Hints for exercises

Exercise 1.1. Make a change of variable $x = x(y)$ and compute the residue in y-coordinate.

Exercise 1.2. Compute the double integral of the form (1.8) by a direct computation.

Exercise 1.3. Use the properties of integration along the fiber:

$$\int_{\delta h} \omega = \int_h \Big(\int_{\mathbb{S}^1} \omega \Big) = \int_h \Big(2\pi i \operatorname*{Res}_M \omega \Big).$$

Exercise 2.1. The integrals in items 1 and 2 can be computed exactly by the residue formula. Integrals in items 3 and 5 can be computed, if we change coordinates as

$$x = \sqrt{t}x', y = \sqrt{t}y', z = \sqrt{t}z'.$$

In item 4 use coordinate changes $x = tx'$ and $x = t^{3/2}x'$.

Exercise 2.2. The main problem in the proof of Theorem 2.2 is that integration is carried out over a ramified homology class depending on t, i.e., we deal with integration over a domain depending on the parameter. Actually, one can fix this domain in some sense. Namely, for a continuous family of homology classes

$$\gamma(t) \in H_k\big(X \setminus S(t)\big)$$

we can find a cycle c in the complement $X \setminus S(t)$, which represents this homology class for all $t\colon\ |t - t_0| < \varepsilon$ [as such c, we can take any cycle realizing the class $\gamma(t_0)$]. Then we have

$$\int_{\gamma(t)} \omega(t) = \int_c \omega(t).$$

This implies that the integral is a holomorphic function of t by standard theorems on integrals of holomorphic forms.

Exercise 2.3. The hint was given in the text of the exercise.

Exercise 2.4. The statement follows from the fact that the point where $\Sigma = \{x^2 + y^2 = 1\}$ and $L_t = \{y = t\}$ meet has coordinates $x = \sqrt{1 - t^2}$. This gives the desired ramification.

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Exercise 4.1. The desired formula for the R-transform is obtained from the formula for the F-transform if we pass to the affine chart $x^0 = 1$ and note that the set Y, which appears in the definition of the cycle $h(p)$, *must* include the projective plane $x^0 = 0$ as one of its components. This implies that the cycle $h(p)$ does not meet the plane $x^0 = 0$ and the integral defining $F_{x \to p}$ can be computed in the chart $x^0 = 1$.

Exercise 4.2. Use local coordinates.

Exercise 5.1. Use the Cauchy formula with respect to the normal variable to the submanifold to express the derivative in terms of the original function.

Exercise 5.2. It suffices to use formulas for derivatives of parametric integrals from the Appendix to Chapter 2.

Exercise 5.3. The proof is straightforward.

Exercise 7.1. Use local coordinates in the affine chart.

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