac{\partial}{\partial t} - \alpha\frac{\partial}{\partial w}\right)P(w,t;\alpha) = D\frac{\partial^2}{\partial w^2}P(w,t;\alpha), \quad P(w,0;\alpha) = \delta(w), \tag{C.35}$$

where $D = \sigma^2\tau_0$ is the diffusion coefficient. The solution to this equation has the form of the Gaussian distribution

$$P(w,t;\alpha) = \frac{1}{2\sqrt{\pi Dt}}\exp\left\{-\frac{(w+\alpha t)^2}{4Dt}\right\}. \tag{C.36}$$

The corresponding integral distribution function defined as the probability of the event that $w(t;\alpha) < w$ is given by the formula

$$F(w,t;\alpha) = \int\limits_{-\infty}^{w} dwP(w,t;\alpha) = \Pr\left(\frac{w}{\sqrt{2Dt}} + \alpha\sqrt{\frac{t}{2D}}\right), \tag{C.37}$$

where function $\Pr(z)$ is the *probability integral* (5.4). In this case, the typical realization curve of the Wiener random process with shear is the linear function of time

$$w^*(t;\alpha) = -\alpha t.$$

In addition to the initial value, supplement Eq. (C.35) with the boundary condition

$$P(w,t;\alpha)|_{w=h} = 0, \quad (t > 0). \tag{C.38}$$

This condition breaks down realizations of process $w(t;\alpha)$ at the instant they reach boundary h. For $w < h$, the solution to the boundary-value problem (C.35), (C.38) (we denote it as $P(w,t;\alpha,h)$) describes the probability distribution of those realizations of process $w(t;\alpha)$ that survived instant t, i.e., never reached boundary h during the whole temporal interval. Correspondingly, the norm of the probability density appears not unity, but the probability of the event that $t < t^*$, where t^* is the instant at which process $w(t;\alpha)$ reaches boundary h for the first time

$$\int_{-\infty}^{h} dwP(w,t;\alpha,h) = P(t < t^*). \tag{C.39}$$

Introduce the integral distribution function and probability density of random instant at which the process reaches boundary h

$$\begin{aligned} F(t;\alpha,h) = P(t^* < t) = 1 - P(t < t^*) = 1 - \int_{-\infty}^{h} dwP(w,t;\alpha,h), \\ P(t;\alpha,h) = \frac{\partial}{\partial t}F(t;\alpha,h) = -\frac{\partial}{\partial w}P(w,t;\alpha,h)|_{w=h}. \end{aligned} \tag{C.40}$$

If $\alpha > 0$, process $w(t;\alpha)$ moves on average out of boundary h; as a result, probability $P(t < t^*)$ (C.39) tends for $t \to \infty$ to the probability of the event that process $w(t;\alpha)$ never reaches boundary h. In other words, limit

$$\lim_{t\to\infty} \int_{-\infty}^{h} dwP(w,t;\alpha,h) = P\left(w_{\max}(\alpha) < h\right) \tag{C.41}$$

is equal to the probability of the event that the process absolute maximum

$$w_{\max}(\alpha) = \max_{t\in(0,\infty)} w(t;\alpha)$$

is less than h. Thus, from Eq. (C.41) follows that the integral distribution function of the absolute maximum $w_{\max}(\alpha)$ is given by the formula

$$F(h;\alpha) = P\left(w_{\max}(\alpha) < h\right) = \lim_{t\to\infty} \int_{-\infty}^{h} dwP(w,t;\alpha,h). \tag{C.42}$$

After we solve boundary-value problem (C.35), (C.38) by using, for example, the reflection method, we obtain

$$P(w,t;\alpha,h) \\ = \frac{1}{2\sqrt{\pi Dt}}\left\{\exp\left[-\frac{(w+\alpha t)^2}{4Dt}\right] - \exp\left[-\frac{h\alpha}{D} - \frac{(w-2h+\alpha t)^2}{4Dt}\right]\right\}. \quad \text{(C.43)}$$

Substituting this expression in Eq. (C.40), we obtain the probability density of instant t^* at which process $w(t;\alpha)$ reaches boundary h for the first time

$$P(t;\alpha,h) = \frac{1}{2Dt\sqrt{\pi Dt}}\exp\left\{-\frac{(h+\alpha t)^2}{4Dt}\right\}.$$

Finally, integrating Eq. (C.43) over w and setting $t \to \infty$, we obtain, in accordance with Eq. (C.42), the integral distribution function of absolute maximum $w_{\max}(\alpha)$ of process $w(t;\alpha)$ in the form

$$F(h;\alpha) = P\left(w_{\max}(\alpha) < h\right) = 1 - \exp\left\{-\frac{h\alpha}{D}\right\}. \quad \text{(C.44)}$$

Consequently, the absolute maximum of the Wiener process has the exponential probability density

$$P(h;\alpha) = \left\langle \delta\left(w_{\max}(\alpha) - h\right)\right\rangle = \frac{\alpha}{D}\exp\left\{-\frac{h\alpha}{D}\right\}.$$

The Wiener random process offers a possibility of constructing other processes convenient for modeling different physical phenomena. In the case of positive quantities, the simplest approximation of such kind is the logarithmic-normal (lognormal) process. Consider this process in greater detail.

C.4.3 Logarithmic-Normal Random Process

We define the lognormal process (logarithmic-normal process) by the formula

$$y(t;\alpha) = e^{w(t;\alpha)} = \exp\left\{-\alpha t + \int_0^t d\tau z(\tau)\right\}, \quad \text{(C.45)}$$

where $z(t)$ is the Gaussian white noise process with the parameters

$$\langle z(t)\rangle = 0, \quad \left\langle z(t)z(t')\right\rangle = 2\sigma^2\tau_0\delta(t-t').$$

The lognormal process satisfies the stochastic equation

$$\frac{d}{dt}y(t;\alpha) = \{-\alpha + z(t)\}\,y(t;\alpha), \qquad y(0;\alpha) = 1.$$

The one-time probability density of the lognormal process is given by the formula

$$P(y,t;\alpha) = \langle \delta\left(y(t;\alpha)-y\right)\rangle = \left\langle \delta\left(e^{w(t;\alpha)}-y\right)\right\rangle$$
$$= \frac{1}{y}\left\langle \delta\left(w(t;\alpha)-\ln y\right)\right\rangle = \frac{1}{y}\, P(w,t;\alpha)\big|_{w=\ln y},$$

where $P(w,t;\alpha)$ is the one-time probability density of the Wiener process with a drift, which is given by Eq. (C.36), so that

$$P(y,t;\alpha) = \frac{1}{2y\sqrt{\pi Dt}}\exp\left\{-\frac{(\ln y+\alpha t)^2}{4Dt}\right\}$$
$$= \frac{1}{2y\sqrt{\pi Dt}}\exp\left\{-\frac{\ln^2\left(ye^{\alpha t}\right)}{4Dt}\right\}, \tag{C.46}$$

where $D = \sigma^2\tau_0$.

Note that the one-time probability density of random process $\widetilde{y}(t;\alpha) = 1/y(t;\alpha)$ is also lognormal and is given by the formula

$$P(\widetilde{y},t;\alpha) = \frac{1}{2\widetilde{y}\sqrt{\pi Dt}}\exp\left\{-\frac{\ln^2\left(\widetilde{y}e^{-\alpha t}\right)}{4Dt}\right\}, \tag{C.47}$$

which coincides with Eq. (C.46) with parameter α of opposite sign. Correspondingly, the integral distribution functions are given, in accordance with Eq. (C.37), by the expressions

$$F(y,t;\alpha) = P\left(y(t;\alpha)<y\right) = \Pr\left(\frac{1}{\sqrt{2Dt}}\ln\left(ye^{\pm\alpha t}\right)\right), \tag{C.48}$$

where $\Pr(z)$ is the probability integral (5.4)

Figure 5.1 show the curves of the lognormal probability densities for $\alpha/D = 1$ and dimensionless times $\tau = Dt = 0.1$ and 1. Figure 5.2 shows these probability densities at $\tau = 1$ in logarithmic scale along the abscissa.

Structurally, these probability distributions are absolutely different. The only common feature of these distributions consists in the existence of long flat *tails* that appear in distributions at $\tau = 1$; these tails increase the role of high peaks of processes $y(t;\alpha)$ and $\widetilde{y}(t;\alpha)$ in the formation of the one-time statistics.

Having only the one-point statistical characteristics of processes $y(t;\alpha)$ and $\widetilde{y}(t;\alpha)$, one can obtain a deeper insight into the behavior of realizations of these processes on the whole interval of times $(0,\infty)$. In particular,

(**1**) The lognormal process $y(t;\alpha)$ is the Markovian process and its one-time probability density satisfies the Fokker–Planck equation

$$\left(\frac{\partial}{\partial t}-\alpha\frac{\partial}{\partial y}y\right)P(y,t;\alpha)=D\frac{\partial}{\partial y}y\frac{\partial}{\partial y}yP(y,t;\alpha),\quad P(y,0;\alpha)=\delta(y-1). \tag{C.49}$$

From Eq. (C.49), one can easily derive the equations for the moment functions of processes $y(t;\alpha)$ and $\widetilde{y}(t;\alpha)$; solutions to these equations are given by the formulas $(n=1,\ 2,\dots)$

$$\langle y^n(t;\alpha)\rangle=e^{n(n-\alpha/D)Dt},\quad \langle\widetilde{y}^n(t;\alpha)\rangle=\left\langle\frac{1}{y^n(t;\alpha)}\right\rangle=e^{n(n+\alpha/D)Dt}, \tag{C.50}$$

from which follows that moments exponentially grow with time.

From Eq. (C.49), one can easily obtain the equality

$$\langle\ln y(t)\rangle=-\alpha t.$$

Consequently, parameter α can be rewritten in the form

$$-\alpha=\frac{1}{t}\langle\ln y(t)\rangle\quad\text{or}\quad\alpha=\frac{1}{t}\langle\ln\widetilde{y}(t)\rangle. \tag{C.51}$$

Note that many investigators give great attention to the approach based on the Lyapunov analysis of stability of solutions to deterministic ordinary differential equations

$$\frac{d}{dt}\mathbf{x}(t)=A(t)\mathbf{x}(t).$$

This approach deals with the upper limit of problem solution

$$\lambda_{\mathbf{x}(t)}=\overline{\lim_{t\to+\infty}}\frac{1}{t}\ln|\mathbf{x}(t)|$$

called the characteristic index of the solution. In the context of this approach applied to stochastic dynamic systems, these investigators often use statistical analysis at the last stage to interpret and simplify the obtained results; in particular, they calculate statistical averages such as

$$\left\langle\lambda_{\mathbf{x}(t)}\right\rangle=\overline{\lim_{t\to+\infty}}\frac{1}{t}\langle\ln|\mathbf{x}(t)|\rangle. \tag{C.52}$$

Parameter α is the *Lyapunov exponent* of the lognormal random process $y(t)$.

(**2**) From the integral distribution functions, one can calculate the typical realization curves of lognormal processes $y(t;\alpha)$ and $\widetilde{y}(t;\alpha)$

$$y^*(t)=e^{\langle\ln y(t)\rangle}=e^{-\alpha t},\quad \widetilde{y}^*(t)=e^{\langle\ln\widetilde{y}(t)\rangle}=e^{\alpha t}, \tag{C.53}$$

which are the exponentially decaying curve in the case of process $y(t;\alpha)$ and the exponentially increasing curve in the case of process $\widetilde{y}(t;\alpha)$.

Consequently, the exponential increase of moments of random processes $y(t;\alpha)$ and $\widetilde{y}(t;\alpha)$ are caused by deviations of these processes from the typical realization curves $y^*(t;\alpha)$ and $\widetilde{y}^*(t;\alpha)$ towards both large and small values of y and $\widetilde{y}$.

As it follows from Eq. (C.50) at $\alpha/D = 1$, the average value of process $y(t;D)$ is independent of time and is equal to unity. Despite this fact, according to Eq. (C.48), the probability of the event that $y < 1$ for $Dt \gg 1$ rapidly approaches the unity by the law

$$P\left(y(t;D) < 1\right) = \Pr\left(\sqrt{\frac{Dt}{2}}\right) = 1 - \frac{1}{\sqrt{\pi Dt}} e^{-Dt/4},$$

i.e., the curves of process realizations run mainly below the level of the process average $\langle y(t;D)\rangle = 1$, which means that namely large peaks of the process govern the behavior of statistical moments of process $y(t;D)$. n Here, we have a clear contradiction between the behavior of statistical characteristics of process $y(t;\alpha)$ and the behavior of process realizations.

(**3**) The behavior of realizations of process $y(t;\alpha)$ on the whole temporal interval can also be evaluated with the use of the p-majorant curves $M_p(t,\alpha)$. We call the majorant curve the curve $M_p(t,\alpha)$ for which inequality $y(t;\alpha) < M_p(t,\alpha)$ is satisfied for all times t with probability p, i.e.,

$$\mathsf{P}\left\{y(t;\alpha) < M_p(t,\alpha) \text{ for all } t \in (0,\infty)\right\} = p.$$

The above statistics (C.44) of the absolute maximum of the Wiener process with a drift $w(t;\alpha)$ makes it possible to outline a wide enough class of the majorant curves. Indeed, let p be the probability of the event that the absolute maximum $w_{\max}(\beta)$ of the auxiliary process $w(t;\beta)$ with arbitrary parameter β in the interval $0 < \beta < \alpha$ satisfies inequality $w(t;\beta) < h = \ln A$. It is clear that the whole realization of process $y(t;\alpha)$ will run in this case below the majorant curve

$$M_p(t,\alpha,\beta) = Ae^{(\beta-\alpha)t} \tag{C.54}$$

with the same probability p. As may be seen from Eq. (C.44), the probability of the event that process $y(t;\alpha)$ never exceeds majorant curve (C.54) depends on this curve parameters according to the formula

$$p = 1 - A^{-\beta/D}.$$

This means that we derived the one-parameter class of exponentially decaying majorant curves

$$M_p(t,\alpha,\beta) = \frac{1}{(1-p)^{D/\beta}} e^{(\beta-\alpha)t}. \tag{C.55}$$

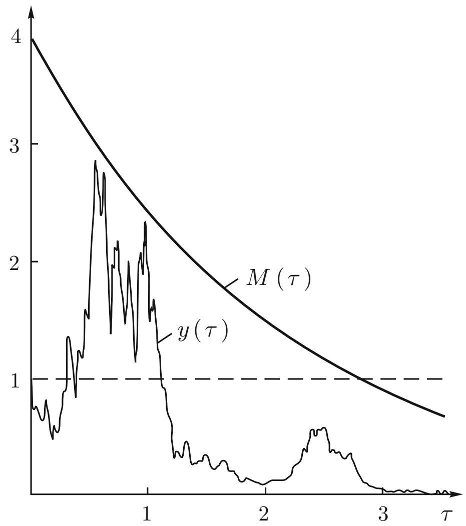

Fig. C.2 Schematic behaviors of a realization of process $y(t; D)$ and majorant curve $M(\tau)$ (C.56)

Notice the remarkable fact that, despite statistical average $\langle y(t; D)\rangle = 1$ remains constant and higher-order moments of process $y(t; D)$ are exponentially increasing functions, one can always select an exponentially decreasing majorant curve (C.55) such that realizations of process $y(t; D)$ will run below it with arbitrary predetermined probability $p < 1$. In particular, inequality ($\tau = Dt$)

$$y(t; D) < M_{1/2}(t, D, D/2) = M(\tau) = 4e^{-\tau/2} \tag{C.56}$$

is satisfied with probability $p = 1/2$ for any instant t from interval $(0, \infty)$.

Figure C.2 schematically shows the behaviors of a realization of process $y(t; D)$ and the majorant curve (C.56). This schematic is an additional fact in favor of our conclusion that the exponential growth of moments of process $y(t; D)$ with time is the purely statistical effect caused by averaging over the whole ensemble of realizations.

Note that the area below the exponentially decaying majorant curves has a finite value. Consequently, high peaks of process $y(t; \alpha)$, which are the reason of the exponential growth of higher moments, only insignificantly contribute to the area below realizations; this area appears finite for almost all realizations, which means that the peaks of the lognormal[1] process $y(t; \alpha)$ are sufficiently narrow.

[1] Sentence by S.I. Vavilov from his paper [2, p. 584].

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