Appendix A

A.1 Sketch of the Field Theory Derivation

The quantised Hamiltonian, Eq. (3.2), is here derived from a composite bosonic quantum field on a curved background. Denoting the field components $\varphi_J(x^0, \vec{x})$, $J = 1, \ldots, N$ (and with $\hbar = 1$ for simplicity) the action of a quantum field reads [1, 2]

$$S = \int d^4x\, \sqrt{-g}\left(\sum_J g^{\mu\nu}\partial_\mu\varphi_J\partial_\nu\varphi_J + \sum_{J,K} M^2_{JK}c^2\varphi_J\varphi_K\right), \tag{A.1}$$

where M_{JK} is the mass-matrix, which is positive and can be assumed symmetric without loss of generality; $g^{\mu\nu}$, $\mu, \nu = 0, \ldots, 3$ is the space-time metric with signature $(-, +, +, +)$; and $\sqrt{-g}$ is the root of the determinant of $g_{\mu\nu}$. Einstein summation convention is employed. Assuming static and symmetric metric $\partial_{x^0} g^{\mu\nu} = 0$, and $g^{0i} = 0$, $i = 1, 2, 3$, the Euler-Lagrange equations reads

$$\left(\sqrt{-g}^{-1}\partial_\mu\sqrt{-g}\, g^{\mu\nu}\partial_\nu - \hat{M}^2c^2\right)\Phi = 0, \tag{A.2}$$

where we introduced a vector notation for the filed $\Phi^T := (\varphi_1, \ldots, \varphi_N)$ and the operator $\hat{M}$ for the mass-matrix M_{JK}. The first term in Eq. (A.2) is the curved space-time Laplacian [3], which written in terms of covariant derivatives ∇_μ gives

$$\left(g^{\mu\nu}\nabla_\mu\nabla_\nu - \hat{M}^2c^2\right)\Phi = 0. \tag{A.3}$$

In terms of the four-momentum operators $\hat{P}_\mu = i\nabla_\mu$ Eq. (A.3) yields the familiar dispersion relation $g^{\mu\nu}\hat{P}_\mu\hat{P}_\nu = -\hat{M}^2c^2$, where the rest mass-energy $c\hat{P}_0$ is given by the mass-matrix $\hat{M}c^2$, as e.g. for the neutrinos.

It is elementary to diagonalize Eq. (A.3) in the field components. Denoting by m_a, $a = 1, \ldots, N$ the eigenvalues of $\hat{M}$ (we assume non-degenerate spectrum for simplicity) we immediately obtain the Klein-Gordon equation for the corresponding

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field $\tilde{\varphi}_a$: $(g^{\mu\nu}\nabla_\mu\nabla_\nu - m_a^2c^2)\tilde{\varphi}_a = 0$. For small energies, where particle creation and annihilation effects are negligible, the Klein-Gordon field can be treated as particle in first quantization and with the Ansatz $\tilde{\varphi}_a = e^{i[c^2S_0(x)+S_1(x)+c^{-2}S_2(x)+\ldots]}$ one obtains that the solution satisfies Schrödinger-like equation with relativistic corrections [4, 5]. In particular, for a post-Newtonian metric $g_{00} = -(1 + 2\phi(x)/c^2 + 2\phi^2(x)/c^4)$, $g_{ij} = \delta_{ij}(1 - 2\phi(x)/c^2)$ following Ref. [5], for each component $\tilde{\varphi}_a$ one obtains

$$i\frac{\partial}{\partial t} = m_ac^2 + \frac{\hat{P}^2}{2m_a} + m_a\phi(x) - \frac{\hat{P}^4}{8m_a^3c^2} + \frac{m_a\phi^2(x)}{2c^2} + \frac{3}{2m_ac^2}\mathcal{F}(\hat{P}, \phi(x)), \quad \text{(A.4)}$$

where $x^0 = ct$ and $\mathcal{F}(\hat{P}, \phi(x)) := \phi(x)\hat{P}^2 + \left[\hat{P}\phi(x)\right]\hat{P} + \frac{1}{2}\left[\hat{P}^2\phi(x)\right]$; notation: $[\hat{P}\phi(x)]$ denotes that $\hat{P}$ as a differential operator acts only on $\phi(x)$.

In the low-energy limit the field components correspond to internal states of the composite particle with different rest mass-energies m_ac^2. From Eq. (A.4) and linearity of quantum theory thus follows that the Hamiltonian for the external as well as internal states of the composity particle reads

$$\hat{H} = \hat{M}c^2 + \frac{\hat{P}^2}{2\hat{M}} + \hat{M}\phi(x) - \frac{\hat{P}^4}{8\hat{M}^3c^2} + \frac{\hat{M}\phi^2(x)}{2c^2} + \frac{3}{2\hat{M}c^2}\mathcal{F}(\hat{P}, \phi(x)),$$

Operator $\hat{M}c^2$ describes the total energy in the rest frame of the particle. $\hat{M}$ (as well as any other operator) can be split into a constant part $\propto \hat{I}$ and the remaining dynamical part which we denoye $\hat{H}_{int}$, i.e. we can always write $\hat{M} = m\hat{I}_{int} + H_{int}/c^2$. Note that such defined parameter m can be identified with the mass-parameter of the particle, whereas the operator $\hat{H}_{int}$ is the particle's rest frame Hamiltonian, which drives the dynamics of the internal degrees of freedom. Leaving only the lowest order terms in H_{int}/mc^2 in the Hamiltonian $\hat{H}$ yields

$$\begin{aligned}\hat{H} = mc^2 + \frac{\hat{P}^2}{2m} + m\phi(x) + \hat{H}_{int}\left(1 - \frac{\hat{P}^2}{2m^2c^2} + \frac{\phi(x)}{c^2}\right) \\ - \frac{\hat{P}^4}{8m^3c^2} + \frac{m\phi^2(x)}{2c^2} + \frac{3}{2mc^2}\mathcal{F}(\hat{P}, \phi(x)),\end{aligned} \quad \text{(A.5)}$$

Appendix B

B.1 Evaluation of the Expected Photon Number

Consider the light source for the interferometer to be an ideal single photon source, which produces pulses propagating in the $+x$ (horizontal) direction each containing only one photon. In the local shell-frame on the earth's surface this can be represented by the quantum state (5.22). The use of plane-wave propagation is assumed justified by the paraxial approximation of a Gaussian spatial mode.

Detection of the horizontal output mode of the interferometer is assumed to be broadband (i.e. a frequency band-width much greater than the source) and time integrated (detection time is much longer than the pulse width) and thus can be described by an operator

$$a_o^\dagger a_o = \int \frac{d\tau_{r'}}{2\pi} \int d\nu e^{i\frac{\nu}{c}(x_{r'}-c\tau_{r'})} b_\nu^\dagger \int d\nu' e^{-i\frac{\nu'}{c}(x_{r'}-c\tau_{r'})} b_{\nu'} = \int d\nu\, b_\nu^\dagger b_\nu, \tag{B.1}$$

where $b_{\nu'}$ are single wave-number boson annihilation operators for the output mode. The expectation value $\langle a_o^\dagger a_o \rangle$ against the initial state (5.22) is calculated by finding the Heisenberg evolution of the detection operators. Quite generally the evolved single wave-number operators are of the form

$$b_\nu = \frac{1}{2} a_\nu (e^{-i\nu\phi_1} - e^{-i\nu\phi_2}) + \frac{1}{2} \upsilon_\nu (e^{-i\nu\phi_1} + e^{-i\nu\phi_2}), \tag{B.2}$$

where υ_ν are single wave-number boson annihilation operators from which the unoccupied input modes of the interferometer are constructed. Because they are initially in their vacuum state they will not contribute to the expectation value. The phases ϕ_i, $i = 1, 2$ are acquired propagating along the corresponding paths γ_i of the interferometer. Continuity at the mirror boundaries between the mode operator expressions along the different paths is assumed.

By symmetry, the contribution to the phases ϕ_i coming from the propagation along the radial part of the path is the same for both trajectories, as they are both

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evaluated over an equal time-interval and have the same lengths as measured by a distant observer. Because they are common they will be eliminated since only the phase difference $\Delta\phi := \phi_1 - \phi_2$ will contribute to the final expression. From Eq. (5.22) and the metric Eq. (5.16) follows that the phases read $\phi_1 = \frac{1}{c}(l - c\tau_{r+h})$, $\phi_2 = \frac{1}{c}(l - c\tau_r)$ and thus

$$\Delta\phi = \Delta\tau \,, \tag{B.3}$$

with $\Delta\tau$ given by the Eq. (5.21). Moreover, the locally measured radial distance h between the paths is found via

$$h_r = \int_r^{r+h} \frac{dr'}{\sqrt{1 - \frac{2GM}{c^2 r'}}} \,. \tag{B.4}$$

Evaluating the photon number expectation value for the one-photon state $|1\rangle_f = \int d\nu f(\nu) e^{i\frac{\nu}{c}(x_r - c\tau_r)} a^\dagger_\nu |0\rangle$, Eq. (5.22), yields

$$\langle a^\dagger_o a_o \rangle = \langle 1|_f \int d\nu \; b^\dagger_\nu b_\nu \; |1\rangle_f = \int d\nu |f(\nu)|^2 \frac{1}{4} |1 - e^{i\nu\Delta\tau}|^2 \,, \tag{B.5}$$

which is the same result as that of Eq. (5.23) for a single photon normalized wave packet.

Appendix C

C.1 Toy Models

Both experimental proposals discussed in Chap. 5 are formulated within the framework quantum mechanics on curved background. No effects specific to this theory have been experimentally verified so far—bridging this gap remains the principal motivation behind the present work. Quite generally, new physics is expected only at the scale where gravity itself could no longer be described as a classical theory. However, the tension between quantum mechanics and general relativity is of conceptual nature. Both theories stress that only operationally well defined notions may have physical meaning and this concerns also the notion of time (or proper time in general relativity). However, in contrast to general relativity, in quantum mechanics any degree of freedom of a physical system can be in a superposition and thus becomes undefined (beyond the classical probabilistic uncertainty). More generally—the theory allows for physical states that cannot be described within any local realistic model. If this applies to the degrees of freedom on which our operational treatment of time relies—the latter becomes classically undefined. This can be the case even when space-time itself can still be described classically, like in the proposals discussed in this thesis. One could, however, take an opposite view and assume that whenever space-time itself is classical, the time for any system, that constitutes an operationally defined clock, should admit a classical description as well. The tension between these two views motivates the investigation of theoretical frameworks alternative to quantum mechanics in curved space time. Here we sketch an explicit example of such an alternative toy theory, which can be tested by the experiments proposed in Chap. 5.

Relevant for the present problem is how the physical degrees of freedom evolve in a presence of time dilation. In the standard approach such evolution results in entanglement between the spatial mode of the wavefunction and other degrees of freedom. There is no well-defined time that such degrees of freedom experience and even a Bell-type experiment can be designed in which any local realistic model of time can be refuted. This entanglement results from the coupling $\hat{H}_{vis} = \hat{H}_s \frac{V(\hat{r})}{c^2}$, see

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Eq. (5.29), and is the reason for the drop in the interferometric visibility (for both massive and massless cases). All so far observed gravitational effects can, however, be explained with one of two possible effective forms of such an interaction, which reproduce only specific features of (5.29) and correspond to different physical effects.

The effective coupling $\hat{H}_{phase} = \langle \hat{H}_s \rangle \frac{V(\hat{r})}{c^2}$, see Eq. (5.28), reproduces correctly the gravitational phase shift effect of the standard theory, but not the time dilation. Applying the operator (5.28) to the state of the clock degree of freedom $|\tau\rangle$ in a spatial superposition of two locations r_1 and r_2, $|\Psi\rangle = \frac{1}{\sqrt{2}}(|r_1\rangle + |r_2\rangle)|\tau\rangle$, we get

$$\hat{H}_{phase}|\Psi\rangle = \frac{\langle \hat{H}_s \rangle}{\sqrt{2}} \left(\frac{V(r_1)}{c^2}|r_1\rangle + \frac{V(r_2)}{c^2}|r_2\rangle \right) |\tau\rangle,$$

where $\langle \hat{H}_s \rangle = \langle \tau | \hat{H}_s | \tau \rangle$. The full evolution in such a toy model is given by the Hamiltonian $\hat{H}_s + \hat{H}_{ps}$, which yields

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}\left(\hat{H}_s + \hat{H}_{phase}\right)t}|\Psi\rangle = \frac{1}{\sqrt{2}}\left[e^{-\frac{i}{\hbar}\left(\langle \hat{H}_s \rangle \frac{V(r_1)}{c^2}\right)t}|r_1\rangle + e^{-\frac{i}{\hbar}\left(\langle \hat{H}_s \rangle \frac{V(r_2)}{c^2}\right)t}|r_2\rangle \right]|\tau(t)\rangle, \tag{C.1}$$

where $|\tau(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_s t}|\tau\rangle$ and thus the evolution of the clock degree of freedom $|\tau\rangle$ does not depend on the position r. Hence, the coupling (5.28) predicts no general relativistic time dilation and no drop in the interferometric visibility—the clock degree of the freedom remains factorised from the spatial modes. However, each mode acquires a phase proportional to the gravitational potential and an effective mass defined by $\langle \hat{H}_s \rangle$, hence this effective coupling reproduces the relative phase shift measured in interference experiments. For a particle of mass m, $\hat{H}_s = mc^2 + \hat{H}_{int}$ (to first order in $1/c^2$), where $\hat{H}_{int}$ is the Hamiltonian of the internal degrees of freedom and thus one obtains $m + \frac{\langle \hat{H}_{int} \rangle}{c^2}$ for the effective mass. The first term is simply the Newtonian mass, while the second is a relativistic correction. For a single photon mode with a frequency ω, $\hat{H}_s = \hbar\omega a^\dagger a$ and therefore the whole contribution to the phase shift comes from $\frac{\langle \hat{H}_s \rangle}{c^2} = \frac{\hbar\omega}{c^2}$. Thus, any measurement of the gravitational phase shift for photons would represent a signature of a non-Newtonian effective mass. However, no measurement of the phase shift (in the massive or massless case) could represent a measurement of the time dilation, since the phase shifts are explainable by the coupling (5.28) which does not cause clocks at different potentials to tick at different rates.

A different effective coupling can explain all classical general relativistic effects observed so far. It includes an effective gravitational potential $\langle V(\hat{r}) \rangle$—gravitational potential smeared over the support of the wavefunction of a single physical system. Such a coupling reads

$$\hat{H}_{loc} = \hat{H}_s \frac{\langle V(\hat{r}) \rangle}{c^2} \tag{C.2}$$

and it accounts for gravitational experiments in which the relevant degrees of freedom are sufficiently well localized. These include not only classical tests of general relativity [6–10], but also experiments measuring the time dilation between two *localized* atomic clocks, each at a different gravitational potential.

More generally, one can construct a toy model by combining the above effective couplings $\hat{H}_{phase}$ and $\hat{H}_{loc}$, for example

$$\hat{H}_i^{eff} = \hat{H}_s \left(1 + \frac{\langle V(\hat{r})\rangle}{c^2}\right) + \frac{\Delta N_i}{\langle \hat{N}_i \rangle}\left(\langle \hat{H}_s \rangle \frac{V(\hat{r})}{c^2} - \hat{H}_s \frac{\langle V(\hat{r})\rangle}{c^2}\right), \tag{C.3}$$

which governs the evolution of the ith mode of a quantum state (one can associate different modes to e.g., different paths of a Mach-Zehnder interferometer). $\hat{N}_i$ is the number operator in mode i and ΔN_i is its standard deviation. The parameter $\frac{\Delta N_i}{\langle \hat{N}_i \rangle}$ quantifies how well the quantum state is localised. It vanishes for Fock states (e.g. for a pair of atoms or photons, each in one, localised mode), and in the limit of large coherent states (for a coherent state $|\alpha\rangle$ in mode i we have $\frac{\Delta N_i}{\langle \hat{N}_i \rangle} = \frac{1}{|\alpha|} \rightarrow 0$ for $\alpha \rightarrow \infty$), which for photons corresponds to classical light. In both cases the Hamiltonian (C.3) reduces to $\hat{H}_s \left(1 + \frac{\langle V(\hat{r})\rangle}{c^2}\right)$. In the other limit, when the parameter $\frac{\Delta N_i}{\langle \hat{N}_i \rangle} = 1$, which is the case for a single particle in a superposition of two modes, the effective Hamiltonian reduces to $\hat{H}_s + \langle \hat{H}_s \rangle \frac{V(\hat{r})}{c^2}$. The toy model (C.3) predicts no drop in the interferometric visibility for a particle in a spatial superposition (since in the relevant limit the energy *operator* $\hat{H}_s$ does not couple to the potential) but is still consistent with the experiments carried out so far. (Moreover, Eq. (C.3) can be generalised beyond the above weak energy limit).

The difference between the standard extension of quantum mechanics to curved space-time and the toy model (C.3) can only be tested with a quantum system from which the time can be read out and which is put in a coherent spatial superposition at different gravitational potentials. Even though this model is artificial (e.g., it shares the difficulties of all quantum nonlinear models) it highlights the conceptual difference between gravitational phase shift experiments and measurements of the visibility loss. (While the former only probe the semi-classical coupling of energy to the gravitational potential, the latter directly test the full quantum form of such a coupling). Most importantly, the toy model emphasises the necessity of probing quantum mechanics in curved space-time: the results of current experiments cannot necessarily be extrapolated to this regime.

Appendix D

D.1 Mixed Internal States as Clocks

The time dilation induced coupling between internal degrees of freedom and the centre of mass causes decoherence of the latter. The complementarity principle between the visibility V of interference and the which-path information D is given by the inequality $V^2 + D^2 \leq 1$. The equal sign holds for pure states, i.e. for well-defined clocks as considered in Chap. 5. For mixed states, it is possible to have loss of visibility with no accessible which-path information, as is the case here (as well as in most other decoherence models such as in quantum Brownian motion [11, 12]). Mixed state of internal degrees of freedom can be seen as a mixture of clock-states that each measure the proper time along their path. To highlight this, consider for example a particle with a single 2-level internal degree of freedom which is in a clock-state, i.e. in a superposition of the ground and excited state with transition frequency $\omega = 1/t_\perp$ and an arbitrary relative phase ϕ: $|E_\phi\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle + e^{i\phi}|e\rangle\right)$. If the particle moves along a path with overall proper time τ, the internal state will evolve to $|E_\phi(\tau)\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle + e^{i\phi+\omega\tau}|e\rangle\right)$. For the particle in superposition along two paths with proper time difference $\Delta\tau$, the internal clock state will therefore acquire which-path information, thus leading to a loss in visibility given by $|\langle E_\phi(\tau_1)|E_\phi(\tau_2)\rangle| = |\cos(\omega\Delta\tau/2)|$, independent of the phase ϕ. For a fully mixed internal state (analogous to a thermal state),

$$\rho = \frac{1}{2}\left(|E_\phi\rangle\langle E_\phi| + |E_{\phi+\pi}\rangle\langle E_{\phi+\pi}|\right), \tag{D.1}$$

the relative phase between $|g\rangle$ and $|e\rangle$ is unknown and thus no which-path information is available, but it still results in a drop in visibility. The above state can be equivalently written in the basis

$$\rho = \frac{1}{2}\left(|g\rangle\langle g| + |e\rangle\langle e|\right). \tag{D.2}$$

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Since it represents the same state, it will cause the same loss of visibility, the two situations (D.1) and (D.2) cannot be discriminated. The states $|g\rangle$ and $|e\rangle$ individually, however, are not clock-states, thus no time dilation can be read out directly. The interpretation of the visibility drop in this representation is the phase scrambling between $|g\rangle$ and $|e\rangle$ due to the red shift, since the states $|e\rangle$ acquire a different phase than the states $|g\rangle$. Irrespective of the state representation, time dilation causes loss of coherence of composite particles with internal degrees of freedom.

Appendix E

E.1 The Lowest Energy Limit of "Clock" Hamiltonian

Chapter 7 discusses the regime where kinetic and potential energies of test particles are low, such that relativistic corrections to the external motion of the particle (second line of Eq. (A.5)) are negligible, but where relativistic corrections to the internal dynamics $\hat{H}_{int}(-\frac{\hat{P}^2}{2m^2c^2} + \frac{\phi(x)}{c^2})$ cannot be neglected. In such a regime Eq. (A.5) becomes

$$\hat{H} = mc^2 + \frac{\hat{P}^2}{2m} + m\phi(x) + \hat{H}_{int}\left(\hat{I} - \frac{\hat{P}^2}{2m^2c^2} + \frac{\phi(x)}{c^2}\right), \tag{E.1}$$

(which is the Hamiltonian in Eq. (7.2)).

Below it is shown how Eq. (E.1) can be derived directly from a simple WKB approximation to the Klein-Gordon equation (A.3). It suffices to take the following Ansatz for a solution:

$$\Phi(x^0, \vec{x}) = e^{-i\hat{M}cx^0}\psi(x^0, \vec{x}). \tag{E.2}$$

It gives $\partial_0\partial_0\Phi \approx -\hat{M}^2c^2e^{-i\hat{M}cx^0}\psi - 2i\hat{M}ce^{-i\hat{M}cx^0}\partial_0\psi$, where $e^{-i\hat{M}cx^0}\partial_0\partial_0\psi$ has been neglected as small compared to the other terms. We thus have

$$g^{00}(\hat{M}^2c^2\Phi + 2i\hat{M}ce^{-i\hat{M}cx^0}\partial_0\psi) \approx (g^{ij}\hat{\nabla}_i\hat{\nabla}_j - \hat{M}^2c^2)\Phi.$$

Introducing again $t = x^0/c$ and denoting $\dot{\psi} \equiv \partial_t\psi$ yields

$$i\dot{\Phi} \approx \left(\hat{M}c^2 + \frac{1}{2\hat{M}}g_{00}g^{ij}\hat{\nabla}_i\hat{\nabla}_j + \frac{1}{2\hat{M}}(-g_{00} - 1)\hat{M}^2c^2\right)\Phi. \tag{E.3}$$

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For a post-Newtonian metric introduced above and using again the notions of mass and internal energy defined through $\hat{M} = m\hat{I}_{int} + H_{int}/c^2$, Eq. (E.3) immediately reduces to Eq. (E.1) (when keeping terms of order H_{int}/mc^2).

E.2 Einstein's Hypothesis of Equivalence

This appendix shows, that the conditions derived in Sect. 7.2—imposed by the EEP on the dynamics of a massive system with internal degrees of freedom—are equivalent to conditions stemming directly from requiring the validity of the Einstein's hypothesis of equivalence.

As in the main text, the rest mass-energy operator of a massive system with internal degrees of freedom is denoted by $\hat{M}_r = m_r\hat{I}_{int} + \hat{H}_{int,r}/c^2$ and the inertial mass-energy operator by $\hat{M}_i = m_i\hat{I}_{int} + \hat{H}_{int,i}/c^2$. In an inertial coordinate system (x, t) and in the absence of external gravitational field, the low energy limit of a Hamiltonian of such system reads

$$i\hbar\frac{\partial}{\partial t} = \hat{M}_r c^2 - \frac{\hbar^2}{2\hat{M}_i}\nabla^2, \tag{E.4}$$

where $-i\hbar\nabla \equiv -i\hbar\frac{\partial}{\partial x} = \hat{P}$ is the center of mass momentum operator and where $1/\hat{M}_i \approx \frac{1}{m_i}(\hat{I}_{int} - \hat{H}_{int,i}/m_i c^2)$. Lorentz boost is generated by $\hat{K} = i\hbar t\nabla + i\hbar\frac{x}{c^2}\frac{\partial}{\partial t}$ and to lowest order in the boost parameter v, the resulting new coordinates read $(x' \approx x + vt, t' \approx t + \frac{vx}{c^2})$ [1], thus

$$\begin{cases} \nabla = \nabla' + \frac{v}{c^2}\frac{\partial}{\partial t'}, \\ \frac{\partial}{\partial t} = v\nabla' + \frac{\partial}{\partial t'}, \end{cases} \tag{E.5}$$

The Hamiltonian in Eq. (E.4) transforms into

$$i\hbar\frac{\partial}{\partial t'} = \hat{M}_r c^2 - \frac{\hbar^2}{2\hat{M}_i}\nabla'^2 + i\hbar v\left(\frac{\hat{M}_r}{\hat{M}_i} - 1\right)\nabla' + \mathcal{O}(c^{-4}). \tag{E.6}$$

and is invariant under the Lorentz boost if $\hat{M}_i = \hat{M}_r$. Since the rest mass parameter m_r can be assigned arbitrary value without introducing observable consequences (as long as the gravitational field produced by the system is not considered—which is the case here), the physical requirement imposed by demanding Lorentz invariance in this limit reads $\hat{H}_{int,i} = \hat{H}_{int,r}$, as derived the main text.

Requiring the validity of the Einstein's hypothesis of equivalence—the total physical equivalence between laws of relativistic physics in a non-inertial, constantly accelerated, reference frame and in a stationary frame subject to homogeneous gravity—imposes further conditions. A transformation from the initial inertial frame (x, t) to an accelerated coordinate system ($x'' \approx x + \frac{1}{2}gt^2, t'' \approx t + \frac{gtx}{c^2}$), with g denoting the acceleration, gives (to lowest order):

$$\begin{cases} \nabla = \nabla'' + \frac{gt}{c^2}\frac{\partial}{\partial t''}, \\ \frac{\partial}{\partial t} = gt\nabla'' + \left(1 + \frac{gt}{c^2}\right)\frac{\partial}{\partial t''}. \end{cases} \tag{E.7}$$

Schrödinger equation Eq. (E.4) transforms under Eq. (E.7) into

$$i\hbar\frac{\partial}{\partial t''} = \hat{M}_r c^2 - \hat{M}_r gx + i\hbar gt\left(\frac{\hat{M}_r}{\hat{M}_i} - 1\right)\nabla'' - \frac{\hbar^2}{2\hat{M}_i}\nabla''^2. \tag{E.8}$$

For a massive particle subject to a homogeneous gravitational potential $\phi(x) = gx$ its coupling to gravity is given by its gravitational charge—the total gravitational mass-energy $\hat{M}_g = m_g\hat{I}_{int} + \hat{H}_{int,g}/c^2$, where m_g describes the gravitational mass parameter and $\hat{H}_{int,g}$ contribution to the mass from internal energy. The Hamiltonian of such a system reads

$$i\hbar\frac{\partial}{\partial t'} = \hat{M}_r c^2 - \hat{M}_g gx - \frac{\hbar^2}{2\hat{M}_i}\nabla'^2. \tag{E.9}$$

Thus, for the validity of the Einstein's Hypothesis of Equivalence in addition to $\hat{H}_{int,i} = \hat{H}_{int,r}$ it is also required that $\hat{M}_g = \hat{M}_i$—in full agreement with the derivation in the main text. Moreover when the hypothesis of equivalence holds, the Hamiltonians of a composed quantum system subject to weak gravity reduces to the Hamiltonian in Eq. (7.2).

E.3 Fully Classical Test Theory of the EEP

In classical physics Hamiltonian of a composite system is a function of phase space variables of the centre of mass (Q, P) and of the internal degree of freedom (q, p) with the internal mass-energies $M_\alpha = m_\alpha c^2 + E_\alpha$ and reads

$$\tilde{H}^C_{test} = M_r + \frac{P^2}{2M_i} + M_g\phi(Q) \approx m_r c^2 + E_r + \frac{P^2}{2m_i} + m_g\phi(Q) - E_i\frac{P^2}{2m_i c^2} + E_g\frac{\phi(Q)}{c^2}. \tag{E.10}$$

Time evolution of a classical variable is obtained from its Poisson bracket with the total Hamiltonian: $d/dt = \{\cdot, \tilde{H}^C_{test}\}_{PB}$. The acceleration of the center of mass Q reads

$$\ddot{Q} = -M_g M_i{}^{-1}\nabla\phi(Q)\ , \tag{E.11}$$

where ∇ is derivative with respect to Q. Equation (E.11) recovers the result that free fall is universal if $M_g = M_i = 1$ (or more generally, M_g/M_i can be any positive number, the same for all physical systems, but such a numerical factor would just redefine the gravitational potential).

The time evolution of the internal variable q (keeping only first order terms in $H_{int,\alpha}/m_\alpha c^2$) reads

$$\dot{q}(Q, P) = \dot{q}_r - \dot{q}_i \frac{P^2}{2m_i^2 c^2} + \dot{q}_g \frac{\phi(Q)}{c^2}, \tag{E.12}$$

where $\dot{q}_\alpha := \{q, H_\alpha\}_{PB}$ are in principle different velocities. The gravitational time dilation factor $\Delta\dot{q}/\dot{q} := \frac{\dot{q}(Q+h,P)-\dot{q}(Q,P)}{\dot{q}(Q,P)}$ reads

$$\Delta\dot{q}/\dot{q} \approx \frac{\dot{q}_g}{\dot{q}_r}\frac{\nabla\phi(Q)h}{c^2}, \tag{E.13}$$

and it reduces to that predicted by general relativity $\Delta\dot{q}/\dot{q} \approx \frac{\nabla\phi(Q)h}{c^2}$ if $H_{int,r} = H_{int,g}$. Similarly, universality of special relativistic time dilation is recovered if $H_{int,r} = H_{int,i}$.

Conditions for the validity of the EEP (and the number of parameters to test) are the same in the fully classical case above and in the model H^C_{test} which describes a system with quantised centre of mass degrees of freedom. Since the EEP imposes equivalence conditions on the mass-energies of the system, it is the quantisation of the internal energy which is relevant for the difference between the classical and the quantum formulation of the EEP.

E.4 Lagrangian Formulation of the Test Theory

Lagrangian formulation of the test theory is obtained from the Legendre transform of the test Hamiltonian. The derivation is valid for both the classical and the quantum model; we will thus write for brevity $H_{test} = m_r c^2 + H_{int,r} + \frac{P^2}{2m_i} + m_g\phi(Q) - H_{int,i}\frac{P^2}{2m_i^2 c^2} + H_{int,g}\frac{\phi(Q)}{c^2}$.

For the centre of mass degree of freedom the canonically conjugate velocity is given by

$$\dot{Q} = \frac{\partial H_{test}}{\partial P} = \frac{P}{m_i}\left(1 - \frac{H_{int,i}}{m_i c^2}\right).$$

We formally introduce position q and momentum p of the internal degrees of freedom, which dynamics is given by the Hamiltonians $H_{int,\alpha} = H_{int,\alpha}(q, p)$. The conjugate internal velocity is thus defined as $\dot{q} = \frac{\partial H_{test}}{\partial p}$ and reads

$$\dot{q} = \frac{\partial H_{int,r}}{\partial p} - \frac{\partial H_{int,i}}{\partial p}\frac{P^2}{2m_i^2 c^2} + \frac{\partial H_{int,g}}{\partial p}\frac{\phi(Q)}{c^2}.$$

Lagrangian of the test theory can now be obtained through the Legendre transform of H_{test}: $L_{test} := P\dot{Q} + p\dot{q} - H_{test}$. We first introduce the total internal Lagrangians L_α via the Legendre transform of the total internal mass-energies $m_\alpha c^2 + H_{int,\alpha}$:

$$L_\alpha := \frac{\partial H_{int}}{\partial p} p - m_\alpha c^2 - H_{int,\alpha} \equiv -m_\alpha c^2 + L_{int,\alpha},$$

which leads the test Lagrangian in the form:

$$L_{test} = L_r - L_i \frac{\dot{Q}^2}{2c^2} + L_g \frac{\phi(Q)}{c^2}. \tag{E.14}$$

Note, that $-m_\alpha c^2$ is the non-dynamical part of the internal Lagrangian and $L_{int,\alpha}$ is its dynamical part—in a full analogy to the Hamiltonian picture where mc^2 is the non-dynamical and $H_{int,\alpha}$ the dynamical part of the internal mass-energy. The conditions for the validity of the EEP derived in the main text for the internal Hamiltonians now translate to $L_i = L_r = L_g$. Indeed, when the internal dynamics is universal $L_\alpha \equiv L_0$ the Eq. (E.14) reduces to

$$L_{test} \xrightarrow{L_\alpha \equiv L_0} L = L_0(1 - \frac{\dot{Q}^2}{2c^2} + \frac{\phi(Q)}{c^2}). \tag{E.15}$$

Equation (E.15) is the lowest order approximation to the dynamics of a particle in space-time given by e.g. the Schwarzschild metric. Indeed, $L \approx L_0\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$ with metric elements $g_{00} \approx -(1 + 2\phi(x)/c^2)$, $g_{ij} \approx c^{-2}\delta_{ij}$, $g_{0i} = g_{i0} = 0$ $i, j = 1, 2, 3$. In the limit $L_0 \approx -mc^2$ the non-relativistic Lagrangian of a massive particle in Newtonian potential is recovered $L \approx -mc^2 + m\dot{Q}^2/2 - m\phi(Q)$.

In contrast to thus far considered test theories of the EEP for composed systems, which only incorporate internal (binding) energy as fixed parameters, test theory given by the Lagrangian in Eq. (E.14) incorporates the dynamics of the associated degrees of freedom.

E.5 Quantum Test of the Classical WEP

Assume that WEP holds but only in the Newtonian limit, $m_i = m_g \equiv m$, and that LLI is valid ($\hat{H}_{int,r} = \hat{H}_{int,i}$) but $\hat{H}_{int,i} \neq \hat{H}_{int,g}$. In particular, we restrict to classical violations of the WEP, i.e. $[\hat{H}_{int,i}, \hat{H}_{int,g}] = 0$. For an internal energy eigenstate $|E_j\rangle$ we have $\hat{M}_i|E_j\rangle = M_{1,i}|E_j\rangle$ and $\hat{M}_g|E_j\rangle = M_{j,g}|E_j\rangle$ where $M_{j,\alpha} = m + E_{j,\alpha}/c^2$. From Eq. (7.6) we obtain $\ddot{\hat{Q}}|E_j\rangle = -g_j|E_j\rangle$ $(j = 1, 2)$ where $g_j = gM_{j,g}/M_{j,i}$ where we assumed homogeneous gravitational field g. Parameters describing possible violations are $\eta_j := M_{j,g}/M_{j,i}$ (which can be seen as the diagonal elements of the matrix $\hat{\eta}$ introduced in Sect. 7.3). When $\eta_1 \neq \eta_2$ for some two internal states

$|E_1\rangle$, $|E_2\rangle$ the centre of mass will have the free-fall acceleration that depends on the internal state. Consider now a coherent superposition of the two internal energy eigenstates, semi-classically localised at some height h:

$$|\Psi(0)\rangle = 1/\sqrt{2}(|E_1\rangle + |E_2\rangle)|h\rangle. \tag{E.16}$$

Under free-fall it evolves into

$$|\Psi(t)\rangle = 1/\sqrt{2}(e^{i\phi_1}|E_1\rangle|h_1\rangle + e^{i\phi_2}|E_2\rangle|h_2\rangle), \tag{E.17}$$

where $h_j = h - 1/2 g_j t^2$, $j = 1, 2$ is the position of the centre of mass correlated with the internal state $|E_j\rangle$ after time t of free fall and $\phi_j(t)$ is the free propagation phase for a particle with a total mass $M_{j,i}$ under gravitational acceleration g_j, which can be found e.g. in [13]. Initial superposition in a presence of classical violations evolves into an entangled state, with the internal degree of freedom entangled to the position. As a result the reduced state of the internal degrees of freedom $\hat{\rho}_{int}(t)$ becomes mixed: $\hat{\rho}_{int}(t) := \text{Tr}\{|\Psi(t)\rangle\langle\Psi(t)|\} = 1/2(|E_1\rangle\langle E_1| + |E_2\rangle\langle E_2| + e^{i\phi_1 - i\phi_2}\langle h_2|h_1\rangle|E_1\rangle\langle E_2| + h.c)$. The amplitude of the off-diagonal elements

$$\mathcal{V} := |\langle h_2|h_1\rangle| \tag{E.18}$$

quantifies the coherence of the reduced state and it decreases with the position amplitudes becoming distinguishable, in agreement with the quantum complementarity principle for pure states, see e.g. [14]. When the position amplitudes become orthogonal we have $\mathcal{V} = 0$ and the reduced state becomes maximally mixed. The classical violations of the WEP and the superposition principle of quantum mechanics thus entail decoherence of any freely falling system into its internal energy eigenbasis.

Since we assumed the validity of the LLI but a violation of the WEP we shall also observe a related violation of the LPI. Indeed, a coherent superposition of different energy states evolves in time and thus constitutes a "clock". A frequency of such a "clock" is given by the inverse of the energy difference between the superposed states. The internal state in Eq. (E.16) when trapped at a height h evolves in time at a rate $\omega(h) = \omega(0)(1 + (E_{2,g} - E_{1,g})/(E_{2,i} - E_{1,i})gh/c^2)$ where $\omega(0) = (E_{2,i} - E_{1,i})/\pi\hbar$, in violation of the LPI. In case of no violations this rate would read $\omega(h)_{GR} = \omega(0)(1 + gh/c^2)$. An anomalous frequency dependence on the system's position in the laboratory frame $\omega(h)$ would be the only consequence of the classical violations of the LPI for classical clocks. However, for a quantum "clock" there is an additional effect: The final state of the internal degree of freedom in Eq. (E.17) is stationary (because it becomes fully mixed). Classical violations discussed above thus result in a decoherence of any time evolving state, a "clock" into a stationary mixture.

Decoherence effect and entanglement between internal and external degrees of freedom, that would arise as a result of the classical violations of the WEP, cannot be described within a fully classical theory. Quantum test theory of the EEP is therefore necessary in order to describe all effects of the EEP violations on quantum systems, even if the violations themselves are assumed to be classical.

Realisation of such a quantum test of the classical WEP in principle takes place in interferometric experiments where atoms propagating in the two arms of the interferometer are in different energy eigenstates (Raman beam-splitting). As an example we consider a recent experiment performed by the group of P. Bouyer [15]. In this experiment Mach-Zehnder interferometer with ^{87}Rb was operated during a ballistic flight of an airplane with the aim to provide a proof of principle realisation of an inertial sensor in microgravity. We approximate the centre of mass position of the atoms by a Gaussian distribution $\langle x|h_j\rangle \propto e^{-(h_j-x)^2/2l_c^2}$ where l_c is the coherence length of the atom's wave-function. Assuming small violations the visibility in Eq. (E.18) can be approximated to $\mathcal{V} \approx 1-(\Delta\eta\frac{gT^2}{l_c})^2$, where $\Delta\eta = |\eta_1-\eta_2|$. From the experimental parameters estimated in [15]: $\mathcal{V} \approx 0.65$, $T = 20$ ms and estimating $l_c \approx 10$ µm we can infer a bound $\Delta\eta < 8 \times 10^{-3}$.

Appendix F

F.1 Proof of the Bell Theorem for Temporal Order

More formally, the thesis of the theorem 1 in Sect. 8.2 can be expressed in the following way: assume that the observers a_1, b_1, c_1 are given as an input a bit i_1 and produce, through their local operations and measurements, a single bit of output o_1, while the observers a_2, b_2, c_2 have a bit i_2 as input and a bit o_2 as output. The observer d produces an arbitrary variable z as a result of a local measurement. Then, given the assumptions of the theorem, the conditional probability

$$P\left(o_1, o_2 | i_1, i_2, z\right) \tag{F.1}$$

does not violate Bell inequalities for any value of z.

By assumption, no initially entangled state is shared by the three groups of observers. Since quantum mechanics is valid locally, and the three regions are space-like separated, any shared resource can be described as a tri-partite separable state $\rho^{S_1 S_2 d}$, where S_1 represents the subsystem on which the operations at events A_1, B_1, C_1 are performed, S_2 is the subsystem on which the operations at events A_2, B_2, C_2 are performed, and d is the part of the system on which a measurement at D is performed. Because of assumption (4), the operations in each region are always performed in a specific order, although this order might depend on a classical variable λ defined on some space-like surface in the past. Thus, for each λ, the sequence of the three operations performed at events A_1, B_1, C_1 is represented by a POVM $\{E^{\lambda}_{o_1}(i_1)\}_{o1}$; similarly a POVM $\{F^{\lambda}_{o_2}(i_2)\}_{o_2}$ describes the operations performed at events A_2, B_2, C_2, and $\{G_z\}_z$ represents the D measurement.

More generally, the value of λ might be modified by a local operation. For example, given a λ such that A_1 is before B_1 and C_1, the order between the latter two can depend on the operation performed in A_1. However, the choice of operation on A_1 can only depends on the local input i_1. Thus, for each value of i_1 the result of the three operations is still described by a POVM $\{E^{\lambda}_{o_1}(i_1)\}_{o_1}$. Even if, for a given i_1, $B_1 \preceq C_1$ with some probability $0 < q(i_1) < 1$, (and $C_1 \preceq B_1$ with

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M. Zych, *Quantum Systems Under Gravitational Time Dilation*,
Springer Theses, DOI 10.1007/978-3-319-53192-2

probability $1-q(i_1)$), the resulting effect can be described by the mixture of POVMs $\{q(i_1)E_{o_1}^{\lambda,\, B_1 \preceq C_1}(i_1) + [1-q(i_1)]E_{o_1}^{\lambda,\, C_1 \preceq B_1}(i_1)\}_{o_1}$, which is still a POVM.

The probability distribution resulting from the local measurements is thus given by:

$$P\,(o_1, o_2, z|i_1, i_2) = \int d\lambda\, p(\lambda) \mathrm{Tr}\left[E_{o_1}^{\lambda}(i_1) \otimes F_{o_2}^{\lambda}(i_2) \otimes G_z\, \rho^{S_1 S_2 d}\right], \tag{F.2}$$

where $p(\lambda)$ is an arbitrary probability distribution for λ. Consider in particular a product state $\rho^{S_1 S_2 d} = \rho_1^{S_1} \otimes \rho_2^{S_2} \otimes \rho_3^{d}$. Then (F.2) takes the form

$$P\,(o_1, o_2, z|i_1, i_2) = \int d\lambda\, p(\lambda) \mathrm{Tr}\left[E_{o_1}^{\lambda}(i_1)\rho_1^{S_1}\right] \mathrm{Tr}\left[F_{o_2}^{\lambda}(i_2)\rho_2^{S_2}\right] \mathrm{Tr}\left[G_z \rho_3\right]$$

and the conditional probability (F.1) is

$$P\,(o_1, o_2|i_1, i_2, z) = \int d\lambda\, p(\lambda) P(o_1|i_1, \lambda) P(o_2|i_2, \lambda), \tag{F.3}$$

where $P(o_1|i_1, \lambda) = \mathrm{Tr}\left[E_{o_1}^{\lambda}(i_1)\rho_1^{S_1}\right]$ and $P(o_2|i_2, \lambda) = \mathrm{Tr}\left[F_{o_2}^{\lambda}(i_2)\rho_2^{S_2}\right]$. Since the probability distribution (F.3) satisfies the hypothesis of Bell's theorem [16], it cannot violate any Bell inequality. For an arbitrary separable state, the conditional probability (F.1) will be a convex combination of probabilities of the form (F.3), so it will still respect the hypothesis of Bell's theorem and will not allow any violation of Bell's inequalities.

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