Appendix A
Useful Constants and Conversion Factors

$\mathfrak{R} = k_B N_A$	8.314472 J/mol K
k_B	$1.3806505 \times 10^{-23}$ J/K = 8.617339×10^{-5} eV/K
N_A	6.0221415×10^{23}
V_{molar}	22.4141 L/mol @ 273.15 K, 101,325 Pa
M_{air}	28.9644 gm/mol
g	9.80665 m/s^2
$c = (\varepsilon_o \mu_o)^{-1/2}$	2.997925×10^8 m/s
ε_o	$8.854187817620 \times 10^{-12}$ F m^{-1}
μ_o	$4\pi \times 10^{-7}$ H m^{-1}
h	6.626069×10^{-23} J s = 4.135667×10^{-15} eV s
σ	5.6697×10^{-8} W/m^2-K^4 = $[k_B{}^4 \pi^2 / 60\hbar^3 c^2]$
1 psi	6894.8 Pa = 2.307 ft H_2O (1 psf = 47.88 Pa)
1 torr	133.32 Pa (1″ H_2O = 249.1 Pa)
1 ft^3	28.317 L
1 gal	3.785411784 L = 231 in.^{3}
1 gpm	63.088 cm^3/s (1 cfm = 4.72×10^{-4} m^3/s)
1 cal	4.184 J
1 W	3.413 BTU/h
1 ton	3517 W (≡12,000 BTU/h)
1 hp	746 W

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Appendix B
Resonator Quality Factor

"*A man with one watch knows the time; a man with two is never sure*."

The quality factor (Q) of a resonator is a dimensionless measure of the "sharpness" of a resonance. One of the greatest sources of its utility is that we have many equivalent ways of expressing the Q. This variety allows us to connect the most convenient experimental method to the parameter of interest and provides us with a "second watch" if we want to make a self-consistency check of our results, either theoretically or experimentally.

Caution: The results summarized below assume that the resonance under consideration is "isolated," so there are no other resonances that might be sufficiently close in frequency that they would affect the amplitude or phase of the resonance being considered.

Q-multiplier. For a given force F or pressure p, the magnitude of the response at the resonance frequency of the ith mode, ω_i, $i = 0, 1, 2, 3, \ldots \infty$, will be "amplified" by the quality factor Q_i of that mode:

$$Q_i = \left| \frac{p(\omega_i)}{p(\omega = 0)} \right| \tag{B.1}$$

Energy storage-to-dissipation ratio. The time-averaged power dissipated is written in terms of the energy stored and the average power dissipated$<\Pi_{\text{dissipated}}>$ follows:

$$Q = 2\pi \frac{E_{\text{stored}}}{E_{\text{dissipated/cycle}}} = \frac{\omega E_{\text{stored}}}{\langle \Pi_{\text{dissipated}} \rangle_t} \tag{B.2}$$

Lumped elementstorage-to-loss ratio. For a mass–spring system with natural frequency ω_o, mass m, and stiffness s; or a similar electrical circuit with inductance L and capacitance, C, or an acoustical compliance C and acoustical inertance L:

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$$Q = \frac{\omega_o m}{R_m} = \frac{k}{\omega_o R_m} = \frac{1}{\omega_o R_{dc} C} = \frac{\omega_o L}{R_{dc}}$$

Half-power bandwidth. If the frequencies of the −3 dB points are f_+ and f_-, then the full −3 dB bandwidth of the resonance, $\Delta f = f_+ - f_-$, is related to the quality factor, Q, by the resonance frequency, $f_o = (f_+ f_-)^{1/2}$ written as follows:

$$Q = \frac{f_o}{f_+ - f_-} = \frac{\omega_o}{\omega_+ - \omega_-} \quad \text{where} \quad f_o = \sqrt{f_+ f_-} = \frac{\sqrt{\omega_+ \omega_-}}{2\pi} \tag{B.3}$$

Rate of phase change with frequency atresonance. If the phase shift between force (or pressure) and velocity (or volume flow rate) is expressed as ϕ in radians, or θ in degrees, then:

$$Q = \frac{\omega_o}{2} \frac{d\phi}{d\omega}\bigg|_{\omega_o} = \frac{f_o}{2} \frac{d\phi}{df}\bigg|_{f_o} = f_o \frac{\pi}{360^\circ} \frac{d\theta}{df}\bigg|_{f_o} = \frac{f_o}{114.6^\circ} \frac{d\theta}{df}\bigg|_{f_o} \tag{B.4}$$

Free decay rate. If the time required for the amplitude of the oscillations to decay to $e^{-1} \cong 0.368$ of their value is $\tau = \beta^{-1} = (2m)/R_m$, then,

$$Q = \frac{1}{2} \omega_o \tau = \pi \tau f_o \tag{B.5}$$

Similarly, the Q is expressed as 2π times the number of cycles required for the ***energy*** to decay by e^{-1}, or π times the number of cycles required for the ***amplitude*** to decay by e^{-1}. More generally,

$$Q = \frac{\pi N}{\ln[x]} \tag{B.6}$$

N is the number of cycles for the amplitude to decay by a factor of x.

Reflection coefficient. In a standing-wave resonator, the standing wave can be represented as the superposition of a right- and left-going traveling waves. If the left-going wave is reflected with an amplitude that is $R < 1$ times the right-going wave amplitude, the coefficient of the right-going wave would be given by the infinite geometric series $1 + R + R^2 + R^3 + \cdots$ and the left-going wave would have an amplitude that is R times that infinite sum. The resulting quality factor, Q_n, of the nth mode of the resonator can be expressed in terms of the reflection coefficient R and the mode number n:

$$Q_n = n\pi \frac{\sqrt{R}}{1 - R} \tag{B.7}$$

Pole-zero resonance fit. Many modern spectrum analyzers allow a resonance to be fit by a pole-zero function. A single resonance will have two complex poles that are complex conjugates, $a \pm jb$. The resonance frequency is $f = (a^2 + b^2)^{1/2} \cong b$, if the damping is small ($a \ll b$).

$$Q = \frac{-1}{2a}\sqrt{a^2 + b^2} \cong \frac{-b}{2a} \tag{B.8}$$

Loss tangent and damping factor. In the characterization of elastomers used as vibration isolators, it is common to define a frequency dependent, complex elastic modulus E^*. The complex modulus has a real part E' and an imaginary part E'' such that $E^* = E' + jE'' \cong E(1 + j\delta)$, where we choose to define a "loss tangent," tan δ, that is the inverse of the quality factor.[1]

$$Q = \frac{1}{\tan \delta} = \frac{E'}{E''} = \frac{1}{2\zeta} \tag{B.9}$$

The damping factor, ζ, is the ratio of the mechanical resistance to the critical value of the mechanical resistance, $R_m^{\text{crit}} = 2(km)^{1/2} = 2m\omega_o$.[2]

[1]For example, see J.C. Snowdon, *Vibration and Shock in Damped Mechanical Systems* (Wiley, 1968).

[2]W.T. Thomson, *Theory of Vibration with Applications*, 2nd ed. (Prentice-Hall, 1981); ISBN 0-13-914523-0.

Appendix C
Bessel Functions of the First Kind

Bessel's equation:

$$x^2 \frac{d^2 J_m(x)}{dx^2} + x \frac{d\, J_m(x)}{dx} + (x^2 - m^2) J_m(x) = 0 \tag{C.1}$$

$$\frac{1}{x}\frac{d}{dx}\left(x \frac{dJ_m(x)}{dx}\right) + \left(1 - \frac{m^2}{x^2}\right) J_m(x) = 0 \tag{C.2}$$

Series expansions:

$$J_m(x) = \frac{1}{m!}\left(\frac{x}{2}\right)^m - \frac{1}{1!(m+1)!}\left(\frac{x}{2}\right)^{m+2} + \frac{1}{2!(m+2)!}\left(\frac{x}{2}\right)^{m+4} - \cdots \tag{C.3}$$

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \tag{C.4}$$

$$J_1(x) = \frac{x}{2} - \frac{2x^3}{2 \cdot 4^2} + \frac{3x^5}{2 \cdot 4^2 \cdot 6^2} - \cdots \tag{C.5}$$

$$J_2(x) = \frac{x^2}{2 \cdot 2^2} - \frac{x^4}{2 \cdot 3 \cdot 2^4} + \frac{x^6}{2 \cdot 3 \cdot 4 \cdot 2^6} - \cdots \tag{C.6}$$

Asymptotic forms for large argument:

$$\lim_{x \to \infty} [J_1(x)] = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \tag{C.7}$$

Addition theorem:

$$1 = J_0(x) + 2J_2(x) + 2J_4(x) + 2J_6(x) + \cdots \tag{C.8}$$

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Relationships to trigonometric functions:

$$\sin\ x = 2J_1(x) - 2J_3(x) + 2J_5(x) - 2J_7(x) + \cdots \tag{C.9}$$

$$\cos\ x = J_0(x) - 2J_2(x) + 2J_4(x) - 2J_6(x) + \cdots \tag{C.10}$$

$$\cos(x\ \cos\ \theta) = J_0(x) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(x)\cos(2k\theta) \tag{C.11}$$

$$\cos(x\ \sin\ \theta) = J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\ \cos\ (2k\theta) \tag{C.12}$$

$$\sin(x\ \sin\ \theta) = 2\sum_{k=0}^{\infty} J_{2k+1}(x)\ \sin\ [(2k+1)\theta] \tag{C.13}$$

$$\sin(x\cos\theta) = 2\sum_{k=1}^{\infty}(-1)^k J_{2k+1}(x)\ \cos[(2k+1)\ \theta] \tag{C.14}$$

Integral representations:

$$J_m(x) = \frac{(x/2)^m}{\sqrt{\pi}\ \Gamma(m+1/2)}\int_0^{\pi}\cos(x\cos\theta)\,d\theta \tag{C.15}$$

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\ \sin\theta)\ d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(x\ \cos\theta)\,d\theta \tag{C.16}$$

$$J_m(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\ \sin\theta\ - m\theta)\,d\theta \tag{C.17}$$

Recurrence relations:

$$J_{m-1}(x) + J_{m+1} = \frac{2m}{x}J_m(x) \tag{C.18}$$

$$J_{m-1}(x) - J_{m+1}(x) = 2\frac{dJ_m(x)}{dx} \tag{C.19}$$

$$\frac{dJ_m(x)}{dx} = J_{m-1}(x) - \frac{m}{x}J_m(x) \tag{C.20}$$

$$\frac{dJ_m(x)}{dx} = -J_{m+1}(x) + \frac{m}{x}J_m(x) \quad ((C.21))$$

Derivatives:

$$\frac{dJ_m(x)}{dx} = \frac{1}{2}\left[J_{m-1}(x) - J_{m+1}(x)\right] \quad (C.22)$$

$$\frac{dJ_0(x)}{dx} = -J_1(x) \quad (C.23)$$

$$\frac{d}{dx}\left[x^m J_m(x)\right] = x^m J_{m-1}(x) \quad (C.24)$$

$$\frac{d}{dx}\left[x^{-m} J_m(x)\right] = -x^{-m} J_{m+1}(x) \quad (C.25)$$

Integrals:

$$\int J_1(x)\, dx = -J_0(x) \quad (C.26)$$

$$\int xJ_0(x)\, dx = xJ_1(x) \quad (C.27)$$

$$\int x^{p+1} J_p(x)\, dx = x^{p+1} J_{p+1}(x) \quad (C.28)$$

$$\int J_0^2(x)\, x\, dx = \frac{x^2}{2}\left[J_0^2(x) + J_1^2(x)\right] \quad (C.29)$$

$$\int J_m^2(x)\, x\, dx = \frac{x^2}{2}\left[J_m^2(x) - J_{m-1}(x)J_{m+1}(x)\right] \quad (C.30)$$

$$\int J_m(ax)J_m(bx)\, x\, dx = \frac{x}{a^2 - b^2}\left[bJ_m(ax)J_{m-1}(bx) - aJ_m(bx)J_{m-1}(ax)\right] \quad (C.31)$$

Roots of Bessel functions (15 digits)

The nth roots of $J_m(x) = 0$

$m \backslash n$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$m=0$	2.40482555769577	5.52007811028631	8.65372791291101	11.7915344390142	14.9309177084877
$m=1$	3.83170597020751	7.01558666981561	10.1734681350627	13.3236919363142	16.4706300508776
$m=2$	5.13562230184068	8.41724414039986	11.6198411721490	14.7959517823512	17.9598194949878
$m=3$	6.38016189592398	9.76102312998166	13.0152007216984	16.2234661603187	19.4094152264350
$m=4$	7.58834243450380	11.0647094885011	14.3725366716175	17.6159660498048	20.8269329569623
$m=5$	8.77148381595995	12.3386041974669	15.7001740797116	18.9801338751799	22.2177998965612
$m=6$	9.93610952421768	13.5892901705412	17.0038196678160	20.3207892135665	23.5860844355813
$m=7$	11.0863700192450	14.8212687270131	18.2875828324817	21.6415410198484	24.9349278876730
$m=8$	12.2250922640046	16.0377741908877	19.5545364309970	22.9451731318746	26.2668146411766
$m=9$	13.3543004774353	17.2412203824891	20.8070477892641	24.2338852577505	27.5837489635730
$m=10$	14.4755006865545	18.4334636669665	22.0469853646978	25.5094505541828	28.8873750635304

The nth roots of $J_{m'}(x) = 0$

$m \backslash n$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$m=0$	3.83170597020751	7.01558666981561	10.1734681350627	13.3236919363142	16.4706300508776
$m=1$	1.84118378134065	5.33144277352503	8.53631636634628	11.7060049025920	14.8635886339090
$m=2$	3.05423692822714	6.70613319415845	9.96946782308759	13.1703708560161	16.3475223183217
$m=3$	4.20118894121052	8.01523659837595	11.3459243107430	14.5858482861670	17.7887478660664
$m=4$	5.31755312608399	9.28239628524161	12.6819084426388	15.9641070377315	19.1960288000489
$m=5$	6.41561637570024	10.5198608737723	13.9871886301403	17.3128424878846	20.5755145213868
$m=6$	7.50126614468414	11.7349359530427	15.2681814610978	18.6374430096662	21.9317150178022
$m=7$	8.57783648971407	12.9323862370895	16.5293658843669	19.9418533665273	23.2680529264575
$m=8$	9.64742165199721	14.1155189078946	17.7740123669152	21.2290626228531	24.5871974863176
$m=9$	10.7114339706999	15.2867376673329	19.0045935379460	22.5013987267772	25.8912772768391
$m=10$	11.7708766749555	16.4478527484865	20.223031416817	23.7607158603274	27.1820215271905

Appendix D
Trigonometric Functions

Euler's formula

$$e^{jx} = \cos x + j \sin x \tag{D.1}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \qquad \sin x = \frac{e^{jx} - e^{-jx}}{2j} \tag{D.2}$$

Addition and subtraction

$$\cos \alpha = \pm \sin \left(\alpha \pm \frac{\pi}{2}\right) \qquad \sin \alpha = \pm \cos \left(\alpha \mp \frac{\pi}{2}\right) \tag{D.3}$$

$$\sin^2\alpha + \cos^2\alpha = 1 \tag{D.4}$$

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \tag{D.5}$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \tag{D.6}$$

$$\sin \alpha \pm \sin \beta = 2 \sin \left(\frac{\alpha \pm \beta}{2}\right) \cos \left(\frac{\alpha \mp \beta}{2}\right) \tag{D.7}$$

$$\cos \alpha + \cos \beta = 2 \cos \left(\frac{\alpha + \beta}{2}\right) \cos \left(\frac{\alpha - \beta}{2}\right) \tag{D.8}$$

$$\cos \alpha - \cos \beta = -2 \sin \left(\frac{\alpha + \beta}{2}\right) \sin \left(\frac{\alpha - \beta}{2}\right) \tag{D.9}$$

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Products and powers

$$\cos\alpha\cos\beta = {}^1\!/\!_2[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \tag{D.10}$$

$$\sin\alpha\sin\beta = -{}^1\!/\!_2[\cos(\alpha+\beta) - \cos(\alpha-\beta)] \tag{D.11}$$

$$\sin\alpha\cos\beta = {}^1\!/\!_2[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \tag{D.12}$$

$$\sin^2\alpha = {}^1\!/\!_2(1 - \cos 2\alpha) \tag{D.13}$$

$$\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha) \tag{D.14}$$

$$\cos^2\alpha = {}^1\!/\!_2(1 + \cos 2\alpha) \tag{D.15}$$

$$\cos^3\alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha) \tag{D.16}$$

Appendix E
Hyperbolic Functions

$$e^x = \cosh x + j \sinh x \tag{E.1}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2} \tag{E.2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \quad \coth x = \frac{1}{\tanh x} \tag{E.3}$$

Series expansions

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{E.4}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \tag{E.5}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \tag{E.6}$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \quad \left(x^2 < \frac{\pi^2}{4}\right) \tag{E.7}$$

$$\sinh^{-1} x = x - \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots \quad \left(x^2 < 1\right) \tag{E.8}$$

$$\cosh^{-1} x = \ln(2x) - \frac{1}{2}\frac{1}{2x^2} - \frac{1 \cdot 3}{2 \cdot 4}\frac{1}{4x^4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{1}{6x^6} - \cdots \quad \left(x^2 < 1\right) \tag{E.9}$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \quad \left(x^2 < 1\right) \tag{E.10}$$

Addition and subtraction

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$$\cosh^2\alpha - \sinh^2\alpha = 1 \tag{E.11}$$

$$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \cosh\alpha\sinh\beta \tag{E.12}$$

$$\cosh(\alpha \pm \beta) = \cosh\alpha\cosh\beta \pm \sinh\alpha\sinh\beta \tag{E.13}$$

$$\tanh(\alpha \pm \beta) = \frac{1 + \tanh\alpha\tanh\beta}{\tanh\alpha + \tanh\beta} = \frac{\sinh 2\alpha \pm \sinh 2\beta}{\cosh 2\alpha + \cosh 2\beta} \tag{E.14}$$

$$\sinh\alpha \pm \sinh\beta = 2\sinh\left(\frac{\alpha \pm \beta}{2}\right)\cosh\left(\frac{\alpha \mp \beta}{2}\right) \tag{E.15}$$

$$\cosh\alpha + \cosh\beta = 2\cosh\left(\frac{\alpha + \beta}{2}\right)\cosh\left(\frac{\alpha - \beta}{2}\right) \tag{E.16}$$

$$\cosh\alpha - \cosh\beta = 2\sinh\left(\frac{\alpha + \beta}{2}\right)\sinh\left(\frac{\alpha - \beta}{2}\right) \tag{E.17}$$

Products and powers

$$\sinh\alpha\sinh\beta = \tfrac{1}{2}[\cosh(\alpha + \beta) - \cosh(\alpha - \beta)] \tag{E.18}$$

$$\cosh\alpha\cosh\beta = \tfrac{1}{2}[\cosh(\alpha + \beta) + \cosh(\alpha - \beta)] \tag{E.19}$$

$$\sinh\alpha\cosh\beta = \tfrac{1}{2}[\sinh(\alpha + \beta) + \sinh(\alpha - \beta)] \tag{E.20}$$

$$\sinh 2\alpha = 2\sinh\alpha\cosh\alpha \tag{E.21}$$

$$\sinh 3\alpha = 3\sinh\alpha + 4\sinh^3\alpha \tag{E.22}$$

$$\cosh 2\alpha = 2\cosh^2\alpha - 1 \tag{E.23}$$

$$\cosh 3\alpha = 4\cosh^3\alpha - 3\cosh\alpha \tag{E.24}$$

$$\sinh^2\alpha = \tfrac{1}{2}(\cosh 2\alpha - 1) \tag{E.25}$$

$$\sinh^3\alpha = \frac{1}{4}(\sinh 3\alpha - 3\sinh\alpha) \tag{E.26}$$

$$\cosh^2\alpha = \tfrac{1}{2}(\cosh 2\alpha - 1) \tag{E.27}$$

$$\cosh^3\alpha = \frac{1}{4}(\cosh 3\alpha + 3\cosh\alpha) \tag{E.28}$$

Functions of complex arguments

$$\begin{aligned} \sin jx &= \cosh x \quad \sinh jx = j\sin x \\ \sin x &= -j\sinh jx \quad \sinh x = -j\sin jx \end{aligned} \tag{E.29}$$

$$\begin{aligned} \cos jx &= \cosh x \quad \cosh jx = \cos x \\ \cos x &= \cosh jx \quad \cosh x = \cos jx \end{aligned} \tag{E.30}$$

$$\tan jx = -j\tanh jx \quad \tanh jx = -j\tan jx \tag{E.31}$$

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